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SELF-FINANCING OF AN R & D PROJECT

by

Morton I. Kamien and Nancy L. Schwartz

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Managerial Economics and Decision Sciences
Graduate School of Management
Northwestern University
Evanston, Illinois 60201

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The firm must determine whether new product development is worthwhile and, if so, an introduction date and spending plan (employing only internally generated funds) that will maximize its discounted expected profits. Since competing development plans are unknown and development of a new product already preempted by another is assumed worthless, profits can be known only probabilistically.

The Model

The firm earns profits at the constant rate $\pi \geq 0$ per unit time from the sales of its current product. These profits will continue until the product is displaced by the firm's new product or appearance of a rival substitute product. The class of potential rivals is large and diffuse, possibly including some firms that are currently in the same line of business, firms in other businesses, and newcomers. The firm knows neither the composition of this group of potential rivals, nor precisely when a rival product will be introduced, nor by whom. Its beliefs regarding the introduction date of a rival product are summarized by a probability distribution $F(t)$, where $F(t)$ is the probability that a rival product will appear by time t . In particular we assume the exponential form $F(t) = 1 - e^{-ht}$. The conditional probability of rival product introduction at any time t , given that it has not yet appeared, is $F'/(1-F) = h$, a constant, and the expected introduction date of the rival product is $1/h$. Thus the parameter h , often called the hazard rate, reflects the intensity of innovational rivalry perceived by the firm in the sense that a higher value of h is associated with the expectation of more imminent introduction of a rival product.

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Introduction

Industrial R & D projects are typically financed internally from the firm's current profits and accumulated funds. This occurs for several reasons. First, external financing may be difficult to obtain without substantial tangible assets associated with the R & D to be claimed by the lender if the project fails. Second, the firm might be reluctant to reveal detailed information about the project that would make it attractive to outside lenders, fearing leaks to potential rivals. Although a firm may be able to obtain external financing for an R & D project if it has a good record in such ventures, we restrict attention in this paper to situations in which funding is entirely internal. Analysis of a similar problem with external financing available was conducted in [3]. We focus upon the optimal development plan for a new product when there may be others with similar projects, the circumstances under which financing is a binding constraint, and the impact of the financing constraint upon the development plan both with and without potential rivals in new product development.

We envision a firm contemplating developing a new product to replace its current one to enhance profits. Alternatively, it recognizes that its product might be displaced by some rival product at an unknown future date and hopes to resist possible loss of profits by developing and marketing a new product of its own. The Industrial Research Institute refers to the class of R & D projects characterized by the latter description as in "support of existing business", see Brown [2].

Since the probability a rival product will not have appeared by time t is e^{-ht} , the expected present value of profits from continued indefinite sale of the firm's existing product is

$$(1) \int_0^{\infty} e^{-(r+h)t} \pi dt = \pi / (r+h)$$

where $r > 0$ denotes the discount rate, the earnings rate on cash balances. The profit stream available from product innovation is assumed to have value P when discounted to the moment of introduction. In order for new product development to be attractive, the reward from innovation must exceed the expected profit from failing to do so. We assume therefore that

$$(2) P > \pi / (r+h)$$

Development of the new product requires accumulation of effective development effort in amount A . This is achieved by efficacious expenditure of money through time. The contribution to effective effort of any expenditure rate $y(t)$ is governed by a monotone increasing, concave function. In particular, letting $z(t)$ denote the amount of effective development effort accumulated by time t , we assume that

$$(3) z'(t) = y^{\frac{1}{2}}(t), \quad z(0) = 0, \quad z(t) = A$$

Effective effort is initially zero, accumulates as the square root of development spending (reflecting diminishing returns to faster spending found in empirical studies by Mansfield [5] and others), and must equal A for successful development to be achieved by time T .

Let $R(t)$ denote the firm's cash balance at time t . It is augmented by interest earnings at rate r on the principal R and by profit π on the current product; it is diminished by expenditures $y(t)$ on R&D.

Hence movement of R is described by

$$(4) \quad R'(t) = rR(t) + \pi - y(t), \quad R(0) = R_0 \geq 0$$

Combining our assumptions about rewards from the current and new products, the development function, and cash balances, we can now state the firm's expected profit maximization problem. A planned introduction date $T^* > 0$ and development expenditure plan $y^*(t) \geq 0, 0 \leq t \leq T^*$ are to be chosen to

$$(5) \quad \text{maximize} \quad \int_0^T e^{-(r+h)t} (\pi - y(t)) dt + e^{-(r+h)T} P$$

subject to (3), (4), and

$$(6a) \quad R(t) \geq 0, \quad 0 \leq t \leq T$$

The objective functional (5) reflects the firm's profit from its existing product and expenditures on new product development so long as no new rival product has appeared, as well as the reward from the new product provided no rival product appears before T . If a rival product does appear, the firm receives and spends nothing thereafter. A solution in which $T^* \rightarrow \infty$ indicates that new product development is not worthwhile.

Although nonnegative cash balances are required at all times, we now show that it suffices to require only

$$(6b) \quad R(T) \geq 0$$

The proof rests on a result to be developed later in Proposition I that the solution to (3)-(5), (6b) involves $y'(t) > 0$.

Lemma. If $y(t)$ is a continuous function with $y'(t) > 0$ for

$0 \leq t \leq T$, and if (4) holds, then (6b) implies

$$(7) \quad R(t) > 0, \quad 0 < t < T$$

Proof. First suppose $R_0 > 0$. Assume the hypotheses of the Lemma hold but (7) does not. Let $0 < t_0 < T$ be the first moment that $R = 0$. Then the left hand derivative $R'(t_0) \leq 0$, and since $y' > 0$, $R(t_0) = 0$, and π constant, it follows from (4) that R can only decrease further. Repeating this argument with $R < 0$, R must decrease still further and indeed $R(t) < 0$ for $t_0 < t \leq T$. This contradicts (6b). Hence (6b) can be satisfied only if (7) holds.

Now suppose the hypotheses of the Lemma hold with $R_0 = 0$. If $R'(0) > 0$, the preceding paragraph applies with t_0 the first $t > 0$ that $R = 0$. If $R'(0) < 0$, then once R becomes negative, the above reasoning leads to the conclusion that (6b) cannot be satisfied unless (7) is. Finally, if $R(t) = 0$ for some initial time period, then from (4), $y = \pi$ and so $y' = 0$ during that period, contradicting the assumption that $y' > 0$; hence this case cannot occur.

The General Solution

To solve the problem posed in (3) - (5), (6b), we associate multipliers λ and γ with constraints (3) and (4) respectively and form the Hamiltonian

$$H = e^{-(r+h)t} (\pi - y) + \lambda y^{\frac{1}{2}} + \gamma(rR + \pi - y)$$

Then an optimal solution in which $T^* < \infty$ must satisfy (3), (4), (6b) and (8)-(12):

$$(8) \quad \partial H / \partial y = - e^{-(r+h)t} + \lambda / 2y^{\frac{1}{2}} - \gamma = 0$$

$$(9) \quad \lambda' = - \partial H / \partial z = 0 \quad \text{so that } \lambda \text{ is constant}$$

$$(10) \quad \gamma' = - \partial H / \partial R = - r\gamma \quad \text{so that } \gamma(t) = ke^{-rt} \quad \text{where}$$

$$(11) \quad k \geq 0, \quad kR(T) = 0$$

$$(12) \quad H(T) = (r+h)e^{-(r+h)T} P$$

Conditions (11) follow from the required nonnegativity of $R(T)$; see Arrow-Kurz [1]. Expression (12) is the transversality condition. Since (3)-(5) is concave in y and R , these necessary conditions are also sufficient for optimality. Existence of a solution can be established by means employed in [4].

The solution to the necessary conditions can be summarized in the following four equations. Substituting from (10) into (8) gives

$$(13) \quad y^{\frac{1}{2}}(t) = \lambda e^{(r+h)t} / 2(1+ke^{ht})$$

Substituting from (10), (11), and (13) into (12) gives

$$(14) \quad (1+ke^{hT}) (\pi + y(T)) = (r+h)P$$

In addition (3), (4) and (6b) must be satisfied:

$$(15) \quad \int_0^T y^{\frac{1}{2}}(t) dt = A$$

$$(16) \quad k[R_0 + (1-e^{-rT}) \pi / r - \int_0^T e^{-rt} y(t) dt] = 0$$

where (11) has been employed in deriving (16). With $y(t)$ specified in (13), the three equations (14)-(16) jointly determine the three nonnegative constants T , λ , and k . Nonnegativity of λ is implied by (13).

Before pursuing detailed analysis of the solution, we can immediately obtain a qualitative property of the optimal spending path from (13).

Proposition I.

If development is optimally undertaken, the optimal R & D expenditure plan $y^*(t)$ satisfies

$$(17) \quad 2r \leq y'(t)/y(t) \leq 2(r+h) , \quad 0 < t < T$$

Proof.

Logarithmic differentiation of (13) yields

$$(18) \quad y'/2y = r+h-khe^{ht}/(1+ke^{ht}) = (r+h+rke^{ht})/(1+ke^{ht})$$

from which (17) easily follows.

Proposition I has important consequences. First, any nonnull solution of (3)-(5), (6b) does involve $y'(t) > 0$, so the hypotheses of the Lemma will be satisfied. Hence a solution to (3)-(5), (6b) is also a solution to (3)-(6a); nonnegativity of the cash balance at T assures its nonnegativity throughout. Second, since an optimal nonnull plan satisfies (7), the cash balance will not be zero before project completion. In particular, a solution in which R & D spending just equals current profits cannot be optimal for any interval of time.

It also follows from (17) that if there is no innovational rivalry ($h=0$), then $y'/y = 2r$; the proportionate growth rate of development spending is twice the discount rate. If the cash constraint is binding, then its effect must be to lower the absolute spending rate to maintain feasibility, thereby extending the development period. However, the general shape of the spending plan is unaffected by the presence of a cash constraint. In contrast, if there is innovational rivalry ($h > 0$), then the proportionate growth rate of spending is affected by the cash constraint. If the constraint is inactive (so $k = 0$) then, from (18), the spending

growth rate is $2(r+h)$. Otherwise the proportionate spending growth rate decreases with increasing severity of the constraint (measured by k). Combining the observations of the last two paragraphs, we note that an active cash constraint will affect the level of R & D spending and (if $h > 0$) its temporal pattern but the actual cash balance will not be zero before project completion.

Cash Constraint Inactive

If the cash constraint is not binding, then $k = 0$ so (13) reduces to

$$(19) \quad y^{\frac{1}{2}}(t) = \lambda e^{(r+h)t}/2$$

Substitute (19) into (15), integrate, and rearrange to

$$(20) \quad \lambda/2 = (r+h)A/(e^{(r+h)T}-1)$$

so that

$$(21) \quad y^{\frac{1}{2}}(T) = (r+h)A/(1-e^{-(r+h)T})$$

Substituting (21) into (14) with $k = 0$ gives

$$1-e^{-(r+h)T^*} = (r+h)A/((r+h)P - \pi)^{\frac{1}{2}}$$

(22) or equivalently,

$$T^* = -(r+h)^{-1} \ln [1-(r+h)A/((r+h)P-\pi)^{\frac{1}{2}}]$$

Time T^* will be positive and finite provided

$$(23) \quad (r+h)P-\pi > (r+h)^2 A^2$$

If (23) does not hold, then the project should not be undertaken.

Substituting from (22) into (20) gives

$$(24) \quad \lambda/2 = ((r+h)P-\pi)^{\frac{1}{2}} - (r+h)A$$

which may be combined with (19) to obtain the optimal expenditure plan

$$(25) \quad y^*(t) = [((r+h)P-\pi)^{\frac{1}{2}} - (r+h)A]^2 e^{2(r+h)t}, \quad 0 \leq t \leq T^*$$

The cash balance over the development period may be found by substituting from (25) into (4) and integrating:

$$(26) \quad R(t) = e^{rt} [R_0 + (1 - e^{-rt})\pi/r - (e^{(r+2h)t} - 1) \lambda^2 / 4(r+2h)]$$

where λ is given in (24). To verify that the cash constraint is inactive, one need only check that $R(T^*) \geq 0$; thus use (22) and (24) to evaluate (26) at T^* , thereby generating a relationship that holds among the parameters when the cash constraint is inactive.

$$(27) \quad R_0 \geq (b^{r/(r+h)} - b^2) ((r+h)P - \pi / (r+2h) - (1 - b^{r/(r+h)})\pi / r$$

where

$$(28) \quad b \equiv 1 - (r+h)A / ((r+h)P - \pi)^{\frac{1}{2}}$$

We now show when undertaking new product development is optimal

Proposition II.

Suppose (27) obtains. Then the R & D project is optimally undertaken (following (22) and (25)) if and only if (23) holds.

Proof.

Condition (23) is clearly necessary for T^* in (22) to be positive and finite. To show that it is a sufficient condition, we show that if (23) holds, then the maximized value in (5) following (25) exceeds (1).

Thus it must be shown that

$$\int_0^{T^*} e^{-(r+h)t} (\pi - y^*(t)) dt + e^{-(r+h)T^*} P - \pi / (r+h) > 0$$

Evaluating the left side with the aid of (22) and (25) yields

$$[((r+h)P - \pi)^{\frac{1}{2}} - (r+h)A]^2 / (r+h) > 0$$

establishing the desired result.

Thus, in sum, the optimal policy is given by (25) and (22) provided (23) and (27) hold. It is evident that the development period is prolonged as either required effort A or current profits π increase and is shortened as the innovational reward P rises. So long as the cash constraint is inactive, the sole impact of current profits is on the attractiveness of the innovation. The larger current profits, the smaller the net gain from innovation (for fixed gross innovational reward P).

A consequence of Proposition II is that new product development may be undertaken either in pursuit of improved profits or as a defensive measure against possible losses due to rival entry. Without potential rivalry, a new product will be developed only if it is expected to yield greater profits than the current one ($P > \pi/r$). However, a project that would be rejected if there were no fear of rival entry ($P < \pi/r$, $h = 0$) may nevertheless be undertaken if the possibility of such rival preemption were recognized ($h > 0$) and (23) holds. Intuitively, the explanation is that the possibility of rival preemption reduces the expected value of the current product since the expected duration of receipts falls. This enhances the relative value of the new project so it may be undertaken in support of the current line of business as a defensive measure.

Having found the optimal solution when the cash constraint is inactive, we now show that this case encompasses a remarkably broad range of circumstances.

Proposition III. If

$$(29) \quad P \leq 2\pi / (r+h)$$

so the reward associated with the new product does not exceed twice the expected profit from the existing product, then its development can be fully financed at the optimal rate without impedence by the cash constraint.

Proof.

From (17) the optimal spending rate rises over the development period. Therefore if $y^*(T^*) \leq \pi$, then surely $y^*(t) \leq \pi$ for all $0 \leq t \leq T^*$, so spending according to (25) can always be covered by current receipts. But, from (25), $y^*(T^*) = (r+h)P - \pi \leq \pi$ provided (29) holds, establishing the Proposition.

Proposition III is interesting for several reasons. First, it indicates that for a class of possible innovations, new product development can proceed without impedence by cash requirements regardless of the effective effort A required or the initial cash balance R_0 . If (29) holds and development is worthwhile (23), then the optimal development schedule is sufficiently leisurely to keep the cash constraint inactive. In other words, because the reward from the new product is modest relative to that of the current product, its development is so prolonged that its difficulty is unimportant. Of course, if the difficulty A is so large that (23) is violated, then the project is rejected.

Second, condition (29) indicates that a firm earning high profits from its current product or facing little innovational rivalry (h small) is better able to finance new product development from current profits than one earning low profits or facing intense rivalry. However, a newcomer (not producing the current product and for whom therefore $\pi = 0$) nevertheless may develop a superior product more rapidly than the incumbent. A newcomer facing the same parameters will choose to develop more rapidly because his potential net innovational reward P exceeds the incumbent's $P - \pi / (r+h)$. However, a newcomer needs a substantial initial cash balance in order for development not to be impeded by a cash constraint. The

required initial cash can be determined by setting $\pi = 0$ in (27) - (28):

$$(30) \quad R_0 \geq [a^{r/(r+h)} - a^2] (r+h)P / (r+2h)$$

where

$$(31) \quad a \equiv 1 - A((r+h)/P)^{\frac{1}{2}}$$

If (30) is satisfied, then an entrant will develop faster than an otherwise identical firm already in the market. However, if (30) fails, then the would-be entrant's speed of development will be constrained by its cash R_0 and so its pace may or may not exceed the incumbent's. The cash required in (30) increases with P ; a larger reward encourages faster, costlier development. It increases with A for A small, as increased development effort requires more cash. However, for large A , this effect is more than offset by the impact of lessened profitability in reducing the development pace; cash required then decreases with A .

While the substantial initial cash balance required of the new-comer may pose a barrier to the individual innovator, it need not hamper entry of a firm currently in another line of business. This may help explain why the innovator of a superior product in a particular line of business is often a firm formerly in another line. It also emphasizes the point made earlier that the current firms in a market may not be the only potential innovators of a new product.

Third, since in many instances the expected rewards from the new product will not be more than twice as high as the expected profits from the current product, the solution with the cash constraint inactive should have considerable applicability. Moreover, the cash constraint can always be rendered inactive by a sufficiently large initial cash balance. Of course (29) is only a sufficient condition that cash not be a constraint; (27) can be satisfied for a far broader range of parameter configurations.

Cash Constraint Active

If (27) fails, then the cash constraint (6b) will be active. Although an explicit solution for the three parameters k , T , and λ appears unavailable, implicit differentiation of the system of three equations (14)-(16) determining them can be conducted. Before reporting the results of that investigation, we provide the explicit solution for the special case of $h = 0$, no innovational rivalry. That case also approximates situations in which the appearance of a rival product seems remote.

In case $h = 0$, (27)-(28) reduces to

$$(32) \quad (R_0 + rA^2)/r^{\frac{1}{2}}A \geq (P-2\pi/r)/(P-\pi/r)^{\frac{1}{2}}$$

Thus we now consider the case that $h = 0$ and (32) fails, so the financing constraint is active and $k > 0$. Substitute (13) into (15) and integrate:

$$(33) \quad \lambda/2(1+k) = rA/(e^{rT}-1)$$

Then from (16) and (33)

$$(34) \quad R_0 + (1-e^{-rT})\pi/r = (\lambda/2(1+k))^2(e^{rT}-1)/r = rA^2/(e^{rT}-1) .$$

Ignoring the middle portion, (34) may be viewed as a quadratic equation in e^{rT} . Solving and taking the larger root (since $e^{rT} > 1$ is required) gives

$$(35) \quad e^{rT*} = [R_0 + 2\pi/r + rA^2 + ((R_0 + rA^2)^2 + 4\pi A^2)^{\frac{1}{2}}]/2(R_0 + \pi/r)$$

Thus we have an explicit expression for the development period in case there is no rivalry but the cash constraint is active. An explicit expression for the optimal spending rate $y^*(t)$ can be obtained by substitution from (33) and (35) into (13). The development period varies directly with A but inversely with R_0 and π . It is independent of the reward P . The role of π is completely confined to its effect on the ability to finance and acts like an increase in initial R_0 . This is the opposite of its impact when cash was not scarce; then π 's effect was as a reduction in the net reward for innovation.

Next we state a counterpart to Proposition II.

Proposition IV.

If $h = 0$, then a necessary and sufficient for the firm to undertake new product development is

$$(36) \quad P - \pi/r > rA^2$$

Proof.

If (32) holds, the conclusion follows immediately from Proposition II. Now suppose that (32) fails. Then for the project to be worthwhile, it must yield expected rewards that exceed profits available in its absence so

$$\int_0^T e^{-rt} (\pi - y^*) dt + e^{-rT} P > \pi/r$$

But since the cash constraint is tight, we have

$$\int_0^T e^{-rt} (\pi - y^*) dt = -R_0$$

Substituting this equation into the inequality above, one sees that it is worthwhile to proceed providing

$$(37) \quad P > e^{rT} (R_0 + \pi/r)$$

Thus, we must show that under the conditions of the Proposition, (36) implies (37) and also that (37) implies (36).

We first show that (36) implies (37). From (35) and the failure of (32) it follows after some manipulation that

$$e^{rT} (R_0 + \pi/r) < \pi/r + A(rP - \pi)^{\frac{1}{2}}$$

But using the bound on A in (36):

$$e^{rT} (R_0 + \pi/r) < \pi/r + A (rP - \pi)^{\frac{1}{2}} < \pi/r + P - \pi/r = P$$

which is (37).

On the other hand if (37) holds, then substitution for e^{rT^*} from (35) into (37) yields

$$2(P-\pi/r) > R_0 + rA^2 + [(R_0 + rA^2)^2 + 4\pi A^2]^{\frac{1}{2}}$$

Since $R_0 \geq 0$ and $4\pi A^2 \geq 0$ it follows that

$$2(P-\pi/r) > rA^2 + [(rA^2)^2]^{\frac{1}{2}} = 2rA^2$$

So (36) is satisfied.

In the absence of innovational rivalry, a lack of financing (small R_0) may retard development, but, according to Proposition IV, the condition (36) governing whether the project is sufficiently attractive to be undertaken is independent of the financial resources available. In other words, without innovational rivalry, product improvements will not be bypassed solely because of limited cash.

If the cash constraint is active and there is innovational rivalry, then explicit expression for the three constants T , k , and λ on which y^* depends is not available. However, implicit differentiation of equations (14)-(16) indicates that, at least for h small, the optimal development period T will be prolonged as the required development effort increases but shortened as either the initial cash R_0 or profits from the current product π increase. In addition, if $h = 0$, then a larger reward P hastens development.

Thus it appears that an increase in the expected benefit from innovation P generally hastens development, although there is one interesting exception. If the cash constraint is tight and there is no innovational rivalry, then P does not affect the pace of development; good and excellent projects may proceed equally rapidly just governed by cash availabilities and required effort.

Summary

We have analyzed the problem of a firm contemplating new product development. It may anticipate higher profit from an improved product or fear loss of profits were its current product to be displaced by a superior rival product . If, however, a rival product is introduced prior to the firm's own new product, the defense will be unsuccessful and the firm will also lose the resources devoted to development.

We sought an R & D program to maximize the present expected value of profits, assuming that the firm must finance development entirely from its cash reserves and internally generated profits. The problem was solved by optimal control techniques. We showed that a state variable path constraint could in this case be replaced by a terminal state constraint. That is, requiring a nonnegative cash balance at the moment R & D is completed insured that the cash balance would be nonnegative throughout the development period. Furthermore, the cash balance turns out to be strictly positive throughout the development period, except possibly at the terminal time. This result rests on our findings that the expenditure on new product development optimally increases through time. The constraint, when active, does affect the entire spending plan although in no case will expenditures just match receipts for more than a single moment.

We were able to characterize the optimal solution and developed a number of interesting findings. For a large class of R & D projects, including those for which the rewards from the new product are not more than double the expected profits from the current product, the self-financing constraint does not impede development.

Profit from the current product plays two roles; it contributes toward the available cash to finance new product development, but also reduces the attractiveness of introducing a new product that replaces it. The net impact depends on whether the cash constraint is active. So long as financing is not an active constraint, larger current profits retard new product development through their effect on reducing the net gain from innovation. In contrast, if the cash constraint is active, then the role of current profits in providing cash dominates and incremental current profits hasten new product development.

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