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OPTIMAL EXHAUSTIBLE RESOURCE DEPLETION WITH ENDOGENOUS TECHNICAL CHANGE

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Introduction

The prospect of imminent exhaustion of some natural resources, especially those employed in energy generation, has prompted calls for conservation through reduction or even cessation of economic growth. A natural framework for analyzing such proposals is a hybrid offspring of Ramsey's optimal growth model [8] and Hotelling's model of exhaustible resources [4]. Specifically, Anderson [1], Solow [10], and Stiglitz [11], among others, have recently studied optimal growth models modified to incorporate production requirements for an essential exhaustible resource. It has been shown that while resource irreplenishability limits growth in per capita consumption, this limitation may be offset by technical progress along with increasing capital accumulation and substitution.

Reliance upon technical progress, in particular, to offset the constraining irreplenishability of a vital natural resource is based on its role as a major source of past economic growth. Along with discovery of the past importance of technical progress has come awareness that it proceeds neither smoothly nor without effort. It is affected by the resources devoted to it. Unpredictability in technical progress results from our partial ignorance of the principles underlying natural and social phenomena. The randomness can be reduced at a cost, but it cannot be eliminated.

These features of technical advance have typically not been incorporated in most of modern growth theory, wherein progress is regarded as proceeding steadily, costlessly, and exogenously. In contrast, a few papers have considered more abrupt and significant changes in technology used. Smith [9] assumes that two alternate technologies are available from the outset, one requiring an exhaustible resource at low initial cost and the other employing an inexhaustible factor at high initial cost. He shows that maximum output will be achieved by using only the exhaustible resource technology at the beginning but gradually supplementing and then replacing it with the high cost alternative.

Dasgupta and Heal [2] considered the possibility that a new technology will eventually appear that does not employ limited natural resources. The technical advance could be costlessly achieved but the date of its availability was both random and exogenous. In a subsequent paper, Dasgupta, Heal and Majumdar [3] extended this model by making development of the new technology endogenous. Their model is very similar to and consistent with the one independently developed by us and described below.

Our view is that technical advance is in large part neither costless nor exogenous [6]. The rate and direction of technical progress are influenced in the long run by the economic resources allotted to it, guided by the quest for profits and government policy. In this paper, we follow the framework used by Dasgupta and Heal [2] while making technical advance endogenous. We omit steady technical progress that reduces unit factor requirements in favor of emphasizing a drastic technical change that relaxes the limitation imposed by resource irreplenishability. The date this new technology becomes available is unknown, but we assume it is affected by research effort. In particular, it is supposed that the probability of R&D completion is a known nondecreasing function of cumulated research effort. Effort accumulates by devoting the single produced good to R&D. There are decreasing returns to compression of the development period. We assume the conditional probability of successful completion with incremental effort (given it is incomplete) is a nondecreasing function of total R&D effort accumulated. Any particular R&D approach need not have this property; indeed, a single project may be characterized by a conditional probability of completion that rises for a while with incremental effort but eventually peaks and falls. However, at the aggregative level of our analysis, the conditional probability of success might well be a nondecreasing function of effective effort.

The population is assumed stationary and labor employment is suppressed. A single multipurpose good is produced by means of a reproducible factor, capital, and an exhaustible resource. Output may be divided among current consumption (yielding immediate utility), research on the new technology, or augmentation of the productive capital stock. The objective is to maximize the expected discounted stream of utility from consumption.

We show that R&D effort need not begin immediately; there may be a period in which output is divided between augmentation and consumption alone. Eventually, however, the search for the alternative technology begins. The single-humped temporal pattern of consumption noted by Dasgupta and Heal is followed. The rate of R&D also tends to rise initially although it too eventually falls (if not successful) as exhaustible resource depletion exerts an increasing drag on the economy. Finally we show that incorporation of extraction costs that rise as the remaining resource supply shrinks does not alter the temporal patterns noted. Since optimal control problems with more than one state variable are generally quite difficult to analyze, it is of some methodological interest that we have provided a qualitative characterization of a problem with three state variables and three controls.

The Model

Let $C(t)$ be the aggregate consumption rate at time t and $U(C(t))$ be the instantaneous utility derived therefrom. We assume utility is an increasing, strictly concave, twice differentiable function with

$$U'(0) = \infty, \; U' > 0, \; U'' < 0 \tag{1}$$

We later specify that elasticity of marginal utility is constant and greater than unity; i.e.

$$U(C) = - C^{1-\eta}, \quad \eta > 1 \tag{2}$$

The production rate depends on the stock K(t) of productive capital and the rate of exhaustible resource usage R(t). The production function F(K,R) is twice differentiable and homogenous of degree one. Hence

$$F(K,R) = Kf(y) \tag{3}$$

where

$$y \equiv R/K, \quad f(y) \equiv F(1,R/K) \tag{4}$$

and further

$$f(0) = 0, \; f'(0) = \infty, \; f' > 0, \; f'' < 0, \; \lim_{y \to \infty} f(y) < \infty \tag{5}$$

The amount S(t) of exhaustible resource remaining at time t diminishes with employment in production according to

$$S'(t) = -R(t) = -K(t)y(t), \quad S(0) = S_0 > 0, \; S(t) \geq 0 \tag{6}$$

With no possibility of R&D or new technology, the problem is to choose nonnegative paths of consumption C(t), productive capital K(t), resource stock S(t), and resource usage R(t) (equivalently, factor proportion y(t)) for $t \geq 0$ to maximize the discounted utility stream

$$\int_0^\infty e^{-\delta t} U(C(t))dt \tag{7}$$

subject to (6) and

$$K'(t) = K(t)f(y(t)) - C(t), \; K(0) = K_0 > 0, \; K(t) \geq 0 \tag{8}$$

The constant discount rate δ is strictly positive. Equation (8) indicates that the single composite output may be either consumed or used to augment the capital stock. Capital investment is reversible in that the capital stock shrinks if consumption exceeds current production. Capital stock is, however, bounded below by zero. Problem (6)-(8) was thoroughly analyzed by Dasgupta and Heal in Section 1 of their paper and needs no further discussion here.

Now suppose that a new technology could be developed in which production no longer required the exhaustible resource S nor even capital K. We need not be explicit about the technology's characteristics except to specify that one could reliably estimate the maximum value W of the discounted utility stream from the time T the new technology becomes available forward.

Thus

$$W \equiv \max \int_T^{\infty} e^{-\delta(t-T)} U(C(t))dt \qquad (9)$$

subject to appropriate constraints

The new technology need not be implemented immediately upon availability nor must exhaustible resource use diminish or cease upon employment of the new technology. It may be optimal to continue using the old one at a modified rate and to gradually employ the new one as it becomes economic. We suppose the optimization in (9) takes these considerations into account. The maximum value W could therefore depend on the stocks of capital and exhaustible resource remaining at T, but that dependence is supposed small enough to be ignored. (Dasgupta and Heal provide an extended discussion of the plausibility of this assumption.) Consequently, W is independent of the (unknown) time T since the horizon is infinite and both the functions and initial conditions are stationary. Whenever the utility function takes the form in (2), we also assume $W < 0$ for conformability.

The actual temporal pattern of consumption after the new technology appears at T may follow a rising and then falling path or a sequence of such rises and declines. An innovation expands the productive ability of the economy, so consumption may rise at first. Eventually new resource constraints may exert limitations that reduce consumption and trigger search for further technical advance. We envisage a sequence of innovations and possibly a wave-like consumption path after T, but we shall be concerned here with just a single innovation and the optimal path to its attainment. The relevant information about the economy after T is summarized in equation (9). Although the temporal stream of utility after T is unlikely to be constant, it is equivalent to a constant utility stream of δW.

At this point we depart from Dasgupta and Heal; rather than assume, as they do, that the new technology will appear exogenously and without cost, we suppose it can appear only as a result of successful R&D requiring resources diverted from consumption or capital investment. Let $m(t)$ be the rate at which the composite good is allotted to R&D. The effectiveness with which the composite good contributes towards bringing the R&D to fruition depends on its rate of application. We assume decreasing returns to compression of the development period. The R&D rate and the growth of cumulative effective effort $z(t)$ devoted to the project by time t are related by a bounded, concave, monotone increasing twice differentiable function $g(m(t))$

$$z'(t) = g(m(t)), \qquad z(0) = 0 \tag{10}$$

where

$$g(0) = 0,\ 0 < g' < \infty,\ g'' < 0 \tag{11}$$

Let $\varphi(z)$ be the probability the R&D will be successfully completed by the time cumulative effective effort is z. The function φ is assumed twice continuously differentiable, satisfying

$$\varphi(0) = 0,\ \varphi'(0) = 0,\ \varphi' \geq 0,\ \lim_{z\to\infty} \varphi(z) = 1 \tag{12}$$

Define

$$h(z) \equiv \varphi'(z)/(1-\varphi(z)) \tag{13}$$

as the completion rate or conditional probability of completion. Note h(z)dz is approximately the probability of completion with incremental effort dz, given the project is incomplete when cumulative effort is z. We further assume that

$$h'(z) \geq 0 \quad \text{for} \quad 0 \leq z \leq \bar{z} \tag{14}$$

where $\bar{z}$ is the smallest value of z for which completion is certain:

$$\varphi(z) \begin{cases} < 1 \text{ for } 0 \leq z < \bar{z} \\ = 1 \text{ for } \bar{z} \leq z \end{cases} \tag{15}$$

$\bar{z}$ need not be finite. Supposition (14) encompasses the case that the R&D requires at least effort $\underline{z}$; i.e. $h(z) = \varphi(z) = 0$ for $0 \leq z < \underline{z}$. The formulation of the R&D sector in (10)-(15) is based on the model in [5].

Now we can state the optimization problem. Since R&D will reach fruition at an unknown time, it is necessary to devise a contingency plan for the period until $\bar{z}$ will be attained and the new technology assured. The plan will actually be followed only until the random time T at which the R&D is complete, after which there is a new maximization problem and associated plan (represented by (9)).

At t, utility U(C(t)) is received, provided the old technology is still in use--which has probability $1 - \varphi(z(t))$. In addition, with probability

$$d\,\varphi(z(t)) = \varphi'(z(t))z'(t)dt = \varphi'(z(t))g(m(t))dt$$

the new technology will become available during (t,t+dt), providing a future utility stream with discounted value W at time t. Thus the expected discounted utility stream to be maximized through nonnegative choice of consumption rate

$C(t)$, R&D rate $m(t)$, and factor proportions $y(t)$ is

$$\int_0^{\infty} e^{-\delta t}[U(C)\{1-\varphi(z)\} + \varphi'(z)g(m)W]dt \tag{16}$$

with dependence of variables on t suppressed. The maximization is subject to constraints:

$$K' = Kf(y) - C - m \tag{17}$$

$$K(0) = K_0 \qquad K(t) \geq 0$$

$$S' = -Ky \tag{18}$$

$$S(0) = S_0 \qquad S(t) \geq 0$$

$$z' = g(m) \tag{19}$$

$$z(0) = 0$$

Equation (17) is similar to (8), except the composite good may be used in R&D as well as for consumption and capital investment. Equations (18) and (6) are identical as are (19) and (10).

Although both K and S must be nonnegative always, we need only require

$$\lim_{t\to\infty} K(t) \geq 0 \qquad \lim_{t\to\infty} S(t) \geq 0 \tag{20}$$

To see this, note that if S were to become zero or negative, it could not later increase from that value; see (18). Hence nonnegativity of S as $t \to \infty$ likewise insures its nonnegativity always. Similarly, while K can both rise and fall over time, once K becomes zero (or negative), it cannot then increase; see (17) and (5). Thus nonnegativity of K as $t \to \infty$ likewise insures its nonnegativity always.

<u>Solution</u>: Necessary conditions and interpretations

Problem (16)-(19) with nonnegativity restrictions on all variables can be viewed as one of optimal control. We introduce current value multiplier functions

$\lambda(t)$, $\mu(t)$, and $\gamma(t)$ associated with differential equations (17), (18), and (19) respectively. Define the current value Hamiltonian

$$H = U(C)\{1 - \varphi(z)\} + \varphi'(z)g(m)W + \lambda\{Kf(y) - C - m\} - \mu Ky + \gamma g(m) \quad (21)$$

If C, m, y, K, S, z comprise an optimal solution, then there must be functions λ, μ, and γ such that these nine variables simultaneously satisfy (17)-(19) and

$$\partial H/\partial C = U'(C)\{1-\varphi(z)\} - \lambda \leq 0 \qquad C\,\partial H/\partial C = 0 \quad (22)$$

$$\partial H/\partial m = \varphi'(z)g'(m)W - \lambda + \gamma g'(m) \leq 0 \qquad m\,\partial H/\partial m = 0 \quad (23)$$

$$\partial H/\partial y = \lambda Kf'(y) - \mu K \leq 0 \qquad y\,\partial H/\partial y = 0 \quad (24)$$

We shall assume that the limiting form of the finite horizon transversality conditions hold in this problem as the horizon is extended indefinitely. Hence we also have

$$\lambda' = \delta\lambda - \partial H/\partial K = \delta\lambda - \lambda f(y) + \mu y \quad (25)$$

$$\lim_{t\to\infty} e^{-\delta t}\lambda(t) \geq 0 \qquad \lim_{t\to\infty} e^{-\delta t}\lambda(t)K(t) = 0 \quad (26)$$

$$\mu' = \delta\mu - \partial H/\partial S = \delta\mu \quad (27)$$

$$\lim_{t\to\infty} e^{-\delta t}\mu(t) \geq 0 \qquad \lim_{t\to\infty} e^{-\delta t}\mu(t)S(t) = 0 \quad (28)$$

$$\gamma' = \delta\gamma - \partial H/\partial z = \delta\gamma + U(C)\varphi'(z) - \varphi''(z)g(m)W \quad (29)$$

$$\lim_{t\to\infty} e^{-\delta t}\gamma(t) \geq 0 \qquad \lim_{t\to\infty} e^{-\delta t}\gamma(t)z(t) = 0 \quad (30)$$

We have assumed indefinitely large marginal utility of consumption as $C \to 0$ and indefinitely large marginal productivity of the exhaustible resource as its usage approaches zero. Consequently, in view of (1), $C > 0$ for all t, and, in view of (5) and (24), $y > 0$ for all t. Further, the capital stock K cannot vanish in finite time since that would force C to zero (see (17)) which has been ruled out. Hence (22) and (24) reduce to

$$U'(C)\{1 - \varphi(z)\} = \lambda \tag{31}$$

$$\lambda f'(y) = \mu \tag{32}$$

respectively, while (23) is equivalent to

$$\begin{array}{ll} \text{either (i) } m > 0 \text{ satisfies } & [\varphi'(z)W + \gamma]g'(m) = \lambda \\ \text{or (ii) } m = 0 \quad \text{and} & [\varphi'(z)W + \gamma]g'(0) \leq \lambda \end{array} \tag{33}$$

These first order conditions may be readily interpreted. First note that λ, μ, and γ are the marginal values of the capital stock, natural resource reserves, and cumulative effective R&D effort respectively. According to (31), the expected marginal utility of the composite product in consumption should just equal its marginal value in capital investment. Condition (32) requires the marginal value of the exhaustible resource to be the same in use as in reserve. According to (33), the marginal value of the produced good in R&D has two parts. The first is the total contribution of the new technology multiplied by the probability the marginal R&D brings it forth, while the second is the value of the increment in cumulative effective R&D effort. There is no R&D when its marginal value is below the composite good's value in other uses.

Interpretation of (33) may be helped by integrating (29) with (30):

$$\gamma(t) = -\varphi'(z)W + \int_t^{\infty} e^{-\delta(s-t)}\varphi'(z(s))[\delta W - U(C(s))]ds \tag{34}$$

so (33) can be written as

$$g'(m)\int_t^{\infty} e^{-\delta(s-t)}\varphi'(z(s))[\delta W - U(C(s))]ds \leq \lambda \tag{35}$$

with equality holding in case $m > 0$. Now it is easy to see that if the composite product is devoted to R&D, then its marginal expected value in this use must equal its marginal value in capital investment. Also, note that

$$\gamma(0) = \int_0^{\infty} e^{-\delta s}\varphi'(z(s))[\delta W - U(C(s))]ds$$

Thus $\gamma(0) = 0$ if R&D is never undertaken since then $\varphi' = 0$ always; and $\gamma(0) > 0$ if the expected net utility from innovation is positive.

For future use, substitute from (32) for μ into (25):

$$\lambda'/\lambda = \delta - (f(y) - yf'(y)) \tag{36}$$

Thus the marginal valuation of capital is rising whenever the discount rate exceeds the marginal product of capital, and is falling when the inequality is reversed.

To determine the temporal behavior of y, differentiate (32) totally with respect to time, use (36) to eliminate λ' and (27) and (32) to eliminate μ' and μ yielding,

$$y' = f'(f-yf')/f'' < 0 \tag{37}$$

The ratio of exhaustible resource to capital in production falls over time. Note that the differential equation followed by the factor proportion y depends solely on the production function.

In similar fashion, to find the temporal behavior of consumption, differentiate (31) totally with respect to time, substitute from (19) and (36), divide through by λ, using (31), and rearrange slightly:

$$-U''C'/U' = f(y) - yf'(y) - \delta - g(m)h(z) \tag{38}$$

Consumption rises or falls according as the right side of (38) is positive or negative. We shall analyze (38) further after looking at the R&D decision.

Beginning the Search for a New Technology

The criterion (33) for choice of the R&D rate m may be written as

$$q(t) \equiv [\varphi'(z)W + \gamma]g'(m) - \lambda \leq 0 \tag{39}$$

where m is chosen so that $q = 0$ if possible and $m = 0$ otherwise. Since $z(0) = 0$ and $\varphi'(0) = 0$, we have at $t = 0$

$$\text{Either (i) } m(0) = 0 \text{ and } \gamma(0)g'(0) - \lambda(0) < 0 \tag{40}$$

$$\text{or (ii) } q(0) = \gamma(0)g'(m(0)) - \lambda(0) = 0$$

Suppose there is an interval $0 \le t \le t_0$ during which $m(t) = 0$ and $q(t) < 0$. Then since $m = z = 0$ during this period, and since $\varphi'(0) = g(0) = 0$, (29) reduces to $\gamma' = \delta\gamma$ so that

$$\gamma(t) = \gamma_0 e^{\delta t}, \quad 0 \le t \le t_0 \qquad \text{where} \quad \gamma_0 = \gamma(0)$$

and

$$q(t) = e^{\delta t}[\gamma_0 g'(0) - \mu_0/f'(y)] < 0 \qquad 0 \le t \le t_0 \tag{41}$$

where (27) has been solved to give

$$\mu(t) = \mu_0 e^{\delta t} \text{ where } \mu_0 = \mu(0)$$

and used with (32) to eliminate λ from (39). The sign of $q(t)$ is the sign of the square bracketed expression in (41); that expression increases over time since y declines (see (37)) and f' is a decreasing function of its argument.

Thus, there will be a $t_0 < \infty$ at which $q(t_0) = 0$ and R&D begins. We have supposed that $\gamma_0 > 0$; the innovation would have positive value.

We have shown that R&D may, but need not, begin immediately at $t = 0$. Before the onset of R&D, the behavior of the economy is identical to that of Dasgupta and Heal (Section 1), since our model is equivalent to theirs in the absence of R&D. Thus this period needs no further discussion here. Eventually, however, the diminishing supply of exhaustible resource will exert pressure to begin the search for an alternate technology; that is, time t_0 will be attained. We next consider a time span beginning when R&D starts.

While the Search Goes On

While $m > 0$, the equality of (33i) holds so it may be differentiated totally with respect to time;

$$[\varphi'' z' W + \gamma']g' + [\varphi' W + \gamma]g'' m' = \lambda'$$

Substituting from (29) for γ', using (33i) to eliminate γ, and (36) to eliminate λ' yields (recalling (19) and rearranging)

$$-\lambda g'' m' / g' = \lambda(f - yf') - (\delta W - U)\varphi' g'$$

Now divide through by λ, using (31), to obtain finally

$$-g'' m' / g' = [f(y) - yf'(y)] - [\delta W - U(C)] g'(m) h(z) / U'(C) \tag{42}$$

Because of (11), the R&D rate increases (decreases) when the right side of (42) is positive (negative).

We are studying a period when all three control variables are positive. From (37) it is evident that y is declining, while the behaviors of C and m are given by (38) and (42). We know also from (18) and (19) that S must be decreasing while z is increasing. To examine the behavior of C and m we consider the projections of the optimal path and phase diagram boundaries in C-m space through time. This procedure is similar to the one we employed in [7] although only one phase diagram boundary shifted through time there.

Behaviors of C and m separately

Let t_m denote the moment R&D begins. Thus

$$t_m = \begin{cases} t_0 & \text{if (40i) holds} \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

Through study of (38) we can partially characterize the behavior of C'. We summarize it as

Proposition 1 Behavior of C'

a. If $h(z) = 0$, then sign C' = sign $(f - yf' - \delta)$

b. If $f - yf' \leq \delta$ at s, then $C'(t) < 0$ for all $t > s$.

c. If $h(z) > 0$ and $f - yf' > \delta$, then

$$C'(t) \lesseqgtr 0 \text{ as } m(t) \gtreqless m^0(t)$$

where $m^0(t)$ is implicitly defined by

$$g(m^0(t)) = [f - yf' - \delta]/h(z)\big|_t \tag{44}$$

<u>Proof</u>.

a. follows immediately from (38)

b. follows immediately from (38)

c. In this case, for given values of $y(t)$, $z(t)$, there is at most one value of m for which $C' = 0$; it satisfies (44). If $m(t) > m^0(t)$, then it follows from (38) that $C' < 0$. The remaining case follows similarly.

The value m^0 specifies a line in the C-m plane that we call the $C' = 0$ locus. To see how this locus moves over time in the C-m plane, differentiate (44) totally with respect to t:

$$g'(m)\ dm^0/dt = -yf''y'/h - (f-yf' - \delta)h'z'/h^2 \tag{45}$$

Since the right side of (45) is negative under the assumptions of case c. and (37) and since $g' > 0$, it follows that the critical value m^0 decreases through time so the $C' = 0$ locus moves down. See Figure 1.

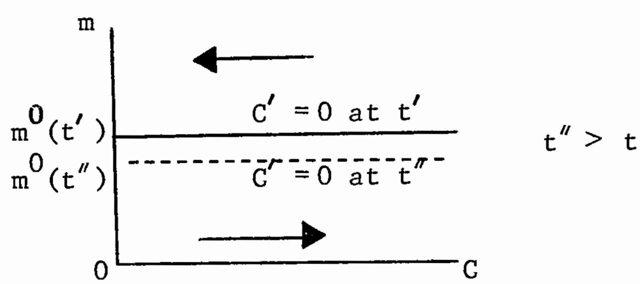

Fig. 1 Behavior of C

Next we examine the behavior of m' through study of (42). Since $h(0) = 0$, it follows that $m'(t_m) > 0$; this sign will be maintained so long as $h(z) = 0$. After the R&D has progressed sufficiently that there is a positive probability of imminent completion, the possibility arises that m' might reverse sign. From (42), for given $y(t)$ and $z(t)$, the combinations of C and m for which $m' = 0$ (assuming $h(z) > 0$) satisfy

$$(f-yf')/h(z)g'(m) = (\delta W - U(C))/U'(C) \equiv G(C) \tag{46}$$

where the right hand equation defines the function G(C). We henceforth assume (2), that U is isoelastic with $\eta > 1$ and $W < 0$. Then

$$G(C) = C(1 + \delta WC^{\eta-1})/(\eta-1) \tag{47}$$

The left side of (46) takes nonnegative values only; the range of values of C for which $G(C) \geq 0$ is

$$0 \leq C \leq (-\delta W)^{-1/(\eta-1)} \equiv C_M \tag{48}$$

The left side of (46) is an increasing function of m, while the right side is concave in C:

$$G'(C) = (1 + \eta\ \delta WC^{\eta-1})/(\eta-1) \tag{49}$$

$$G''(C) = \eta\ \delta\ WC^{\eta-2} < 0 \tag{50}$$

The value of C that maximizes G(C) is

$$C^{\#} = (-\ \eta\ \delta\ W)^{-1/(\eta-1)} \tag{51}$$

and

$$\max G(C) = G(C^{\#}) = (-\ \eta^{\eta}\delta W)^{-1/(\eta-1)} \tag{52}$$

On the other hand, the smallest the left side of (46) can be at any time is

$$(f-yf')/h(z)g'(0)$$

which decreases through time. Combining these observations with (42), we conclude that

$$\text{if } (f-yf')/h(z)g'(0) > G(C^{\#}) \text{ at } t, \text{ then } m'(t) > 0 \tag{53}$$

If the hypothesis in (53) does not hold, then the sign of m' is not readily apparent. We thus turn to the $m' = 0$ locus, defined by (46). To find its shape, view (46) as an implicit definition of the function $m^c(C,t) = m$ and differentiate with respect to C holding t fixed:

$$[-(f-yf')g''/h(g')^2]\ \partial m^c/\partial C = G'(C) \tag{54}$$

Thus $\partial m^c/\partial C$ takes the sign of $G'(C)$, so from (48), (49), and (51)

$$\frac{\partial m^c}{\partial C} \begin{cases} > 0 & \text{for } 0 < C < C^{\#} \\ < 0 & \text{for } C^{\#} < C < C_M \end{cases} \tag{55}$$

To determine the direction of movement of the m-component of a point in the C-m plane, we return to (42); for fixed t and C, we have $m' \gtrless 0$ as $m \gtrless m^c(C,t)$. Hence $m' > 0$ above the $m' = 0$ locus and $m' < 0$ for points below the locus. See Figure 3 below.

Conclusions about m' reached so far may be summarized as

<u>Proposition 2</u> Behavior of m'

a. If $h(z) = 0$, then $m'(t) \geq 0$ with strict inequality in case $m(t) > 0$.

b. If $(f-yf')/h(z)g'(0)\big|_t > G(C^{\#})$, then $m'(t) > 0$.

c. If $(f-yf')/h(z)g'(0)\big|_t < G(C^{\#})$, then

$m'(t) \gtrless 0$ as $m(t) \gtrless m^c(C(t),t)$

Next, to determine how the $m' = 0$ locus shifts over time in the C-m plane, we hold C fixed and differentiate (46) with respect to t.

$$[-(f-yf')g''/h(g')^2]\ \partial m^c/\partial t = yf''y'/hg' + (f-yf')H'z'/h^2g \tag{56}$$

The right side of (56) is positive and the coefficient of $\partial m^c/\partial t$ is positive; hence the m-coordinate of each point of the $m' = 0$ locus increases over time. The $m' = 0$ locus moves up. The C-coordinate of its peak remains fixed at $C^{\#}$. The intercepts of the curve on the m = 0 axis satisfy

$$(f-yf')/h(z)g'(0) = G(C) \tag{57}$$

Since the left side of (57) decreases through time, the smaller root $\underline{C}$ of (57) decreases over time while the larger root $\overline{C}$ grows. See Fig. 2, in which

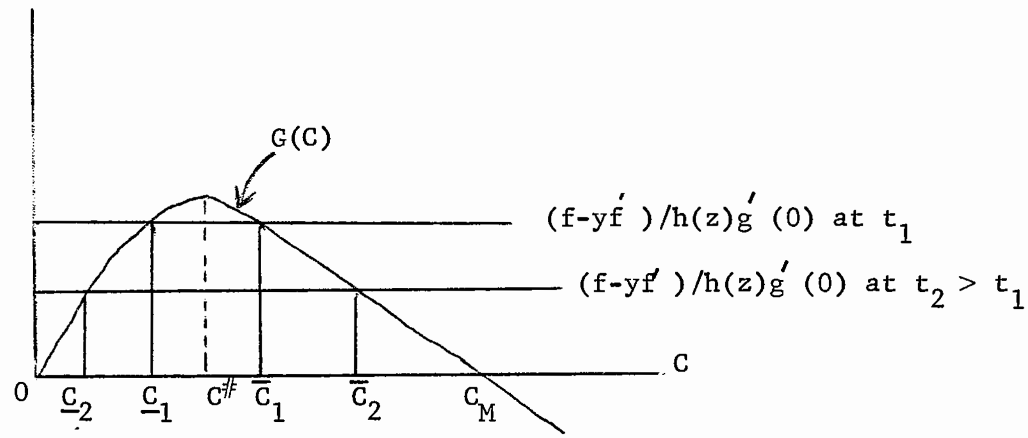

Fig. 2

$\underline{C}_1$, $\overline{C}_i$ are the intercepts of the $m' = 0$ locus on the $m = 0$ axis at time t_i. Thus we obtain Fig. 3.

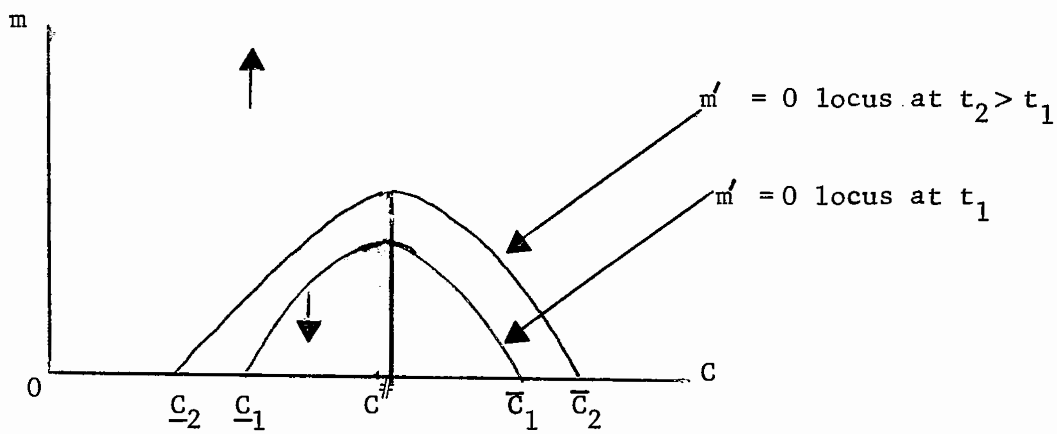

Fig. 3 Behavior of m

Combined Temporal Patterns

Now we combine our analyses of C' and m', summarized in Propositions 1 and 2 and in Figures 1 and 3, to construct the temporal patterns an optimal policy may follow.

First suppose that $f-yf' < \delta$ at t_m. Then, from Prop. 1.b., $C'(t) < 0$ thereafter. Further, from Prop. 2.a., we have $m'(t_m) > 0$. Thus the projection of the optimal path in the C-m plane shows consumption falling while the

R&D rate is rising for some period after t_m. As time passes, the condition of Prop. 2.c. will eventually be satisfied and the $m' = 0$ locus will arise from below. If T or $\bar{z}$ is not attained in the meantime, the $m' = 0$ locus may eventually overtake the optimal path. At the moment of intersection, the optimal path is stationary ($m' = 0$) while the $m' = 0$ locus continues to rise. Thereafter the optimal path lies below the locus and $m' < 0$ as well as $C' < 0$.

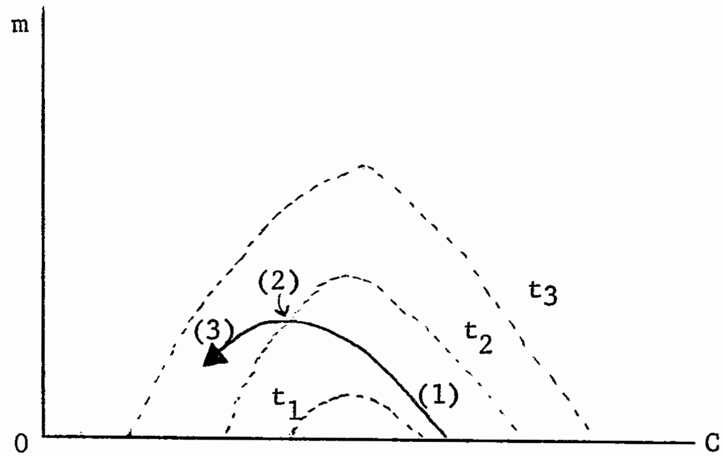

Fig. 4 Path for case of $f-yf' \leq \delta$ at t_m

In Fig. 4, we consider three moments $t_1 < t_2 < t_3$. The position of the $m' = 0$ locus at each of these instants is indicated by dashed curves. The optimal path is shown as the solid curve, with the position along this path at these three moments designated. We have $m'(t_1) > 0$ and $C'(t_1) < 0$. At t_2, the path just touches the moving $m' = 0$ locus; $m'(t_2) = 0$ and $C'(t_2) < 0$. Later, at t_3, the $m' = 0$ locus has risen as shown and both C and m continue to fall towards zero, until T or $\bar{z}$ is attained. Fig. 4 suggests a case in which $t_m > 0$. If $t_m = 0$, then R&D may begin at a positive rate so the optimal path begins within the positive C-m quadrant. The results of Fig. 4 can be displayed as temporal patterns of C(t) and m(t); see Fig. 5. Patterns for the situation that $t_m = 0$ are illustrated in Fig. 6.

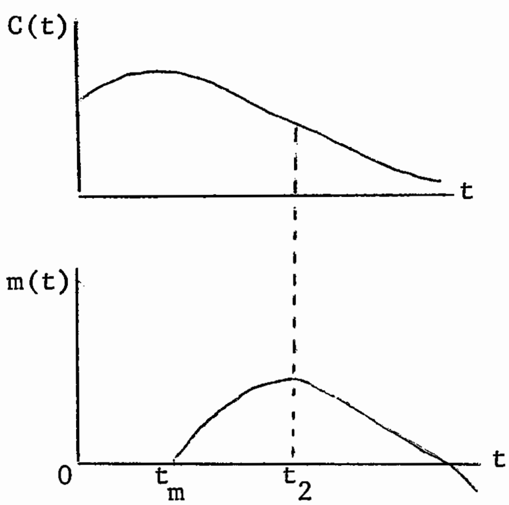

Fig. 5

$f\text{-}yf' \leq \delta$ at $t_m > 0$

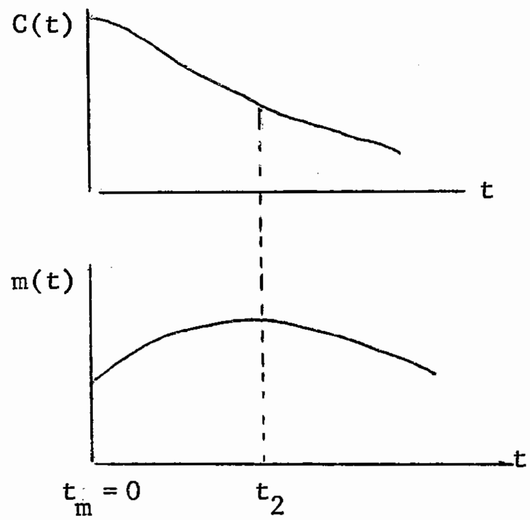

Fig. 6

$f\text{-}yf' \leq \delta$ at $t_m = 0$

Next we take up the cases in which $f\text{-}yf' > \delta$ at t_m. From Props. 1.a. and and 2.a., we have $C'(t_m) > 0$ and $m'(t_m) > 0$. As h(z) increases, the $C' = 0$ locus will eventually appear in the positive C-m quadrant and descend from above, while the $m' = 0$ locus will eventually arise from below. A sketch of the $C' = 0$ and $m' = 0$ loci at a single moment t_4 and of the optimal path up to that time ($0 \leq t \leq t_4$) appears as Figure 7 below. Directional arrows indicate movement consistent with differential equations (38) and (42) at t_4.

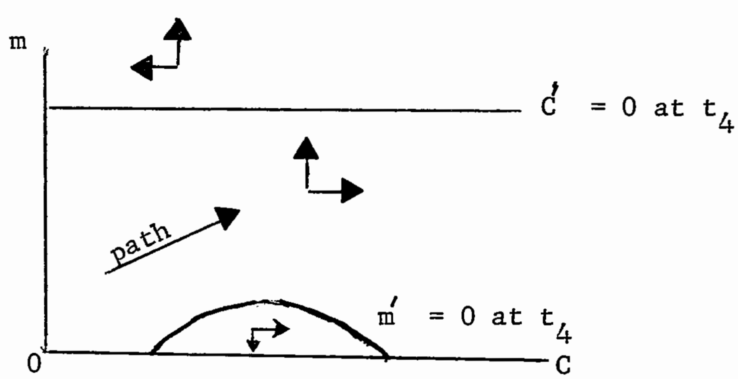

Fig. 7

Case $f\text{-}yf' > \delta$ at t_m, seen at time t_4

The next question is whether the optimal path will be overtaken first by the falling $C' = 0$ locus or the rising $m' = 0$ locus. Consider the two possibilities in turn.

Suppose the rising path in the C-m plane encounters the falling $C' = 0$ locus. At that moment, the path is stationary while the $C' = 0$ locus is falling. The path will subsequently be above the $C' = 0$ locus, so that $C' < 0$ thereafter. If T or $\bar{z}$ is not attained in the meantime, the $m' = 0$ locus may eventually overtake the path, after which both C and m will fall.

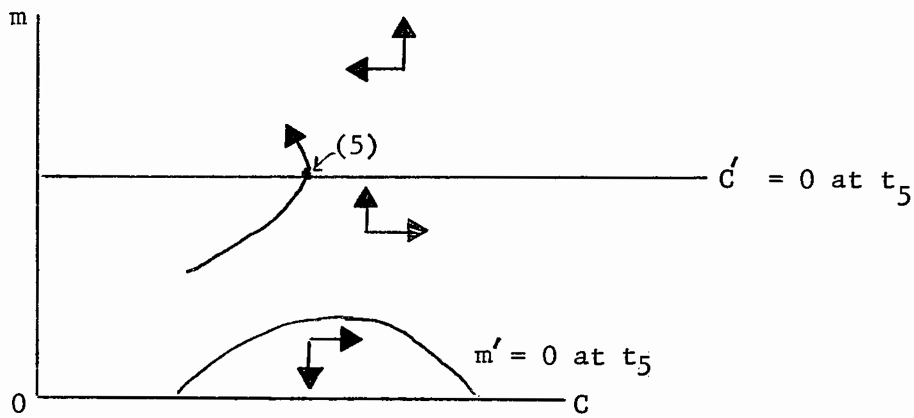

Fig. 8

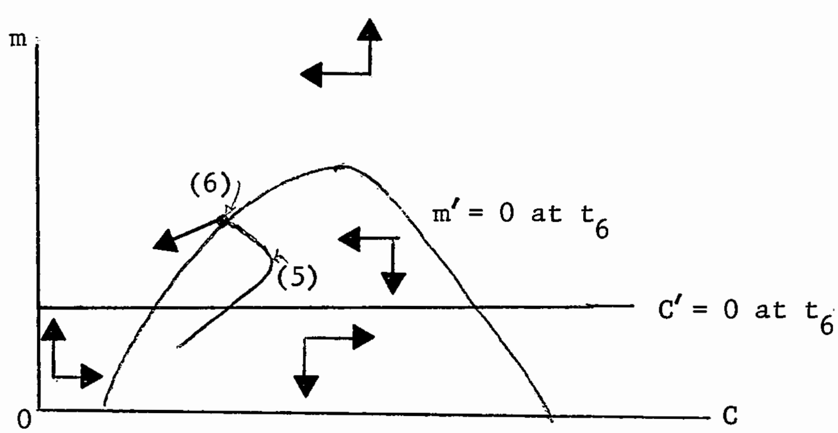

Fig. 9

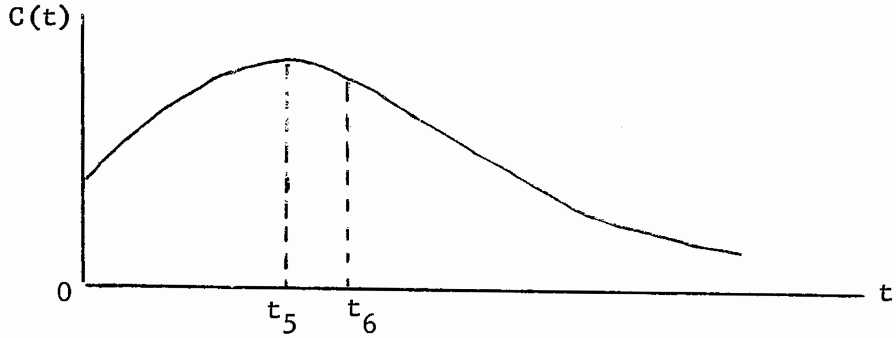

Fig. 10

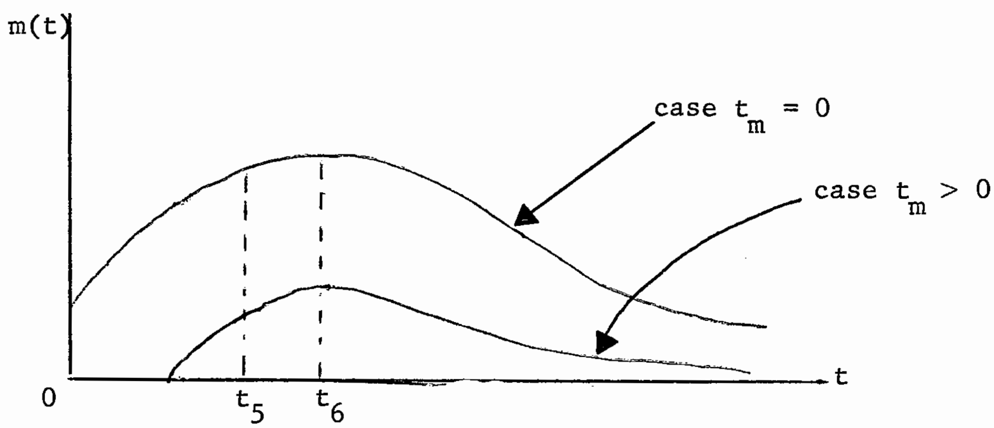

Fig. 11

Figure 8 shows the position of the $C' = 0$ and $m' = 0$ loci at the moment $t_5 > t_4$ that the $C' = 0$ locus touches the path. The path followed for a period of time embracing t_5 is also shown, along with the general directional arrows pertinent at t_5. Figure 9 shows the loci at $t_6 > t_5$ when the path touches the $m' = 0$ locus; it also shows the path taken over some time spanning t_5 and t_6. Figures 10 and 11 illustrate the corresponding temporal profiles of C (in Fig. 10) and of m (in Fig. 11). Figure 11 contains alternative profiles, for the cases in which R&D does and does not begin immediately. Times t_5 and t_6 retain the definitions just provided. Only the direction of movement is depicted; no inferences are made regarding height of the curves or their concavity/convexity. Consumption peaks before R&D expenditures in this case.

Next we consider the alternate situation in which $f - yf' > \delta$ at t_m and the rising optimal path in the C-m plane meets the $m' = 0$ locus before intersecting the $C' = 0$ locus. Then C and m both rise after t_m and until the moment t_a the path meets the $m' = 0$ locus. Next m falls while C continues to rise, until the moment $t_b > t_a$ the path meets the $C' = 0$ locus. Thereafter C and m both fall over time, until T or $\bar{z}$ is attained. This case is illustrated in the Figures below. Details of construction are similar to those in cases developed above. All figures will be truncated at whatever time $\bar{z}$ is attained. The paths are actually followed only so long as T is not achieved.

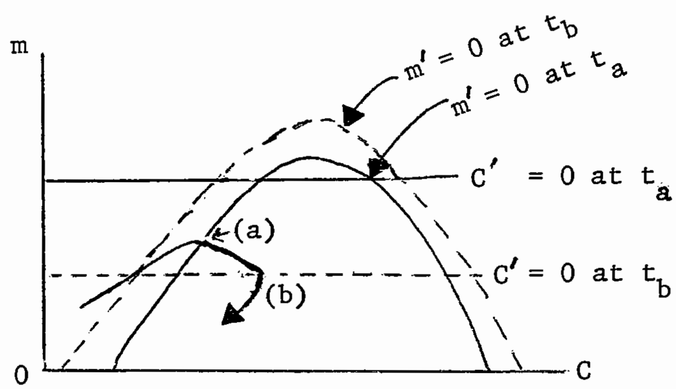

Fig. 12

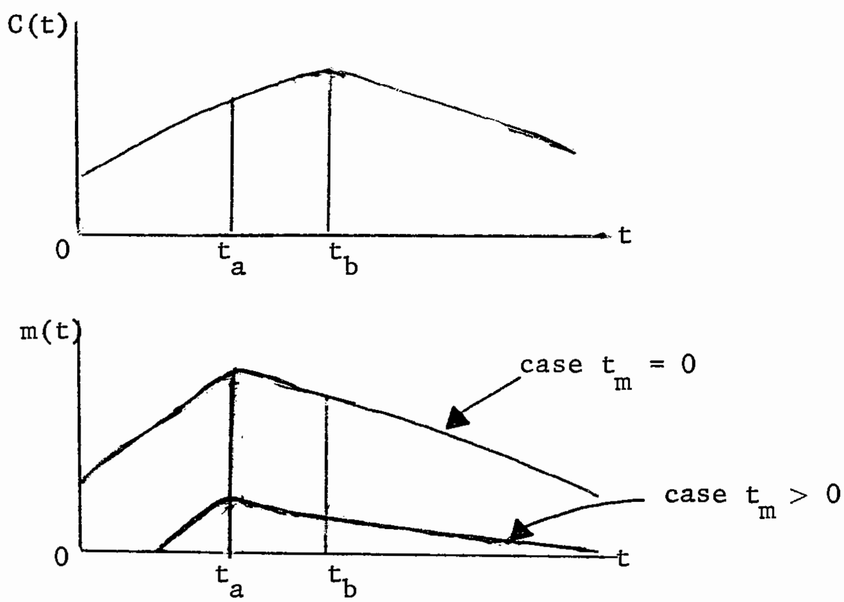

Fig. 13

Extraction Cost

Finally we introduce resource extraction costs that rise as the remaining resource stock falls, assuming extraction cost is proportional to the extraction rate, for a given resource stock. The unit extraction cost is a decreasing convex function of the remaining resource stock. Thus to extract the resource at rate R when the remaining stock is S requires E(S)R of the composite good, where

$$E > 0, \; E' < 0, \; E'' > 0 \tag{58}$$

Problem (16)-(19) must be modified, replacing the differential equation of (17) by

$$K' = Kf(y) - E(S)Ky - C - m \tag{59}$$

Otherwise the problem is as set forth earlier. After modifying the Hamiltanian appropriately, one finds conditions (22), (23), (26), (28)-(30) as before. Conditions (24), (25), and (27) will be replaced respectively by

$$\lambda K[f'(y) - E(S)] - \mu K = 0 \text{ if } y > 0 \tag{60}$$

$$\lambda' = \delta\lambda - \lambda f(y) + \lambda E(S)y + \mu y \tag{61}$$

$$\mu' = \delta\mu + \lambda E'(S)Ky \tag{62}$$

As before C, y, and K will all be positive in an optimal program under our assumptions. Thus from (60)

$$\lambda[f'(y) - E(S)] = \mu \tag{63}$$

Substituting from (63) for μ into (61) yields

$$\lambda'/\lambda = \delta - (f - yf') \tag{64}$$

which is exactly (36). Conditions (31) and (33) are obtained as before, and, in view of (64), differential equations (38) and (42) follow. The temporal behavior of factor proportions y may be found by differentiating (63) totally with respect to t and then employing (62) to eliminate μ' and (63) to eliminate μ and finally (64) to eliminate λ'/λ:

$$-f''y'/(f - yf') = -(f' - E) < 0 \qquad (65)$$

The signing follows from (63), since

$$f'(y) - E(S) > 0 \text{ in the economic region}$$

That is, since f(y) is the output of composite good per unit capital and E(S)y is the resource extraction cost per unit capital, the difference f(y) - E(S)y is the net output per unit capital; for economic sense the marginal net resource product must be positive. Therefore we again conclude that the factor proportion y is temporally falling. Since equations (38) and (42) are unchanged and since y decreases through time, all the previous qualitative results are preserved when resource extraction costs are introduced as in (59).

Summary & Comparison with Micro Model

In sum, we have developed alternate temporal profiles of consumption and R&D, as depicted in Figures 6, 10-11, and 13, to be followed until the advent of the new technology. Since the amount of effective effort required for success is unknown, a contingency plan must be constructed for the worst case, namely that effort $\bar{z}$ is required. The Figures are to be truncated on the right at the moment $\bar{z}$ is reached; this feature has not been explicitly exhibited. The paths are followed only until the time T the new technology is available, after which the paths associated with (9) can be realized.

In every case, both the consumption and R&D rates are single-peaked, and both must eventually decline towards zero (again, except if $\bar{T}$ or $\bar{z}$ is attained in the meantime). The eventual diminution of both activities is attributable to the fact that the resource stock is limited and diminishing, eventually reducing the capacity of the economy, despite its ability to build capital stock (at the expense of consumption and research). The bleak picture of declining consumption imposed by the irreplenishability of an essential resource is brightened by the eventual dawning of a new era after successful development of the new technology. The new era may ultimately have to be supplanted by still another, as other environmental or resource constraints begin to exact their toll. These rises and declines together with the rescues through the creative response have marked the course of human history.

These observations prompt two possible extensions of our model. First, the discounted utility stream available from the new technology may depend on the nature of the technical advance. For example it is unlikely that a nuclear breeder reactor, a fusion reactor, or a means of efficiently harnessing solar energy would all provide equivalent opportunities for future economic growth. Likewise, it is realistic to suppose that the expected costs and risks of developing alternative technologies differ. Incorporation into the model of different potential technologies with different rewards and development costs permits investigation of a number of questions. Is it optimal to develop all possible technologies? If only several out of the many technologies are to be developed, how are these selected? How is the pace affected by the diversity of approaches? If development proceeds in parallel, when are certain development efforts discontinued?

Second, successful development of a new technology may require a sustained commitment of resources above a minimal level. This requirement together with our finding that development of a new technology may optimally not begin at $t = 0$, and the Dasgupta, Heal, and Majumdar [3] observation that resource poor countries postpone technological development relative to resource rich countries, poses a fascinating and important question. If development of a new technology is not initiated by its optimal starting time, might it then be postponed further because its cost, relative to the economy's diminishing resources, has risen? If this can occur, then we may face the possibility of starting development too late in the sense the minimum required resource commitment will be impossible. This element of irreversibility, of being too late with too little, is an implicit fear among those alarmed about the imminent exhaustion of oil reserves.

Finally, our analysis of the aggregate model may be compared with that of a single firm seeking a technical improvement through R&D for profit, acting within a stationary environment. In our study of the latter situation [5], we found that if the R&D effort were worthwhile, then it would always begin immediately. Further, so long as $h' > 0$ as we have assumed in this paper, the planned R&D rate was increasing through time. Actual spending ceased, of course, upon project completion. The difference in the findings about the time to begin the R&D effort is attributable to the temporally changing opportunity cost of R&D in the present paper as contrasted with the constant opportunity cost in the microeconomic model within a stationary environment (both characterizations excepting the time value of resources). The R&D rate increases in the stationary environment as the conditional probability of project completion with incremental R&D is nondecreasing with cumulated effective effort. On the other hand, in the present context, while the conditional probability of project completion with incremental R&D is nondecreasing, the diminishing economic capacity exerts an increasingly stringent limit on what can be done, regardless of its attractiveness.