Discussion Paper No. 24

POISSON PROCESS AND

DISTRIBUTION-FREE STATISTICS, 1

by

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1. **INTRODUCTION**

Suppose that \( Y(t), \ 0 \leq t < \ 
\), is a Poisson process with stationary increments and parameter \( \lambda \). Let \( T \) equal the time \( t \) at which \( Y(t) = t \) for the last time. That is

\[
T = \sup \{ t : Y(t) = t \}
\]

The random variable \( T \) assumes the values \( 0, 1, \ldots, \infty \). We have the following basic facts which are easily verified:

\[
P(Y(t) < t, \text{ all } t > 0) = P(T = 0) = \begin{cases} 1 - \lambda, & \text{if } \lambda < 1 \\ 0, & \text{otherwise} \end{cases}
\]

(See Appendix A)

\[
P(T < \infty) = \begin{cases} 1 & \text{if } \lambda < 1 \\ 0 & \text{otherwise} \end{cases}
\]

\[
P(T = k) = \frac{(\lambda k)^k e^{-\lambda}}{k!} \quad (1 - \lambda), \ k = 0, 1, \ldots \quad \text{if } \lambda < 1
\]

Let \( U \) be a function of \( Y(t) \) with the following property:

The value of \( U \) is completely determined by the function \( Y(t) \) for \( 0 \leq t \leq T \).

(1.3)

Denote

\[
g_n(t) = \frac{Y(n t)}{n}, \quad 0 \leq t \leq 1.
\]

Subject to the condition that \( T = n \), the behavior of \( g_n(t) \) is exactly that
to see that $N_n$ is "distribution free". That is the distribution of $N_n$ does not depend on the form of the cdf $F$, where $F(u) = P(X_t < u)$, as long as $F$ is continuous. We assume here that $F(u) = u$, $0 \leq u \leq 1$.

It is obvious that $N$ is geometrically distributed when $\lambda < 1$. Specifically,

$$P(N \geq k) = \lambda^k, \quad k = 0, 1, \ldots$$  \hspace{1cm} (1.7)

The counterpart of equation (1.4) is

$$P(N \geq k) = \sum_{n=0}^{\infty} P(N_n \geq k) \frac{n^n}{n!} \left(\lambda e^{-\lambda}\right)^n (1 - \lambda)$$

It will be seen in Appendix C that

$$\frac{k}{(1 - \lambda)} = \sum_{n=0}^{\infty} \frac{n^{n-k}}{(n-k)!} (\lambda e^{-\lambda})^n, \quad k = 0, 1, \ldots$$

(This follows from equation (C.1) with $u = 0$.)

Hence the counterpart of the relationship (1.4) is

$$P(N_n \geq k) = \frac{n^{-1}}{(n-k)^{n-k}} \frac{n!}{n^n} \frac{k}{(n-k)!}$$

By a simple application of Stirling's formula, it follows that

$$\lim_{n \to \infty} P\left(\frac{N_n}{n} \geq t\right) = e^{-t^2/2}$$  \hspace{1cm} (1.8)

a result which was first proved by Smirnov [8].

We close this introduction by considering a second example.

*Example 2* Let

$L$ = number of ladder points of $Y(t) - t$, $t \geq 0$. That is $L$ equals the number of times that $Y(t) - t$ achieves positive maxima which exceed all preceding maxima. (See Figure 1.)
Similarly we let \( L_n \) denote the number of ladder points in the empirical cdf \( F_n \).  \( L \) is also geometrically distributed with

\[
P(L \geq k) = \lambda^k
\]

Hence it follows that

\[
P(N_n \geq k) = P(L_n \geq k) = \frac{n!}{(n-k)!} \lambda^k
\]

and

\[
\lim_{n \to \infty} P\left( \frac{N_n}{n} \geq t \right) = e^{-t^2/2}
\]

2. **ONE-SIDED MAXIMA, FIRST APPROACH**

We define

\[
M = \sup_{t \geq 0} Y(t) - t
\]

which is the maximum exceedance of the Poisson process \( Y(t) \) above the straight line \( t \). Suppose that \( \lambda < 1 \). If \( M \geq u \), (where \( u \) is an arbitrary positive number not necessarily an integer) this means that \( Y(\cdot) \) intersects the straight line \( t + u \), necessarily only a finite number of times. The probability that such an intersection last occurs at height \( n \) is

\[
\left( \frac{(n-u)!}{n!} \right) e^{-\lambda} (n-u) (1-\lambda).
\]

Hence

\[
P(M \geq u) = \sum_{n=u}^{\infty} \left( \frac{(n-u)!}{n!} \right) e^{-\lambda} (n-u) (1-\lambda)
\]

\[
\]

From now on throughout this paper it will be assumed that \( F_n \) is the empirical cdf of \( n \) independent random variables, each uniformly distributed on \([0,1]\).
\[ P(N_n \geq u) = \sum_{i+j=n} u^{i,j} \frac{(1-u)^i}{n^i} \frac{(i+u)^j}{n^j} \]  

(2.5)

Equations (2.4), (2.5) are well-known. See [1], [2], [3], [4]. It is easy to give a direct proof of (2.4) which is analogous to that of (2.1) using the relation (8.1) in Appendix B. The argument goes as follows: \( M_n \geq u \) means that \( F_n(t) \) crosses the line \( t = \frac{u}{n} \) somewhere. The probability of a last crossing at height \( i/n \) is the binomial probability

\[ \binom{n}{i} \left( \frac{1}{n} - \frac{u}{n} \right)^i \left( 1 - \frac{1}{n} + \frac{u}{n} \right)^{n-i} \]

times the conditional probability that the part of \( F_n(t) \) traversing between \( \left( \frac{i}{n} - \frac{u}{n}, \frac{i}{n} \right) \) and \( (1,1) \) stays below \( t \leq \frac{u}{n} \). By (8.1), Appendix B this equals

\[ \frac{1 - (1 - \frac{u}{n})}{1 - (\frac{i}{n} - \frac{u}{n})} = \frac{u/n}{(n-i+u)/n} \]

Hence

\[ P(M_n \geq u) = \sum_{i+j=n} \binom{n}{i} \left( \frac{1}{n} - \frac{u}{n} \right)^i \left( 1 - \frac{1}{n} + \frac{u}{n} \right)^{n-i} \frac{u}{n} \]

which is the same as (2.4). In Section 3 we will give a more intuitive explanation of (2.1) and (2.4).

3. ONE-SIDED MAXIMA, SECOND APPROACH

We first need a technical preliminary about the Poisson process which is interesting in its own right. Define

\[ J = \begin{cases} \text{value of } Y(t) - t \text{ at the first instant that } Y(t) > t \text{ if this takes place,} \\ 0, \text{ otherwise.} \end{cases} \]
Figure 2
In other words, \( M \) is a sum of a geometric number of uniformly distributed random variables.

It is well-known that if \( U_1, U_2, \ldots \) are independent random variables, each uniformly distributed on \([0,1]\), then

\[
P(\frac{\sum U_i}{n} < u) = \sum_{i=0}^{n} \frac{(-1)^i [c(u-i)]^n}{n!}
\]

where

\[
c(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x > 0 
\end{cases}
\]

Hence we can derive the distribution of \( M \) once again, using Theorem 3.3, by the calculation

\[
P(M < u) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(-1)^i [c(u-i)]^n}{(n-i)!} \frac{\lambda^n}{n!} (1-\lambda)
\]

\[
= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{(\lambda c(u-i)) e^{-\lambda}}{e^{\lambda}} (1-\lambda)
\]

which agrees with (3.2).

The Laplace transform of \( M \) is

\[
\mathcal{L}(e^{-\alpha M}) = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \frac{(1-\lambda)^r}{(1-e^{-\alpha})}
\]

\[
= \frac{\lambda (1-\lambda)}{\alpha - \lambda(1-e^{-\alpha})}
\]
conditionally distributed as $n$ independent random variables, each uniformly distributed on $[0, 1]$.

(b) $M_n$ is distributed as

$$U_1^+ \ldots + U_{L_n}^+$$

where $U_1, U_2, \ldots$ is a sequence of independent, uniform $[0, 1]$ random variables, independent of $L_n$.

It is now easy to determine the asymptotic distribution of $M_n$.

**Theorem 3.5**

$$\lim P \left( \frac{M_n}{\sqrt{n}} \geq t \right) = e^{-2t^2}$$

**Proof:**

$$\frac{M_n}{\sqrt{n}}$$

is distributed as

$$\frac{U_1^+ \ldots + U_{L_n}^+}{\sqrt{n}} = \frac{U_1^+ \ldots + U_{L_n}^+}{L_n} \frac{L_n}{\sqrt{n}}$$

Since $L_n$ converges in probability to $\frac{a}{n}$ as $n \to \infty$ (Example 2, Introduction) it follows that $(U_1^+ \ldots + U_{L_n}^+)/L_n$ converges in probability to $\frac{a}{2}$. Hence (by Example 2, Introduction)

$$\lim P \left( \frac{M_n}{\sqrt{n}} \geq t \right) = \lim P \left( \frac{L_n}{\sqrt{n}} \geq \frac{a}{2} \right) = e^{-\left(2t^2\right)/2}$$

which completes the proof. The above result was first proved by Kolmogorov [5].

Let us consider now the random variables $I_1^1, I_2^1, \ldots$ defined earlier. This is the succession of excesses of $Y(t)$ over the line $t$. The counterparts for empirical cdf's is $I^{(1)}_1, I^{(2)}_2, \ldots$ the succession of excesses of $n(F_n(t) - t)$
\[ N(1) = N, \quad L(1) = L, \quad N_n(1) = N, \quad L_n(1) = L_n. \]

It should be emphasized, however, that the random variable \( T \) still refers to last crossing of the line \( t \) (c.f. Section 1.). Equivalent to (1.1) is the relationship

\[ P(Y(t) < \alpha t \text{ for all } t > \alpha) = \begin{cases} 
1 - \frac{1}{\rho} & \text{if } \lambda < \rho \\
0 & \text{otherwise},
\end{cases} \]

Hence \( N(\alpha) \) is geometrically distributed with

\[ P(N(\alpha) \geq k) = \left( \frac{\lambda}{\alpha} \right)^k, \quad k = 0, 1, \ldots \]

Since

\[ P(N(\alpha) \geq k) = P(N \geq k) / \rho^k \]

it follows that for \( \alpha > 1 \),

\[ P(N_n(\alpha) \geq k) = P(N_n \geq k) / \alpha^k = \frac{1}{\rho^k} \frac{n!}{(n-k)! n^k} \quad k = 0, 1, 2, \ldots \quad (4.1) \]

Before going further, let us point out one interesting consequence of (4.1); namely,

\[ P(N_n(\alpha) \geq 1) = \frac{1}{\rho} \quad (4.2) \]

But this is just

\[ 1 - P(F_n(t) < \alpha t, \text{ all } 0 < t \leq 1) \]

which we know equals \( 1/\rho \) by (8.1), Appendix B. Thus (4.1) generalizes that assertion.

We now see an asymptotic result for \( N_n(\alpha) \) by letting \( \rho \) vary with \( n \), as follows.
To obtain a version of Theorem 3.4 for $M_\mathbb{N}$, we consider the random variables

\[ l_1, l_2, \ldots, j_1, j_2, \ldots \]

\[ l_1^{(n)}, l_2^{(n)}, \ldots, j_1^{(n)}, j_2^{(n)}, \ldots \]

where the role of the line $t$ is now played by $\mathbb{N}$. In other words, $l_1, l_2, \ldots$ is the succession of excesses of $Y(t)$ over the line $\mathbb{N}$; $j_1, j_2, \ldots$ is the succession of increases of $Y(t) - \mathbb{N}$ at the succession of ladder points of $Y(t) - \mathbb{N}$. (Strictly speaking the notation for the $l_1$'s and $j_1$'s should indicate their dependence on $\mathbb{N}$ but we suppress the $\mathbb{N}$ in the interest of typography.) Since when $\lambda = 1$, the conditional distributions of the jumps as described in Theorems 3.2 and 3.4 do not depend on the actual value of $\lambda$ as long as $\lambda < 1$, it follows similarly that the following is true:

**Theorem 4.1**

\[
P(I_1, \ldots, I_k) \equiv A \mid N(\mathbb{N}) \geq k)
\]

\[
P((J_1, \ldots, J_k) \equiv A \mid L(\mathbb{N}) \geq k)
\]

\[
P((l_1^{(n)}, \ldots, l_k^{(n)}) \equiv A \mid N_n(\mathbb{N}) \geq k)
\]

\[
P((j_1^{(n)}, \ldots, j_k^{(n)}) \equiv A \mid L_n(\mathbb{N}) \geq k)
\]

\[
P((U_1, \ldots, U_k) \equiv A)
\]

where $U_1, \ldots, U_k$ are independent random variables each uniformly distributed on $[0,1]$. For (a) and (b) we need to assume that $\rho \geq \lambda$. For (c) and (d) we need $\rho \geq 1$.

It follows now, as in Theorem 3.6, that
Lemma 5.1 Suppose \( \lambda < 1 \).

\[
P((-\infty, Y(t) \cdot t < s; 0 \leq t \leq T) \cup (T = 0))
\]

\[
= \frac{P(M < s) \cdot P(M < r)}{P(M < r + s)}
\]

Proof: Define

\[
R_1(r,s) = P(Y(t) \text{ hits } t-r \text{ and avoids } t+s \text{ en route})
\]

\[
R_2(r,s) = P(Y(t) \text{ hits } t+s \text{ and avoids } t-r \text{ en route})
\]

Since \( \lambda < 1 \),

\[
P(Y(t) \text{ hits } t - r) = 1 = R_1(r,s) + R_2(r,s)
\]

By the regenerative nature of the Poisson process,

\[
P((Y(t) \text{ hits either } t - r \text{ or } t+s; 0 \leq t \leq T) \cap (T > 0))
\]

\[
= R_2(r,s) + R_1(r,s) (1 - P(M < r)).
\]

We also have

\[
P(M < s) = R_1(r,s) P(M < r + s)
\]

Hence,

\[
P((-\infty, Y(t) \cdot t < s, 0 \leq t \leq T) \cup (T = 0))
\]

\[
= 1 - P((Y(t) \text{ hits either } t - r \text{ or } t+s, 0 \leq t \leq T) \cap (T > 0))
\]

\[
= 1 - R_2(r,s) + R_1(r,s) (1 - P(M < r))
\]

\[
= R_2(r,s) P(M < r)
\]

\[
= \frac{P(M < s) \cdot P(M < s)}{P(M < r + s)}
\]

which completes the proof.
Substituting in the right side of (3.3) and cancelling out the \((1-\lambda)\)'s gives (5.1) with \(e^{-\lambda} = e^{-1}\). Since \(e^{-\lambda}/e^{-1}\) defines a 1-1 map of \([0,1]\) onto itself this gives (5.1). Similarly, if we use (2.2) and substitute in (5.3) this gives (5.2).

From (5.2) it follows that the \(Q_n(r,s)\) terms satisfy the following difference equations:

\[
\sum_{\begin{subarray}{c} i+j=n \\ j\leq r+s \end{subarray}} \binom{i}{r} \binom{j}{s} \frac{(1-r-s)^j}{j!} \frac{(i-i-s)^i}{i!} = \begin{cases} \sum_{\begin{subarray}{c} i+j=n \\ i\leq r \end{subarray}} \binom{i}{r} \binom{j}{s} \frac{(1-r-s)^j}{j!} \frac{(i-i-s)^i}{i!} & \text{if } n \leq r + s \\ 0 & \text{if } n > r + s \end{cases}
\]

These can be used for iterative numerical computation of the \(Q_n(r,s)\). For \(r = s\) the above difference equations are similar to ones developed by other methods by Massey [6].
APPENDIX B

Let \( F_n(t) \), \( 0 \leq t \leq 1 \) be an empirical cdf of \( n \) independent random variables, each uniformly distributed over \([0,1]\). Then if \( \gamma > 1 \)

\[
P(F_n(t) < \gamma t \text{ all } t \in [0,1]) = 1 - \frac{1}{\gamma}
\]

(S.1)

This is easily proved by induction on \( n \) as follows. Let \( U \) be the largest of the \( n \) points selected in \([0,1]\). Then \( U \) has the integrating density \( n u^{n-1} \) in \([0,1]\). By the induction hypothesis

\[
P(F_n(t) < \gamma t \text{ all } t \mid U = u) = \begin{cases} \frac{(u - (n-1)/\gamma)}{\gamma} & \text{if } u > 1/\gamma \\ 0 & \text{otherwise} \end{cases}
\]

Hence

\[
P(F_n(t) < \gamma t \text{ all } t) = \int_0^1 \left( \frac{u- (n-1)/\gamma}{\gamma} \right) (1/u) n u^{n-1} \, du = 1 - \frac{1}{\gamma}
\]

Since (S.1) clearly holds for \( n = 1 \), this completes the proof. Another proof of (S.1) appears in Section 4 (See 4.1).
\[(n + u)^{n-k} = (n + u)^{n-k+1} (n - k + k + u)\]

Hence, from (C.1)
\[\left(\frac{u}{n} \right)^{n-k+1} \left(1 - \frac{u}{n} \right)^n = \left(\frac{(n+u)^{n-k+1}}{(n-k+1)} \right) \left(\frac{(n+u)^{n-k+1}}{(n-k+1)} \right)^n + (k+u) \frac{(n+u)^{n-k+1}}{(n-k+1)} \left(\frac{(n+u)^{n-k+1}}{(n-k+1)} \right)^n\]

The first expression on the right can be evaluated by (C.1) to equal
\[\left(\lambda, e^\gamma\right) \lambda^k \left(\frac{u+1}{1 - \gamma}\right)\]

Hence considering the second expression on the right,
\[\left(\frac{u+1}{n} \right)^{n-k+1} \left(\frac{1}{n-k+1} \right)^n = \left(\frac{e^{\lambda - \gamma} - \lambda e^{-\gamma}}{\gamma} \right)^{n-k+1} \left(\frac{e^{\lambda - \gamma} - \lambda e^{-\gamma}}{\gamma} \right)^{n-k+1} \left(\frac{u+1}{1 - \gamma}\right)\]

which proves (C.2) also in the case that \(u \geq 0\). It follows that the left sides of (C.1) and (C.2) have absolutely convergent power series expansions in \(u\) for all \(u\). Hence from considerations of analyticity both sides of (C.1) and (C.2) are equal for all \(u\).