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Social Welfare Functions when Preferences are Convex and Continuous: Impossibility Results

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Abstract

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The paper shows that if the class of admissible preference orderings is restricted in a manner appropriate for economic models, then Arrow's impossibility theorem for social welfare functions continues to be valid. Specifically if the space of alternatives is R_+^n, $n \geq 2$, where each dimension represents a different public good and if each person's preferences are restricted to be convex, continuous, and strictly monotonic, then no social welfare function exists that satisfies unanimity, independence of irrelevant alternatives, and non-dictatorship.

Social Welfare Functions when Preferences are Convex and Continuous: Impossibility Results

by

Ehud Kalai, Eitan Muller, and Mark A. Satterthwaite

1. Introduction

Arrow [1] showed with his impossibility theorem that for a set of at least three alternatives no social welfare function exists satisfying unanimity (U), independence of irrelevant alternatives (IIA), and non-dictatorship (ND) if admissible preferences are not a priori restricted in some manner. If the variety of preference orderings that are admissible is restricted sufficiently, then social welfare functions do exist that satisfy U, IIA, and ND. Black [2] originally characterized and Arrow [1,pp.75-80] extensively discussed the condition of single-peakedness, which is the best known of the restrictions on the set of admissible preferences that are sufficient to make majority rule into a social welfare function satisfying U, IIA, and ND. Subsequently a great deal of research, culminating in a paper by Sen and Pattanaik [8], was done to determine what restriction on admissible preferences is both sufficient and necessary for majority rule to be a social welfare function satisfying Arrow's three conditions.[1]

The substantive conclusion of this literature is that in order to use majority rule as a valid Arrow type social welfare function the set of admissible preferences must be restricted to a class that is much smaller than any class that economic theory can justify a priori. Kramer [5] decisively confirmed this conclusion by showing that if the set of alternatives being ordered by the social welfare function is a multidimensional subset of Euclidean space and if individuals' admissible preferences are restricted to be representable by continuously differentiable, quasi-concave utility functions, majority rule fails as an Arrow social welfare function whenever the profile of preferences do not contain a majority of individuals who are unanimous in their ordering of the entire space of alternatives. In other words, any disagreement, no matter how minor, is almost certain to make majority rule inconsistent.

The weakness of these published results is that they describe the properties of majority rule. The power and appeal of Arrow's theorem is that it rules out construction of any social welfare function satisfying U, IIA, and ND, not just social welfare functions based on majority rule. Our purpose in this paper is to confirm our intuition and show that the negative conclusions derived for the special case of majority rule generalize for the case of public goods into true impossibility results.[2]

We proceed as follows, First we derive a condition on the set of admissible preferences that, if satisfied, guarantees that no social welfare function satisfying U, IIA, and ND can be constructed.[3] Second, we apply this condition to two different restrictions on admissible preferences. We let the set of alternatives be the psoitive orthant of R^n where each axis represents a public good. The first restriction we consider is that individuals' preferences be continuous and convex. We show that if and only if $n \geq 2$, where n is the dimensionality of the set of alternatives, then no social welfare function satisfying U, IIA, and ND exists. The second restriction we consider is that individual preferences be continuous, convex and strictly monotonic. For this restriction we show that if $n \geq 1$, then no social welfare function satisfying U, IIA, and ND exists. Therefore, because economic theory generally can not justify restrictions stronger than these, we are justified in stating that Arrow's assumption of no restriction on the set of admissible preferences is not critical. The paper concludes with a short analysis of two very strong restrictions on admissible preferences each of which are sufficient to allow construction of social welfare functions satisfying U, IIA, and ND.

The main limitation on our results is that we prove them only for the purely public goods case. To illustrate, suppose a society is selecting the level of public expenditure for three programs: defense, health, and parks. Presumably each individual has continuous and convex preferences over this three dimensional choice space. Our results state that no means exists for constructing a group preference ordering over such a choice space without violating U, IIA, or ND.

If, however, private goods are included as additional dimensions of the choice space, then our proofs no longer apply. The reason is that the presence of private goods requires that each individual's set of admissible preferences differs from every other individual's set of admissible preferences. For example, a reasonable restriction on person one's preferences is that he be indifferent among allocations that are identical except for the amount of private goods person two receives. Obviously this restriction is as unreasonable to place on person two's preferences as it is reasonable to place on person one's preferences. Consequently our assumption that all individuals have identically restricted sets of admissible preferences fails.

Too much should not be made of this limitation of our results to the purely public goods case because this case has interest in its own right. Ruys [7] has studied the existence of equilibria within economies containing only public goods. Campbell [3] has argued that legislatures very seldom make decisions concerning the exact quantity of private goods that an individual receives. Instead they make decisions concerning the rules by which individuals may pursue the acquisition of private goods. These rules are public goods; therefore the type of model described in this paper is an acceptable description of the decision problem legislatures face. In a similar manner Zeckhauser and Weinstein [9] in their study of the shape of Pareto optimal regions have argued that if the financing mechanism for public goods is included in the analysis and if an equilibrium exists within the private goods market, then a functional relationship exists between each public goods bundle and each individual's private goods bundle. Consequently each individual can calculate his utility level for any possible public good allocation and, as a result, individuals have well defined preferences over the possible bundles of public goods. Therefore each individual's private goods bundle need not be included explicitly in the analysis.

2. The Model

Let $I = \{1, \ldots, m\}$ be the set of m individuals in the society. Let Σ be the set of all complete, reflexive, and transitive preference relations that may be defined on the set of alternatives A, $|A| \geq 3$. An element of Σ is denoted by $\precsim$ with strict preference and indifference being denoted by $\prec$ and $\sim$ respectively. Let $\mathcal{P}$, a fixed, non-empty subset of Σ, be the set of possible preference relations that are admissible as preferences for the individuals within I. Thus $\precsim_i \in \mathcal{P}$ represents the preferences of individual $i \in I$.

The product $\mathcal{P}^m$ is the set of admissible preference m-tuples where each point in $\mathcal{P}^m$ is a list describing the preferences of the individuals within I. We call such a list $\precsim_I = (\precsim_1, \ldots, \precsim_m)$ a <u>profile</u>. Two preference relations, $\precsim$ and $\precsim'$ agree on a subset B of A if, for every pair $x, y \in B$,

$$x \precsim y \quad \Leftrightarrow \quad x \precsim' y.$$

We denote agreement on B by $\precsim|_B = \precsim'|_B$. Two profiles, $\precsim_I = (\precsim_1, \ldots, \precsim_m)$ and $\precsim'_I = (\precsim'_1, \ldots, \precsim'_m)$, agree on $B \subset A$ if, for all $i \in I$, $\precsim_i|_B = \precsim'_i|_B$.

A <u>social welfare function</u> (SWF) on $\mathcal{P}$ is a function $f: \mathcal{P}^n \to \Sigma$. An <u>Arrow SWF</u> (ASWF) is a SWF that satisfies the conditions of unanimity and independence of irrelevant alternatives.

<u>Unanimity</u> (U). Suppose, for some $\precsim_I \in \mathcal{P}^m$, $f(\precsim_I) = \precsim$. If $x, y \in A$, and $x \prec_i y$ for all $i \in I$, then $x \prec y$.

<u>Independence of Irrelevant</u> Alternatives (IIA). If, two profiles $\precsim_I, \precsim'_I \in \mathcal{P}^m$ agree on B, then $\precsim|_B = \precsim'|_B$ where $\precsim = f(\precsim_I)$ and $\precsim' = f(\precsim'_I)$.

A SWF f has a dictator on the set $B \subset A$ if an individual $i \in N$ exists such that, for every profile $\precsim_I = (\precsim_1, \ldots, \precsim_m) \in \mathcal{P}^m$ and every pair $x, y \in B$, $y <_i x$ implies $y < x$ where $\precsim = f(\precsim_I)$. A family $\mathcal{P}$ is called dictatorship enforcing if every ASWF on $\mathcal{P}^m$ has a dictator on the set A.

Example A (Arrow's Theorem). If $|A| \geq 3$ and $\mathcal{P} = \Sigma$, then $\mathcal{P}$ is dictatorship enforcing.

Example B. If $\mathcal{P}$ is any subset of Σ with $|\mathcal{P}| = 1$, then $\mathcal{P}$ is dictatorship enforcing.

Example B is a direct consequence of condition U.

3. A Basic Theorem

In this section we state a simple theorem that is very useful in determining whether any particular $\mathcal{P} \subset \Sigma$ is a dictatorship enforcing family of preference relations. Throughout $\mathcal{P}$ represents a fixed, nonempty subset of Σ. A pair of distinct alternatives $x, y \in A$ is trivial (relative to $\mathcal{P}$) if all the relations in $\mathcal{P}$ agree on the set $\{x,y\}$. A set of three distinct alternatives $\{x,y,z\}$ is a free triple if, for every $\precsim \in \Sigma$, there exists $\precsim' \in \mathcal{P}$ such that

$$\precsim|_{\{x,y,z\}} = \precsim'|_{\{x,y,z\}},$$

i.e. $\mathcal{P}$ admits all possible orderings of the three alternatives. Two, non-trivial pairs $B = \{x,y\}$ and $C = \{w,z\}$ are strongly connected if:

a. $|B \cup C| = 3$;

c. $B \cup C$ is a free triple.

In other words, B and C are strongly connected if they share an element in common and together form a free triple. Two pairs B and C are connected if a finite sequence of pairs

$$B=B_1, B_2, \ldots, B_{n-1}, B_n=C$$

exist such that B_i and B_{i+1} are strongly connected for each i=1, 2, ..., n-1. Finally a family $\mathcal{P}$ is saturating if (a) the set A contains at least two non-trivial pairs and (b) every non-trivial pair $B \subset A$ is connected to every other non-trivial pair $C \subset A$.

Theorem 1. Every saturating family $\mathcal{P}$ is dictatorship enforcing.

Proof. The first step is to show that if a non-trivial pair B is strongly connected to another non-trivial pair C, then an individual $j \in I$ exists who is dictator on $D = \{B \cup C\}$. Since, by hypothesis, B and C are strongly connected, D is a free triple. Arrow's theorem may be applied to this triple: an individual $j \in I$ exists who is a dictator on D.[4]

The second step is to show that if an individual $j \in I$ is dictator on a pair B_i and a second pair B_{i+1} exists to which B_i is strongly connected, then j is also dictator on B_{i+1}. Step one implies that since B_i and B_{i+1} are strongly connected an individual $k \in I$ exists who is dictator on $D_i = \{B_i \cup B_{i+1}\}$. Suppose that $j \neq k$. Let $B_i = \{x,y\}$ and consider a profile $\precsim_I \in \mathcal{P}$ such that $x >_j y$ and $x <_k y$. Such a profile exists because D_i is a free triple. Let $\precsim = f(\precsim_I)$. Because j is dictator on B_i, $x >_j y$. But, because k is also dictator on B_i, $x <_j y$, which is a contradiction. Therefore individual j is dictator on B_{i+1} as well as B_i.

The third step is to form an inductive chain and to prove that if two pairs B and C are connected, then an individual $j \in I$ exists who is dictator on both. Because B and C are connected a finite sequence of pairs

$$B=B_1, B_2, \ldots , B_i, B_{i+1}, \ldots , B_n=C$$

exists such that B_i and B_{i+1} are strongly connected for all $i=1, \ldots, n-1$. Step one implies that some individual $j \in I$ is dictator on $D_1 = \{B_1 \cup B_2\} = B$. Step 2 implies that he must also be dictator on $D_2 = \{B_2 \cup B_3\}$, $D_3 = \{B_3 \cup B_4\}$, etc. Therefore individual j is dictator on C as well as on B.

The last step is to note two facts. First, because $\mathcal{P}$ is saturating, at least two non-trivial pairs exist and each is connected with every other, non-trivial pair. Consequently, an individual j exists who is dictator over them. Second, if a pair $B = \{x,y\}$ is trivial, then individual j, along with every other individual $i \in I$, is dictator on B. Hence individual j is dictator on all pairs, trivial and non-trivial. Q.E.D.

4. Dictatorship Enforcing Families

In this section we show that two families of preference relations, which are common within economics, are dictatorship enforcing because they are saturating. In both cases the set of alternatives A consists of R^n_+, the non-negative orthant of n-dimensional, Euclidean space. An alternative x is therefore an n-dimensional vector $(x^1, \ldots , x^n)$ whose components are non-negative. The first dictatorship enforcing family that we consider is the collection $\mathcal{P}_n^o$ of all convex, continuous preference orderings defined on R_+^n, $n \geq 2$. This class of admissible preferences is precisely the class that Zeckhauser and Weinstein [9] identify as occurring in societies where the choice of the public goods bundle is considered jointly with the method for financing the public goods bundle. The second, more restrictive family is the collection $\mathcal{P}_n^*$ of all convex, continuous, and strictly monotonic preference relations defined on R_+^n, $n \geq 1$. It is the class of admissible preferences that is appropriate for a committee whose task is to divide a fixed budget among several worthy programs.

A preference relation is convex if, for every alternative $x \in R_+^n$, the set $\{y \in R_+^n \mid x \prec y\}$ is convex. A family $\mathcal{P}_n \subset \Sigma$ is convex if every $\precsim \in \mathcal{P}_n$ is convex. A preference relation $\precsim \in \Sigma$ is continuous if it can be represented by a continuous utility function on R_+^n. A preference relation $\precsim \in \Sigma$ is strictly monotonic if, for any pair of distinct alternatives $x, y \in R_+^n$, $x \leq y$ implies $x \prec y$.[5]

The proofs that the families $\mathcal{P}_n^o$ and $\mathcal{P}_n^*$ are dictatorship enforcing make use of both linear and concentric preference relations. A preference relation $\precsim \in \Sigma_n$ is linear if and only if a vector $p = (p_1, \ldots, p_n) \in R_n$ exists such that, for all pairs $x, y \in R_+^n$, $x \precsim y$ if and only if $\langle p,x\rangle \leq \langle p,y\rangle$ where $\langle p,x\rangle = \sum_{i=1}^{n} p_i x^i$, the inner product of p and x. Three observations follow directly from this definition. First, if a linear preference relation is parameterized by the vector $p \in R^n$, then the indifference surface containing a specific point $x' \in R_+^n$ is the plane

$\{x \in R^n_+ | \langle p,x \rangle = \langle p,x' \rangle\}$. Second, every linear preference relation is convex. Third, a linear preference relation $\precsim$ with parameter vector $p \in R^n$ is strictly monotonic if and only if $p > 0$.

A preference relation $\precsim \in \Sigma$ is <u>concentric</u> if and only if a vector $p = (p_1, \ldots, p_n) \in R^n$ exists such that, for all pairs $x, y \in R^n_+$, $x \precsim y$ if and only if $\|x-p\| \leq \|y-p\|$ where $\|x-p\| = [\sum_{i=1}^{n} (x^i - p_i)^2]^{\frac{1}{2}}$, the distance from x to p. Again three observations follow directly from the definition. First, if $p \in R^n$ is the parameter for a concentric preference relation $\precsim \in \Sigma$, then the indifference surface containing a specific point $x' \in R^n_+$ is the hollow sphere $\{x \in R^n_+ \mid \|x-p\| = \|x'-p\|\}$ with center at p. Second every concentric preference relation is convex. Third, if three points $x,y,z \in R^n_+$ are not colinear, then a concentric preference relation $\precsim \in \mathscr{P}^o_n$ exists such that $x \sim y \sim z$.[7] Three points are <u>colinear</u> if a scaler λ exists such that $z = (1-\lambda)x + \lambda y$.

Theorem 2. The family $\mathscr{P}^o_n$ of convex, continuous preference relations on R^n_+ is dictatorship enforcing if and only if $n \geq 2$.

Proof. If n=1, then the requirement of convexity is the well known requirement of single-peakedness. When preferences are single-peaked, then majority rule is an ASWF.[6] Therefore, if n=1, then $\mathscr{P}^o_1$ is, as the theorem requires, not dictatorial enforcing.

If $n \geq 2$, then we can show that $\mathscr{P}^o_n$ is saturating and, consequently, dictatorial by Theorem 1. First we show that every distinct pair of alternatives $(x,y) \in R^{2n}$ is non-trivial. Pick a vector $p' \in R^n$ such that $\langle p',x\rangle \neq \langle p',y\rangle$. Suppose, with no loss of generality, that $\langle p'x\rangle < \langle p',y\rangle$. Let $p'' = -p'$. Therefore $\langle p'',x\rangle > \langle p'',y\rangle$. Let $\precsim' \in \mathcal{P}_n^o$ and $\precsim'' \in \mathcal{P}_n^o$ be the linear preference relations that p' and p'' respectively define. Therefore $x <' y$ and $y <'' x$, which proves that x and y are non-trivial.

The proof's second step is to show that every pair of distinct points $(x,y) \in R_+^{2n}$ is connected to a pair of reference points. Let these reference points be $e_1 = (e_{11}, \ldots, e_{1n}) \in R^n$ and $e_2 = (e_{21},\ldots, e_{2n}) \in R^n$ where $e_i = (e_{i1}, \ldots, e_{in}) \in R^n$ has the property that $e_{ij} = 1$ if $i=j$ and $e_{ij}=0$ if $i\neq j$. There are three cases to consider.

Case 1. The pair $(x,y) \in R_+^{2n}$ has the property that x, e_1, and e_2 are not colinear. This case is depicted for n=2 by Figure 1. Our goal is to show that the pair (x,y) is strongly connected to the pair (e_1,x) and that the pair (e_1,x) is strongly connected to the pair (e_1,e_2). Therefore we must show that the two triples (y,x,e_1) and (x,e_1,e_2) are free.

Since x is not colinear with e_1 and e_2, the points x, y, and either e_1 or e_2 are not colinear. Suppose, without loss of generality, that x,y, and e_2 are not colinear. Therefore, as we pointed out when we defined concentric preference relations, a concentric preference relation $\precsim \in \mathcal{P}_n^o$ exists such that $x \sim y \sim e_2$. Let p be the parametrization of $\precsim$. Concentric preference relations also exist such that x,y, and e_2 may be ordered in any other manner. For example, let us construct a concentric relation $\precsim' \in \mathcal{P}_n^o$ such that $x \precsim' y \precsim' e_2$. Let $r_x = x-p$, $r_y = y-p$ and $r_e = e_2-p$ be the vectors from p to the points x,y and e_2 respectively. Pick a very small scalar $\varepsilon > 0$. Let

$$x^o = p + (1+\varepsilon)r_x$$

and

$$e_2^o = p + (1-\varepsilon)r_e.$$

If we pick ε small enough, then a concentric preference relation $\precsim' \in \mathcal{P}_n^o$ exists such that $x^o \sim' y \sim' e_2^o$ and $x \prec y \prec e_2$ because, relative to the appropriately chosen sphere through x^o, y, and e_2^o, the point x lies within the interior and e_2^o lies in the exterior. Thus (y,x,e_2) is a free triple. Similarly, because x, e_1, and e_2 are not colinear, (x, e_1, e_2) is a free triple. Therefore (x,y) is connected to (e_1, e_2).

Case 2. The pair $(x,y) \in R_+^{2n}$ has the property that y, e_1, and e_2 are not colinear. This case is identical to case 1 with the roles of x and y switched.

Case 3. The pair $(x,y) \in R_+^{2n}$ has the property that x, y, e_1, and e_2 are all colinear. Let the point o be the origin of R_+^n. The same techniques used for case 1 suffice to show that each of the triples (x, y, o), (y, o, e_1), and (o, e_1, e_2) are free. Therefore (x,y) is connected to (e_1, e_2).

The proof's third step is to observe that because each of any two arbitrary pairs $(x',y') \in R_+^{2n}$ and $(x'',y'') \in R_+^{2n}$ are connected to the reference pair (e_1,e_2), the two pairs are connected to each other. That is, any two pairs within R_+^{2n} are connected with each other. Therefore the family $\mathcal{P}_n^o$ is saturating and, by Theorem 1, dictatorship enforcing. Q.E.D.

Theorem 3. The family $\mathcal{P}_n^*$ of convex, strictly monotonic, continuous preference relations on R_+^n is dictatorship enforcing for all $n \geq 1$.

Proof. If n=1, then $\mathcal{P}_1^*$ consists of one element and every individual is a dictator. If $n \geq 3$, then the proof, without any loss of generality, may be constructed with explicit reference to linear preference relations only. If $n=2$, then the proof is more difficult; it requires that a more general class of convex, strictly monotonic, continuous preference relations be referred to. Therefore we first spell out the proof for the $n \geq 3$ case and then sketch the proof for the $n=2$ case.

The first step is to show that a pair $(x,y) \in R_+^{2n}$ is non-trivial if and only if neither $x \geq y$ nor $y \geq x$. If $x \geq y$, then strict monotonicity implies that $y \prec x$ for all $\precsim \in \mathcal{P}_n^*$. Identical reasoning applies to the $x \leq y$ case. Therefore if $x \geq y$ or $y \geq x$, the pair (x,y) is trivial. If neither $x \leq y$ nor $y \geq x$, then a pair of components $(i,j) \in N \times N$ must exist such that $x^i > y^i$ and $x^j < y^j$. Linear preference relations $\precsim' \in \mathcal{P}_n^*$ and $\precsim'' \in \mathcal{P}_n^*$ exist such that $x \precsim' y$ and $y \precsim'' x$. These two relations are constructed as follows.

First, we show that if neither $x \geq y$ nor $y \geq x$, then a linear $\precsim \in \mathcal{P}_n^*$ exists such that $x \sim y$. The requirement therefore is to find a vector $p = (p_1, \ldots, p_n) \in R^n$, $p > 0$, such that

$$(1) \qquad \langle p,x \rangle = \langle p,y \rangle.$$

Because neither $x \geq y$ nor $y \geq x$, a pair of indices $(i,j) \in N \times N$ exist such that $x^i > y^i$ and $x^j < y^j$. Impose the restriction, without loss of generality, that $\sum_{i=1}^{n} p_i = 1$. Equation (1) may therefore be solved for p_i:

$$(2) \qquad p_i = \frac{[1 - \sum_{k\neq i}^{k\neq j} p_k](y_j - x_j) + \sum_{k\neq i}^{k\neq j} p_k(y_k - x_k)}{[(x_i - y_i) + (y_j - x_j)]}$$

The denominator is positive and the numerator can be made positive by picking each component $p_k (k=1, 2, \ldots, n; k\neq i, k\neq j)$ such that it is positive and sufficiently close to zero. Therefore p_i can be made positive and, consequently a $p \in R^n$ exists such that $p > 0$ and (1) is satisfied.

Given that a relation $\precsim \in \mathcal{P}_n^*$ exists such that $x \sim y$, a relation $\precsim' \in \mathcal{P}_n^*$ may easily be constructed such that either $x <' y$ or $y <' x$. For example, in order to construct $\precsim'$ such that $x <' y$, pick a point $x^* > x$ that preserves the inequalities $x^{*i} > y^i$ and $x^{*j} < y^j$. Construct, as above, a linear $\precsim' \in \mathcal{P}_n^*$ such that $x^* \sim y$. Strict monotonicity and transitivity then implies that $y > x$. Therefore the claim that $(x,y) \in R^{2n}$ is a non-trivial pair if and only if neither $x \geq y$ nor $y \geq x$ is true.

The second step is to show, in much the same manner as we did in the proof of Theorem 2, that any non-trivial pair $(x,y) \in R_+^{2n}$ is connected to the reference pair $(e_1,e_2) \in R_+^{2n}$.[8]

Observation 1. If (x,y) is a non-trivial pair, then a linear $\precsim \in \mathcal{P}_n^*$ exists such that $x \sim y$. We proved this observation immediately above in the proof's first step.

Observation 2. If a linear $\precsim \in \mathcal{P}_n^*$ exists such that $w \sim x \sim y$ for a triple $(w,x,y) \in R^{3n}$ of non-colinear points, then (w,x,y) is a free triple. Given that $w \sim x \sim y$ for $\precsim \in \mathcal{P}_n^*$, an ordering $\precsim' \in \mathcal{P}_n^*$ such that $w <' x <' y$ may be constructed as follows. Pick points $w^* \in R^n_+$ and $y \in R^n_+$ such that $w^* > w$, $y > y^*$, and the distances $\| w^* - w \|$ and $\| y - y^* \|$ are small. If w^* and y^* are chosen close enough to w and y respectively, then continuity guarantees that a $\precsim'' \in \mathcal{P}^*_n$ exists such that $w^* \sim'' x \sim'' y^*$. Consequently, by transitivity and monotonicity, $w \prec'' x \prec'' y$.

Observation 3. If a triple $(w,x,y) \in R^{3n}$ is composed of points that each lie on a distinct axis, then a linear $\precsim \in \mathcal{P}^*_n$ exists such that $w \sim x \sim y$. Without loss of generality, let $w = \zeta_w e_1$, $x = \zeta_x e_2$, and $y = \zeta_y e_3$ where ζ_w, ζ_x, and ζ_y are strictly positive scalars. If $\precsim' \in \mathcal{P}^*_n$ is a linear ordering parameterized by the vector $p = (p_1, \ldots, p_n)$ where $p_1 = \zeta_w^{-1}$, $p_2 = \zeta_x^{-1}$, $p_3 = \zeta_y^{-1}$, and $p_k = 1$ (k=4, 5, ..., n), then it satisfies the requirement $w \sim' x \sim' y$.

Given these three observations we can show that the non-trivial pair (x,y) is connected to the reference pair (e_1,e_2). Observation 1 states that a linear $\precsim \in \mathcal{P}^*_n$ exists such that $x \sim y$. Let p be the vector that parameterizes $\precsim$. Pick an index $i \in N$ and a point $z_1 = \zeta_1 e_i$ on axis i such that:

a. $\langle p,x \rangle = \langle p,z_1 \rangle = \langle p,y \rangle$ and

b. x, y, and z_1 are not colinear,

i.e. $x \sim y \sim z_1$. Such a pair $i \in N$ and $z_1 \in R^n_+$ exists because $p > 0$ and $n \geq 3$. In fact, $\zeta_1 = \langle p,x \rangle \div p_i$. Observation 2 implies that (x,y,z_1) is a free triple.

The construction that led to the choice of z_1 implies that (y,z_1) is a non-trivial pair. Therefore, in exactly the same manner that we picked the index i and the point z_1, we may pick a second, distinct index $j \in N$, a point $z_2 = \zeta_2 e_j$, and a vector $p > 0$ such that

a. $\langle p,y \rangle = \langle p,z_1 \rangle = \langle p,z_2 \rangle$ and

b. y, z_1, and z_2 are not colinear.

Therefore (y, z_1, z_2) is a free triple.

The points z_1 and z_2 are non-trivial. Therefore, as before, pick an index $k \in N$, a point $z_3 = \zeta_3 e_k$, and a vector $p > 0$ such that

a. $\langle p,z_1 \rangle = \langle p,z_2 \rangle = \langle p,z_3 \rangle$,

b. z_1, z_2, and z_3 are not colinear,

c. $k=1$ if $i \neq 1$ and $j \neq 1$,

d. $k=2$ if $\{i \neq 1$ or $j \neq 1\}$ and $\{i \neq 2$ and $j \neq 2\}$, and

e. $k=3$ otherwise.

The triple (z_1,z_2,z_3) is free.

By construction an $\ell \in \{1,2,3\}$ exists such that $z_\ell = \zeta_\ell e_1$. Without loss of generality suppose that $z_3 = \zeta_3 e_1$. By construction $z_1 \neq \zeta_1 e_2$ or $z_2 \neq \zeta_2 e_2$. Suppose, again without loss of generality, that $z_2 \neq \zeta_2 e_2$. Let $z_4 = e_2$, the second reference point. Pick a vector $p > 0$ such that

$$\langle p,z_2 \rangle = \langle p,z_3 \rangle = \langle p,z_4 \rangle.$$

Observation 3 guarantees that this construction is possible. Since z_2, z_3, and z_4 all lie on different axes.they cannot be colinear. Therefore (z_2,z_3,z_4) is a free triple. Let $z_5 = e_1$, the first reference point. Observation 3 states that a vector $p > 0$ exists such that $\langle z_2,p \rangle = \langle z_4,p \rangle = \langle z_5,p \rangle$. Therefore (z_2,z_4,z_5) is a free triple.

The product of this procedure is the following collection of free triples: (x,y,z_1), (y,z_1,z_2), (z_1,z_2,z_3), (z_2,z_3,z_4), and

(z_2,z_4,z_5). From this collection a sequence of pairs may be extracted: $B_1 = (x,y)$, $B_2 = (y,z_1)$, $B_3 = (z_1,z_2)$, $B_4 = (z_2,z_3)$, $B_5 = (z_2,z_4)$, and $B_6 = (z_4,z_5) = (e_1,e_2)$. Inspection shows that the pairs B_i and B_{i+1} are strongly connected for i=1, 2, ..., 5. Thus the terminal pairs (x,y) and (e_1,e_2) are connected. Therefore every non-trivial pair is connected to the reference pair and the family $\mathcal{P}_n^*$ is saturating. Consequently, by Theorem 1, $\mathcal{P}_n^*$ is dictatorship enforcing for $n \geq 3$.

The case of n=2 may be proved using same program of showing that every non-trivial pair $(x,y) \in R_+^4$ is connected to the reference pair (e_1,e_2). The difference is that when n=2 linear preference relations cannot be used to show that a point $z_1 \in R_+^2$ exists such that (x,y,z_1) is a free triple. The family $\mathcal{Q}_2$ of piecewise linear preference relations, however, can be used to show that (x,y) is contained within a free triple and therefore can be used to prove the theorem for n=2. The preference relation $\precsim \in \Sigma_2$ is an element of $\mathcal{Q}_2$ if and only if a vector $q = (q_1,q_2,q_3,q_4) \in R^4$ exists such that, for all pairs $(x,y) \in R_+^4$, $x \precsim y$, if and only if

$$q_1x^1 + q_2(x^2+q_4) + q_3\text{Min}[x^1,x^2+q_4]$$
$$\leq q_1y^1 + q_2(y^2+q_4) + q_3\text{Min}[y^1,y^2+q_4].$$

If $q_1 > 0$, $q_2 > 0$, and $q_3 > 0$, then $\precsim \in \mathcal{Q}_2$ is both convex and strictly monotonic. Figure 2 shows the type of family of indifference curves that an element of $\mathcal{Q}_2$ generates. In the figure the triple (x,y,z_1) are indifferent with each other. If we perturb the elements of (q_1,q_2,q_3,q_4), then the indifference curves can be shifted sufficiently to achieve any desired ordering of (x,y,z_1); therefore (x,y,z_1)

is a free triple. Given this technique for constructing free triples, the remainder of the proof for the n=2 case exactly parallels the proof for the $n \geq 3$ case. Q.E.D.

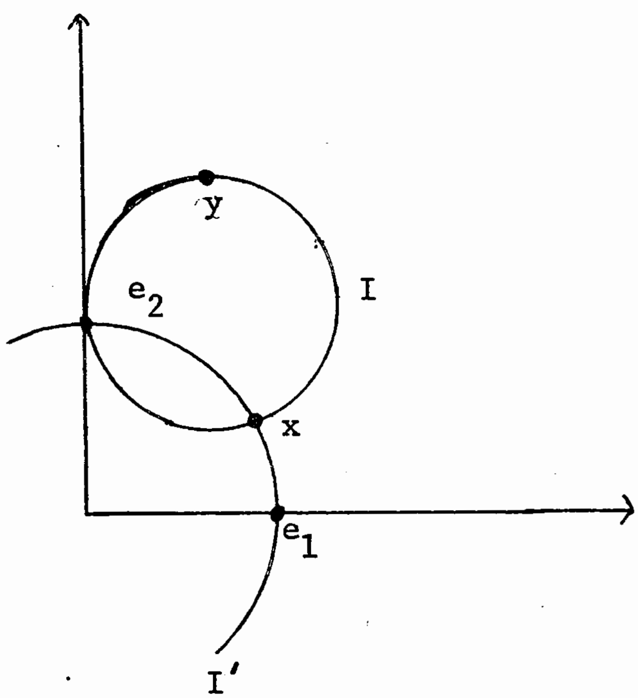
5. A Democratic Family of Preference Relations

In proving Theorem 3 for the case of n=2 we had to use the family $\mathcal{Q}_2$ of piecewise linear preference relations instead of linear preference relation. The reason for this is simply that the family $\mathcal{P}_2^+$ of linear, monotonic preference relations is not dictatorship enforcing when n=2. Nevertheless, as can be seen from the proof of Theorem 3, the family of linear, strictly monotonic preferences is dictatorship enforcing when $n \geq 3$.

Formally, a preference relation $\precsim$ is contained in $\mathcal{P}_2^+$ if and only if a scalar $a > 0$ exists such that, for any pair $(x,y) \in R_+^4$, $x \precsim y$ if and only if $ax^1 + x^2 \leq ay^1 + y^2$. Given that $\mathcal{P}_2^+$ is the family of admissible preference relations, let the preferences $(\precsim_1, \ldots, \precsim_n)$ of the n individuals within the society be described by the vector $(a_1, a_2, \ldots, a_n)$ where a_i is the parameter that describes the linear preferences of person i. Finally let $a = A_M(a_1, \ldots, a_n)$ be the median value of the vector $(a_1, \ldots, a_n)$. A valid non-dictatorial ASWF defined on the family $\mathcal{P}_2^+$ is this: $\precsim = f_M(\precsim_1, \ldots, \precsim_n)$ where $\precsim$ is that linear preference relation whose parameter a is $A_M(a_1, \ldots, a_n)$.

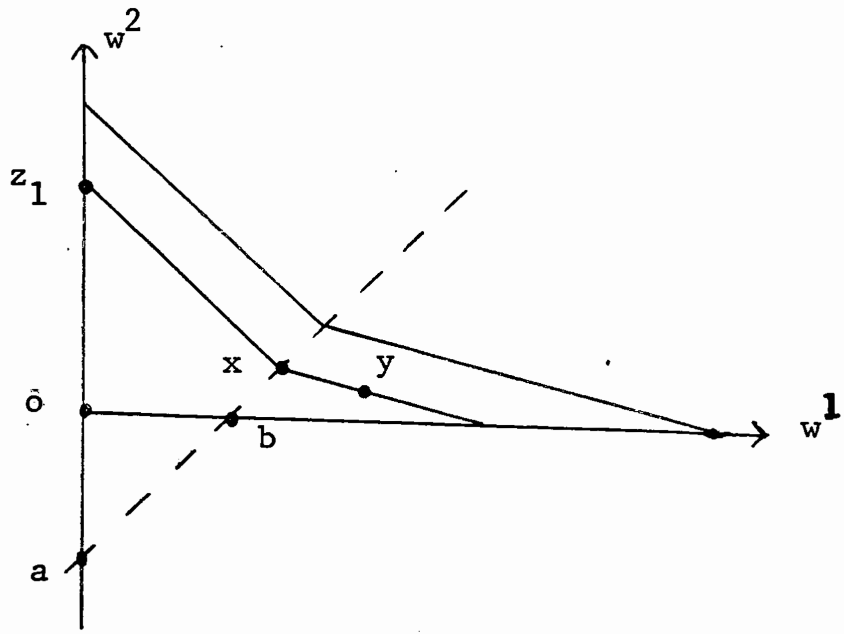
This positive result applies also to the family $\mathcal{P}_2^{++}$ of Cobb-Douglas preference relations. A preference relation $\precsim$ is an element of $\mathcal{P}_2^{++}$ if and only if a positive scalar α exists such that, for all pairs $(x,y) \in R^4_+$, $x \precsim y$ if and only if $(x^1)^\alpha(x^2)^{1-\alpha} \leq (y^1)^\alpha(y^2)^{1-\alpha}$. In this case, let the ASWF be defined as $\precsim = f_{CD}(\precsim_1, \ldots, \precsim_n)$ where $\precsim$ is that Cobb-Douglas preference relation whose parameter α is $A_M(\alpha_1, \ldots, \alpha_n)$. The reason for this positive result is, of course, that a Cobb-Douglas preference relation is a linear preference relation if a logarithmic scale is applied to the space of alternatives $A \equiv R^n_+$.

Figure 1



Explanation. I is the indifference curve generated by the concentric preference relation $\precsim$ such that $x \sim y \sim e_2$. I' is the indifference curve generated by the concentric preference relation $\precsim''$ such that $x \sim e_1 \sim e_2$.

Figure 2



Explanation. The lengths of the line segments oa and ob are equal to the value of the paramenter q_4 (in drawing the diagram we have assumed a positive value for q_4). The region below the dotted diagonal contains all points $x = (x^1, x^2)$ such that $x^1 > x^2 + q_4$.

The segment z,x of the indifference curve has slope $-(q_1+q_3)/q_2$ and the segment x,y of the indefference curve has slope $-q_1/(q_2+q_3)$.

FOOTNOTES

1. The paper of Sen and Pattanaik [8] is focused on the existence of choice sets, a question somewhat more general than this paper's question concerning the existence of a social welfare function. Existence of a choice set requires that the social ordering be quasi-transitive; existence of a social welfare function requires that the social ordering be fully transitive.
2. Subsequent to the writing of this paper we have discovered that Maskin [6] was also working on this question using a different approach. His paper condiders the case of purely private goods while our paper considers the case of purely public goods. His work and our work reported here were done independently of each other.
3. Maskin [6] and Kalai and Muller [4] have separately developed necessary and sufficient conditions that characterize those classes of admissible preferences for which a social welfare function satisfying U, IIA, and ND exists. The condition developed here is implied by their conditions. Our condition is, however, much simpler because it addresses a less ambitious question.
4. This can be seen as follows. Let $\precsim_I^\circ$ and $\precsim^\circ$ represent preference relations that are defined solely over the free triple D. Define $f_D(\precsim_I^\circ) = \precsim^\circ$ such that if $\precsim_I^\circ|_D = \precsim_I|_D$ for some $\precsim_I \in \mathcal{P}^m$, then $\precsim^\circ|_D = \precsim|_D$ where $\precsim = f(\precsim_I)$. In other words, f_D is constructed to agree with f on D. Since D is a free triple, Arrow's theorem applies to f_D and a person $j \in I$ exists who is dictator. Because f and f_D agree on D person j is also a dictator on D within f.

5. The notation $x \geq y$ means that each component of the vector x is at least as great as the corresponding component of vector y.

6. See Arrow [1, pp.75-80] for a discussion of single-peakedness.

7. The parameter vector θ such that $\| x - p \| = \| y - p \| = \| z - p \|$ may be determined as follows. Pick $i \in N$ and $j \in N$, $i \neq j$, such that the determinant

$$\begin{vmatrix} 2(y^i - x^i) & 2(y^j - x^j) \\ 2(z^i - x^i) & 2(z^j - x^j) \end{vmatrix}$$

does not vanish. Such a pair (i,j) must exist because x,y, and z are not colinear. Set $p_k = 0$ for all $k \in \{N - \{i,j\}\}$. Calculate p_i and p_j by solving the two linear equations:

$$2(y^i - x^i)p_i + 2(y^j - x^j)p_j = \sum_{k=1}^{n} [(y^k)^2 - (x^k)^2]$$

$$2(z^i - x^i)p_i + 2(z^j - x^j)p_j = \sum_{k=1}^{n} [(z^k)^2 - (x^k)^2]$$

8. Recall that $e_i = (e_{i1}, e_{i2}, \ldots, e_{in})$ where $e_{ij} = 1$ if $i=j$ and $e_{ij}=0$ if $i \neq j$.

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