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EXISTENCE AND OPTIMALITY OF EQUILIBRIUM IN LABOUR-MANAGED ECONOMIES */

by

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*/ I gratefully acknowledge J. Dreze for introducing me to this subject and the interesting discussions with him gave rise to the main definition in the paper. I am also indebted to D. Schmeidler who simplified the proof of Theorem 3 and to T. Ichiishi for fruitful discussions.

ABSTRACT

The formation of firms is explained due to the existence of some commodities which have the three properties:

-- cannot be transferred,

-- cannot be produced, and

-- cannot be marketed.

These properties give rise to the formation of Labour-Managed economies for which we define an equilibrium. Under that definition no coalition has the incentive to withdraw and form a new firm operating in the market with the existing prices (for the marketed goods).

We investigate the existence and optimality of equilibrium allocations in two cases:

(i) when workers may belong to more than one firm (i.e., "moonlighting" is not excluded), and

(ii) when every individual may work in one and only one firm.

We consider also the Replica Economy, and prove the equivalence of the core and equilibria allocations.

I. Introduction

In this paper the formation of firms is explained due to the existence of some commodities (like initiative, skill, imagination, connections etc.), which have the three properties:

(1) cannot be transferred (i.e., individually specific),

(2) cannot be produced (but, rather, serve as inputs), and

(3) cannot be marketed, (i.e., the firm does not face a price per unit in which it can acquire these commodities).

Clearly, types of labour (or more generally, "time") share the first two properties. The third property is the main assumption on labour in Labour-Managed economies [4, 7, 9, 10]; i.e., labour has no price ("wage") but rather receives a share in the value added. (A detailed description of labour-managed economies, which concentrates on this very issue is given in [4]). As Ward [10] notes:

> "In Illyria a single class of inputs, labour, is singled out for special treatment. The distinctive features of Illyrian behaviour stem entirely from this fact."

We shall therefore refer to commodities with the above three properties as types of labour. However, the model presented here is valid for any economy with commodities (possibly other than labour) that have properties (1) - (3). For example, the shares in Arrow-Debreu economy are actually "types of labour." (Section III).

A labour-managed economy is an economy where production is carried out in firms organized by workers who get together and form collectives or partnerships. These firms hire non-labour inputs, including capital, and sell outputs under the assumed objective of maximizing the welfare of the members. Since labour is not marketed, when a group of people (coalition) forms a firm, it faces a technology set which uses only the amount (and types) of labour its members have. It is only natural, therefore, to use the production coalition economy (defined by Hildenbrand [6]). Under that approach, each group of workers S faces a given technology set, $Y(S)$. The correspondence $Y(\cdot)$ assigning a production set to every coalition describes fully the production possibilities of the economy.

In a forthcoming paper by J. Dreze and the author, an analogy between local public goods and labour-managed economies has been established. Thus the definition we use here for an equilibrium is closely related to that of a structural equilibrium in [5]. In particular, since workers can choose the firm(s) they want to work in, we require that in equilibrium no mobility will occur. There is no incentive for any coalition S to withdraw from the existing firm structure, produce by itself $y(S)$, distribute the derived value added (evaluated at the prevailing prices for the marketed goods) in such a way that each of its members is made better off.

Two types of equilibrium are considered: When individuals are allowed to belong to more than one firm (i.e., when "moonlighting" is allowed) we call the derived partition a <u>firm structure</u>. When individuals are allowed to work in one and only one firm, the derived structure is called a <u>coalition structure</u>. In Section IV we prove, under standard assumptions, the existence of an equilibrium with a firm structure. In Section VII we prove, using the (necessary) additional "balancedness assumption on the technology sets", that an equilibrium with a coalition structure exists.

Section V focuses on optimality properties of equilibria with firm structures. It is proved that for every labour-managed economy there exists an equilibrium allocation which is Pareto optimum. However, it is shown that there are economies whose equilibrium allocation is not Pareto optimum. Moreover, equilibrium allocations may even be non-optimal relative to the existing firm structure! Recalling that individuals are allowed to work in more than one firm (hence, no indivisibility exists), these results are rather surprising. It follows that there are equilibrium allocations that cannot be supported by efficiency prices for labour.

The above results are in contrast to Meade's claim that both labour-management and entrepreneurial management

> "will lead to the same Pareto optimal equilibrium situations in the long run, provided that there is perfect mobility of factors and that there is perfect competition" [7, p. 404].

Using Dreze's formulation, the purpose of [4] is to establish Meade's claim with a maximum of generality and rigor, and indeed Dreze accomplishes his goal. The difference between our results and those in [4, 7, 9] stems, obviously, from the different definitions of equilibrium used in the above models. Naturally, I view the definition of equilibrium in this paper as intuitively appealing.

In Section VI we show that the source of nonoptimality is due to the inappropriateness of the "core" for coalition production economies. It fails to allow a group of people, S, to form and produce in "sub-firms" and share among themselves the total output thus derived. To enable such actions we define the concept of the super-additive technology, which may be of interest also in contexts different from the one presented here. Using this concept, we prove in Section IX that the core of the replica economy shrinks to the set of equilibrium allocations.

In Section VIII, the optimality of equilibrium allocations with coalition structures (rather than firm structures) is investigated. We show that in contrast to the results of Section V, there may exist no equilibrium allocation which is Pareto optimum. On the other hand, again in contrast to the case of firm structures, every equilibrium allocation is Pareto optimum relative to its equilibrium coalition structure.

The model, definitions and assumptions are presented in Section II.

II. The Model

There are m individuals, g marketed goods and ℓ types of labour. By definition labour is individually specific hence the commodity space is $R^{g+m\ell}$. Let $a \in R^{g+m\ell}$, denote:

$$a_c = (a_1, \ldots, a_g)$$

$$a_L = (a_{g+1}, \ldots a_{g+m\ell})$$

$$a_{L,i} = (a_{g+(i-1)\ell+1}, \ldots, a_{g+i\ell}), \; i=1,\ldots,m \; .$$

$M = \{1,\ldots,m\}$ denotes the set of individuals each endowed with $w^i \in R^{g+m\ell}$ and a utility function defined over his consumption set X^i. Since labour cannot be transferred, $x^i \in X^i$ implies $x^i_{L,j} = 0$ for all $j \neq i$. Similarly, since i does not endow j's labour, $w^i_{L,j} = 0$ $\forall j \neq i$.

Using the coalition production approach, each group of individuals, S, $S \subset M$, faces a production set $Y(S) \subset R^{g+m\ell}$. Again, since labour cannot be transferred nor is it marketed, $y \in Y(S)$ implies that y_c is produced by using only the labour members of S own, i.e., $y \in Y(S)$ implies $y_{L,i} = 0$ $\forall i \notin S$. Moreover, since labour cannot be produced, $y_L \leq 0$.

$Y(S)$ can be derived from a more primitive technology $\tilde{Y}(S) \subset R^{g+\ell}$. Indeed, since types of labour are defined solely by their productive characteristics, technologically there can be no distinction between the same type of labour provided by different individuals. Thus, $\tilde{y} \in \tilde{Y}(S)$ states that $\tilde{y}_c$ can be produced by S using the amount of labour $\tilde{y}_L \in R^{\ell}$ (regardless which individuals contribute it). $\tilde{Y}(S)$ represents the <u>potential</u> production possibilities available to S if and when forms. However, as labour cannot be hired, we have to add the restriction that whenever individual $i \in M$ contributes (ex-post) some labour to S, he must belong to that coalition. We therefore "extend" the technology $\tilde{Y}(S)$ to $Y(S)$ in the following way:

$$Y(S) = \{y \in R^{g+m\ell} \mid y_{L,j} = 0 \ \forall j \notin S,\ y_L \leq 0 \text{ and } (y_c, \sum_{i \in S}(y_{L,i})) \in \tilde{Y}(S)\}$$

Remark 1: We could also impose the restriction:

$$y \in Y(S) \Rightarrow y_L \geq \sum_{i \in S}(x_L^i - w_L^i) \text{ for some } x^i \in X^i,\ i \in S\ ,$$

i.e., the maximum amount of labour required does not exceed the maximum amount of labour S can afford to contribute to production. However, in equilibrium this condition holds, eliminating the need for the above restriction.

Definition 1: A firm structure is a nonempty collection $\{S_k\}$ of subsets of M, such that $S_i \neq S_j \ \forall i \neq j$.

Note that individuals may work in (belong to) more than one firm, (i.e., "moonlighting" is not excluded). If individuals are restricted to work in one and only one firm, the firm structure forms a partition of M, and we refer to such firm structures as coalition structures. Formally,

Definition 2: A coalition structure is a firm structure F such that for all S_i, $S_j \in F$, $S_i \cap S_j = \emptyset$, $i \neq j$.

We shall first investigate the existence of an equilibrium and its optimality properties when a firm structure is formed and then, in Sections VII, VIII we investigate the more restricted case of coalition structures.

Denote the price simplex for the marketed goods by Δ, i.e.,

$\Delta = \{p \in R_+^g \mid \Sigma p_i = 1\}$,

Definition 3: A structural competitive equilibrium (s.c.e.) consists of a price system $p \in \Delta$, consumption bundles $x^i \in X^i$, $i \in M$, a firm structure F, production plans $y(S) \in Y(S)$, $S \in F$, and a value added distribution $\{t_{is}\}$ $i \in M$, $S \in F$, such that:

(i) $\sum_{i \in M} x^i = \sum_{i \in M} w^i + \sum_{S \in F} y(S)$,

(ii) There is no:

$S_o \subset M$, $\bar{x}^i \in X^i$, $i \in S_o$, $y(S_o) \in Y(S_o)$ such that

$$\sum_{i \in S_o} p\bar{x}^i_c \leq p \sum_{i \in S_o} w^i_c + p\, y_c(S_o), \quad \sum_{i \in S_o} \bar{x}^i_L - \sum_{i \in S_o} w^i_L = y_L(S_o) ,$$

and $u^i(\bar{x}^i) \gg u^i(x^i)$ $\forall$ $i \in S_o$,

(iii) For all $i \in M$, $p\, x^i_c \leq p\, w^i_c + \sum_{\substack{S \in F \\ i \in S}} t_{is}$,

(iv) For all $S \in F$, $\sum_{i \in S} t_{is} \leq p\, y_c(S)$,

(v) For all $S \in F$, $p\tilde{y}_c(S) \leq py_c(S)$, $\forall$ $(\tilde{y}_c(S), y_L(S)) \in Y(S)$.

Conditions (i), (iii) and (iv) need no comments. Condition (v) is the equivalent of profit maximization. It states that in each firm $S \in F$ the value added, given the labour vector $y_L(S)$, is maximized. (Since labour has no price, without the constraint on $y_L(S)$ a firm may desire to use an infinite amount of labour, being a free input). Condition (ii) assures that no mobility will occur. There is no incentive for any coalition S_o to produce by itself $y(S_o)$, distribute the value added $py_c(S_o)$ in such a way that every member of S_o can purchase (in the market) a vector $\bar{x}^i_c$ whereby his utility is increased.

Assumptions;

For each $i \in M$,

(a.1) X^i is a closed convex subset of $R_+^{g+m\ell}$ and

$x^i \in X^i$ implies $x^i_{L,j} = 0 \ \forall j \neq i$,

(a.2) u^i is a continuous quasi-concave function on X^i ,

(a.3) $w^i \in X^i$ and $\exists \ \hat{x}^i \in X^i$ with $\hat{x}^i_c \ll w^i_c$,

(a.4) There is no satiation in consumption in X^i ;

i.e., for any $x \in X^i$ and for any neighborhood U of x in

X^i, there exists an $\tilde{x} \in U$, $\tilde{x}_L = x_L$, such that $u^i(\tilde{x}) > u^i(x)$.,

For every $S \subset M$,

(b.1) $Y(S)$ is a closed convex subset of $R^{g+m\ell}$ and $y \in Y(S)$ implies

$y_L \leq 0$, $y_{L,i} = 0 \ \forall i \notin S$,

(b.2) $0 \in Y(S)$,

and for $Y = \sum_{S \subset M} Y(S)$

(c.1) Y is closed ,

(c.2) $Y \cap \{-Y\} \subset \{0\}$, and

(c.3) $Y \supset R_-^{g+m\ell}$.

Theorem 1: Under assumptions (a) - (c) there exists a structural competitive equilibrium.

We shall prove the theorem after the next section which may give some more insight and motivation to our model.

III. The Arrow-Debreu Economy

A good example of a non-marketed goods economy is provided by the standard A - D economy, where there are m individuals each endowed with $\tilde{w}^i \in R^g_+$. The consumption set is $\tilde{X}^i \subset R^g_+$, over which the utility function $\tilde{u}^i$ is defined. There are J firms with the technology sets $\tilde{Y}^j \subset R^g$, $j = 1,\ldots, J$. Each firm j is owned by the individuals who hold its shares. The shares θ_{ij}, $j = 1,\ldots, J$ individual $i \in M$ holds are a-priori given and are not marketed, i.e., $\ell = J$. For all j, $\sum_{i\in M} \theta_{ij} = 1$, with $\theta_{ij} \geq 0$ for all $i \in M$.

A competitive equilibrium consists of a price vector $p \in \Delta$, consumption bundles $\tilde{x}^i \in \tilde{X}^i$, $i \in M$, and production plans $\tilde{y}^j \in \tilde{Y}^j$, $j = 1,\ldots, J$ such that

(α) $p\tilde{x}^i \leq p\tilde{w}^i + \sum_{j=1}^{J} \theta_{ij} p\tilde{y}^j$, and $\tilde{u}^i(\hat{x}^i) > \tilde{u}^i(\tilde{x}^i)$ implies

$p\hat{x}^i > p\tilde{w}^i + \sum_{j=1}^{J} \theta_{ij} p\tilde{y}^j$, $i = 1,\ldots,m$,

(β) $p\tilde{y}^j = \text{Max } p\tilde{Y}^j$, $j = 1,\ldots, J$, and

(γ) $\sum_{i\in M} \tilde{x}^i = \sum_{i\in M} \tilde{w}^i + \sum_{j=1}^{J} \tilde{y}^j$.

The natural way to embed this economy in our model is by defining the following:

$$w^i = (w^i_c , w^i_L) \in R^{g+m\ell},\ w^i_c = \tilde{w}^i,\ w^i_{L,i} = (\theta_{i1},\ldots, \theta_{iJ}) \text{ and } w^i_{L,j} = 0, \forall j \neq i ,$$

$$X^i = \{x \in R^{g+mJ}_+ \mid x_c \in \tilde{X}^i,\ x_L = 0\}$$

$$u^i(x^i) = \tilde{u}^i(x^i_c) \quad \text{(i.e., the utility is independent of the non-marketed goods.)}$$

$$Y^j = \{y \in R^g \times R^{mJ}_- \mid y = t\hat{y},\ \hat{y}_c \in \tilde{Y}^j,\ \sum_{i\in M} (\hat{y}_{L,i})_j \leq -1,\ 0 \leq t \leq 1\} .$$

[$(\sum_{i\in M} (\hat{y}_{L,i})_j$ denotes the total amount of labour (shares) of type (firm) j which serves as an input in $\hat{y}$]. Let $Y = \{y | y = \sum_{j=1}^{J} y^j,\ y^j \in Y^j\}$.

Then, the coalition production technology is given by:

$$Y(S) = \{y \in Y \mid y_L \geq - \sum_{i\in S} w_L^i \} \quad .$$

While in a "capitalistic economy" a firm is associated with a technology set, in a labour-managed economy a firm is identified with its workers (in A-D model, with its shareholders). Hence, the same production planes produced in both economies in the same technological way may give rise to different firm structures. Here, the set of workers in firm j (defined by the technology set $\tilde{Y}^j$) is: $S_j = \{i\in M | \ \theta_{ij} > 0\}$. Define:

$$\tilde{F} = \{S_1,\ldots S_J\} \text{ and } F = \{S_k \in \tilde{F} | \ S_i \neq S_j \ , i\neq j \},$$

i.e., F is the firm structure derived from the collection $\tilde{F}$.

For $S\in F$, $y^j \in Y^j$, $j = 1\ldots, J$, denote:

$$j(S) = [j | S_j = S] \text{ and } y(S) = \sum_{j(S)} y^j \ .$$

Clearly, $\sum_{j=1}^{J} y^j = \sum_{S\in F} y(S)$.

When the J firms "form" in the A-D model, the firm structure F is realized in the associated labour-managed economy.

The following theorem proves that the A-D economy thus embedded is a special case of our model.

Theorem 2: Under assumptions (a) - (c), $(p, \{\tilde{x}^i\}_{i\in M}, \{\tilde{y}^j\}_{j=1}^{J})$ is a competitive equilibrium if and only if $(p, \{x^i\}_{i\in M}, F, \{y(S)\}_{S\in F}, \{t_{iS}\})$ is a structural competitive equilibrium with:

For all $i\in M$: $x_c^i = \tilde{x}^i$, $x_L^i = 0$,

For all $S\in F$: $y_c(S) = \tilde{y}(S)$, for $j\in j(S)$ $(y_{L,i}(S))_j = -\theta_{ij}$ and

$(y_{L,i}(S))_j = 0$ $j \notin j(S)$.

For all $i\in M$, $S\in F$: $t_{is} = \sum_{j\in j(S)} \theta_{ij}\, p\tilde{y}^j$.

Proof: Let $(p, \{\tilde{x}^i\}_{i\in M}, \{\tilde{y}^j\}_{j=1}^{J})$ be a competitive equilibrium, and let $(\{x^i\}_{i\in M}, \{y(S)\}_{S\in F}, \{t_{iS}\})$ be defined as above. Clearly, $\sum_{i\in M} x_L^i = \sum_{i\in M} w_L^i + \sum y_L(S)$ $(=0)$, and by (γ) $\sum_{i\in M} x_c^i = \sum_{i\in M} w_c^i + \sum_{S\in F} y_c(S)$. Hence, (i) is fulfilled. It is easily verified that for all $T\subset M$ $Y(T) = \sum_{i\in T} Y(\{i\})$, and for all $i\in M$, $Y_c(\{i\}) = \sum_{j=1}^{J}\theta_{ij}\tilde{Y}_j$. (In fact, we could use this formulation to represent the production coalition technology sets.) By (β), for all $y_c(T) \in Y_c(T)$, $T\subset M$, $py_c(T) \leq \sum_{i\in T}\sum_{j=1}^{J}\theta_{ij}\, p\tilde{y}^j$, which establishes (v) . (iii) and (iv) follow immediately from (2) and the definition of t_{iS} . Suppose $u^i(\bar{x}^i) > u^i(x^i)$ (hence, $\tilde{u}^i(\bar{x}_c^i) > \tilde{u}^i(x_c^i)$), $\forall\ i \in S_o \subset M$. By (α), $p\bar{x}_c^i > pw_c^i + \sum_{j=1}^{J}\theta_{ij}\, p\tilde{y}^j$, and using the previous inequality we get $\sum_{i\in S} p\bar{x}_c^i > \sum_{i\in S} p\, w_c^i + py_c(S_o)\ \forall\ y_c(S_o) \in Y_c(S_o)$, thus (ii) is also fulfilled.

Let $(p, \{x^i\}_{i\in M}\ \{y(S)\}_{S\in F}, \{t_{is}\})$ be a s.c.e. for the associated labour-managed economy. From $x^i \in X^i$ we have $x^i_L = 0$, $\forall i\in M$. As $y\in Y(S)$ implies $y = \sum_{j\in j(S)} y^j$, $y^j \in Y^j$, we can associate with $\{y(S)\}_{S\in F}$ the vectors $\{y^j\}_{j=1}^{J}$, $y^j \in Y^j$. Define for $i\in M$, $\tilde{x}^i = x^i_c$ and for $j = 1,..,J$, $\tilde{y}^j = y^j_c$.

Since $\forall T\subset M$, $Y_c(T) = \sum_{i\in T} \sum_{j=1}^{J} \theta_{ij} \tilde{Y}^j$, (v) implies (β) and (ii), (iii) with assumption (a.4) imply that $\sum_{S\in F} t_{is} \geq \sum_{j=1}^{J} \theta_{ij} p\tilde{y}^j$. (Note that $i\notin S \Rightarrow t_{is} = 0$.)

Summing (iv) over all $S\in F$ yields:

$$\sum_{S\in F} p\tilde{y}_c(S) = \sum_{j=1}^{J} p\tilde{y}^j \geq \sum_{S\in F} \sum_{i\in S} t_{is} = \sum_{i\in M} \sum_{S\in F} t_{is} \geq \sum_{i\in M} \sum_{j=1}^{J} \theta_{ij} p\tilde{y}^j = \sum_{j=1}^{J} p\tilde{y}^j ,$$

hence, for all $i\in M$ $\sum_{S\in F} t_{is} = \sum_{j=1}^{J} \theta_{ij} p\tilde{y}^j$. The first part of (α) is implied by (iii) and the second is derived by (ii).

Q.E.D.

IV. Existence of Equilibrium Firm Structures:

We shall now prove theorem 1 by converting our labour-managed economy to a standard A-D model in which (in contrast to the previous section) all commodities, labour included, are marketed, and thus have prices. Consider an A-D economy with m individuals and with 2^m-1 firms with the technology sets $Y(S)$, $S\subset M$. Each firm $Y(S)$ is owned through a share system $\{\theta_{is}\}$ where $\theta_{is}\geq 0$ for all $i\in M$, $\theta_{is}=0$ for $i\notin S$, and $\sum_{i\in S}\theta_{is}=1$. (i.e., only members of the coalition S can own shares in the firm $Y(S)$). The consumption sets, initial endowments and preferences are identical to those in the original (labour managed) economy. Under assumption (a)-(c) [3] there exists a price vector $q\in R_+^{g+mJ}$, consumption bundles $x^i\in X^i$, $i\in M$, and production plans $y(S)\in Y(S)$, $S\subset M$, such that:

$$(\alpha)\quad qx^i\leq qw^i+\sum_{S\subset M}\theta_{is}\,qy(S)\ ,\ \text{and}\ \ u^i(\bar{x}^i)>u^i(x^i)\ \text{implies}$$
$$q\bar{x}^i>qw^i+\sum_{S\subset M}\theta_{is}\,qy(S)\ ,\ i=1,\ldots,m\ ,$$

$$(\beta)\quad qy(S)=\operatorname{Max} qY(S)\quad \forall S\subset M\ ,\ \text{and}$$

$$(\gamma)\quad \sum_{i\in M}x^i=\sum_{i\in M}w^i+\sum_{S\subset M}y(S)\ .$$

By (β) and (c.3), $q\geq 0$, and by (α) and (a.4), $q_c\geq 0$. Denote $\hat{q}=\frac{q}{\|q_c\|}$. Without loss of generality $\hat{q}=q$.

Define: $p=q_c$, $F=\{S\subset M\mid y(S)\neq 0\}$, and

$t_{is}=\theta_{is}\,qy(S)-q_{L,i}\,y_{L,i}(S)$, $i\in M$, $S\subset M$.

Note that for $i\notin S$ $\theta_{is}=0$ and $y_{L,i}(S)=0$, hence $t_{is}=0$.

Since $x^i \in X^i$, from (γ) we have:

$$x^i_{L,i} = w^i_{L,i} + \sum_{S\in F} y_{L,i}(S) \text{ , and by } (\alpha)$$

$$q_c x^i_c + q_{L,i}\, x^i_{L,i} \le q_c w^i_c + q_{L,i}\, w^i_{L,i} + \sum_{S\in F} \theta_{is}\, qy(S) \text{ .}$$ Thus (iii) holds.

By the definition of $Y(S)$ and $\{\theta_{is}\}$,

$$\sum_{i\in S} t_{is} = qy(S) - \sum_{i\in S} q_{L,i}\, y_{L,i}(S) = qy(S) - q_L\, y_L(S) = py_c(S) \text{ .}$$

Therefore, (iv) holds. Clearly (γ) coincides with (i) and (β) implies (v).

It is left to be shown that (ii) holds as well.

Let $S_o \subset M$, $\bar{x}^i \in X^i$ with $u^i(\bar{x}^i) > u^i(x^i)\ \forall\ i\in S_o$, $y(S_o)\in Y(S_o)$ with

$y_L(S_o) = \sum_{i\in S_o} \bar{x}^i_L(S_o) - \sum_{i\in S_o} w^i_L(S_o)$. By ($\alpha$), ($\beta$) and the definitions of F

and $\{\theta_{is}\}$, $\sum_{i\in S_o} q\bar{x}^i > \sum_{i\in S_o} qw^i + \sum_{i\in S_o} \sum_{S\subset M} \theta_{is} qy(S) \ge \sum_{i\in S_o} qw^i + qy(S_o)$, thus

$$\sum_{i\in S_o} p\bar{x}^i_c > \sum_{i\in S_o} pw^i_c + py_c(S_o) \text{ .}$$

In other words, we have proved

<u>Theorem 3</u>: Every competitive equilibrium for the associated A-D economy is an equilibrium for the original labour-managed economy.

<u>Corollary 1</u>: Theorem 1.

Note that by theorem 3, assumptions (a)-(c) can be relaxed (see, for example, [8]).

V. Optimality of Equilibrium Firm Structures

Having proved existence, the question of optimality of a s.c.e. naturally arises.

Definition 4: (a) Let F be a firm structure. An F - attainable allocation is a vector $(\{x^i\}_{i\in M}, \{y(S)\}_{S\in F})$ such that

(1) $x^i \in X^i$ for all $i \in M$,

(2) $y(S)\in Y(S)$ for all $S\in F$, and

(3) $\sum_{i\in M} x^i = \sum_{i\in M} w^i + \sum_{S\in F} y(S)$.

(b) Let F be a firm structure. A [strong] Pareto optimum relative to F is an F-attainable allocation $(\{x^i\}_{i\in M}, \{y(S)\}_{S\in F})$ for which there exists no other F-attainable allocation $(\{\bar{x}^i\}_{i\in M}, \{\bar{y}(S)\}_{S\in F})$ such that $u^i(\bar{x}^i) > u^i(x^i)\ \forall\ i \in M$. $[u^i(\bar{x}^i) \geq u^i(x^i)\ \forall i\in M$ and $u^i(\bar{x}^i) > u^i(x^i)$ for at least one $i\in M]$.

(c) An attainable allocation is an F-attainable allocation for some firm structure F.

(d) A [strong] Pareto optimum is an attainable allocation $(\{x^i\}_{i\in M}, \{y(S)\}_{S\in F})$ for which there exists no other attainable allocation $(\{\bar{x}^i\}_{i\in M}\ \{\bar{y}(S)\}_{S\in \tilde{F}})$ such that $u^i(\bar{x}^i) > u^i(x^i)\ \forall i\in M\ [u^i(\bar{x}^i) \geq u^i(x^i)$ $\forall i\in M$ and $u^i(\bar{x}^i) > u^i(x^i)$ for some $i\in M]$.

Clearly, every strong Pareto optimum is Pareto optimum, but the converse is not necessarily true. (By assuming "resource relatedness" of individuals, or monotonicity of preference, the two definitions coincide.)

An immediate corollary of theorem 3 is:

Corollary 2: For every labour-managed economy (which satisfies assumptions (a)-(c)) there exists a s.c.e. whose allocation is a strong Pareto optimum.

Proof: Under assumptions (a)-(c), every competitive allocation in A-D economy is a strong Pareto optimum. By Theorem 3 the conclusion follows.

The proof of Theorem 2 together with Theorems 1 and 3 suggest that:

- Every s.c.e. allocation is a (strong?) Pareto optimum.
- Every s.c.e. allocation can be sustained by a system of personalized prices for labour which serve as efficiency prices for these commodities.

The following example, however, disproves the above two conjectures.

Example 1: There are three individuals, each endowed with one type of non-marketed commodity. The utility of each individual depends solely upon the amount of the (one) marketed good he consumes, (x_1^i), and is monotonic increasing in this good. To simplify notation we shall not make labour individually specific and consider the commodity space R^4. (It is trivial to extend the dimension from 4 to $1 + 3 \times 3 = 10$.)

Let $w^1 = (0,1,0,0)$, $w^2 = (0,0,2,0)$, $w^3 = (0,0,0,1)$, and let the technology sets, for producing the marketed good, be given by:

$$Y(\{1,2\}) = \{y \in R^4 | y_1 \leq 2 \text{ Min } [-y_2, -y_3, 1], y_4 = 0\},$$

$$Y(\{2,3\}) = \{y \in R^4 | y_1 \leq \text{Min } [-y_3, -y_4, 1], y_2 = 0\},$$

$$Y(\{1,2,3\}) = \{y \in R^4 | y_1 \leq 2.6 \text{ Min } [-y_2, -\tfrac{y_3}{2}, -y_4, 1]\},$$

$$Y(S) = \{0\} \quad \text{otherwise} .$$

Define: $x^1 = x^2 = (1.2,0,0,0)$, $x^3 = (0.2,0,0,0)$.

Clearly, $\{x^i\}_{i \in M}$ is a structural competitive allocation, for $F = \{1,2,3\}$, $p = 1$, $t_{iM} = x_1^i$, $i \in M$. [In particular, there is no coalition $S \subset M$, such that $y \in Y(S)$ with $y_c \geq \sum_{i \in S} x_c^i$.*/ Hence, (ii) is fulfilled.]

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*/ Throughout the paper this formulation is short for: there is no $y \in Y(S)$ such that $y_c \geq \sum_{i \in S} x_c^i$ and $y_L = \sum_{i \in S} x_L^i - \sum_{i \in S} w_L^i$, $x^i \in X^i$ $\forall i \in S$.

However, by forming the firm structure $\tilde{F} = (\{1,2\}, \{2,3\})$, there exists $\tilde{y} \in \sum_{S \in \tilde{F}} Y(S)$ with $\tilde{y}_c = 3 > 2.6$, and therefore $\{x^i\}_{i=1}^3$ is not Pareto optimum. (Obviously, not a strong Pareto optimum.)

Suppose (p_2, p_3, p_4) are efficiency prices for labour which sustain $(\{x^i\}_{i=1}^3, \{y(S)\}_{S \in F}), (p_1 = 1)$. Then:

$$y(\{1,2\}) = 0 \Rightarrow p_2 + p_3 \geq 2$$

$$y(\{2,3\}) = 0 \Rightarrow p_3 + p_4 \geq 1$$

$$(2.6, -1, -2, -1) \in Y(M) \Rightarrow p_2 + 2p_3 + p_4 \leq 2.6 .$$

Contradiction. Hence, there are no efficiency personalized prices for labour which sustain the s.c.e. allocation $(\{x^i\}_{i=1}^3, \{y(S)\}_{S \in F})$.

In fact, we shall now show that it is possible for a s.c.e. allocation not to be even Pareto optimum relative to the equilibrium firm structure! .

Example 2: Consider the economy of Example 1, the initial endowments being: $w^1 = (0,1,0,0)$, w^2 $(0,0,1,0)$, $w^3 = (0,0,0,1)$, and with the following technology sets:

$$Y(\{1,2\}) = \{y \in R^4 | y_1 \leq -(2y_2 + y_3), \quad y_4 = 0\},$$

$$Y(\{2,3\}) = \{y \in R^4 | y_1 \leq -(2y_3 + y_4), \quad y_2 = 0\},$$

$$Y(\{1,3\}) = \{y \in R^4 | y_1 \leq -(2y_4 + y_2), \quad y_3 = 0\},$$

$Y(S) = 0$ otherwise.

Define: $F = [\{1,2\}, \{2,3\}, \{1,3\}]$,

$$x^i = (\tfrac{7}{4}, 0,0,0), \ i = 1,2,3$$

$$y(\{1,2\}) = (\tfrac{7}{4}, -\tfrac{3}{4}, -\tfrac{1}{4}, 0) ,$$

$$y(\{2,3\}) = (\tfrac{7}{4}, 0, -\tfrac{3}{4}, -\tfrac{1}{4}) ,$$

$$y(\{1,3\}) = (\tfrac{7}{4}, -\tfrac{1}{4}, 0, -\tfrac{3}{4}) .$$

$$\forall S \in F, \ t_{is} = \tfrac{7}{8} \quad i = 1, 2, 3 .$$

Since for all $S \subset M$, Max $y_c(S) \leq 2 + 1 = 3$, and $x_c^i + x_c^j = \frac{7}{4} + \frac{7}{4} = 3.5 > 3$, it is clear that $(1, \{x^i\}_{i=1}^3, F, \{y(S)\}_{S \in F}, \{t_{is}\})$ is a s.c.e. Neverthe-less, the

F - attainable allocation $(\{\bar{x}^i\}, \{\bar{y}(S)\}_{S \in F})$, given by:

$\bar{x}^i = (2, 0, 0, 0)$, $i = 1, 2, 3$,

$\bar{y}(\{1, 2\}) = (2, -1, 0, 0)$,

$\bar{y}(\{2, 3\}) = (2, 0, -1, 0)$, and

$\bar{y}(\{1, 3\}) = (2, 0, 0, -1)$,

proves that $(\{x^i\}_{i=1}^3, \{y(S)\}_{S \in F})$ is not Pareto optimum relative to F .

VI. The Super-Additive Technology

Examples 1 and 2 above indicate that the concept of the core (or condition (ii) in Def. 3) is not appropriate for coalition production economies. It fails to allow a group of people, S, to form and produce in "sub-firms" $\{S_j\}$ $S_j \subset S$ $j=1,\ldots,J$, and share among themselves the total output thus derived. Condition (ii) requires only that no S, by acting on its own as one firm, can benefit by deviating from the equilibrium firm structure. This, as we shall see, is the source of nonoptimality of some equilibrium allocations.

Definition 5: $(\{x^i\}_{i \in S}, \{y(S_j)\}_{S_j \in z})$ is an attainable program for S (realized via z) if $x^i \in X^i$ $\forall$ $i \in S$, $S_j \subset S$, $y(S_j) \in Y(S_j)$ $\forall S_j \in z$ and

$$\sum_{i \in S} x_L^i = \sum_{i \in S} w_L^i - \sum_{S_j \in z} y_L(S_j) .$$

Definition 6: The super-additive technology of coalition S, $\tilde{Y}^*(S)$ is given by:

$$\tilde{Y}^*(S) = \{ \sum_{S_j \in z} y(S_j) \mid \exists \ \{x^i\}_{i \in S} \ .\ni. \ (\{x^i\}_{i \in S}, \{y(S_j)\}_{S_j \in z}) \text{ is an attainable program for } S \}.$$

That is, $\tilde{Y}^*(S)$ allows S to produce in sub-firms, provided the total amount of labour involved is affordable by S. By remark 1, however, we could replace $\tilde{Y}^*(S)$ by: $Y^*(S) = \sum_{S_j \subset S} Y(S_j)$. If condition (ii) is replaced by:

(ii*) There is no $S_o \subset M$, $\bar{x}^i \in X^i$ $i \in S_o$, $y(S_o) \in Y^*(S_o)$

$$\text{such that } \sum_{i \in S_o} P\bar{x}_c^i \leq \sum_{i \in S_o} Pw_c^i + Py_c(S_o), \quad \sum_{i \in S_o} \bar{x}_L^i - \sum_{i \in S_o} w_L^i = y_L(S_o) \text{ and}$$

$$u^i(\bar{x}^i) > u^i(x^i) \ \forall \ i \in S_o,$$

it follows that every s.c.e.* (s.c.e with (ii*) replacing (ii)) is Pareto optimum. To realize that, replace S_o by M,in (ii*).

It may well be argued that every coalition S does in fact face the technology $Y^*(S)$ since it can always decide to produce in sub-firms and then distribute the output. In that case the coalition structure $\{M\}$ always forms since $Y^*(M) \supset \sum_{S \subset M} Y(S)$, and the interesting question is which sub-firms form.

The following Theorem shows that replacing (ii) by (ii*) enables us to concentrate only on equilibrium allocations without considering the distribution of the value-added involved.

Theorem 4: Let $p \in \Delta$ be a price vector, $x^i \in X^i$, $i \in M$, consumption bundles, F a firm structure and $y(S) \in Y(S)$, $S \in F$, production plans, such that (i) and (ii*) hold. Assuming nonsatiation (a.4) there exists a value added distribution $\{t_{is}\}$ $i \in M$, $S \in F$, such that $(p, \{x^i\}_{i \in M}, F, \{y(S)\}_{S \in F}, \{t_{is}\})$ is a s.c.e.* (hence s.c.e.).

Proof: Since $p \in \Delta$, $p \geq 0$, we have by (i)

$$(1) \quad p \sum_{i \in M} x_c^i = p \sum_{i \in M} w_c^i + p \sum_{S \in F} y_c(S) .$$

Choose $\{t_{is}\}$ such that:

$$(2) \quad \sum_{\substack{S \in F \\ i \in S}} t_{is} \equiv px_c^i - pw_c^i , \quad t_{is} = 0 \quad i \notin S , \ i \in M, \ S \in F .$$

We shall first show that (iv) of Definition 3 is fulfilled. Suppose not, i.e., $\exists\, \hat{S} \in F$ such that

$$(3) \quad \sum_{i \in \hat{S}} t_{i\hat{s}} > py_c(\hat{S})$$

By (1), (2) and (3):

$$\sum_{S \in F} py_c(S) = \sum_{i \in M} (px_c^i - pw_c^i) = \sum_{i \in M} \sum_{S \in F} t_{is} = \sum_{S \in F} \sum_{i \in S} t_{is} > \sum_{\substack{S \in F \\ S \neq \hat{S}}} \sum_{i \in S} t_{is} + py_c(\hat{S}) .$$

Hence, $\{\hat{S}\} \neq F$. Therefore, there exists $\bar{S} \in F$, such that

(4) $\quad py_c(\bar{S}) > \sum_{i \in \bar{S}} t_{i\bar{s}}$

Define: $\hat{F} = [S \in F \mid \sum_{i \in S} t_{is} > py_c(S)]$,

$\bar{F} = [S \in F \mid \sum_{i \in S} t_{is} < py_c(S)]$.

We proved that $\hat{F} \neq \emptyset$ implies $\bar{F} \neq \emptyset$. To complete the proof we need the following definitions:

Definition 4: We shall say that <u>i is connected to j</u> if there are individuals $i_o, i_1, \ldots, i_k$ and coalitions $S_o, \ldots, S_k, S_{k+1}$, with $i_o = i$, $i_k = j$, such that

$$(i_{t-1}, i_t) \in S_t,\ S_t \in F,\ t = 1, \ldots, k,\ i_o \in S_o \in F,\ i_k \in S_{k+1} \in F .$$

Definition 5: <u>A coalition G is connected to a coalition H</u>, if there exist $i \in G$, $j \in H$ such that i is connected to j .

Definition 6: $\{S_1, \ldots S_k, S_{k+1}\}\ \{i_o, \ldots, i_k\}$ is called a <u>minimal connection for H and G</u> if it connects H and G , and there is no other connection of G and H with less than $k+1$ coalitions.

Since F is a finite collection of coalitions, each has a finite number of members, it follows that minimal connections exist.

For each $i \in M$ define:

$$K_i = [j \in M \mid j \text{ is connected to } i] .$$

[Note that $i \in K_i \quad \forall i \in M$] .

Denote:

$$\hat{K} = \bigcup_{\substack{i \in S \\ S \in \hat{F}}} K_i, \quad \bar{K} = \bigcup_{\substack{i \in S \\ S \in \bar{F}}} K_i .$$

By (3) $\hat{K} \neq \emptyset$ and by (4) $\bar{K} \neq \emptyset$.

Distinguish between the two cases:

I. $\bar{K} \cap \hat{K} = \emptyset$.

In this case we shall show that $\bar{K}$ violates (ii*). Let $G = [S \in F \mid S \subset \bar{K}]$. $G \neq \emptyset$ since $\bar{S} \in G$. Moreover, $S \in G$ implies $S \cap \hat{K} = \emptyset$, hence, for all $i \in S \in G$, $\{S \in F \mid i \in S\} \cap \hat{F} = \emptyset$.

By definition of $\hat{F}$; for all $S \in G$, $\sum_{i \in S} t_{is} \leq py_c(S)$.

Since $\bar{S} \in G$, we get:

$$(5) \quad \sum_{S \in G} py_c(S) > \sum_{S \in G} \sum_{i \in S} t_{is} = \sum_{i \in \bar{K}} \sum_{S \in G} t_{is} = \sum_{i \in \bar{K}} (px_c^i - pw_c^i) .$$

By the local nonsatiation, there exists, for all $i \in \bar{K}$, $\tilde{x}^i$ with $\tilde{x}_L^i = x_L^i$ $u^i(\tilde{x}^i) > u^i(x^i)$ and by (5) $\sum_{i \in \bar{K}} (p\tilde{x}_c^i - pw_c^i) \leq \sum_{S \in G} py_c(S)$. By (i), $\sum_{i \in \bar{K}} \tilde{x}_L^i - \sum_{i \in \bar{K}} w_L^i = \sum_{S \in G} y_L(S) \in Y^*(\bar{K})$, hence, $\bar{K}$ violates (ii*). Contradiction.

II. $\bar{K} \cap \hat{K} \neq \emptyset$.

Let $\{S_o, S_1, \ldots, S_{k+1}\}$ $\{i_o, i_1, \ldots, i_k\}$ be a minimal connection of $i_o \in S_o \in \hat{F}$ to $i_k \in S_{k+1} \in \bar{F}$. Define:

$$\tilde{t}_{i_t S_t} = t_{i_t S_t} - \varepsilon , t = 0, \ldots, k ,$$

$$\tilde{t}_{i_t S_{t+1}} = t_{i_t S_{t+1}} + \varepsilon, \; t = 0, \ldots, k ,$$

$$\tilde{t}_{iS} = t_{iS} \quad \text{otherwise.}$$

(Due to the minimality of the connection, $\tilde{t}$ is well defined.)

Since $i_t \in S_t \cap S_{t+1}$, $t = 0, \ldots, k$, $\sum_{S \in F} \tilde{t}_{is} = \sum_{S \in F} t_{iS}$ for all $i \in M$.

Thus, by (2)

$$(6) \quad \sum_{S \in F} \tilde{t}_{iS} = px_c^i - pw_c^i \ , \ i \in M \ .$$

Moreover, for all $S \in F$, $S \neq S_o$, $S \neq S_{k+1}$

$$(7) \quad \sum_{i \in S} \tilde{t}_{iS} = \sum_{i \in S} t_{iS} \ .$$

Choose $\epsilon = \text{Min}\ [(\sum_{i \in S_o} t_{iS_o} - py_c(S_o)),\ (py_c(S_{k+1}) - \sum_{i \in S_{k+1}} t_{iS_{k+1}})]$.

As $S_o \in \hat{F}$, $S_{k+1} \in \bar{F}$, $\epsilon > 0$. With $\tilde{t}_{iS}$ replacing t_{iS} , in view of (7), the number of coalitions in either $\hat{F}$ or $\bar{F}$ reduces. Repeating this process for $\tilde{t}_{iS}$, (since, by (6), (2) holds), after a finite number of steps either $\bar{K} \cap \hat{K} = \emptyset$ which is case I or else $\bar{F}$ or $\hat{F}$ is empty. If $\hat{F}$ is empty, (iv) is satisfied. If $\bar{F}$ is empty, by (1), $\hat{F}$ is empty. Hence, in any case we proved that (i) and (ii*) imply (iii) and (iv).

Since obviously (ii*) implies (ii), to complete the proof of the theorem we have to show that (v) holds as well. Suppose not, i.e.,

$\exists\ (\tilde{y}_c(S_o), y_L(S_o))$ with $py_c(S_o) < p\tilde{y}_c(S_o)$, $S_o \in F$. Then M violates (ii*), since $\tilde{y} = \sum_{S \neq S_o} y(S) + \tilde{y}(S_o) \in Y^*(M)$. By (a.4) there exists $\bar{x}^i \in X^i$, $\bar{x}^i_L = x^i_L$, with $u^i(\bar{x}^i) > u^i(x^i)$ $i \in M$, and by choosing $\bar{x}^i$ close enough to x^i, by (1):

$$\sum_{i \in M} p\bar{x}^i_c \leq \sum_{i \in M} pw^i_c + p\tilde{y}_c \ .$$

Q.E.D.

[This Theorem is essential for the proof of the equivalence of core and s.c.e.* allocations when the replica economy is considered. (Section IX)].

Although every s.c.e.* allocation is Pareto optimum, there may exist no labour-prices that support it. For example, consider the following economy:

$M = \{1,2\}$; $w^1 = (0,1,0)$, $w^2 = (0,0,1)$; $u^i(x^i) = x_c^i = x_1^i$, $i \in M$;

$Y(\{1,2\}) = \{y \in R^3 | y_1 \leq - y_2 - y_3\}$, $Y(\{i\}) = 0$, $i \in M$.

It can be easily verified that $x^1 = (\frac{1}{2},0,0)$, $x^2 = (1\frac{1}{2},0,0)$ $[y=(2,-1,-1)]$ is a s.c.e.* allocation. However, the only competitive allocation supported by labour prices is $\bar{x}^1 = \bar{x}^2 = (1,0,0)$ $[p_1 = p_2 = p_3, \bar{y} = (2,-1,-1)\}$.

As will be proved in Section IX, when replicating the economy such examples do not exist.

VII. Existence of Equilibrium Coalition Structures

Theorem 1 proves the existence of a firm structure which brings about structural competitive equilibrium. In particular, individuals may exist who participate in more than one firm. Though as shown above, in the standard A-D economy this type of structure is formed, the question that naturally arises is whether for every economy that fulfills assumptions (a)-(c) there exists a structural competitive equilibrium, the firm structure of which is a coalition structure. The following example will shed some light upon this problem.

Example 3: Consider the economy of example 2 with the following technology sets:

$Y(\{i,j\}) = [y \in R^4 \mid y = t\,(1, -e^i - e^j)\ 0 \leq t \leq 1]$ (e^i denotes the i-th unit vector in R^3)

$Y(S) = \{0\}$ otherwise.

In essence, this is the "game of pairs", where every two individuals can produce one unit of x_1 if they work "full time," or any convex combination of this production and the $\{0\}$ vector.

If $\{x^i\}_{i=1}^3$ is a structural competitive allocation, we must have */ $\sum_{i=1}^{3} x_1^i = 1\frac{1}{2}$ which is realized by forming the firm structure $F = (\{1,2\}, \{2,3\}, \{1,3\})$, and $y(S) = (\frac{1}{2}, -\frac{1}{2}(e^i + e^j))$ for all $\{i,j\} \in F$. No coalition structure can give rise to such an allocation since for any coalition structure B, $y \in \sum_{S \in B} Y(S)$ implies $y_1 \leq 1$, hence $\sum_{i \in M} x_1^i \leq 1$.

Most models will include this type of economy as a special case, since, in particular, the utilities here do not depend upon the non-marketed goods.

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*/Since by (ii) $x_1^i + x_1^j \geq 1\ \forall\ i, j$.

It therefore follows that if we want to get a positive result as to the formation of coalition structures rather than firm structures, we shall have to impose rather strong restrictions either on the utilities of the individuals (e.g., if i works for more than one firm, $u^i(x^i) \leq u^i(w^i)$, for all attainable x^i's), or on the technology sets.

Example 3 indicates that if ad-hoc restrictions on the preference ordering are not made, a necessary condition for an equilibrium coalition structure to exist (in general) is that the technology sets be balanced in the following way:

Define $\hat{Y} = \bigcup \sum_k \gamma_k Y(S_k)$ where $\{S_k\}$ is a balanced collection with $\{\gamma_k\}$ as its weights*/ and the union is taken over all balanced collections. Then,

(d) For any $\hat{y} \in \hat{Y}$ there exists a coalition structure $B(\hat{y})$ such that $\hat{y} = \sum_{S \in B(\hat{y})} y(S)$.

To see that (d) is a necessary assumption, note that in the special case of one marketable commodity (hence $p = 1$) and with $u^i(x^i) = x^i_c \; \forall i \in M$ (e.g., Examples 1-3), condition (ii) requires that an equilibrium allocation be in the core of the economy. Since, by the definition of the utility functions the market game is a game with side-payments, balancedness of the technology sets is a necessary condition. We shall now show that it is also a sufficient condition for an economy to have a s.c.e. whose firm structure is also a coalition structure. To prove this claim we make use of Bohm's result [1]. It will also become clear why an analogous proof to that given for Theorem 1 cannot be used in this case.

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*/ A set of coalitions $\{S_k\}_{k=1}^t$ is called balanced, if there exist "weights" $\{\gamma_k\}_{k=1}^t$ such that for all k, $\gamma_k \geq 0$ and for all $i \in M$, $\sum_{\substack{k \\ i \in S_k}} \gamma_k = 1$.

Theorem 5: Under assumptions (a)-(d) there exists an equilibrium coalition structure.

Proof: We replace the set $\bigcup_F \sum_{S\in F} Y(S)$ (where F is a firm structure) in [·1], by the set $\hat{Y}$. Clearly, $\hat{Y}$ is convex. Since in the proof of the theorem Bohm makes use only of the balancedness condition, his proof carries over to our case. Therefore, there exist prices $q \in R_+^{g+m\ell}$, consumption bundles $x^i \in X^i$, $i \in M$, a production plan $\hat{y} \in \hat{Y}$, and $(t^1,..,t^m) \in R^m$ such that:

(1) $\sum_{i\in M} x^i = \sum_{i\in M} w^i + \hat{y}$

(2) For all $i\in M$: $qx^i \leq qw^i + t^i$, and $u^i(\bar{x}^i) > u^i(x^i)$ implies $q\bar{x}^i > qw^i + t^i$

(3) $\sum_{i\in S} t^i \geq qy(S)\ \forall\ S\subset M$

(4) $q\hat{y} = \sum_{i\in M} t^i$

(5) $qy(B_j) = \text{Max}\ q.y(B_j)\ j = 1,.., J,$

where $B = \{B_1,...,B_J\}$ is the coalition structure such that $\hat{y} = \sum_{j=1}^{J} y(B_j)$. (Such a B exists by (d)).

Combining (3) and (4) we get: $py(B_j) = \sum_{i\in B_j} t^i$ $j = 1,...,J$. By (a.4) and (2), $q_c \neq 0$. Define:

$$p = \frac{q_c}{\|q_c\|}\ ,\ t_{ij} = t^i - q_L\left(w_L^i - x_L^i\right)\ i \in B_j\ .$$

By (1), (3), (4) and (5) it is left to be shown that (ii) holds, in order to conclude that $(p,\{x^i\}_{i\in M}, B,\{y(S)\}_{S\in B},\{t_{ij}\})$, is a s.c.e.

Suppose that for some $S\subset M$ $u^i(\bar{x}^i) > u^i(x^i)\ \forall i\in S$, $y(S) \in Y(S)$ and

$y_L(S) = \sum_{i\in S} \bar{x}_L^i - \sum_{i\in S} w_L^i$. By (2) and (3)

$$\sum_{i\in S} q\,\bar{x}^i > \sum_{i\in S} qw^i + \sum_{i\in S} t^i \geq \sum_{i\in S} qw^i + qy(S), \text{ hence}$$

$$\sum_{i\in S} p\bar{x}_c^i > \sum_{i\in S} pw_c^i + py_c(S)\ .$$

Q.E.D.

VIII. Optimality of Equilibrium Coalition Structure

In contrast to Corollary 2 there are economies (which fulfill assumptions (a)-(d)) for which every equilibrium allocation associated with an equilibrium coalition structure is not Pareto optimum.

Example 4: Consider the economy of Example 1 with $\hat{Y}$ replacing $Y(M)$. For our three-person economy, the relevant balanced collections are: $(\{1,2\}, \{3\})$, $(\{2,3\}, \{1\})$, $(\{1,2,3\})$, $(\{1,2\}, \{2,3\})$ with the weights $(1,1)$, $(1,1)$, (1), $(\frac{1}{2},\frac{1}{2})$, respectively. Hence $\hat{Y} = \text{conv}\,[Y(\{1,2\}) \cup Y(\{2,3\}) \cup Y(\{1,2,3\})]$.

If $(\{x^i\}_{i=1}^3, y)$ is Pareto optimum, $\sum_{i \in M} x_c^i = y_c = 3$. However, for any coalition structure B, $\text{Max} \sum_{S \in B} y_c(S) \leq 2.6$.

Example 4 implies that we could not prove Theorem 5 in an analogous way to the proof of Theorem 1. However, every equilibrium allocation associated with a coalition structure is Pareto optimum relative to that structure. This is in contrast with the case of firm structures (Example 2).

Theorem 6: Let B be an equilibrium coalition structure with $(\{x^i\}_{i \in M}, \{y(s)\}_{S \in B})$ the equilibrium allocation. Then, this allocation is Pareto optimum relative to B.

Proof: Suppose not, i.e., there exists a B-attainable allocation $(\{\bar{x}^i\}_{i \in M}, \{\bar{y}(S)\}_{S \in B})$ with $u^i(\bar{x}^i) > u^i(x^i)\ \forall\ i \in M$. Since $p \geq 0$, $\sum_{i \in M} p\bar{x}_c^i = \sum_{i \in M} pw_c^i + \sum_{S \in B} p\bar{y}_c(S)$, and as $(p, \{x^i\}_{i \in M}, B, \{y(S)\}_{S \in B}, \{t_{is}\})$ is a s.c.e., by (ii): for all $S \in B$: $\sum_{i \in S} p\bar{x}_c^i > \sum_{i \in S} pw_c^i + p\bar{y}_c(S)$. Summing over all $S \in B$ we get a contradiction.

Q.E.D.

IX. The Replica Economy

In section VI we noted that not every s.c.e.*, (i.e., s.c.e. with (ii*) replacing (ii) in Def. 3), can be supported by personalied efficiency prices for labour. We now show that this is in contrast with the situation in the replica economy.

Let M_r denote the r-fold replica of M, M_r consists of mr individuals which will be indexed by the pair (i,t), $i=1,..,m$, $t=1,..,r$. For all $i \in M$, each (i,t), $t=1,..,r$ is of type i in the sense that $x^{(i,t)} = x^i$, $u^{(i,t)} = u^i$, $w^{(i,t)} = w^i$, and for all $S \subset M$, $i \in S$, $Y(S) = Y(S/\{i\} \cup \{i,t\})$. [Note that we do not make labour a named commodity, otherwise the commodity space will tend to infinity, ($g + rm\ell \to \infty$ as $r \to \infty$). Lemma 1 below enables us to use this formulation.] By the definition of the super-additive technology, and using assumption (b.2) the following properties hold:

($\underline{P}$ 1) $Y_r^* \equiv Y^*(M_r) = r \cdot Y_1^* \equiv r \cdot Y^*(M)$.

($\underline{P}$ 2) $S \subset M_r$ implies $Y^*(S) \subset Y_r^*$.

($\underline{P}$ 3) For any coalition S with $k_i(S)$ members of type $i, k_i(S) \geq 1$ for all $i \in M$,

$\text{Min}\ [k_i(S) \mid i \in M]\ Y_1^* \subset Y^*(S)$.

The goal of this section is to prove the existence of efficiency prices (for the marketed goods as well as for labour) that support any s.c.e.* allocation which belongs to the replica economy E_r for all $r = 1,2,...,$ (Theorem 7 below). To this end we consider the super-additive core of the market game (M_r, V^*) generated by E_r, where for all $S \subset M_r$:

$$V^*(S) = [\{u^{i,t}\}_{(i,t) \in M_r} \in R_+^{mr} \mid \forall (i,t) \notin S,\ u^{i,t} = 0\ .$$

$$\forall (i,t) \in S \quad u^{i,t} \leq u^{i,t}(x^{i,t}) \ , \ x^{i,t} \in X^i$$

$$\sum_{(i,t)\in S} x^{i,t} = \sum_{(i,t)\in S} w^{i,t} + y(S), \ y(S) \in Y^*(S)] \ .$$

We shall denote by C_r^* the core allocations of this game and by

(a') assumptions (a) with (a.2) replaced by:

(a.2') u^i is a continuous strictly concave function on X^i .

Lemma 1: Under assumptions (a')-(c), an allocation in C_r^*, $r \geq 2$ assigns the same consumption bundle to all consumers of the same type.

Proof: Standard.

By Lemma 1 we can denote a core allocation for the sequence economies E_r by the vector $x = (x^1, \ldots, x^m) \in R^{(g+\ell m)m}$ (rather than $\tilde{x} = \{\tilde{x}^{i,t}\} \in R^{(g+m\ell)mr}$) .

Lemma 2: Under assumptions (a')-(c), for every $x = (x^1, \ldots, x^m) \in \bigcap_{r=1}^{\infty} C_r^*$ there exists a price system $\varphi \in R_+^{g+\ell m}, \varphi \neq 0$ such that for all E_r :

(1) $r \sum_{i\in M} x^i = r \sum_{i\in M} w^i + y^r \ , \ y^r \in Y^*(M_r)$,

(2) $\varphi x^i \leq \varphi w^i$ and $u^i(\bar{x}^i) > u^i(x^i)$ implies $\varphi \bar{x}^i > \varphi w^i$,

(3) $\varphi y^r = 0 \geq \varphi Y^*(S) \ \forall S \subset M_r$

Proof: This is a direct application of the Theorem in [2].

Theorem 7: Under assumptions (a')-(c), every s.c.e.* allocation for E_r, $r = 1,2,\ldots$, can be supported by an efficiency price system $\varphi \in R_+^{g+m\ell}$, $\varphi \neq 0$. (i.e., (1), (2) and (3) of Lemma 2 hold.)

Proof: Let $\{\bar{x}^{i,t}\}$ be a s.c.e.* allocation for E_r and let $p \in \Delta$ be the equilibrium price system. Define the game (M_r, V_P^*): For $S \subset M_r$, $S \neq M_r$:

$$V_P^*(S) = [\{u^{i,t}\}_{(i,t)\in M_r} \in R_+^{mr} \mid \forall\ (i,t)\notin S,\ u^{i,t} = 0\ ,$$

$$\forall (i,t)\in S \quad u^{i,t} \le u^{i,t}(x^{i,t}),\ x^{i,t} \in X^i\ ,$$

$$\sum_S p x_c^{i,t} = \sum_S p w_c^{i,t} + p y_c(S),$$

$$\sum_S x_L^{i,t} = \sum_S w_L^{i,t} + y_L(S)\ ,\ y(S)\in Y^*(S)\ ]\ ,$$

and

$$V_P^*\ (M_r) = V^*(M_r)\ .$$

Denote the core allocations of this game by $C_r(V_P^*)$. By Theorem 4 $(\bar{x}^{i,t}) \in C_r\ (V_P^*)$. Clearly for all $S\subset M_r$ $V_P^*\ (S) \supset V^*(S)$ and since $V_P^*(M_r) = V^*(M_r)$ it follows that $C_r(V_P^*) \subset C_r^*$. Hence $\{\bar{x}^{i,t}\} \in C_r^*$ $r=1,2,\ldots,$ and Lemma 2 concludes the proof.

Q.E.D.

In fact, following the pattern of the proof of Theorem 5, and using Lemma 2, every $x\in\bigcap_{r=1}^{\infty} C_r^*$ is a s.c.e.* allocation for every E_r, $r=1,2,\ldots$. Hence, C_r^* shrinks to the set of s.c.e.* allocations for every E_r, $r=1,2,\ldots$, as $r\to\infty$ (i.e., the analog of the "equivalence theorem" holds).

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