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THE INCENTIVES FOR PRICE-TAKING BEHAVIOR WHEN CONSUMERS HAVE INCOMPLETE INFORMATION

by

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0. INTRODUCTION

There appear to be two fundamental incentive properties of the Walrasian market mechanism. First, in economies with a finite number of agents, consumers may be able to benefit by departing from competitive behavior. Second, as the number of consumers becomes large, the gains from non-competitive behavior typically go to zero. The second proposition has been established by Roberts and Postlewaite [8, p. 120, 124]. The first was established by Hurwicz [5, p. 330] who constructed an economy such that any decentralized allocation mechanism which selects Pareto-optimal, individually rational allocations has the property that one agent can gain in that economy by departing from the rules of the mechanism. Although it is generally accepted that in most economies a consumer may be able to manipulate Walrasian prices to his benefit, it remains an open question as to how extensive this phenomenon is. 1/

In the model of Hurwicz and that of Roberts and Postlewaite, it is assumed that consumers have enough information or can collect enough information to be able to compute a good non-competitive response. For the Walrasian mechanism, this requires at least local knowledge of the aggregate excess demand function of the other consumers. While such an informational assumption strengthens the Roberts-Postlewaite Theorems since competitive behavior occurs even if consumers know everything, 2/ it seems to weaken the Hurwicz theorem since one might conjecture that with less information consumers may be reluctant to depart from

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1/ Aside from the environment which Hurwicz constructed, it can be shown, for example, that gains from non-competitive behavior arise in any exchange economy with Cobb-Douglas preferences and initial endowments which are not Pareto-optimal.

2/ See [8, p. 117 especially footnote 4].

competitive behavior. Countering this conjecture is the plausible argument that finding a response which is better than the competitive response need not involve any information about other players. To see this, suppose that each consumer has chosen his strategy but that one consumer is allowed to alter his while the others keep theirs fixed. If there is a better strategy for that consumer one would expect that he would eventually find it. [3/] If one accepts this view of strategy selection then the Hurwicz theorem applies independently of any informational assumption. However, we would argue that the consumer who searches for a better strategy in the manner described is getting something for free. In particular, if he happens to choose a strategy which is worse than the competitive response, he is not penalized for (i.e., stuck with) his choice.

In this paper, we examine the impact on the incentive properties of the Walrasian mechanism of the assumptions that consumers possess incomplete information about the potential gains from non-competitive behavior and that they can not revise their strategy once it has been chosen. Under these conditions one might expect that price-taking behavior would be more likely to occur than under the assumption of complete information. In fact this is not true. Our conclusion, simply stated, is that <u>the introduction of incomplete information changes none of the incentive properties of the Walrasian mechanism</u>.

We begin in Section 1 by introducing the model of behavior, under complete information. In Section 2, we extend the model to one of behavior under incomplete information. Section 3 contains the theorems which relate incentive compatibility

[3/] That is, if the original choices do not constitute a Nash equilibrium (under complete information) then eventually a consumer will discover and use a better strategy even though he has limited information.

under complete information to incentive compatibility under incomplete information. An example which illustrates the model and theorems is presented in Section 4. Some concluding remarks about extensions of these results to a wider class of mechanisms and environments and about the relationship of these results to those in other papers are contained in Section 3.

1. Complete Information

1.a. CONSUMERS, ECONOMIES, AND COMPETITIVE EQUILIBRIA

To simplify the presentation I will consider only exchange economies with a fixed finite number, L, of commodities. In these economies a consumer is characterized by an admissible consumption set $X \subseteq R^L$ (the L-dimensional Euclidean space), a preference relation $\succsim$ on X, and an initial endowment vector $w \in R^L$. We will assume throughout that $\succsim$ is a continuous, complete preorder, that X is closed (with respect to the usual topology on R^L) and bounded from below.

An alternative representation of a consumer, which we will find useful, is to represent the pair $(X, \succsim)$ as a subset of R^{2L}. (See, e.g., Hildenbrand (1971).) We let $G = \{(x,x') \in X \times X \mid x \succsim x'\}$. Given G we can extract $\succsim$ and X where $X = \{x \in R^L \mid (x,x) \in G\}$. Assuming that $\succsim$ is continuous and X closed is equivalent to assuming that G is closed. Let $\mathcal{G}$ be the set of all those subsets of R^{2L} which are representations of continuous preorders on closed admissible consumptions sets. A particular consumer is then simply an element of the set $A \equiv \mathcal{G} \times R^L$ where an element of A is the pair (G,w). We will write, for consumer $a \in A$, $\succsim(a)$, $X(a)$, $w(a)$ respectively for his preferences, admissible consumptions, and endowment.

Let $P = \{p \in R^L \mid p_\ell \geq 0$ for $\ell = 1,\ldots,L,\ \Sigma_{\ell=1}^L p_\ell = 1\}$ be the standard (price) simplex in R_+^L. The competitive (price taking) response rule of a consumer is the set of trades which are preference maximizing given the budget determined by those prices.

Formally, we consider a mapping $C: A \times P \to R^L$. Given $a \in A$, $p \in P$ we let

$$C(a,p) = \{z \in R^L \mid z = x - w(a),\ x \text{ is } \succsim(a) \text{ maximal on } X(a) \text{ subject to } px \leq pw(a)\}.$$

$C(a,p)$ is called the competitive response of a to p and the correspondence $C(a,\cdot)$ is called the <u>competitive response rule</u> of consumer a.

A pure exchange economy is simply a collection of consumers, $\{a_i\}_{i\in I}$, where I is an index set and $a_i \in A$ for all $i \in I$. Given an economy $e = \{a_i\}_{i\in I}$, we say that p is a <u>competitive equilibrium price</u> whenever

$$0 \in \Sigma_{i\in I}\, C(a^i,p).$$

We let $W(e)$ be the set of all competitive equilibrium prices for e.

For any one consumer a_i, in the economy e, his set of possible competitive outcomes is

$$f_c^i(e) = \{x^i \in X(a_i) \mid x^i - w(a_i) \in C(a_i,p), \text{ for some } p \in W(e) \text{ and } \exists\, x^k \text{ for all } k \neq i \text{ such that } \Sigma_{k\in I}(x^k - w(a_k)) = 0 \text{ and } x^k - w(a_k) \in C(a_k,p)\}.$$

That is, $f_c^i(e)$, is the set of all consumptions for i which are consistent with a competitive equilibrium allocation for the economy e.

1.b. INCENTIVES FOR NON-COMPETITIVE BEHAVIOR

It has been well documented that consumers in a market economy e may be able to improve their situation by following non-competitive behavior (see Hurwicz [5] or Ledyard [6]). In particular, it is possible that (say) consumer a in e could behave <u>as if</u> he is really a^*. If he were to do so, the set of possible outcomes he could attain is $f_c^1(e^*) = f_c^1[a^*,a_2,\ldots,a_N]$. In many cases it is possible for 1 to choose a^* such that for any $\bar{x}^1 \in f^1(e)$ and any $x^1 \in f^1(e^*)$ it is true that $x^1 \succ (a)\ \bar{x}^1$. In this case we would say that consumer 1 has an incentive to misrepresent his characteristics.[4/] That is, he acts as if he is competitive, but he uses the response rule $C(a_1^*, \cdot)$ instead of the "correct" rule $C(a_1, \cdot)$.

An alternative formulation of the above situation which actually allows for a wider range of possible misbehavior can be found in Roberts and Postlewaite [8]. There they consider a class of non-competitive response rules for each agent. We adopt this approach and allow consumer $a \in A$ to use any rule (correspondence) $S(.)$ which satisfies: given $a \in A$, $p \in P$,

(1) $z \in S(p) \Rightarrow z = x - w(a)$ for some $x \in X(a)$,

(2) $z \in S(p) \Rightarrow p \cdot z \leq 0$,

and (3) S is upper-semi continuous, convex valued and non-empty at P

That is, consumer a may respond in any way he desires to prices so long as he (1) is able to survive if he completes the trade z and (2) can financially complete the trade.[5/] We let $\mathscr{S}(a)$ be the set of all such correspondences from P to R^L. We note in passing that if $a' \in A$, $w(a') = w(a)$, and $X(a') = X(a)$ then $C(a', \cdot) \in \mathscr{S}(a)$. That is, misrepresentation of preferences is still a possible non-competitive behavior. However, there are many $S(\cdot) \in \mathscr{S}(a)$ which do not arise from $\succsim$ - maximization, (e.g., those $S(\cdot)$ which violate the strong axiom of revealed preference).

4/ Hurwicz [5] and Ledyard [6] require all misbehavior to be on this form. McFadden [7] also imposed a similar restriction which he called "orthodox pseudo-competitive" behavior.

5/ (3) can be deleted at the cost of possible non-existence of market clearing prices. This lack of existence does not alter the theorems of Section 3 but does require some model changes which tend to obscure the main results.

Given an economy e and a set of response rules $S = \{S^i(\cdot)\}_{i \varepsilon I}$, such that $S^i \varepsilon \mathscr{A}(a_i)$ for all $i \varepsilon I$, we say that a price $p \varepsilon P$ is <u>market clearing</u> if $0 \varepsilon \Sigma_{i \varepsilon I} S^i(p)$. We let $Q(S)$ be the set of all market clearing prices[6] given the response rules S. If $S^i(\cdot) = C(a_i, \cdot)$ for all i, then $Q(S) = W(e)$, the set of competitive equilibrium prices. The set of possible outcomes for agent a_i in e given S is

$$f^i(S,e) = \{x^i \varepsilon R^L \mid x^i - w(a_i) \varepsilon S^i(p),$$

$$\Sigma_{k \varepsilon I}(x^k - w(a_k)) = 0,\ x^k - w(a_k) \varepsilon S^k(p)$$

for $k \varepsilon I$ and $p \varepsilon Q(S)\}$.

As in the case of misrepresentations, it is possible that there is some $S^1 \varepsilon \mathscr{A}(a_1)$ such that $x^1 > (a_1)\ \bar{x}^1$ for all $x^1 \varepsilon f^1(S,e)$ and all $\bar{x}^1 \varepsilon f^1(S/C(a_1,\cdot),e)$ where $S/C(a_1,\cdot)$ is the collection of response rules derived from S by replacing S^1 with $C(a_1,\cdot)$. If such S^1 exists then 1 would have an incentive to behave non-competitively.

The model we have been examining leads naturally to a game theoretic model of non-competitive behavior within the institutional setting of a Walrasian market system. In this game given an economy $\{a_i\}_{i \varepsilon I}$, the <u>players</u> are the consumers. The <u>admissible strategy</u> set of i is $\mathscr{A}(a_i)$ and the <u>payoff function</u> of $i \varepsilon I$ is $V^i(\{S^i\}_{i \varepsilon I})$ where $V^i(S) = u^i[f^i(S,e)]$ with $u^i(x^i) \geq u^i(\bar{x}^i)$ iff $x^i \succsim (a_i)\ \bar{x}^i$. We note that $f^i(S,e)$ may not be a single allocation (e.g. when $Q(S)$ is not single-valued) which will mean that $V^i(\cdot)$ is not really well-defined. This is a difficulty we will confront in the next section.

Supposing, for now, that f^i is a function for each i, we say that

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6/ If $S \varepsilon \mathscr{A}(a_i)$ for all i, $Q(S) \neq \emptyset$.

the Walrasian system is individually incentive compatible in the class of economies, E, if $\bar{S} = [C(a_1, \cdot),\ldots,C(a_N, \cdot)]$ is a Nash equilibrium of the above game for all $e \in E$. That is, incentive compatibility obtains if, given price-taking behavior of the other consumers, no consumer can, through non-competitive behavior, induce the system to select an allocation he prefers to that he obtains if he takes prices as given.

1.c. A SLIGHT REVISION

As indicated above, the fact that the payoff to i, $V^i(S)$, from the vector of response correspondences, $(S^1,\ldots,S^N) = S$, may not be single-valued creates difficulties in using a game theoretic approach to analyze incentives. Roberts and Postlewaite [8] neatly sidestep this issue. However, the introduction of incomplete information requires us to confront it. We resolve the problem in a way which is naive but which is consistent with the model we introduce in the next section.

We will simply assume that each agent presumes that a single market clearing price is selected randomly given $Q(S)$. We do this by introducing a function γ^i from the set of subsets 7/ of P to the space $\mathcal{M}(P)$ of probability measures on P. We assume that there exists a function $U: A \times R^L \to R^1$ such that $U(a,\cdot)$ represents $\geq(a)$ on $X(a)$, and such that $U(a,\cdot)$ is a Von-Neumann-Morgenstern utility function for a.

We define

$$U^i[e,\gamma] = \int_P U(a^i, w(a^i) + S^i(p))\,d\gamma^i[Q(S)]$$

to be the utility consumer i receives in economy e from the joint strategy

7/ In general, we would expect that $\gamma^i(Q)$ is concentrated on Q.

S under randomization γ^i.

With this approach we can now define incentive compatibility in an unamgibuous manner.

Definition 1.1: Let $E \equiv \{(e,\gamma)\}$ be a set of exchange economies paired with a vector of randomization rules. The Walrasian mechanism is individually incentive compatible in E if, for all $(e,\gamma) \in E$, $\langle C(a^i,\cdot)\rangle_{i\in I}$ is a Nash equilibrium of the game with payoff functions

$$V^i(S) = \int U^i(a^i, w(a^i) + S^i(P))d\gamma^i[Q(S)]$$

and strategy spaces $\mathscr{S}^i(a^i)$.

That is, incentive compatibility obtains if, for each $i \in I$,

$$V^i(C) \geq V^i(C/S^i) \quad \text{for all} \quad S^i \in \mathscr{S}^i(a^i).$$

2. Incomplete Information [8/]

To model consumer behavior when there is incomplete information we consider the natural extension of the game, introduced in Definition 1.1, to one with incomplete information.

Let $Y \equiv \times_{i \in I} Y^i$ where an element $y^i \in Y^i$ is to be interpreted as the type of consumer i where the concept of type includes endowments, preferences, and beliefs about the economy. Thus, $y \in Y$ is a vector of types of consumers. We assume throughout that Y is a metric space. Associated with each $y \in Y$ is a complete information economy,

$$e_y = \langle\{a(y^i),\ S(y^i)\}_{i\in I}\rangle$$

8/ This section is based on the model developed by Harsayani [4].

where $a(y^i)$ are the true preferences and endowments of i if he is of type y^i and $S(y^i) \in \mathscr{S}(a(y^i))$ is i's response correspondence if he is of type y^i. Also associated with each $y \in Y$ and each i is $\gamma_y^i \in \mathscr{M}(P)$ where $\gamma_y^i(B)$ describes i's a priori beliefs concerning the probability of the occurrence of a particular equilibrium price when B is the set of market clearing prices.

To close the model, for each i there is a probability measure $\eta^i \in \mathscr{M}(Y)$ which describes his a priori beliefs concerning the likelihood of a vector of types y.

We let $\mathscr{E} \equiv \langle I, (e_y), (\gamma_y^r), (\eta^r)\rangle$ and call $\mathscr{E}$ an exchange economy under incomplete information. The payoff to $r \in I$ in the economy $\mathscr{E}$ is

$$U^r(\mathscr{E}) = \int_Y U^r(e_y, \gamma_y^r)\, d\eta^r = \int U[a(y^r), w[a(y^r)] + S(y^r)(p)]\, d\gamma_y^r[Q(e_y)]\, d\eta^r .$$

To ensure that U^r is well-defined, we assume throughout that for all $S \in \mathscr{S}(a(y^r))$ $U[a(y^r), w[a(y^r)] + S(y^r)(p)]$ is bounded and integrable over Y and P.

In the complete information model in section 1, the incentive for non-competitive behavior was modeled as the payoff from using, as a strategy, a response rule S different from the competitive response C. In an economy with incomplete information this corresponds to choosing, for each y^r, a response rule in $\mathscr{S}[a(y^r)]$. Of particular interest are best replay strategies which in this model are strategies which maximize the conditional expected gain. We let $\eta^r(\cdot|y^r)$ be the conditional measure on Y, given $y^r \in Y^r$, derived from η^r. Further let $U^r(\mathscr{E}, y^r) \equiv \int U^r(e_y, \gamma_y^r)\, d\eta^r(\cdot|y^r)$. A function $\beta^r: Y^r \to \mathscr{S}$ such that $\beta^r(y^r) \in \mathscr{S}[a(y^r)]$ for all $y^r \in Y^r$ is called r's uniform best replay in $\mathcal{E}$ if for all $y^r \in Y^r$, $U^r(\mathcal{E}/\beta^r,y^r) \geq U^r(\mathcal{E}/S^r,y^r)$ for all S^r such that $S^r(y^r) \in \mathcal{A}(a(y^r))$ for all $y^r \in Y^r$. A vector of strategies $(\beta^1,\ldots\beta^I)$ is called a Bayes equilibrium if for each r, β^r is a uniform best replay in $(\mathcal{E}/\beta)$ where $(\mathcal{E}/\beta)$ is the economy $\mathcal{E}$ with S^r replaced by β^r for each $r \in I$.

We can now introduce the concept of incentive compatibility under incomplete information.

Definition 2.1: The Walrasian mechanism is incentive compatible over the class of incomplete information economies $\hat{\mathcal{E}} \equiv \{\mathcal{E}\}$ if for each $\mathcal{E} \in \hat{\mathcal{E}}$ it is true that the vector of competitive response rules $\{C(a(y^r),\cdot)\}_{r \in I}$ is a Bayes equilibrium.

3. Incentive Compatibility

In this section, we explore the relationship between incentive compatibility in complete information and incomplete information economies. Our conclusion (summarized in Theorem 3.6 and Remark 3.7) is that if enough prior beliefs must be covered then the Walrasian mechanism is incentive compatible under incomplete information if and only if it is incentive compatible for all underlying complete information economies.

We begin by stating an obvious fact.

Theorem 3.1: Let $\mathcal{E}$ be an exchange economy under incomplete information. If the Walrasian mechanism is incentive compatible in (e_y,γ_y) for all $y \in Y$ then the Walrasian mechanism is incentive compatible in ε .

Proof: Incentive compatibility in (e_y, γ_y) implies that, for each $\tilde{r} \in I$,

(1) $U^r(e_y/C^r_y,\gamma^r_y) \geq U^r(e_y/S^r_y,\gamma^r_y)$ for all $S^r_y \in \mathscr{S}(a(y^r))$ where

$C^r_y \equiv C[a(y^r),\cdot]$ and $U^r(e_y/S^r_y,\gamma^r_y) \equiv \int_P U[a(y^r),w[a(y^r)]$

$+ S^r_y(p)]d\gamma^r[Q(e_y/S^r_y)]$.

Since (1) holds for all $y \in Y$, it follows that, for all $y^r \in Y^r$,

$$\int U^r(e_y/C^r_y,\gamma^r_y)d\eta^r(\cdot|y^r) \geq \int U^r(e_y/S^r_y,\gamma^r_y)d\eta^r(\cdot|y^r) \text{ for all } S^r_y \in \mathscr{S}[a(y^r)].$$

Q E D

Thus, incentive compatibility for all underlying complete information economies implies incentive compatibility for the incomplete information economy. Of more interest and less obvious is the fact that if one requires incentive compatibility to obtain under incomplete information for a wide enough range of prior beliefs then incentive compatibility must obtain for all underlying complete information economies.

Given a set Y, let $\mathscr{T}(Y) \subseteq \mathcal{M}(Y)$ be the set of measures on Y which are concentrated on a single element y. That is $\eta \in \mathscr{T}(Y)$ if there is $y^o \in Y$ such that $\eta(\{y^o\}) = 1$. Let $C(Y) \subseteq \mathcal{M}(Y)$ be the set of measures on Y which are representable by continuous density functions. That is, $\eta \in C(Y)$ if there is a continuous function $h:Y \to \mathbb{R}^1$ such that for all Borel subsets $B \subseteq Y$, $\eta(B) = \int_B h(y)dy$. We will be interested in the class of incomplete information economies generated from one such economy by varying prior beliefs, $\eta \in \mathcal{M}(Y)$. Given $\mathscr{E}^*$, and $R \subseteq \mathcal{M}(Y)$ we let $\mathscr{E}^*_R$ be the class of incomplete information economies such that $\mathscr{E} \in \mathscr{E}^*_R$ if $\mathscr{E} \equiv \langle I^*, (e^*_y), (\gamma^{r*}_y), \eta^r\rangle$ and $\eta^r \in R$ for all $r \in I$.

<u>Theorem 3.2</u>: Let $\mathscr{E}^*$ be an arbitrary incomplete information economy. If for all $\mathscr{E} \in \mathscr{E}^*_{\mathscr{T}(Y)}$ the Walrasian mechanism is incentive compatible, then

the Walrasian mechanism is incentive compatible in (e_y, γ_y) for all $y \in Y$.

Proof: Suppose not. then there exists $y \in Y$, $r \in I$, and $S \in \mathscr{S}[a(y^r)]$ such that $U^r(e_y/S,\gamma_y^r) > U^r(e_y/C_y^r,\gamma_y^r)$. Consider $\hat{\mathcal{E}} \in \mathcal{E}^*_{\mathcal{J}(Y)}$ where $\hat{\eta}^r(\{y\}) = 1$. Then

$$\int U^r(e_y/S,\gamma_y^r)\,d\hat{\eta}^r(\cdot|y^r) > \int U^r(e_y|C_y^r,\gamma_y^r)\,d\hat{\eta}^r(\cdot|y^r),$$

which contradicts the hypothesis of incentive compatibility over $\mathcal{E}^*_{\mathcal{J}(Y)}$.

Q E D

The result in Theorem 3.2 is the trivial observation that if there is an underlying economy in which r can gain from non-competitive behavior and r thinks he is in that economy with probability one, then incentive compatibility cannot obtain even though information is "incomplete" and, even though, in fact, consumer r may be wrong.

Remark 3.3: It could be fairly said that, in requiring incentive compatibility for single point distributions, we have not retained the spirit of incomplete information and that prior beliefs should really be more diffuse. However, it is easy to show that if, in Theorem 3.2, we replace $\mathcal{J}(Y)$ with any set $B \subseteq \mathcal{M}(Y)$ which has the property that if $y \in Y$, there is an $\eta \in B$ such that $\eta(\{y\}) > 0$ then the conclusion of that theorem still follows since $\mathcal{J}(Y) \subseteq B$.

This raises an obvious question. Will the conclusion still hold if we replace $\mathcal{J}(Y)$ with $C(Y)$, the space of measures with continuous densities since for $\eta \in C(Y), \eta(\{y\}) = 0$ for all $y \in Y$?

Theorem 3.4: Let $\mathcal{E}^*$ be an incomplete information economy with the property that for each $\bar{y} \in Y$, if $S^* \in \mathscr{S}[a(\bar{y}^r)]$ then there is a neighborhood N of $\bar{y}$ such that $V^r(y,S^*\mathscr{E}^*)$ is well defined and continuous on N where $V^r(y,S^*,\mathscr{E}^*) \equiv \bar{U}^r(e_y/S^*,\gamma_y^r) - \bar{U}^r(e_y,\gamma_y^r)$ and $\bar{U}^r(e_y,\gamma_y^r) \equiv$ $\int U(a(y_r),w[a(y_r)] + C[a(y^r),p])\, d\, \gamma_y^r[W(e_y)]\, d\, \eta^r$, the payoff to r if all consumers use their competitive response rules. If for all $\mathscr{E} \in \mathscr{E}^*_{C(Y)}$ the Walrasian mechanism is incentive compatible then the Walrasian mechanism is incentive compatible in (e_y,γ_y) for all $y \in Y$.

Proof: Suppose not, then there is $\bar{y} \in Y$ and $\bar{S} \in \mathscr{S}(a(\bar{y}^r))$ such that $V^r(\bar{y},\bar{S},\mathscr{E}) > 0$. By continuity there is a neighborhood $\bar{N}$ of $\bar{y}$ such that $V_r(y,\bar{S},\mathscr{E}) > 0$ for all $y \in \bar{N}$. Choose an $\eta^r \in C(Y)$ which has the continuous density $h(\cdot)$ such that $h(y) > 0$ for $y \in \bar{N}$ and $h(y) = 0$ otherwise.[9] Then $\int U^r(e_y/\bar{S},\gamma_y^r) - U^r(e_y/C_y^r,\gamma_y^r)\, d\, \eta^r(\cdot|y^r) > 0$, which contradicts the hypothesis of incentive compatibility over $\mathscr{E}^*_{C(Y)}$.

Q E D

Remark 3.4: Theorem 3.4 remains valid even if we replace $C(Y)$ with its subset $C_+(Y)$, the set of measures whose continuous densities are positive over all Y as long as Y has finite Lebesque measure. To see this consider the choice of $\eta^r \in C(Y)$ in the proof. Let $h^*_\delta(y) = h(y) + \delta$ on $\bar{N}$ and $h^*_\delta(y) = \delta > 0$ otherwise. It is easy to show that there is a small enough $\delta > 0$ such that if η^r_δ has the continuous density $g_\delta(y) = h^*_\delta(y)/\delta \cdot \int_Y dy$ then $\int V^r(y,\bar{S},\mathscr{E})g_\delta(y)\, d\, y > 0$ which will establish the result.

Remark 3.5: In Theorem 3.4 we relied heavily on the property of $\mathscr{E}^*$ that $V^r(y,\bar{S},\mathscr{E})$ was continuous in a neighborhood of $\bar{y}$. The following example shows

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9/ That this can be done follows from Urysohn's Lemma.

that there are $\mathcal{E}^*$ which satisfy this condition and, therefore, that the theorem is non-vacuuous. We use the two commodity exchange economies with Cobb-Douglas preferences for each consumer. Let $u(\theta,x) = (x_1 + w_1)^{\alpha}(x_2 + w_2)^{1-\alpha}$ where $\theta = (\alpha,w_1,w_2)$ and x is the net trade of consumer θ. The competitive response rule of consumer θ is $x_1(\theta,p) = [(1-q)/q]\,\alpha\, w_2 - (1-\alpha)w_1$ where $[q,(1-q)] \equiv p$, and $x_2(\theta,p) = -[q/(1-q)]\, x_1(\theta,p)$. We can represent an exchange economy as $(\theta^1,\ldots,\theta^I)$ and $Q(\theta) = \{p \in P | \Sigma_i x_1(\theta^i,p) = 0\}$. Hence $Q(\theta) = [q(\theta),1-q(\theta)]$ where $q(\theta) = [1 + (\Sigma_i(1-\alpha^i)\, w_1^i/\Sigma_i\alpha^i w_2^i)]^{-1}$. If $\theta \in \Theta \equiv \{(\alpha,w_1,w_2) |\ 1 > \alpha > 0,\ w_1 > 0,\ w_2 > 0\}$ then $q(\theta)$ is well-defined. Let $Y^i \equiv \Theta$ for all i where $a(y^i) = \theta^i$ and let $\mathcal{A}(a(y^i)) \equiv \mathcal{A} = \{S:P \to R |$ $S(\cdot) \equiv x(\theta,\cdot)$ for some $\theta \in \Theta\}$ for all $y^i \in Y^i$. Thus, $\mathcal{A}$ consists of all response rules which could be generated by some $\theta \in \Theta$. With some abuse of notation we let $\mathcal{A} \equiv \Theta$. Since $Q(\theta)$ is unique we assume that, for all $y \in Y$, $\gamma_y^r[Q(y)](B) = 1$ if $p(y) \in B$. Then $V^r(y,S,\mathcal{E}) \equiv U[y^r,x(s,Q[y/S])]$ $-U[y^r,x(y^r,Q[y])]$. Since U is continuous in (θ,x), x is continuous in (θ,p), and $q(\cdot)$ is continuous in θ, it follows easily that V^r is continuous in y for all $y \in \times_{i=1}^{I} Y^i \equiv \Theta^{(I)}$.

Combining Theorems 3.2 and 3.1 we get

<u>Theorem 3.5</u>: The Walrasian mechanism is incentive compatible over a class $\hat{\mathcal{E}}$ of incomplete information economics with the property that $\mathcal{E} \in \hat{\mathcal{E}}$ implies $\mathcal{E}_{\mathcal{J}(Y)} \subseteq \hat{\mathcal{E}}$ if and only if the Walrasian mechanism is incentive compatible in all underlying complete information economies.

<u>Remark 3.6</u>: Similar equivalence theorems can be stated for the cases considered in Theorem 3.4, Remark 3.3 and Remark 3.5.

4. An Example

In this section, we present an example of an incomplete information economy which we hope serves several purposes. First, it provides a specific example of the model and the concepts of Bayes equilibrium and incentive compatibility under incomplete information. Second, it shows how unlikely it is that incentive compatibility obtains under complete information. Third, it shows how unlikely it is that incentive compatibility obtains under incomplete information, unless prior beliefs are severely restricted.

Suppose that Mr. 1 knows Mr. 2 has preferences which can be represented by a Cobb-Douglas utility function, $u^2 = x_1^{\gamma} x_2^{1-\gamma}$, and an initial endowment of (b,b). However, he only knows that $\gamma \in [0,1]$ but not its precise value. Suppose also that Mr. 2 knows Mr. 1 has preferences which can be represented by $u^1 = \alpha x_1 + (1-\alpha)x_2$ and an endowment of (a,a). However, he only knows that $\alpha \in [0,1]$.

Given α and γ, the market clearing price is $(q,1-q)$ where $q = \alpha$ Mr. 1's allocation is $[a + b - (b\gamma/\alpha), a + b - b(1-\gamma)/(1-\alpha)]$. Mr. 2's allocation is $[b\gamma/\alpha, b(1-\gamma)/(1-\alpha)]$. Under complete information, Mr. 1's gain from preference misrepresentation is $V^1(\alpha,\gamma,\alpha^*) \equiv \alpha^*[a + b - (b\gamma/\alpha)] + (1-\alpha^*)[a + b - b(1-\gamma)/(1-\alpha)]$ where α^* is his true parameter and α is his "reported" parameter. Maximizing with respect to α implies $\alpha^2/(1-\alpha)^2 = \gamma\alpha^*/(1-\gamma)(1-\alpha^*)$. Hence, $\alpha = \alpha^*$ only if $\gamma = (1-\gamma) = \frac{1}{2}$. Thus, almost always Mr. 1 can gain from a non-competitive response.

Mr. 2's payoff is $V^2(\alpha,\gamma,\gamma^*) \equiv [b\gamma/\alpha]^{\gamma^*}[b(1-\gamma)/(1-\alpha)]^{(1-\gamma^*)}$ where γ^* is true parameter. It is easy to show that γ^* is a maximizer over all γ. That is Mr. 2 cannot gain from misrepresentation -- due to Mr. 1's linear indifference curves.

Remark 4.1: This example shows how rarely the Walrasian mechanism is incentive compatible in this class of environments. In particular only if $\gamma^* = \frac{1}{2}$ does incentive compatibility obtain. (This corresponds to the theorems in Ledyard [6] in that the core of this economy is single-valued only when $\gamma^* = \frac{1}{2}$.)

We now model the incomplete information situation. Let $Y^i = (0,1)$ and let $\eta^1 = \eta^2 = \eta$ be a measure on $Y = Y^1 \times Y^2$ representable by a continuous density function $f(y^1,y^2)dy^1dy^2$. We let $\beta^i:(0,1) \to (0,1)$ be a preference misrepresentation strategy. To interpret if $\beta^1(\frac{1}{2}) = 2/3$ then when Mr. 1's true utility is $u^1 = \frac{1}{2} x_1 + \frac{1}{2} x_2$ he will act as if it is $2/3\ x_1 + 1/3\ x_2$. We can therefore write the payoff to i of the joint strategy β given y^i as $W^i(\beta,y^i)$ where

$$W^1(\beta,y^1) = \int_0^1 V^1[\beta^1(y^1),\beta^2(y^2),y^1]\ f(y^1,y^2)dy^2$$

and

$$W^2(\beta,y^2) = \int_0^1 V^2[\beta^1(y^1),\beta^2(y^2),y^2]\ f(y^1,y^2)\ dy^1.$$

It is fairly easy to establish that (β^{*1},β^{*2}) is a Bayes-equilibrium if $\beta^{*2}(y^2) = y^2$ and $\beta^{*1}(y^1) = z$ where $z^2/(1-z)^2 = A\ y^1/B(1-y^1)$, $A = \int y^2 F(y^1,y^2)\ dy^2$, $B = \int (1-y^2)f(y^1,y^2)\ dy^2$.

Remark 4.2: The fact that $\beta^{*2}(y^2) = y^2$ for all y^2 follows also from Theorem 3.1 and the fact that for all (α^*,γ^*), $V^2(\alpha^*,\gamma^*,\gamma^*) \geq V^2(\alpha^*,\gamma,\gamma^*)$ for all $\gamma \in (0,1)$.

Remark 4.3: It follows from the form of $\beta^{*1}(y^1)$ that $\beta^{*1}(y^1) = y^1$

only if y^1 equals the conditional expected value of y^2 given y^1. For any beliefs η such that for a set of positive measure $y^1 \neq \int y^2 f\, dy^2$, Mr. 1 has an incentive to follow non-competitive behavior. Thus, only for very special η can it be true that the Walrasian mechanism is incentive compatible. A simple example where incentive compatibility does not obtain can be constructed by assuming y^1 and y^2 are independently and identically distributed.

5. Some Concluding Remarks

In this section are collected some observations about the relationship between the results of this paper and those of others.

Remark 5.1: In his article on incentive compatibility, Hurwicz [5] established the fundamental proposition that there is no decentralized adjustment mechanism which is individually rational, yields Pareto-optimal allocations and is incentive compatible if the class of economies is large enough. This implies in particular that the Walrasian mechanism is not incentive compatible over such a class. The example of Section 4 of this paper as well as the theorems of Section 3 indicate, therefore, that even with incomplete information the Walrasian mechanism will not be incentive compatible if the class of economies is large enough.

Remark 5.2: No special properties of the Walrasian mechanism were used in the proofs of any theorems in this paper. This suggests that all the theorems would hold if we substitute any arbitrary decentralized, privacy preserving [10/]

[10/] A mechanism is privacy preserving if, for example, the outcome depends solely on the reported information of the agents, and if each agent knows only his own component of the true situation (y^i in our terminology).

allocation mechanism for the Walrasian process. Such an extension of the results is reasonably straight-forward; however we will not present it here.

Remark 5.3: We considered models with only private goods in this paper. However, none of the theorems depend on this assumption. This again suggests that all the theorems would hold for classes of environments with externalities and public goods. Again, such a generalization is not presented here.

Remark 5.4: In a recent paper, d'Aspremont and Gerard-Varet [1] claim to have discovered an allocation mechanism (based on one used by Groves and Loeb [3]) which is incentive compatible for a class of incomplete information economies with public goods. This result appears to contradict the suggestions for generalization contained above in Remarks 5.2 and 5.3 for the following reason. The mechanism developed by Groves and Loeb, which does induce agents to submit truthful information, does not lead to Pareto-optimal allocations since a portion of resources must usually be destroyed.[11/] If the mechanism is adjusted to save these resources, incentive compatibility is usually destroyed. d'Aspremont and Gerard-Varet appear to have overcome this difficulty (see proposition 5 in their paper) through the use of incomplete information. Thus although the mechanism is not usually incentive compatible under complete information it appears to be incentive compatible under incomplete information for a wide range of prior beliefs. This would be contrary, for example, to the appropriate generalization of theorem 3.4. However, this apparent conflict can be resolved by noting that their outcome rule, which determines the allocation as a function of the reported response rules of the agents, is a function of

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11/ See Groves and Ledyard [2] for more on this difficulty.

the true prior beliefs of the agents (η^r in our terminology). Thus, strictly speaking their mechanism is not privacy preserving and, therefore, does not come under the generalization referred to in Remark 5.2. Indeed, since the outcome for their rule does depend on the prior beliefs of the agents then it must be implicitly assumed that all agents know each others priors. However, under privacy knowledge of these priors is gained only through communication and, therefore, the possibility exists than an agent can gain by manipulating the d'Aspremont-Gerard-Varet scheme through misrepresentation of his prior beliefs. If this is true then their mechanism is incentive compatible only if all prior beliefs are a priori known which seems to contradict the spirit of incomplete information and decentralization.

Remark 5.5: In their article on the incentives for price-taking behavior in large economies, Roberts and Postlewaite [8] show that usually the gains from non-competitive behavior go to zero as the number of consumers grows large. In particular they consider a sequence of (complete information) economies $\{e^k\}_{k=1}^{\infty}$ such that $I^k \to \infty$ and show under certain additional assumptions that for any $\varepsilon > 0$ and each continuous utility, u, there is k^* such that $k \geq k^*$ implies that for each x which an agent obtains through misrepresentation in e^k there is a competitive allocation y to him in e^k such that $u(y) > u(x) - \varepsilon$. Loosely speaking for $e \in \{e^k\}_{k \geq k^*}$ the gain from non-competitive behavior is less than ε. Referring to the proof of Theorem 3.1, if we replace (1) with $U^r(e_y/S_y^r, \gamma_y^r) < U^r(e_y/\gamma_y^r) - \varepsilon$ then it follows that $U^r(\mathcal{E}/S^r) < U^r(\mathcal{E}/C^r) - \varepsilon$. That is, if the gain to non-competitive behavior is less than ε in all underlying complete information economies then the gain from non-competitive behavior is less than ε in the incomplete information economy. It is also easy to see that all other theorems in section 3 can be amended in this fashion. Thus, loosely speaking, if enough prior beliefs must be covered then gains from non-competitive behavior are always small under incomplete information if and only if those gains are always small under complete information.

Remark 5.6: The conclusion to be drawn from all these remarks is that, unless one severely restricts the class of possible prior beliefs or assumes that all prior beliefs are known to everyone (a violation of privacy), incomplete information changes none of the essential incentive properties of the Walrasian mechanism in particular and decentralized allocation mechanisms in general.