I gratefully acknowledge J. Dreze for introducing me to this subject and the interesting discussions with him gave rise to the main definition in the paper. I am also indebted to D. Schmeidler who simplified the proof of Theorem 3 and to T. Ichinishi for fruitful discussions.
ABSTRACT

The formation of firms is explained due to the existence of some commodities which have the three properties:

-- cannot be transferred,
-- cannot be produced, and
-- cannot be marketed.

These properties give rise to the formation of Labour-Managed economies for which we define an equilibrium. Under that definition no coalition has the incentive to withdraw and form a new firm operating in the market with the existing prices (for the marketed goods).

We investigate the existence and optimality of equilibrium allocations in two cases:

(i) when workers may belong to more than one firm (i.e., "moonlighting" is not excluded), and
(ii) when every individual may work in one and only one firm.

We consider also the Replica Economy, and prove the equivalence of the core and equilibria allocations.
I. **Introduction**

In this paper the formation of firms is explained due to the existence of some commodities (like initiative, skill, imagination, connections etc.), which have the three properties:

1. cannot be transferred (i.e., individually specific),
2. cannot be produced (but, rather, serve as inputs), and
3. cannot be marketed, (i.e., the firm does not face a price per unit in which it can acquire these commodities).

Clearly, types of labour (or more generally, "time") share the first two properties. The third property is the main assumption on labour in Labour-Managed economies [4, 7, 9, 10]: i.e., labour has no price ("wage") but rather receives a share in the value added. (A detailed description of labour-managed economies, which concentrates on this very issue is given in [4]). As Ward [10] notes:

"In *Illyria* a single class of inputs, labour, is singled out for special treatment. The distinctive features of *Illyrian* behaviour stem entirely from this fact."

We shall therefore refer to commodities with the above three properties as types of labour. However, the model presented here is valid for any economy with commodities (possibly other than labour) that have properties (1) - (3). For example, the share in Arrow-Debreu economy are actually "types of labour." (Section 11).

A labour-managed economy is an economy where production is carried out in firms organized by workers who get together and form collectives or partnerships. These firms hire non-labour inputs, including capital, and sell outputs under the assumed objective of maximizing the welfare of
the members. Since labour is not marketed, when a group of people
(coalition) forms a firm, it faces a technology set which uses only the
amount (and types) of labour its members have. It is only natural,
therefore, to use the production coalition economy (defined by Hildenbrand
[6]). Under that approach, each group of workers \( S \) faces a given
technology set, \( Y(S) \). The correspondence \( y(\cdot) \) assigning a production
set to every coalition describes fully the production possibilities of
the economy.

In a forthcoming paper by J. Dresch and the author, an analogy between
local public goods and labour-managed economies has been established. Thus
the definition we use here for an equilibrium is closely related to that of
a structural equilibrium in [5]. In particular, since workers can
choose the firm(s) they want to work in, we require that in equilibrium
no mobility will occur. There is no incentive for any coalition \( S \) to
withdraw from the existing firm structure, produce by itself \( y(S) \),
distribute the derived value added (evaluated at the prevailing prices for
the marketed goods) in such a way that each of its members is made better off.

Two types of equilibrium are considered: When individuals are
allowed to belong co more than one firm (i.e., when "moonlighting" is
allowed) we call the derived partition a firm structure. When individuals
are allowed to work in one and only one firm, the derived structure is
called a coalition structure. In Section IV we prove, under standard
assumptions, the existence of an equilibrium with a firm structure. In
Section VII we prove, using the (necessary) additional "balancedness assump-
tion on the technology sets", that an equilibrium with a coalition
structure exists.
Section 7 focuses on optimality properties of equilibria with firm structures. It is proved that for every labour-managed economy there exists an equilibrium allocation which is Pareto optimum. However, it is shown that there are economies whose equilibrium allocation is not Pareto optimum. Moreover, equilibrium allocations may even be non-optimal relative to the existing firm structure. Recalling that individuals are allowed to work in more than one firm (hence, no indivisibility exists), these results are rather surprising. It follows that there are equilibrium allocations that cannot be supported by efficiency prices for labour.

The above results are in contrast to Meade's claim that both labour-management and entrepreneurial management 'will lead to the same Pareto optimal equilibrium situations in the long run, provided that there is perfect mobility of factors and that there is perfect competition' [7, p. 404]. Using Drez's formulation, the purpose of [4] is to establish Meade's claim with a maximum of generality and rigor, and indeed Drez accomplishes his goal. The difference between our results and those in [4, 7, 9] stems, obviously, from the different definitions of equilibrium used in the above models. Naturally, I view the definition of equilibrium in this paper as intuitively appealing.

In Section 6 we show that the source of nonoptimality is due to the inappropriateness of the 'core' for coalition production economies. It fails to allow a group of people, S, to form and produce its "sub-firms" and share among themselves the total output thus derived. To enable such actions we define the concept of the super-additive technology, which may
be of interest also in contexts different from the one presented here. Using this concept, we prove in Section IX that the core of the replica economy shrinks to the set of equilibrium allocations.

In Section VIII, the optimality of equilibrium allocations with coalition structures (rather than firm structures) is investigated. We show that in contrast to the results of Section V, there may exist no equilibrium allocation which is Pareto optimum. On the other hand, again in contrast to the case of firm structures, every equilibrium allocation is Pareto optimum relative to its equilibrium coalition structure.

The model, definitions and assumptions are presented in Section II.
II. The Model

There are $m$ individuals, $g$ marketed goods and $k$ types of labour. By definition labour is individually specific hence the commodity space is $R^{gmk}$. Let $a \in R^{gmk}$, denote:

$$a_c^i = (a_{i1}, \ldots, a_{iM})$$

$$a_L^i = (a_{i1}, \ldots, a_{ik})$$

$M = \{1, \ldots, m\}$ denotes the set of individuals each endowed with $w^i \in R^{gmk}$

and a utility function defined over his consumption set $x_i$. Since labour cannot be transferred, $x_i \in X_i$ implies $x_{ij} = 0$ for all $j \neq i$.

Similarly, since $i$ does not endow $j$'s labour, $x_{ij} = 0$ $\forall j \neq i$.

Using the coalition production approach, each group of individuals, $S$, $\subseteq M$, faces a production set $Y(S) \subseteq R^{gmk}$. Again, since labour cannot be transferred nor is it marketed, $Y(S)$ implies that $Y_c$ is produced by using only the labour members of $S$ own, i.e., $Y(S)$ implies $Y_{ij} = 0$ $\forall i \notin S$. Moreover, since labour cannot be produced, $Y_L \leq 0$.

$Y(S)$ can be derived from a more primitive technology $\bar{Y}(S) \subseteq R^{gmk}$. Indeed, since types of labour are defined solely by their productive characteristics, technologically there can be no distinction between the same type of labour provided by different individuals. Thus, $\bar{Y}(S)$ states that $\bar{Y}_c$ can be produced by $\bar{a}$ using the amount of labour $\bar{Y}_L (\text{regardless which individuals contribute it})$. $Y(S)$ represents the potential production possibilities available to $S$ if and when forms. However, as labour cannot be hired, we have to add the restriction that whenever individual $i \in S$ contributes (ex-post) some labour to $a$, he must belong to that coalition.

We therefore "extend" the technology $Y(S)$ to $\bar{Y}(S)$ in the following way:
\( y(S) = \{ y_{E\cap M} | y_{L,j} = 0 \ \forall y \in S, \ y_L \leq 0 \ \text{and} \ \sum_{i \in S} \ (y_{L,i}) \notin y(S) \} \)

**Remark 1:** We could also impose the restriction:

\( y \in Y(S) = y_L \geq \sum_{i \in S} y_{L,i} \) for some \( x^L \in X \), \( i \in S \),

i.e., the maximum amount of labour required does not exceed the maximum amount of labour \( S \) can afford to contribute to production. However, in equilibrium this condition holds, eliminating the need for the above restriction.

**Definition 1:** A firm structure is a non-empty collection \( \{ S_i \} \) of subsets of \( M \), such that \( S_i \cap S_j = \emptyset \ \forall i \neq j \).

Note that individuals may work in (belong to) more than one firm, (i.e., "moonlighting" is not excluded). If individuals are restricted to work in one and only one firm, the firm structure forms a partition of \( M \), and we refer to such firm structures as coalition structures. Formally,

**Definition 2:** A coalition structure is a firm structure \( F \) such that for all \( S_i, S_j \in F \), \( S_i \cap S_j = \emptyset \), \( i \neq j \).

We shall first investigate the existence of an equilibrium and its optimality properties when a firm structure is formed and then, in Sections VII, VIII we investigate the more restricted case of coalition structures.

Denote the price simplex for the marketed goods by \( \Delta \), i.e.,

\( \Delta = \{ p_a^S | \sum p_a = 1 \} \)

**Definition 2:** A structural competitive equilibrium (s.c.e.) consists of a price system \( p^S \), consumption bundles \( x^L \in X \), \( i \in M \), a firm structure \( F \),
production plans \( y(S) \varepsilon \gamma(S) \), \( SF \), and a value added distribution \( \{ t_{1S} \} \),
\( i \in M, \ v \in SF \), such that:

(i) \( \sum_{i} x_{i} = \sum_{i} z_{i} + \sum_{i} y_{i}(S) \),
\( i \in M, \ i \in S, \ v \in SF \)

(ii) There is no:
\( s_{0} \subseteq M, \ x^{0}_{i} \in x^{0}, \ i \in s_{0}, \ y_{i}(s_{0}) \varepsilon \gamma(s_{0}) \) such that
\( \sum_{i} p x_{i}^{0} \leq p \sum_{i} w_{i}^{0} + p y_{i}(s_{0}), \ i \in s_{0}, \ i \in s_{0}, \ i \in s_{0}, \ i \in s_{0}, \ i \in s_{0}, \ i \in s_{0}, \ i \in s_{0}, \ i \in s_{0} \)
and \( u^{i}(x^{0}) \geq u^{i}(x^{i}) \forall i \in s_{0} \),

(iii) For all \( i \in M, \ p x_{i}^{0} \leq p w_{i}^{0} + \sum_{i} t_{iS} \),
\( i \in S \)

(iv) For all \( v \varepsilon SF, \ \sum_{i} t_{iS} \leq p y_{i}(S) \),

(v) For all \( v \varepsilon SF, \ \sum_{i} p y_{i}(S) \leq p y_{i}(S), \ \forall (y_{i}(S),y_{i}(S)) \varepsilon \gamma(S) \).

Conditions (i), (iii) and (iv) need no comments. Condition (v) is the equivalent of profit maximization. It states that in each firm \( SF \) the value added, given the labour vector \( y_{L}(S) \), is maximised. (Since labour has no price, without the constraint on \( y_{L}(S) \) a firm may desire to use an infinite amount of labour, being a free input). Condition (ii) assures that no mobility will occur. There is no incentive for any coalition \( s_{0} \) to produce by itself \( y(s_{0}) \), distribute the value added \( p y_{i}(s_{0}) \) in such a way that every member of \( s_{0} \) can purchase (in the market) a vector \( x^{i} \) whereby his utility is increased.
Assumptions:

For each $i \in M$, 

(a.1) $x^i$ is a closed convex subset of $\mathbb{R}_{++}^n$ and 

$$x^i \in x^i \text{ implies } x^i_{L^j} = 0 \quad \forall j \neq i,$$

(a.2) $u^i$ is a continuous quasi-concave function on $x^i$, 

(a.3) $w^i \in x^i$ and $\exists x^i \in x^i$ with $\tilde{w}^i \leq w^i$, 

(a.4) There is no satiation in consumption in $x^i$:

i.e., for any $x \notin x^i$ and for any neighborhood $U$ of $x$ in

$x^i$, there exists an $x \in U$, $x^i_L = x_L$, such that $u^i(x) > u^i(x')$.

For every $S \subset M$,

(b.1) $Y(S)$ is a closed convex subset of $\mathbb{R}^n$ and $y \in Y(S)$ implies

$$y_L \leq 0, \quad y_L, i = 0 \quad \forall i \notin S,$$

(b.2) $0 \in Y(S)$, 

and for $Y = \bigcup_{S \subset N} Y(S)$

(c.1) $Y$ is closed ,

(c.2) $\forall \neg \{Y\} \in \{0\}$, and 

(c.3) $\forall \neg \mathbb{R}_{++}^n$.

Theorem: Under assumptions (a) - (c) there exists a structural competitive equilibrium.

We shall prove the theorem after the next section which may give some more insight and motivation to our model.
III. The Arrow-Debreu Economy

A good example of a non-marketed goods economy is provided by the standard $A - D$ economy, where there are $m$ individuals each endowed with $\hat{w}^i \in \mathbb{R}^g_+$. The consumption set is $\bar{X}^i \subset \mathbb{R}^g_+$, over which the utility function $\hat{u}^i$ is defined. There are $J$ firms with the technology sets $\bar{Y}^j \subset \mathbb{R}^g_+$, $j = 1, \ldots, J$. Each firm $j$ is owned by the individuals who hold its shares. The shares $\theta_{i,j}$, $j = 1, \ldots, J$ individual $i \in M$ holds are a-priori given and are not marketed, i.e., $A = J$. For all $j$, $\sum_{i \in M} \theta_{i,j} = 1$, with $\theta_{i,j} \geq 0$ for all $i \in M$.

A competitive equilibrium consists of a price vector $p \in \Delta$, consumption bundles $\hat{x}^i \in \bar{X}^i$, $i \in M$, and production plans $\hat{y}^j \in \bar{Y}^j$, $j = 1, \ldots, J$, such that

(a) $\hat{p}^i \leq \hat{p}^i + \sum_{j=1}^J \theta_{i,j} \hat{p}^j$, and $\hat{u}^i(\hat{x}^i) > \hat{u}^i(\hat{x}^i)$ implies $\hat{p}^i \hat{x}^i < \hat{p}^i + \sum_{j=1}^J \theta_{i,j} \hat{p}^j$, $i = 1, \ldots, m$,

(b) $\hat{p}^j = \text{Max } \hat{p}^j$, $j = 1, \ldots, J$ and

(c) $\hat{u}^i(\hat{x}^i) = \sum_{j=1}^J \theta_{i,j} \hat{u}^j(\hat{y}^j)$.

The natural way to embed this economy in our model is by defining the following:

$$ w^i = (w^i_1, \ldots, w^i_m) \in \mathbb{R}^{g \times m}, u^i \in \mathbb{R}, \hat{w}^i_1 = w^i, \hat{w}^i_1 = 0, \hat{w}^i_1 = 0, \theta_{i,j} \geq 0, \forall j \in M, x^i \in \bar{X}^i, x^i = 0 $$. 

$$ u^i(x^i) = \hat{u}^i(\hat{x}^i) $$. (i.e., the utility is independent of the non-marketed goods.)

$$ \gamma^j = [\gamma^j \in \mathbb{R}^g_+] \text{ y = cy, } \gamma^j \in \gamma^j, \sum_{i \in M} (\gamma^j_{l,i}) \leq 1, 0 \leq \gamma^j \leq 1 \text{.} $$
\[
\{ (x, \hat{y}_L, \hat{y}_J) \} \text{ denotes the total amount of labour (shares) of type (firm) } j \text{ which serves as an input in } y. \] Let \( Y = \{ y | y = \sum_{j=1}^{J} y_j^1 \hat{y}_j^1 \} \).

Then, the coalition production technology is given by:
\[
Y(S) = \{ y \in Y | y_L \geq \sum_{l \in \mathbb{N}} w_l^2 \}.
\]

While in a "capitalistic economy" a firm is associated with a technology set, in a labour-managed economy a firm is identified with its workers (in an A-D model, with its shareholders). Hence, the same production planes produced in both economies in the same technological way may give rise to different firm structures. Here, the set of workers in firm \( j \) (defined by the technology set \( \hat{y}_J \)) is: \( S_j = \{ l \in \mathbb{N} | \theta_{ij} > 0 \} \). Define:
\[
\tilde{F} = \{ S_1, \ldots, S_J \} \text{ and } F = \{ S_k \in \tilde{F} | S_k \neq S_j \}.
\]

i.e., \( F \) is the firm structure derived from the collection \( \tilde{F} \).

For \( S \in F \), \( y_j^J \in \hat{y}_J \), \( j = 1, \ldots, J \), denote:
\[
J(S) = \{ j | S_j = S \} \text{ and } y(S) = \sum_{j=1}^{J} y_j^J.
\]

Clearly, \( \sum_{j=1}^{J} y_j^1 = \sum_{S \in F} y(S) \).

When the \( J \) firms "form" in the A-D model, the firm structure \( F \) is realized in the associated labour-managed economy.

The following theorem proves that the A-D economy thus embedded is a special case of our model.
Theorem 2: Under assumptions (a) - (c), \((p, \{x^i\}_{i \in M}, \{y^i\}_{j \in J})\) is a competitive equilibrium if and only if \((p, \{x^i\}_{i \in M}, F, \{y(S)\}_{S \in F}, \{t_{IS}\})\) is a structural competitive equilibrium with:

For all \(i \in M\):
\[ x^i_c = x^i_L = 0, \]

For all \(S \in F\):
\[ y_c(S) = y(S), \quad \text{for} \quad j \in F(S), \quad (y_i, y_j(S)) = \delta_{ij}, \quad \text{and} \]
\[ (y_{L,i}(S))_j = 0, \quad j \neq i(S). \]

For all \(i \in M, S \in F\):
\[ t_{IS} = \sum_{J \in F(S)} \theta_{ij} p y_j^S. \]

Proof: Let \((p, \{x^i\}_{i \in M}, \{y^i\}_{j \in J})\) be a competitive equilibrium, and let
\((\{x^i\}_{i \in M}, \{y(S)\}_{S \in F}, \{t_{IS}\})\) be defined as above. Clearly,
\[ \sum_{i \in M} x^i_c = \sum_{i \in M} y^i_c + \sum_{S \in F} y_c(S) \]
\((- 0), \) and by (c)
\[ \sum_{i \in M} x^i_c = \sum_{i \in M} \sum_{S \in F} y^i_c(S). \]
Hence, (I) is fulfilled.

It is easily verified that for all \(T \subseteq M\), \(y(T) = \sum_{i \in T} y^i(S)\), and for all
\(i \in T, y_c(T) = \sum_{j \in T} y^i_j\). (In fact, we could use this formulation to represent the production coalition technology sets.) By (p), for all
\(y_c(T) \in \gamma_c(T), \quad T \subseteq M, \quad p y_c(T) \leq \sum_{j \in T} \sum_{i \in T} \theta_{ij} p y_j^R, \)
which establishes (v).

(iii) and (iv) follow immediately from (2) and the definition of \(t_{IS}\).

Suppose \(u^i_c(x^i) > u^i_c(x^i)\) (hence, \(u^i_c(x^i) > u^i_c(x^i), \forall i \in S \in M\).

By (a), \(p x^i_c > p x^i_c + \sum_{j \in T} \theta_{ij} x^i_j, \) and using the previous inequality we get
\[ \sum_{i \in S} p x^i_c > \sum_{i \in S} p x^i_c + p y_c(S), \quad \forall y_c(S) \in \gamma_c(S), \]
thus (iv) is also fulfilled.
Let \( (p, x^i) \in \bigcap \{ y(S) \}_{S \in F} \{ t_{1s} \} \) be a s.c.e. for the associated labour-managed economy. From \( x^j c x^j \) we have \( x^j_{sc} = 0 \), \( W \in G(M) \). As \( y \in Y(S) \) implies \( y = \sum_j y^j \), \( y^j c y^j \), we can associate with \( \{ y(S) \}_{S \in F} \) the vectors \( \{ y^j \}_{j=1}^J, \ y^j c y^j \). Define for \( i \in M \), \( x^i = x^i c x^i \) and for \( j = 1, \ldots, J \), \( y^j w^j \). Since \( \forall \in M \), \( Y_c(T) = \sum_{\epsilon \in T} \sum_{j=1}^J y^j \), \( (v) \) implies (9) and (11), (iii) with assumption (a.4) imply that \( \sum_{S \in F} \sum_{j=1}^J y^j \). (Note that \( i_S = t_{1s} = 0 \).)

Summing (iv) over all \( S \in F \) yields:

\[
\sum_{S \in F} y^j(S) = \sum_{S \in F} t_{1s} \leq \sum_{S \in F} t_{1s} \leq \sum_{S \in F} y^j(S) = \sum_{S \in F} y^j(S)
\]

hence, for all \( i \in M \), \( t_{1s} \leq \sum_{S \in F} y^j(S) \). The first part of (a) is implied by (iii) and the second is derived by (11).

Q.E.D.
IV. Existence of Equilibrium Firm Structures:

We shall now prove theorem 1 by converting our labour-managed economy to a standard A-B model in which (in contrast to the previous section) all commodities, labour included, are marketed, and thus have prices. Consider an A-B economy with \( m \) individuals and with \( 2^m - 1 \) firms with the technology sets \( Y(S), \mathcal{S} \mathcal{M} \). Each firm \( Y(S) \) is owned through a share system \( \{ \theta_{is} \} \) where \( \theta_{is} \geq 0 \) for all \( i \in \mathcal{M}, s \in S \), and \( \sum_{s \in S} \theta_{is} = 1 \). (i.e., only members of the coalition \( S \) can own shares in the firm \( Y(S) \)). The consumption sets, initial endowments and preferences are identical to those in the original (labour managed) economy. Under assumption (a)-(c) \( [3] \) there exists a price vector \( q \in [0,\mathcal{S} \mathcal{M}] \), consumption bundles \( x^i \in \mathcal{X}^i, i \in \mathcal{M} \), and production plans \( y(S) \in Y(S), \mathcal{S} \mathcal{M} \), such that:

\[
\begin{align*}
\text{(c)} & \quad q x^i \leq q y^i + \sum s \theta_{is} y(S), \quad \text{and} \quad u^i(x^i) > u^i(y^i) \quad \text{implies} \\
& \quad \text{\( q x^i > q y^i + \sum s \theta_{is} y(S), \quad i = 1, \ldots, n \),} \\
\text{(b)} & \quad q y(S) = \max \{ y(S) \} \quad \forall y(S), \quad \text{and} \\
\text{(y)} & \quad x^i = y^i + \delta y(S) \\
\quad i \in \mathcal{M} & \quad i \in \mathcal{M} \quad \mathcal{S} \mathcal{M} \\
\end{align*}
\]

By (b) and (c), \( q \geq 0 \), and by (a) and (a.4), \( q \geq 0 \). Denote \( \bar{q} = \frac{q}{||q||} \).

Without loss of generality \( \bar{q} = q \).

Define: \( p = q \), \( P = \{ \mathcal{S} \mathcal{M} | y(S) \neq 0 \} \), and

\( \delta_{is} = \delta_{is} y(S) - q_{i,1} y_{L,i}(s), i \in \mathcal{M}, s \in \mathcal{S} \mathcal{M} \).

Note that for \( 1 \in S \) \( \delta_{is} = 0 \) and \( y_{L,i}(s) = 0 \), hence \( \delta_{is} = 0 \).
Since $x^i \in X^i$, from (γ) we have:

$$x^i_{L,1} = u^i_{L,1} + \gamma y^i_{L,1}(S), \quad \text{and by (α)}$$

$$q^c e + q^i_{L,1} x^i_{L,1} \leq q^c e + q^i_{L,1} w^i_{L,1} + \gamma q^i_{IS} q^i(S). \quad \text{Thus (iii) holds.}$$

By the definition of $y(S)$ and $\{\beta_{IS}\}$,

$$\gamma \sum_{IS} q^i_{IS} = q^i(S) - \sum_{IS} q^i_{L,1} y^i_{L,1}(S) = q^i(S) - q^i_L y^i_L(S) = p^i_C(S).$$

Therefore, (iv) holds. Clearly (γ) coincides with (i) and (β) implies (v).

It is left to be shown that (ii) holds as well.

Let $S_o = x^i \in X^i$ with $u^i(x^i) > u^i(x^i)^\gamma \in S_o$, $y(S_o) \in \gamma(S_o)$ with

$$y^i_{L,1}(S_o) = \sum \frac{y^i}{i} \in S_o, \quad \text{and by (α), (β) and the definitions of } F$$

$$\sum \frac{y^i}{i} \in S_o, \quad \text{and } \text{the definitions of } \{\beta_{IS}\}, \quad \text{and } \sum \frac{y^i}{i} \in S_o.$$
V. Optimality of Equilibrium Firm Structures

Having proved existence, the question of optimality of a s.c.e. naturally arises.

Definition 4: (a) Let $F$ be a firm structure. An $F$-attainable allocation is a vector $\left(\{x^i\}_{i \in \mathcal{I}}, \{y(S)\}_{S \in \mathcal{F}}\right)$ such that

\begin{enumerate}
\item $x^i \in \mathcal{X}^i$ for all $i \in \mathcal{I}$,
\item $y(S) \in \mathcal{Y}(S)$ for all $S \in \mathcal{F}$, and
\item $\sum_{i \in \mathcal{I}} x^i = \sum_{i \in \mathcal{I}} x^i + \sum_{i \in \mathcal{I}} y(S)$.
\end{enumerate}

(b) Let $F$ be a firm structure. A [strong] Pareto optimum relative to $F$ is an $F$-attainable allocation $\left(\{x^i\}_{i \in \mathcal{I}}, \{y(S)\}_{S \in \mathcal{F}}\right)$ for which there exists no other $F$-attainable allocation $\left(\{x'^i\}_{i \in \mathcal{I}}, \{y'(S)\}_{S \in \mathcal{F}}\right)$ such that $y^i(S) > y^i(S)$ \(\forall i \in \mathcal{I}\). \(u^i(x^i) \geq u^i(x'^i)\) and $u^i(y(S)) > u^i(y'(S))$ for at least one $i \in \mathcal{I}$.

(c) An attainable allocation is an $F$-attainable allocation for some firm structure $F$.

(d) A [strong] Pareto optimum is an attainable allocation $\left(\{x^i\}_{i \in \mathcal{I}}, \{y(S)\}_{S \in \mathcal{F}}\right)$ for which there exists no other attainable allocation $\left(\{x^i\}_{i \in \mathcal{I}}, \{y(S)\}_{S \in \mathcal{F}}\right)$ such that $u^i(x^i) > u^i(x^i')$ and $u^i(y(S)) \geq u^i(y(S))$ for some $i \in \mathcal{I}$.

Clearly, every strong Pareto optimum is Pareto optimum, but the converse is not necessarily true. (By assuming "resource relatedness" of individuals, or monotonicity of preference, the two definitions coincide.)

An immediate corollary of theorem 3 is:
Corollary 2: For every labour-managed economy (which satisfies assumptions (4)-(c)) there exists a s.c.e. whose allocation is a strong Pareto optimum.

Proof: Under assumptions (w)-(c), every competitive allocation in A-D economy is a strong Pareto optimum. By Theorem 3 the conclusion follows.

The proof of Theorem 2 together with Theorems 1 and 3 suggest that:
- Every s.c.e. allocation is a (strong?) Pareto optimum.
- Every s.c.e. allocation can be sustained by a system of personalised prices for labour which serve as efficiency prices for these commodities.

The following example, however, disproves the above two conjectures.

Example 1: There are three individuals, each endowed with one type of non-marketed commodity. The utility of each individual depends solely upon the amount of the (one) marketed good he consumes, \( x_1^4 \), and is monotonically increasing in this good. To simplify notation we shall not make labour individually specific and consider the commodity space \( \mathbb{R}^4 \). (It is trivial to extend the dimension from 4 to \( 1 + 3 \times 3 = 10 \).

Let \( w^1 = (0,1,0,0) \), \( w^2 = (0,0,2,0) \), \( w^3 = (0,0,0,1) \), and let the technology set, for producing the marketed good, be given by:

\[
\begin{align*}
Y(1,2) &= \{ y \in \mathbb{R}^4 : y_1 \leq 2 \text{ and } y_2, y_3, y_4 \geq 0 \}, \\
Y(2,3) &= \{ y \in \mathbb{R}^4 : y_1 \leq \text{Min} \{ -y_3, y_4, 1 \}, y_2 = 0 \}, \\
Y(1,2,3) &= \{ y \in \mathbb{R}^4 : y_1 \leq 2.6 \text{ and } y_2, y_3, y_4 \geq 0 \}, \\
Y(S) &= \{ 0 \} \text{ otherwise}. 
\end{align*}
\]

Define: \( x^1 = x^2 = (1,2,0,0,0) \), \( x^3 = (0,2,0,0,0) \).

Clearly, \( \{ x_i^1 \}_{i \in M} \) is a structural competitive allocation, for

\( P = \{ 1,2,3 \}, \; p = 1, \; \epsilon_{p} = x^1_1, \; i \in M. \) [In particular, there is no coalition \( S \subset M \), such that \( y \in Y(S) \) with \( y_c \geq x^1_i, \epsilon_{S} \). Hence, (iii) is fulfilled.]
However, by forming the firm structure \( F = \{\{1,2\}, \{2,3\}\} \), there exists \( y \in \delta F \) with \( \gamma_c = 3 > 2.6 \), and therefore \( \{x^i\}_{i=1}^3 \) is not Pareto optimum.

(Obviously, not a strong Pareto optimum.)

Suppose \( (p_2, p_3, p_4) \) are efficiency prices for labour which sustain \( \{y \}_{i=1}^3 \), \( \{y(S)\} \). Then:

\[
\begin{align*}
y(\{1,2\}) &= 0 = p_2 + p_3 \\
y(\{2,3\}) &= 0 = p_3 + p_4 \\
(2.6, -1, -2, -1) \cdot y(S) &= p_2 + 2p_3 + p_4 \leq 2.6
\end{align*}
\]

Contradiction. Hence, there are no efficiency personalized prices for labour which sustain the s.e.e. allocation \( \{y \}_{i=1}^3 \), \( \{y(S)\} \).

In fact, we shall now show that it is possible for a s.e.e. allocation not to be even Pareto optimum relative to the equilibrium firm structure.

\[ \text{Example 2: Consider the economy of Example 1, the initial endowments being:} \]
\[ \nu^1 = (0,1,0,0), \nu^2 = (0,0,1,0), \nu^3 = (0,0,0,1), \text{ and with the following technology sets:} \]
\[ Y(\{1,2\}) = \{y \in \mathbb{R}^4 \mid y_1 \leq -2y_2 + y_3, y_4 = 0\}, \]
\[ Y(\{2,3\}) = \{y \in \mathbb{R}^4 \mid y_1 \leq -2y_3 + y_4, y_2 = 0\}, \]
\[ Y(\{1,3\}) = \{y \in \mathbb{R}^4 \mid y_1 \leq -2y_4 + y_2, y_3 = 0\}, \]
\[ Y(S) = 0 \text{ otherwise.} \]

Define: \( F = \{\{1,2\}, \{2,3\}, \{1,3\}\} \), \( x^i = (\nu^i, 0,0,0), i = 1,2,3 \)

\[ \begin{align*}
y(\{1,2\}) &= \left(\frac{7}{6}, -\frac{3}{6}, -\frac{1}{6}, 0\right) \\
y(\{2,3\}) &= \left(\frac{7}{6}, 0, -\frac{3}{6}, -\frac{1}{6}\right) \\
y(\{1,3\}) &= \left(\frac{7}{6}, -\frac{3}{6}, 0, -\frac{1}{6}\right) \\
\end{align*} \]

\( \forall \delta F, t_{18} = \frac{7}{8}, 1 = 1, 2, 3. \)
Since for all \( SCM \), \( \max f_c(s) \leq 2 + 1 = 3 \), and \( x_1^1 + x_1^2 = \frac{7}{4} + \frac{7}{4} = 3.5 > 3 \),
it is clear that \( \{1, \{x_i\}_{i=1}^3, F, \{y(S)\}_{S \in F}, \{t_{1s}\} \} \) is a s.c.e. Neverthe-
less, the \( F \) - attainable allocation \( \{(x^i, (y(S))_{S \in F}\}, \) given by:
\[
\begin{align*}
x^i &= (2, 0, 0, 0), \quad i = 1, 2, 3, \\
y \left(\{1, 2\}\right) &= (2, -1, 0, 0), \\
y \left(\{2, 3\}\right) &= (2, 0, -1, 0), \quad \text{and} \\
y \left(\{1, 3\}\right) &= (2, 0, 0, -1), \quad \text{proves that} \quad \{(x^i)_{i=1}^3, \{y(S)\}_{S \in F}\} \quad \text{is not Pareto optimum relative to} \quad F.
\end{align*}
\]
VI. The Super-Additive Technology

Examples 1 and 2 above indicate that the concept of the core (or condition (ii) in Def. 3) is not appropriate for coalition production economies. It fails to allow a group of people, $S$, to form and produce in "sub-firms" $[S_j] S_j \subseteq S, j = 1, \ldots, n$, and share among themselves the total output thus derived. Condition (ii) requires only that no $S_j$, by acting on its own as one firm, can benefit by deviating from the equilibrium firm structure. This, as we shall see, is the source of nonoptimality of some equilibrium allocations.

Definition 5: $(\{x^i\}^i \in S, \{y(S_j)\}^i \in S_j \exists\}$ is an attainable program for $S$

(realized via c) if $x^i \in X$, $y(S_j) \in Y(S_j) \forall S_j \subseteq S$, and

$$\sum_{i \in S} x^i = \sum_{i \in S} y^i \quad \text{and} \quad \sum_{i \in S_j} y^i = y(S_j) \quad \forall S_j \subseteq S.$$

Definition 6: The super-additive technology of coalition $S$, $\gamma^*(S)$ is given by:

$$\gamma^*(S) = \inf \{ \sum_{i \in S} y(S_j) \mid \exists \{x^i\}^i \in S, \{y(S_j)\}^i \in S_j \exists\}.$$

That is, $\gamma^*(S)$ allows $S$ to produce in sub-firms provided the total amount of labour involved is affordable by $S$. By remark 1, however, we could replace $\gamma^*(S)$ by:

$$\gamma^*(S) = \inf \{ \sum_{i \in S} y(S_j) \mid \exists \{x^i\}^i \in S, \{y(S_j)\}^i \in S_j \exists\}.$$

If condition (ii) is replaced by:

$$\forall S_o \subseteq S, \exists \{x^i\}^i \in X, \forall i \in S_o, \gamma(S_o) \subseteq \gamma(S) \text{ such that }$$

$$\sum_{i \in S_o} x^i \leq \sum_{i \in S_o} y(S_o) + y^i \text{ and } \sum_{i \in S_o} x^i = y(S_o) \text{ and } \sum_{i \in S_o} y^i = y^i \text{ and } \forall i \in S_o,$$

it follows that every s.c.e.a (s.c.e with (ii') replacing (ii)) is Pareto optimum. To realize that, replace $S_o$ by $\mathcal{M}, \text{in (ii')}$. 


It may well be argued that every coalition $S$ does in fact face the technology $Y(S)$ since it can always decide to produce in sub-firms and then distribute the output. In that case the coalition structure $[N]$ always forms since $Y(N) = \sum_{S \subseteq M} Y(S) \cdot S \subseteq M$.

The following Theorem shows that replacing (ii) by (i10) enables us to concentrate only on equilibrium allocations without considering the distribution of the value-added involved.

**Theorem 2:** Let $p \in \Delta$, $x^i \in X^i$, $i \in M$, consumption bundles, $F$ a firm structure and $y(S) \in Y(S)$, $S \subseteq F$, production plans, such that (i) and (ii10) hold. Assuming mnestation (a.4) there exists a value added distribution $[t_{is}]_{i \in M, S \subseteq F}$, such that $(p, [x^i]_{i \in M, F}, [y(S)]_{S \subseteq F}, [t_{is}]_{i \in M, S \subseteq F})$ is a s.c.e. (hence s.c.e.).

**Proof:** Since $p \notin \Delta$, $p \geq 0$, we have by (1)

$$p = \sum_{i \in M} x^i = p \cdot \sum_{i \in M} x^i + p \cdot \sum_{S \subseteq F} y(S).$$

Choose $[t_{is}]$ such that:

$$t_{is} = px^i_c - py^i_c, \quad t_{is} = 0 \quad i \notin S, \quad i \in M, \quad S \subseteq F.$$

We shall first show that (iv) of Definition 3 is fulfilled. Suppose not, i.e., $\exists S \in F$ such that

$$\sum_{i \in S} t_{is} > py^i_c(S).$$

By (1), (2) and (3):

$$\sum_{S \subseteq F} py^i_c(S) = \sum_{i \in M} (px^i_c - py^i_c) = \sum_{i \in M} t_{is} = \sum_{S \subseteq F} \sum_{i \in S} t_{is} + py^i_c(S).$$

Hence, $[S] \in F$. Therefore, there exists $S \subseteq F$, such that...
(4) \( py_c(\overline{S}) > \sum_{i \in \overline{S}} t_{i|n} \)

Define: \( \overline{F} = \{ \overline{S} \in F \mid \sum_{i \in \overline{S}} t_{i|n} > py_c(\overline{S}) \} \)

\( \overline{F} = \{ \overline{S} \in F \mid \sum_{i \in \overline{S}} t_{i|n} < py_c(\overline{S}) \} \).

We proved that \( \overline{F} \neq \emptyset \) implies \( \overline{F} \neq \emptyset \). To complete the proof we need the following definitions:

**Definition 4:** We shall say that \( i \) is connected to \( j \) if there are individuals \( i_0, i_1, \ldots, i_k \) and coalitions \( S_0, S_1, \ldots, S_k, S_{k+1} \), with \( i_0 = i, i_k = j \), such that

\((i_{t-1}, i_t) \in S_t, S_t \in F, t = 1, \ldots, k, i_0 \in S_0 \not\in F, i_k \not\in S_{k+1} \in F \).

**Definition 5:** A coalition \( G \) is connected to a coalition \( H \), if there exist \( i \in G, j \in H \) such that \( i \) is connected to \( j \).

**Definition 6:** \( \{ S_1, \ldots, S_k, S_{k+1} \} \) \( \{ i_0, \ldots, i_k \} \) is called a minimal connection for \( H \) and \( G \) if it connects \( H \) and \( G \), and there is no other connection of \( G \) and \( H \) with less than \( k+1 \) coalitions.

Since \( F \) is a finite collection of coalitions, each has a finite number of members, it follows that minimal connections exist.

For each \( i \in M \) define:

\( K_i = \{ j \in M \mid j \text{ is connected to } i \} \).

[Note that \( i \not\in K_i \) \( \forall i \in M \).]

Denote:

\( \overline{K} = \bigcup K_i, \overline{\overline{K}} = \bigcup \overline{K}_i \),

\( \text{ for } i \in \overline{\overline{K}}, \overline{i} \in \overline{\overline{K}} \).

By (3) \( \overline{K} \neq \emptyset \) and by (4) \( \overline{\overline{K}} \neq \emptyset \).
Distinguish between the two cases:

I. \( \hat{x} \cap \hat{K} = \emptyset \).

In this case we shall show that \( \hat{x} \) violates (14). Let \( G = \{ S \in F \mid S \subset \hat{x} \} \).

\( C = \emptyset \) since \( \hat{x} \subseteq C \). Moreover, \( S \in G \) implies \( S \cap \hat{x} = \emptyset \), hence, for all \( i \in S \in G \), \( \{ S \in F \mid i \in S \} \cap \hat{F} = \emptyset \).

By definition of \( \hat{F} \), for all \( S \in G \), \( \sum_{i \in S} t_{iS} \leq \sum_{i \in S} t_{iS} \).

Since \( \hat{x} \subseteq G \), we get:

\[
(3) \quad \sum_{S \in G} p_{y}^{c}(S) > \sum_{S \in G} \sum_{i \in S} t_{iS} = \sum_{i \in \hat{x}} \sum_{S \in G} t_{iS} = \sum_{i \in \hat{x}} \left( p_{x}^{i} - p_{y}^{i} \right).
\]

By the local nonexistence, there exists, for all \( i \in \hat{x}, x^i \) with \( x^i = y^i \),

\( u^i(x^i) > u^i(x^i) \) and by (2) \( \sum_{i \in \hat{x}} \left( p_{x}^{i} - p_{y}^{i} \right) \leq \sum_{S \in G} p_{y}^{c}(S) \). By (1),

\( \sum_{i \in \hat{x}} = \sum_{S \in G} p_{y}^{c}(S) \leq Y^{\prime}(\hat{X}), \) hence, \( \hat{x} \) violates (14). Contradiction.

II. \( \hat{x} \cap \hat{K} = \emptyset \).

Let \( \{ S_{0}, S_{1}, \ldots, S_{k+1} \} \{ t_{0}, t_{1}, \ldots, t_{k} \} \) be a minimal connection of \( i_{0} \in S_{0} \in \hat{F} \) to \( i_{k} \in S_{k+1} \in \hat{F} \). Define:

\[
\begin{align*}
t_{i} & = t_{i}^{t_{i}^{t_{i}t_{i}}}, \quad i = 0, \ldots, k, \\
t_{i} & = t_{i}^{t_{i}^{t_{i}t_{i}t_{i}}} + \varepsilon, \quad i = 0, \ldots, k, \\
t_{i} & = t_{i}^{t_{i}t_{i}}, \quad \text{otherwise.}
\end{align*}
\]

(Due to the minimality of the connection, \( \varepsilon \) is well defined.)

Since \( t_{i} \in S_{i} \cap F_{i+1} \), \( t = 0, \ldots, k \), \( \sum_{S \in F} t_{iS} = \sum_{S \in F} t_{iS} \) for all \( i \in M. \)
Thus, by (2)
\begin{equation}
\forall S \in \mathcal{F} \quad t_{IS} = px^i_c - pv^i_c, \quad i \in \mathcal{N}.
\end{equation}

Moreover, for all \( S \subset \mathcal{F}, S \neq \emptyset, S \neq S_{k+1} \)
\begin{equation}
\forall t_{IS} \in \mathcal{F}
\end{equation}

Choose \( \varepsilon = \min \left( \{ \sum_{S \in \mathcal{F}, S_{k+1} \in \mathcal{F}, S \neq \emptyset} \sum_{i \in S^{k+1}} t_{IS} \} \right) \).

As \( S_0 \in \mathcal{F}, S_{k+1} \in \mathcal{F}, \varepsilon > 0 \). With \( t_{IS} \) replacing \( t_{IS} \), in view of (7),

the number of coalitions in either \( \mathcal{F} \) or \( \overline{\mathcal{F}} \) reduces. Repeating this process

for \( t_{IS} \), (since, by (6), (2) holds), after a finite number of steps either

\( \mathcal{F} \cap \overline{\mathcal{F}} = \emptyset \) which is case (i) or else \( \mathcal{F} \) or \( \overline{\mathcal{F}} \) is empty. If \( \mathcal{F} \) is empty, (iv) is satisfied. If \( \overline{\mathcal{F}} \) is empty, by (1), \( \mathcal{F} \) is empty. Hence, in any case we proved that (i) and (iv) imply (iii) and (iv).

Since obviously (iv) implies (ii), to complete the proof of the theorem we have to show that (v) holds as well. Suppose not, i.e.,

\( \mathcal{F}(\gamma_c(S_0), \gamma_L(S_0)) \) with \( \gamma_c(S_0) < \gamma_c(S_0), S_0 \in \mathcal{F} \). Then \( \mathcal{F} \) violates (iv), since \( \gamma = \sum_{S \in \mathcal{F}} \gamma(S) + \gamma(S_0) \mathcal{C} \) (M). By (a.4) there exists

\( x^* \in \mathcal{X}^i, x^*_L = \mathcal{X}^*_L \), with \( \gamma^*(x^*) > \gamma^*(x^*) \) \( i \in \mathcal{N} \), and by choosing \( x^* \) close

enough to \( x^0 \), by (1):

\[ \sum_{i \in \mathcal{N}} \gamma^*(x^*) \leq \sum_{i \in \mathcal{N}} \gamma^*(x^0) \]  

This Theorem is essential for the proof of the equivalence of core and s.c.e.: allocations when the replica economy is considered. (Section IX).

Although every s.c.e. allocation is Pareto optimum, there may exist no labour-prices that support it. For example, consider the following economy:
\[ M = \{1, 2\}; u^1 = (0, 1, 0), \quad w^2 = (0, 0, 1); \quad u^1(\mathbf{x}) = x^1_C = x^1_L, \quad 1 \in M; \]

\[ Y(\{1, 2\}) = \{y \in \mathbb{R}^3 \mid y_1 \leq y_2 - y_3\}, \quad Y(\{1\}) = 0, \quad \forall \mathbf{n}. \]

It can be easily verified that \( x^1 = (\frac{1}{2}, 0, 0), \quad x^2 = (1, 0, 0) \) \( y = (2, -1, -1) \)

is a S.E.E. allocation. However, the only competitive allocation supported by labour prices is \( \bar{x}^1 = \bar{x}^2 = (1, 0, 0) \) \( p_1 = p_2 = p_3, \quad \bar{y} = (2, -1, 4) \).

As will be proved in Section IX, when replicating the economy such examples do not exist.
VII. Existence of Equilibrium Coalition Structures

Theorem 1 proves the existence of a firm structure which brings about structural competitive equilibrium. In particular, individuals may exist who participate in more than one firm. Though as shown above, in the standard A-D economy this type of structure is formed, the question that naturally arises is whether for every economy that fulfills assumptions (a)-(c) there exists a structural competitive equilibrium, the firm structure of which is a coalition structure. The following example will shed some light upon this problem.

Example 3: Consider the economy of example 2 with the following technology sets:

\[ Y ([i,j]) = \{ y \in \mathbb{R}^n | y = x (1, -e^i - e^j)^t \ 0 \leq t \leq 1 \} (e^i) \text{ denotes the } i\text{-th unit vector in } \mathbb{R}^3 \]

\[ Y (S) = \{ 0 \} \text{ otherwise.} \]

In essence, this is the "game of pairs", where every two individuals can produce one unit of \( x_1 \) if they work full time, or any convex combination of this production and the \( \{ 0 \} \) vector.

If \( \{ X \}_{i=1}^3 \) is a structural competitive allocation, we must have \( \sum_{i=1}^3 x_1^i = 1 \) which is realized by forming the firm structure

\[ F = \left( \{ 1, 2 \}, \{ 2, 3 \}, \{ 1, 3 \} \right), \text{ and } y(S) = \left( \frac{1}{3}, \frac{1}{3} (e^1 + e^3) \right) \text{ for all } \{ i, j \} \subseteq F. \]

No coalition structure can give rise to such an allocation since for any coalition structure \( B, y \in \mathbb{R} \) implies \( y_1 \leq 1 \), hence \( \sum_{i \in B} x_1^i \leq 1. \)

Most models will include this type of economy as a special case, since, in particular, the utilities here do not depend upon the non-marketed goods.

\[ \text{Since by (ii) } x_1^i + x_1^j \geq 1 \ \forall \ i, j. \]
It therefore follows that if we want to get a positive result as to the
formation of coalition structures rather than firm structures, we shall
have to impose rather strong restrictions either on the utilities of the
individuals (e.g., if i works for more than one firm, \( u^i(x^i) \leq u^i(y^i) \),
for all attainable \( x^i(s) \), or on the technology sets.

Example 3 indicates that if ad-hoc restrictions on the preference order-
ing are not made, a necessary condition for an equilibrium coalition structure
to exist (in general) is that the technology sets be balanced in the following
way:

Define \( \hat{Y} = \bigcup_{k \in K} Y(S^k) \) where \( \{S^k\} \) is a balanced collection with
\( \{Y_k\} \) as its weights\(^2\) and the union is taken over all balanced collec-
tions. Then,
(d) For any \( \hat{y} \in \hat{Y} \) there exists a coalition structure \( B(\hat{y}) \) such that
\( \hat{y} = \sum_{k \in K} Y(S^k) \).

To see that (d) is a necessary assumption, note that in the special case
of one marketable commodity (hence \( p = 1 \)) and with \( u^i(x^i) = x^i \forall i \in N \) (e.g.,
Examples 1-3), condition (ii) requires that an equilibrium allocation be in
the core of the economy. Since, by the definition of the utility functions
the market game is a game with side-payments, balancedness of the technology
sets is a necessary condition. We shall now show that it is also a suffi-
cient condition for an economy to have a s.c.e. whose firm structure is also
a coalition structure. To prove this claim we make use of Bohm's result
[1]. It will also become clear why an analogous proof to that given for
Theorem 1 cannot be used in this case.

\(^2\) A set of coalitions \( \{S^k\}_{k=1}^t \) is called balanced, if there exist "weights"
\( \{Y_k\}_{k=1}^t \) such that for all \( k, Y_k \geq 0 \) and for all \( i \in N, \sum_{k \in S^k} Y_k = 1 \).
Theorem 5: Under assumptions (a)-(d) there exists an equilibrium coalition structure.

Proof: We replace the set $\cup \Sigma Y(S)$ (where $F$ is a firm structure) in (1), by

\[ F \neq F \]

the set $Y$. Clearly $Y$ is convex. Since in the proof of the theorem Bohn makes use only of the balancedness condition, his proof carries over to our case. Therefore, there exist prices $q \in \mathbb{R}^{n+M}$, consumption bundles $x^i \in \mathbb{R}^n$, $i \in M$, a production plan $\tilde{y} \in \tilde{Y}$, and $(\tilde{e}^1, \ldots, \tilde{e}^m) \in \mathbb{R}^m$ such that:

1. $\Sigma x^i = \Sigma y^j + \tilde{y}$

2. For all $i \in M$: $q \in \Sigma x^i \leq y_i + \tilde{e}^i$, and $u^1(x^i) > u^1(\tilde{x}^i)$ implies $q \in \Sigma x^i > y_i + \tilde{e}^i$

3. $\Sigma \tilde{e}^j = q(y(S)) \forall S \in M$

4. $q \in \Sigma y_i$ (1) \in M

5. $q(y(S)) = \max \ S \in M q(y(S))$ (1) \in M

where $\tilde{y} = [\tilde{y}_1, \ldots, \tilde{y}_j]$ is the coalition structure such that $\tilde{y} = \Sigma y_i$ (1) \in M (such a $\tilde{y}$ exists by (d)).

Combining (3) and (4) we get: $p(y(S)) = \tilde{e}^j$ (1) \in M, $y = \Sigma y_i$ (1) \in M

By (a4) and (1) \in M

(2), $\sigma \neq 0$. Define:

\[ p = \frac{q \in \Sigma x^i}{|q \in \Sigma x^i|}, \quad e^j = \tilde{e}^j - q \in \Sigma x^i (1) \in M, j = 1, \ldots, J. \]

By (1), (3), (4) and (5) it is left to be shown that (11) holds, in order to conclude that $(p, \Sigma x^i (1) \in M, S, y(S), \Sigma \tilde{e}^j (1) \in M)$, is a s.c.s.

Suppose that for some $S \in M$, $u^1(x^i) > u^1(\tilde{x}^i) \forall i \in S$, $y(S) \notin \tilde{Y}$ and

\[ y^0(S) = \Sigma \tilde{x}^i (1) \in M, i \in S. \]

By (2) and (3)

\[ \tilde{e}^j > \Sigma q \in \Sigma x^i, i \in S u^1(x^i) + q(y(S)), \text{ hence} \]

\[ \Sigma \tilde{x}^i (1) \in S, i \in S u^1(x^i) + q(y(S)), \text{ hence} \]

\[ \Sigma \tilde{x}^i (1) \in S, i \in S u^1(x^i) + q(y(S)), \text{ hence} \]

O.E.D.
VIII. Optimality of Equilibrium Coalition Structure

In contrast to Corollary 2 there are economies (which fulfill assumptions (a)-(d)) for which every equilibrium allocation associated with an equilibrium coalition structure is not Pareto optimal.

Example 4: Consider the economy of Example 1 with \( \hat{Y} \) replacing \( Y(0) \). For our three-person economy, the relevant balanced collections are: \( \{(1,2), (3)\}, \{(2,3), (1)\}, \{(1,2,3)\}, \{(1,2), (2,3)\} \) with the weights \( (1,1), (1,1), (1, \frac{1}{2}, \frac{1}{2}) \), respectively. Hence \( \hat{Y} \) = \text{conv} \( Y((1,2), (2,3)) \) where \( Y((1,2), (2,3)) = Y(1,3) \).

If \( (x^i_{1,1}) \) is Pareto optimal, \( \sum_{i \in M} x^i_{1,1} = c = 3 \). However, for any coalition structure \( B \), \( \max_{s \in \mathcal{S}} \sum_{i \in s} x^i = 2.6 \).

Example 4 implies that we could not prove Theorem 5 in an analogous way to the proof of Theorem 1. However, every equilibrium allocation associated with a coalition structure is Pareto optimal relative to that structure.

This is in contrast with the case of firm structures (Example 2).

Theorem 6: Let \( B \) be an equilibrium coalition structure with \( (x^i)_{i \in M}, (y(s))_{s \in \mathcal{S}B} \) the equilibrium allocation. Then, this allocation is Pareto optimal relative to \( B \).

Proof: Suppose not, i.e., there exists a \( B \)-attainable allocation \( (x^i_{1 \in M}, y(s))_{s \in \mathcal{S}B} \) with \( u^i(x^i) > u^i(x^i) \) \( i \in M \). Since \( \sum_{i \in M} x^i_{1 \in M} = c \), \( \sum_{i \in M} x^i_{1 \in M} + \sum_{s \in \mathcal{S}B} y^i(s) \), and as \( (p, x^i_{1 \in M} \in B, y(s))_{s \in \mathcal{S}B}, (t_{i,s}) \) is a s.c.a., by

\( (i): \) for all \( s \in \mathcal{S}B \), \( \sum_{i \in s} x^i_{1 \in M} > \sum_{i \in s} x^i_{1 \in M} + \sum_{s \in \mathcal{S}B} y^i(s) \). Summing over all \( s \in \mathcal{S}B \) we get a contradiction.

Q.E.D.
IX. The Replica Economy

In section VI we noted that not every s.c.e.#, (i.e., s.c.e. with (1\*) replacing (1) in Def. 3), can be supported by personalized efficiency prices for labour. We now show that this is in contrast with the situation in the replica economy.

Let $M_{r}$ denote the r-fold replica of M, $M_{r}$ consists of mr individuals which will be indexed by the pair $(i,t)$, $i=1,\ldots,m$, $t=1,\ldots,r$.

For all $i \in M$, each $(i,t)$, $t=1,\ldots,r$, is of type $i$ in the sense that $x_{i}^{(i,t)} = x_{i}$, $u_{i}^{(i,t)} = u_{i}$, $w_{i}^{(i,t)} = w_{i}$, and for all $S \subseteq M$, $i \in S$, $Y(S) = Y(S/\{t\} \cup \{1,t\})$. [Note that we do not make labour a named commodity, otherwise the commodity space will tend to infinity, $(g + rMAD = \Rightarrow r \to \infty)$. Lemma 1 below enables us to use this formulation.] By the definition of the super-additive technology, and using assumption (b.2) the following properties hold:

1. $Y_{r}^{*} = Y_{1}^{*}(M_{r}) = rY_{1}^{*} = rY^{*}(M)$.
2. $S \subseteq M_{r}$ implies $Y_{r}^{*}(S) \subseteq Y_{1}^{*}$.
3. For any coalition $S$ with $k_{i}(S)$ members of type $i$, $k_{i}(S) \geq 1$ for all $i \in M$,
   $\min \{k_{i}(S) \mid i \not\in M\} Y_{1}^{*} \subseteq Y^{*}(S)$.

The goal of this section is to prove the existence of efficiency prices (for the marketed goods as well as for labour) that support any s.c.e.# allocation which belongs to the replica economy $E_{r}$ for all $r = 1,2,\ldots$, (Theorem 7 below). To this end we consider the super-additive core of the market game $(M_{r},v_{r})$ generated by $E_{r}$, where for all $S \subseteq M_{r}$:

$v_{r}(S) = \{u^{1,t}_{i} : (i,t) \in M_{r} \times R^{+} \mid w^{1,t}_{i} + S, u_{i}^{1,t} = 0 \}$. 


\[ \forall (i, t) \in S \quad u^i_{1:t} \leq u^i_{1:t}(x^i_{1:t}) \quad \forall i, t \in X^i \]

\[ x^i_{1:t} = \sum_{(i, t) \in S} y(S), y(S) \in \mathbb{R}(S) \]

We shall denote by \( C^e \) the core allocations of this game and by

(a') assumptions (a) with (a.2) replaced by:

(a.2') \( u^i \) is a continuous strictly concave function on \( x^i \).

**Lemma 1:** Under assumptions (a')-(c), an allocation in \( C^e \), \( r \geq 2 \) assigns the same consumption bundle to all consumers of the same type.

**Proof:** Standard.

By Lemma 1 we can denote a core allocation for the sequence economies \( E^e_r \), by the vector \( x = (x^1, \ldots, x^m) \in (g \times m)^m \) (rather than \( x = (x^1, \ldots, x^m) \in (g \times m)^m \)).

**Lemma 2:** Under assumptions (a')-(c), for every \( x = (x^1, \ldots, x^m) \in \cap C^e \),

there exists a price system \( \psi \in \mathbb{R}^{\times m} \) such that

1. \( x^i \) is a \( \psi \)-efficient allocation for \( E^e_r \),

2. \( \psi \leq \psi^* \) and \( u^i(x^i) > u^i(x^i) \) implies \( \psi^i > \psi^i \),

3. \( \psi \) is \( \psi^* \)-efficient.

**Proof:** This is a direct application of the Theorem in [2].

**Theorem 7:** Under assumptions (a')-(c), every s.c.e. allocation for \( E^e_r \), \( r = 1, 2, \ldots \), can be supported by an efficiency price system \( \psi \in \mathbb{R}^{\times m} \), \( \psi \neq 0 \). (i.e., (1), (2) and (3) of Lemma 2 hold.)

**Proof:** Let \( (x^i, T) \) be a s.c.e. allocation for \( E^e_r \) and let \( p \in \Delta \) be the equilibrium price system. Define the game \( (N^e_r, V^e_r) \): For \( S \subset N^e_r \), define
\( V_r^+(S) = \{ u^{i,t} (t, t) \in M_r^+ \mid \forall (i, t) \in S, u^{i,t} = 0 \}, \)

\( W(t, t) \in S \),
\( u^{i,t} \leq u^{i,t} (x^{i,t}), \ y^{i,t} \leq x^{i,t}, \)

\[ \begin{align*}
\zeta^{i,t} &= p^{i,t} x^{i,t} + p y^{i,t}(S), \\
\zeta^{i,t} &= W^{i,t} + y^{i,t}(S) + y^{i,t}(S) \end{align*} \]

\[ S \]

and

\[ V^+_r (M_r) = V^+_r (M_r) \]

Denote the core allocations of this game by \( C^+_r (V^+_r) \). By Theorem 4 \((z^{i,t}) \in C^+_r (V^+_r)\). Clearly for all \( S \in M_r \), \( V^+_r (S) \supset V^+_r (S) \) and since

\( V^+_r (M_r) = V^+_r (M_r) \) it follows that \( C^+_r (V^+_r) \subset C^+_r \). Hence \((z^{i,t}) \in C^+_r \). Hence \((z^{i,t}) \in C^+_r \) \( r=1,2,... \), and Lemma 2 concludes the proof.

Q.E.D.

In fact, following the pattern of the proof of Theorem 5, and using Lemma 2, every \( x \in C^+_r \) is a s.c.e. allocation for every \( E_r, r=1,2,... \).

Hence, \( C^+_r \) shrinks to the set of s.c.e. allocations for every \( E_r, r=1,2,... \), as \( r \to_{\infty} (i.e., \ the \ analog \ of \ the \ "equivalence \ theorem" \ holds). \)
REFERENCES


