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STONE'S CONJECTURE ON FAIR-RETURN PROCESSES: COUNTER EXAMPLE AND RECTIFICATION

by

Prem Prakash

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Taking a discounted-dividend stock valuation model in which the appropriate fair-return for any given future period is itself a random variable, Stone [1975] arrives at a necessary and sufficient condition for stock prices to conform to a fair-return process. The present paper constructs a simple counter-example to show that Stone's condition is, in general, neither necessary nor sufficient as claimed. It is shown that the condition is necessary and sufficient iff (i) the appropriate fair returns for every period are non-random as in Samuelson [1973], or (ii) the expected values of future dividends remain constant under the stochastic process -- a case of limited interest.

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1. INTRODUCTION

Samuelson [1973] shows: If we hypothesize that a stock's present price is set at the expected discounted value of its future dividends, where the future dividends are supposed to be random varaibles generated according to any general (but known) stochastic process, then the stock prices themselves have a martingale property. In arriving at this result, the discount factors are allowed to be possibly different for different periods; but, for each period, the factors are posited to remain constant regardless of the successive realizations of the stochastic process generating the probabilities of the future dividends.

Stone [1975] argues, and rightly so, that, since new information can also influence both perceived risk and required return, constant discount factors are implausible.[1] Accordingly, he considers an extension of the Samuelson model. He allows the discount factors themselves to be random varaibles, thus, allowing for the possibility that period-to-period changes in stock prices are influenced both by shifts in required returns and by shifts in expected dividends. Stone, then, arrives at the conclusion that the stock values constitute a fair-return process if and only if <u>the effective current discount factor over any future time</u>

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interval equals the expected value of the discount factor for the interval given only the current data.[2]

The present paper constructs a simple counter example showing that Stone's condition is, in general, neither necessary nor sufficient as claimed; it also traces where an invalid argument occurs in Stone's [1975] proof. The paper shows that what is correct, however, is that the condition is necessary and sufficient iff (i) the appropriate fair returns for every period are non-random, or (ii) the expected values of the future dividends remain constant under the stochastic process. Case (i) is the same as that of Samuelson [1973]; and Case (ii), while exhibiting a pleasing symmetry to Case (i), is of limited interest.

2. REVIEW

For convenience of cross reference, we will use the same notation as in Stone [1975].

Let $\ldots, Q_{-1}, Q_0, Q_1, \ldots$ be a random vector sequence, to be interpreted as the time sequence of all publicly available information. Time is measured designating the current time as zero.

For any time n, the entire random vector sequence $\ldots, Q_{n-1}, Q_n$ is denoted as $\underline{Q}_n$; when Q_n is still in the future, the time until its realization is stated explicitly immediately after the variable, for instance, thus: $(Q_n, n - m)$; and, for any $T > n + 1$, the joint probability distribution of $(Q_{n+1}, \ldots, Q_{T-1}, Q_T)$ conditional on $\underline{Q}_n$ is denoted as

(1) $$P(Q_T, T - n; Q_{T-1}, T - n - 1; \ldots; Q_{n+1}, 1|\underline{Q_n}) = \text{Prob}\{q_T \leqslant Q_T;\ q_{T-1} \leqslant Q_{T-1}; \ldots; q_{n+1} \leq Q_{n+1}|\underline{Q_n}\}.$$

The marginal distribution of Q_T conditional on $\underline{Q_n}$ is, then, given by

(2) $$P(Q_T, T - n|\underline{Q_n}) = \int_{Q_{T-1}} \cdots \int_{Q_{n+1}} P(dQ_T, T - n; dQ_{T-1}, T - n - 1; \ldots; dQ_{n+1}, 1|\underline{Q_n}),$$

where the integrals are Stieltjes or Lebesgue.

Let $V(n|\underline{Q_n})$ be the value of a common stock at time n, given $\underline{Q_n}$. Posit that the stock value at any time n is given by the present value of expected future dividends, i.e.,

(3) $$V(n|\underline{Q_n}) = \sum_{t=n+1}^{\infty} \left[\prod_{i=n+1}^{t} \lambda_i^{-1}(n|\underline{Q_n})\right] E[D_t|\underline{Q_n}],$$

where $E[D_t|\underline{Q_n}]$ is the expected value of the dividend at time t conditional on $\underline{Q_n}$, and $\lambda_i(n|\underline{Q_n})$ is the discount factor for the time interval $[i-1, i]$ $(i > n)$ conditional on $\underline{Q_n}$.

The stock values are said to constitute a "fair return process" iff the expected change in value for any period gives the current fair return for that period, i.e., iff

(4) $$E[V(n|\underline{Q_n})|\underline{Q_0}] + E[D_n|\underline{Q_0}] = \lambda_n(0|\underline{Q_0})E[V(n - 1|\underline{Q_{n-1}})|\underline{Q_0}],$$

where $E[V(n|\underline{Q_n})|Q_0]$ is the expected value of $V(n|\underline{Q_n})$ conditional only on current data $\underline{Q_0}$.

We may now recapitulate the following claim of Stone [1975]:

<u>Theorem</u> (Stone [1975]):[3] Let the probability distributions be stationary (in the sense that their functional form depends only on available information and the length of time intervals), and let the stock values at all times be given in accord with (3). Then, the stock values conform to a fair-return process if and only if, for all time intervals $[n, t]$ with $n \geq 0$,

$$\text{(5)} \qquad \left[\prod_{i=n+1}^{t} \lambda_i^{-1}(0|\underline{Q_0})\right] P(dQ_t,t|\underline{Q_0}); = \int_{Q_n} \cdots \int_{Q_1} \left[\prod_{i=n+1}^{t} \lambda_i^{-1}(n|\underline{Q_n})\right] P(dQ_t,t;\, dQ_n,n;\, \ldots;\, dQ_1,1|\underline{Q_0})$$

By integrating both sides of (5) over all values of Q_t, Stone [1975] obtains the following as an implication of (5):

$$\text{(6)} \qquad \prod_{i=n+1}^{t} \lambda_i^{-1}(0|\underline{Q_0}) = \int_{Q_n} \cdots \int_{Q_1} \left[\prod_{i=n+1}^{t} \lambda_i^{-1}(n|\underline{Q_n})\right] P(dQ_n,n;\, \ldots;\, dQ_1,1|\underline{Q_0}).$$

3. A COUNTER-EXAMPLE

To keep things simple, take $n = 1$ and $t = 2$. Furthermore, assume that dividends $D_1 = D_T \equiv 0$ for all $T \geq 3$, so that the only dividend flow is at time $t = 2$ -- as, for example, in the case of a futures contract which matures at time $t = 2$. We, thus, get rid of the summation sign in (3), leaving

(7)
$$V(0|\underline{Q}_0) = \lambda_1^{-1}(0|\underline{Q}_0)\ \lambda_2^{-1}(0|\underline{Q}_0)\ E[D_2|\underline{Q}_0]$$

$$V(1|\underline{Q}_1) = \lambda_2^{-1}(1|\underline{Q}_1)\ E[D_2|\underline{Q}_1]$$

Now, let the stochastic process generating the probability distributions for D_2 be as described by the probability tree of Figure 1.

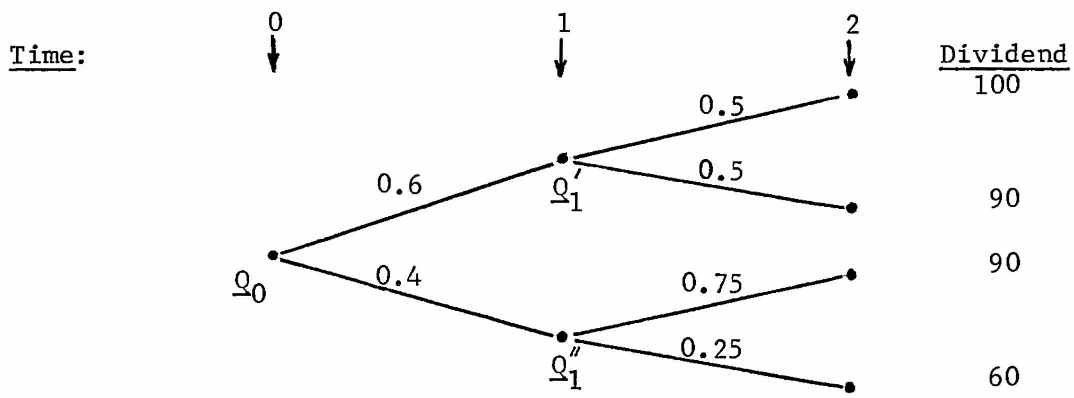

Figure: 1

Then, the following values are readily computed:

$$E[D_2|\underline{Q}_0] = 90.0 \qquad E[D_2|Q_1'] = 95.0 \qquad E[D_2|Q''] = 82.5$$

$$V(0|\underline{Q}_0) = \lambda_1^{-1}(0|\underline{Q}_0)\ \lambda_2^{-1}(0|\underline{Q}_0)\ (90.0)$$

$$V(1|\underline{Q}_1') = \lambda_2^{-1}(1|\underline{Q}_1')\ (95.0)$$

$$V(1|\underline{Q}_1'') = \lambda_2^{-1}(1|\underline{Q}_1'')\ (82.5)$$

Next, recalling $D_1 = 0$, use (4) to note that the stock values constitute a fair return process iff:

$$(0.6 \times 95.0)\ \lambda_2^{-1}(1|\underline{Q}_1') + (0.4 \times 82.5)\ \lambda_2^{-1}(1|\underline{Q}_1'') = 90\ \lambda_2^{-1}(0|\underline{Q}_0)\ .$$

(8) $$\lambda_2^{-1}(0|\underline{Q}_0) = 0.633\ \lambda_2^{-1}(1|\underline{Q}_1') + 0.367\ \lambda_2^{-1}(1|\underline{Q}_1'')$$

Stone's condition (5), on the other hand, implies (6), i e.,

(9) $$\lambda_2^{-1}(0|\underline{Q}_0) = \int_{Q_1} \lambda^{-1}(1|Q_1)\ P(dQ_1,\ 1|\underline{Q}_0)$$
$$= 0.6\ \lambda_2^{-1}(1|\underline{Q}_1') + 0.4\ \lambda_2^{-1}(1|Q_1'')$$

Finally, note that, if (5) holds, then (9) is true, which is not the same as (8), so that stock values do not constitute a fair return process. Conversely, if (8) holds, then (5) cannot be true for, if it were, then (9) would have to be true. This completes the counter example showing that Stone's condition (5) is neither sufficient nor necessary for the stock values (3) to constitute a fair return process (4).

4. A REVIEW OF THE ORIGINAL PROOF

We now pinpoint where an invalid argument is made in the proof of the claimed result in Stone [1975]. By the stock valuation postulate (3) ,

(10) $$E[V(n|\underline{Q}_n)|\underline{Q}_0]$$
$$= \int_{Q_n} \cdots \int_{Q_1} V(n|\underline{Q}_n)P(dQ_n,n;\ \ldots\ ;dQ_1,1|\underline{Q}_0)$$
$$= \sum_{t=n+1}^{\infty} \int_{Q_n} \cdots \int_{Q_1} \prod_{i=n+1}^{t} \lambda_i^{-1}(n|\underline{Q}_n)$$
$$\left[\int_{Q_t} D_t P(dQ_t,t-n|\underline{Q}_n)\right] P(dQ_n,n;\ \ldots\ ;dQ_1,1|\underline{Q}_0)\ .$$

(14)
$$E[V(n|\underline{Q_n})|\underline{Q_0}] = \lambda_n(0|\underline{Q_0})E[V(n-1|\underline{Q_{n-1}})|\underline{Q_0}] - E[D_n|\underline{Q_0}]$$
$$= \sum_{t=n+1}^{\infty}\left[\prod_{i=n+1}^{t}\lambda_i^{-1}(0|\underline{Q_0})\right]E[D_t|\underline{Q_0}]$$

$$= \sum_{t=n+1}^{\infty}\left[\int_{Q_n}\cdots\int_{Q_1}\left[\prod_{i=n+1}^{t}\lambda_i^{-1}(n|\underline{Q_n})\right]\right.$$
$$\left.P(dQ_n,n;\ldots;dQ_1,1|\underline{Q_0})\right]\int_{Q_t}D_tP(dQ_t,t|\underline{Q_0})$$

The question is, when is it that the last line of (14) equals the last line of (10) for arbitrary times n and t, and arbitrary time remaining until the realization corresponding to n?

The answer is quite straightforward. The two expressions must be identical, i.e., it should be possible to factorize the last line of (10) in the form of the last line of (14). This is possible if and only if (15) or (16) are true.

(15)
$$\lambda_i^{-1}(n|\underline{Q_n}) = \lambda_i^{-1}(n-1|\underline{Q_{n-1}}) = \ldots = \lambda_i^{-1}(0|\underline{Q_0})$$

(16)
$$E[D_t|\underline{Q_n}] = E[D_t|\underline{Q_{n-1}}] = \ldots = E[D_t|\underline{Q_0}]$$

If (15) is true, then the product of the discount factors can be factored out of the integrals in (10); and, what remains is readily recognized as $E[D_t|\underline{Q_0}]$. If, on the other hand, (16) is true, then we simply factor $\int_{Q_t} D_tP(dQ_t, t-n|\underline{Q_n}) = E[D_t|\underline{Q_n}]$ out of the remaining n integrals in (10), setting $E[D_t|\underline{Q_n}] = E[D_t|\underline{Q_0}]$. The converse is straightforward.

We have just proven the following

<u>THEOREM</u>: Posit that (I) the random variables (future dividends and discount

Fubini's theorem now allows us to write the following for the right hand term of equation (10):

(11)
$$\sum_{t=n+1}^{\infty}\int_{Q_t}\int_{Q_n}\cdots\int_{Q_1}\left[\prod_{i=n+1}^{t}\lambda_i^{-1}(n|\underline{Q_n})\right] D_t \; P(dQ_t,t;\, dQ_n,n;\, \ldots\,;\, dQ_1,1|\underline{Q_0}).$$

But, in general, D_t cannot validly be placed outside the first n period integrals as in the following statement[4] of Stone [1975]:

(12)
$$\sum_{t=n+1}^{\infty}\int_{Q_t} D_t \left[\int_{Q_n}\cdots\int_{Q_1}\left[\prod_{i=n+1}^{t}\lambda_i^{-1}(n|\underline{Q_n})\right] P(dQ_t,t;\, dQ_n,n;\, \ldots\,;\, dQ_1,1|\underline{Q_0})\right]$$

This, then, is where an invalid argument is made. What is valid, if D_t needs to be placed outside the first n integrals, is the following:

(13)
$$\sum_{t=n+1}^{\infty}\int_{Q_t} D_t \left[\int_{Q_n}\cdots\int_{Q_1}\left[\prod_{i=n+1}^{t}\lambda_i^{-1}(n|\underline{Q_n})\right] P(dQ_n,n;\, \ldots\,;\, dQ_1,1|Q_t,\underline{Q_0})\right] P(dQ_t,t|\underline{Q_0}),$$

which does not admit a substitution using (6).

5. SOME FURTHER RESULTS

We now ask the question, when is (6) a necessary and sufficient condition for the stock values (10) to constitue a fair return process (4)? In other words, using (6), rewrite (4) as follows factors) are generated according to any general, but known stochastic process; (II) the stock values are determined by present discounting the expected future dividends according to (3). Then, for the stock values to constitute a fair returns process (4) such that it is characterized by (6), it is necessary and sufficient that

1. The discount factors for all future periods remain constant under the stochastic process, i.e., (15) is true,

   OR

2. The expected values of dividends at every future time remain constant under the stochastic process, i.e., (16) is true.

The possibility 1. in the above theorem constitutes a restriction on human economic behavior and, so, is interesting. Of course, it is the case studied by Samuelson [1973].

The possibility 2., on the other hand, constitutes a restriction on the dividend generating process, corresponding to the restriction (6) on human economic behavior. This result has a pleasing symmetry with respect to 1. In 1. (or in Samuelson [1973]), the discount factors remain constant, while the expected values of future dividends may change. In 2., on the other hand, the expected values of future dividends remain constant, while the discount factors may change. In the second case also, if the stock values are set at the present discounted expected values of future dividends, then the stock values constitute a fair return process when (and only when) the effective current discount factors for all periods are properly anticipated in the sense of (6).

Possibility 2. being a restriction on the dividend generating process, is of limited interest only. What is much more interesting a question is the following: Let the economic process generating future dividends be unrestricted. Allow the discount factors as well as the dividends to be random variables generated by any general (but known) stochastic process. What restriction on human economic behavior would suffice for the stock prices to constitute a fair return process?

The answer to this question was obtained in Prakash [1974]: For the stock prices to constitute a fair return process when both the dividends and discount factors are random variables, it is sufficient that (i) the market risk preferences are consistent with the von Neumann-Morgenstern axioms, and (ii) the stock values are set at the discounted value of the certainty equivalents of future dividends -- the discounting being done at non-random discount rates (to be interpreted as the market risk-free rates), which may possibly be different for different periods.

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3. SAMUELSON, P.A., "Proof That Properly Discounted Present Values of Assets Vibrate Randomly," The Bell Journal of Economics and Management Science, Vol. 4, No. 2 (Autumn 1973), pp. 369-374

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FOOTNOTES

1. A similar argument is made by Prakash [1974], "Proof That Properly Anticipated Certainty Equivalents Fluctuate Randomly," Discussion Paper No. 113, Nov. 1974 (revised June 1975), The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Ill.

2. This is our interpretation of Stone's [1975] formal statements (2) and (4); it differs somewhat from Stone's. In any case, Stone's formal statements are faithfully reproduced as our (5) and (6).

3. We have put together in this one statement both the sufficiency and necessity claims of Stone [1975], making barest possible changes in the original enunciation of the theorem.

4. See the next to last line of his (5), which follows exactly the same two relations as our (10).