DISCUSSION PAPERS NOS. 210a and 210b

QUASI EQUILIBRIUM IN ABSTRACT ECONOMIES WITHOUT ORDERED PREFERENCES

and

OPTIMAL TAX SCHEMES FOR FINANCING LOCAL PUBLIC GOODS

by

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April 1976

No. 210a

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The purpose of this note is to generalize the powerful result due to W. Shafer and H. Sonnenschein [2], which by "applying standard techniques can be used to prove the existence of equilibrium in economies with interdependent preferences, price-dependent preferences and preferences which may be both non-transitive and non-complete." One particular assumption which is required for such an application is that each individual's initial endowment is an interior point of his consumption set. (This is crucial to assure the continuity of the budget set). However, as Debreu [1] notes: "Assumptions of this type have not been readily accepted on account of strength, and this is in spite of the simplicity that they give to the analysis."

We present here a definition and an existence proof for quasi-equilibrium in abstract economies. This definition when applied to an economy (with the above preferences) coincides with that of Debreu's. Since technically we follow the proof in [2] very closely, we shall only present its outline.

Definition 1: An abstract economy $\Gamma = (X_i, \mathcal{A}_i, P_i)_{i=1}^{i=n}$ is defined by ordered triples $(X_i, \mathcal{A}_i, P_i)$, where $\mathcal{A}_i: \Pi X_j \to X_i$ and $P_i: \Pi X_j \to X_i$ are correspondences.

We denote by N the set $\{1,\ldots,n\}$.

Definition 2: An equilibrium for Γ is a vector $\bar{x} \in X$ such that for all $i \in N$:

(e.1) $x_i \in \mathcal{Q}_i(\bar{x})$, and

(e.2) $P_i(\bar{x}) \cap \mathcal{Q}_i(\bar{x}) = \emptyset$.

Definition 3: Let $\Psi = \{\Psi_1,\ldots,\Psi_n\}$ be a vector of n functions: $\Psi_i\colon X \to R_+$, $i \in N$. A Ψ-quasi-equilibrium for Γ is a vector $\bar{x} \in X = \Pi X_j$, such that for all $i \in N$:

(1) $\bar{x}_i \in \mathcal{Q}_i(\bar{x})$, and

(2) $P_i(\bar{x}) \cap \mathcal{Q}_i(\bar{x}) = \emptyset$ and/or $\Psi_i(\bar{x}) = 0$.

Theorem. Let $\Gamma = (X_i, \mathcal{Q}_i, P_i)_{i=1}^{i=n}$ be an abstract economy, $\Psi_i\colon X \to R_+$, $i \in N$, satisfying for each i,

(a) X_i is a non-empty compact and convex subset of R^ℓ,

(b') Ψ_i is continuous,

(b") $\mathcal{Q}_i(X)$ is a continuous correspondence for all x with $\Psi_i(x) > 0$, and is upper semi continuous for all x with $\Psi_i(x) = 0$,

(b"') for each $x \in X$, $\mathcal{Q}_i(x)$ is non-empty and convex,

(c') P_i has an open graph in $X \times X_i$, and

(c") for each $x \in X$, $x_i \notin H(P_i(x))$, where $H(A)$ denotes the convex hull of A.

Then Γ has a Ψ quasi equilibrium.

For economies (rather than abstract economies), Definition 3 coincides with Debreu's [1] when, for all agents i who are not "consumers" (e.g., firms, government and the market player), $\Psi_i(x) \equiv c > 0$, and for each consumer ℓ, $\Psi_\ell(x) = p\, w^\ell - \text{Min}\, p\, X_\ell$.

Clearly, the theorem can be used to prove the existence of an equilibrium for Γ by adding (plausible) assumptions which imply $\mathcal{O}_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$ even when $\Psi_i(\bar{x}) = 0$. For example, an assumption which guarantees that $\Psi_i(\bar{x}) = 0$ implies $\{\bar{x}^i\} = \mathcal{O}_i(x)$. For an exchange economy, such an assumption is strong monotonicity which implies $p >> 0$, hence $px^i = 0$ if and only if $x^i = 0$. (For a discussion of such assumptions see [1]).

Our result seems to be of particular interest for economies with a tax authority (that either finances production of public goods and/or redistributes income), where even the assumption that every individual's initial endowment is an interior point of his consumption set is not sufficient to guarantee that for each consumer, after-tax income is greater than his "minimum wealth". On the other hand, imposing this latter restriction on the tax authority causes its choice set $(\mathcal{O})$ to be open. Hence the result in [2] cannot be applied.

<u>Proof:</u> By (c'), for each $i \in N$, there is a continuous function $U_i: X \times X_i \to R_+$ such that $U_i(y,x_i) > 0$ if and only if $x_i \in P_i(y)$. Define $V_i(y,x_i) = U_i(y,x_i) \cdot \psi_i(y)$. For each i, define $F_i: X \to X_i$ by $x_i \in F_i(y)$ if and only if x_i maximizes $V_i(y,\cdot)$ subject to $x_i \in \mathcal{A}_i(y)$. If $\psi_i(y) > 0$, $F_i(y)$ is non-empty and has a closed graph [2]. If $\psi_i(y) = 0$, $F_i(y) = \mathcal{A}_i(y)$. Since for all $y \in X$ $F_i(y) \subset \mathcal{A}_i(y)$, by (b") for each y, F_i has a closed graph, and $F_i(y)$ is nonempty. By Kakutani's fixed point theorem there exists $\bar{x} \in X$ such that $\bar{x}_i \in H(F_i(\bar{x}))$ for all i. Since $\mathcal{A}_i(\bar{x})$ is convex, $\bar{x}_i \in \mathcal{A}_i(x)$, $\forall$ $i \in N$. If $\psi_i(\bar{x}) = 0$ (2) holds. If not, $V_i(\bar{x},z_i) > 0$ if and only if $U_i(\bar{x},z_i) > 0$, which implies [2], $P_i(\bar{x}) \cap \mathcal{A}_i(\bar{x}) = \emptyset$.

Q.E.D.

<u>REFERENCES</u>

[1] G. Debreu, "New Concepts and Techniques for Equilibrium Analysis", <u>International Economic Review</u> 3, 1962, 257-272.

[2] W. Shafer and H. Sonnenschein, "Equilibrium in Abstract Economies Without Ordered Preferences", <u>Journal of Mathematical Economies</u> 2, 1975, 345-348.

No. 210b

OPTIMAL TAX SCHEMES FOR FINANCING LOCAL PUBLIC GOODS

by

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ABSTRACT

The existence of a public competitive equilibrium for any arbitrary (non-distortive) tax schemes (in particular -- proportional taxes) for economies with local public goods, is proved. Allowing preferences to be both non-transitive and non-complete enables an explicit introduction of the government as an (additional) agent in the economy. Moreoever, we allow for "spillovers" of the public goods among localities, and for the production sets to depend on the amount of public goods produced in the economy. The only restriction on the tax system is that every individual is able to afford it and that the government's budget never runs a surplus.

Since every equilibrium allocation is Pareto-optimum, every tax schem is optimal (in the sense that its equilibrium allocation is Pareto-optimur

*/ I gratefully acknowledge E. Kalai and P. J. McCabe for their helpful comments on the draft. This work was partially supported by Nation Science Foundation Grant SOC 71-03784 A04 (formerly GS-31346X).

like monotonicity of the preference ordering assumed in [2,5] (for a discussion of such assumptions see [1]) one gets the equivalence of the sets of equilibrium and quasi-equilibrium allocations.

The Model:

There are ℓ private goods, q public goods and n individuals that are partitioned into K exogenous governmental jurisdictions. Denote by $N = \{1,\ldots,n\}$ the set of individuals, and by $B = \{B_1,\ldots,B_K\}$ the set of jurisdictions, then B forms a partition of N, (i.e., $\bigcup_{k=1}^{K} B_k = N, B_\ell \cap B_k = \phi \ \forall \ \ell \neq k$). Each jurisdiction B_k produces public goods for its members using the production set G_k. In order to produce the vector $g^k \in R^q_+$ of public goods, the vector $z^k \in R^\ell_+$ of private goods is required as an input, i.e., $(g^k, -z^k) \in G_k \subset R^q_+ \times R^\ell_-$.

There are J firms which produce private goods, with the production sets $Y_j \subset R^\ell$, owned by the individuals through the share system $\{\theta_{ij}\}$ $(\sum_{i\in N} \theta_{ij} = 1, \ \forall j = 1,\ldots,J, \theta_{ij} \geq 0)$. We also allow for the public goods to enter the production of both private and public goods. For this end, denote the production correspondences:

$$\mathcal{Y}_j: R^{qK}_+ \twoheadrightarrow Y_j \quad \text{and} \quad \mathcal{G}_k: R^{qK}_+ \twoheadrightarrow G_k.$$

When the vector of public goods in the economy is $g = (g^1,\ldots,g^K)$ the feasible production set for firm j (jurisdiction k) is given by $\mathcal{Y}_j(g)(\mathcal{G}_k(g))$.

An individual $i \in N$ consumes private goods x^i, and the public goods produced in the whole economy, i.e., his consumption set is $X_i \subset R_+^{qK+\ell}$. This allows for spillovers of public goods among jurisdictions. Indeed, an individual may consume the public goods a nearby city offers, while he pays taxes to the suburb he is living in. (Obviously we may, in particular, restrict $i \in B_k$ to consume only the public goods g^k provided by his jurisdiction. This is imposed in [5]). We shall assume ((a.1) below) that $(g,0) \in X_i$ for all $i \in N$ and for all $g \in R_+^{qK}$. Each $i \in N$ has an initial endowment $w^i \in R_+^{\ell}$, (public goods are not initially owned), and a preference correspondence P_i which depends upon the public and private goods i consumes, i.e., $P_i: R_+^{qK+\ell} \twoheadrightarrow R_+^{K+\ell}$.
Let $(g,x^i) = (g^1,\ldots,g^K,x^i) \in R_+^{qK+\ell}$, then $P_i(g,x^i) \cap X_i$ is interpreted to be the set of all $(\hat{g},\hat{x}^i) \in X_i$ which consumer i prefers to (g,x^i). (Note that x^i denotes only the consumption of private goods of individual i, while X_i denotes his consumption set of both private and public goods.)

An economy is fully specified by: $\mathcal{E} = (G_k,\mathcal{G}_k,X_i,w^i,P_i,Y_j,\mathcal{Y}_j,B)$. We now define the admissable tax functions. Denote the price simplex (for the private goods) by: $\Delta = \{p \in R_+^{\ell} \mid \Sigma p_i = 1\}$.

<u>Definition 1:</u> Let $e = (g,z,x,y) \in E \equiv R_+^{qK} \times R_+^{\ell K} \times R_+^{\ell n} \times R^{\ell J}$, and let $p \in \Delta$. A <u>feasible tax system for e under p</u> is a vector $\alpha = (\alpha_1,\ldots,\alpha_n) \in R^n$ such that:

(i) for all $i \in N$,

$$\alpha_i \leq p w^i + \text{Max}[0, \sum_{j=1}^{J} \theta_{ij} p y^j] \equiv v^i$$

$$\text{(ii) for all } B_k \in B, \sum_{i \in B_k} \alpha_i = \begin{cases} pz^k, & \sum_{i \in B_k} v^i \geq pz^k \\ \sum_{i \in B_k} v^i, & \sum_{i \in B_k} v^i \leq pz^k \end{cases}$$

Condition (i) assures that each individual can survive in the economy with his after-tax income (as $(g,0) \in X_i$). Since py^j does not necessarily maximize profits in firm j (in which case, by (b.2) $py^j \geq 0$) the second term in (i) is introduced. Note that if we assumed, as in [2 , 5], that $\mathcal{Y}_j(g)$ is a convex cone containing the origin for all $j = 1,\ldots,J$ and for all $g \in R_+^{qK}$ we could replace v^i by pw^i. Condition (ii) guarantees that whenever possible each jurisdiction k finances its own public goods with a balanced budget. If $pz^k > \sum_{i \in B_k} v^i$, $\alpha_i = v^i$ for all $i \in B_k$, and the government has a deficit in its budget. Clearly, this will not be the situation in equilibrium.

<u>Definition 2:</u> A <u>tax scheme</u> for $\mathcal{E}$ is a continuous function $\alpha\colon E \times \Delta \to R^n$ such that for all $e \in E$ and $p \in \Delta$, $\alpha(e,p)$ is a feasible tax system for e under p.

The continuity of α reflects the consistency of the governments, since when allocations and prices are "close", the taxes levied on the individual should not "considerably differ."

Examples for tax schemes are: (to simplify notation we assume that pw^i can be substituted for v^i and that $\sum_{i \in B_k} w^i \gg 0$ for all $B_k \in B$.)

(α.1) Proportional taxation: For $i \in B_k$,

$$\alpha_i(e,p) = \frac{pz^k}{p \sum_{i \in B_k} w^i} \cdot pw^i$$

(We consider only the case in which $pz^k \leq \sum_{i \in B_k} pw^i$. Otherwise, for all tax schemes, $\alpha_i = pw^i \; \forall \; i \in B_k$).

(α.2) Arbitrary after-tax income distribution: Let

$(\beta_1,\ldots,\beta_n) \in R^n_+$ such that $\sum_{i \in B_k} \beta_i = 1$, $k = 1,\ldots,K$. For $i \in B_k$,

$$\alpha_i(e,p) = pw^i - \beta_i\left(\sum_{i \in B_k} pw^i - pz^k\right).$$

Note that (α.1) is the special case where $\beta_i = \dfrac{pw^i}{\sum_{i \in B_k} pw^i}$.

The after-tax income is $pw^i - \alpha_i(e,p) = \beta_i\left(\sum_{i \in B_k} pw^i - pz^k\right)$. For complete equity in jurisdiction k, define $\beta_i = \dfrac{1}{|B_k|}$, $\forall_i \in B_k$.

(α.3) Dictatorial taxation: Let $B_k = \{i_1,\ldots,i_m\}$. Without loss of generality, $i_\ell = \ell \quad \ell = 1,\ldots,m$.

$$\alpha_1(e,p) = \text{Min}\,[pz^k, pw^1]$$

$$\alpha_2(e,p) = \text{Min}\,[pz^k - \alpha_1(e,p), pw^2]$$

$$\vdots$$

$$\alpha_\ell(e,p) = \text{Min}\,[pz^k - \sum_{i=1}^{\ell-1} \alpha_i(e,p), pw^\ell].$$

<u>Definition 3:</u> An <u>attainable allocation</u> is a vector

$$(g,z,x,y) \in R_+^{qK} \times R_+^{\ell K} \times R_+^{\ell n} \times R^{\ell J} ,$$

such that

(1) $(g,x^i) \in X_i$ for all $i \in N$.

(2) $y^j \in \mathcal{Y}_j(g)$ $j = 1,\ldots,J$.

(3) $(g^k, -z^k) \in \mathcal{G}_k(g)$ $k = 1,\ldots,K$.

(4) $\sum_{i\in N} x^i + \sum_{k=1}^{K} z^k - \sum_{j=1}^{J} y^j \leq \sum_{i\in N} w^i$.

The set of all attainable allocations is denoted by A.

<u>Definition 4:</u> Let α be a tax scheme for the economy $\mathcal{E}$. A <u>public competitive quasi equilibrium</u> (p.c.q.e) relative to $(\mathcal{E},\alpha)$ consists of an attainable allocation (g,z,x,y) and a price system $p \in \Delta$ such that:

($\mathcal{E}$.1) For all firms $j = 1,\ldots,J$, y^j maximizes $p\tilde{y}^j$ over $\tilde{y}^j \in \mathcal{Y}_j(g)$.

($\mathcal{E}$.2) For all individual $i \in N$, $px^i \leq pw^i + \sum_{j=1}^{J} \theta_{ij} py^j - \alpha_i(g,z,x,y,p)$.

($\mathcal{E}$.3) For all $i \in N$, $(g,\tilde{x}^i) \in P_i(g,x^i) \cap X_i$ implies

$$p\tilde{x}^i > px^i \text{ and/or } \alpha_i(g,z,x,y,p) = v^i(g,z,x,y,p) .$$

($\mathcal{E}$.4) For every jurisdiction $k = 1,\ldots,K$, $\sum_{i\in B_k} \alpha_i(g,z,x,y,p) = pz^k$.

($\mathscr{E}$.5) For every $k = 1,\ldots,K$, with $\sum_{i\in B_k} pw^i + \sum_{i\in B_k}\sum_{j=1}^{J}\theta_{ij}py^j > 0$ there does not exist $(\overline{g}^k,\overline{z}^k,\{\overline{x}^i\}_{i\in B_k})$ with $(\overline{g}^k, -\overline{z}^k) \in \mathscr{Y}_k(g)$, $p\overline{z}^k \leq \sum_{i\in B_k} pw^i +$

$$+ \sum_{i\in B_k}\sum_{j=1}^{J}\theta_{ij}py^j - \sum_{i\in B_k} p\overline{x}^i$$

and $(\overline{g},\overline{x}^i) \in P_i(g,x^i) \cap X_i$, for all $i \in B_k$,(where $\overline{g}^\ell = g^\ell$ for $\ell \neq k$.)

<u>Definition 5</u>: A <u>public competitive equilibrium</u> (p.c.e.) relative to $(\mathscr{E},\alpha)$ is a p.c.q.e. relative to $(\mathscr{E},\alpha)$ such that for all $i \in N$, $(g,\tilde{x}^i) \in P_i(g,x^i) \cap X_i$ implies $p\tilde{x}^i > px^i$, and ($\mathscr{E}$.5) holds for all $k = 1,\ldots,K$, including those with $\sum_{i\in B_k} pw^i + \sum_{i\in B_k}\sum_{j=1}^{J}\theta_{ij}py^j = 0$.

<u>Definition 6</u>: (i) An attainable allocation (g,z,x,y) is called a <u>Pareto optimum for B_k</u> if there is no other attainable allocation $(\overline{g},\overline{z},\overline{x},\overline{y})$ with $\overline{y} = y$, $\overline{z}^\ell = z^\ell$, $\overline{g}^\ell = g^\ell$ $\forall$ $\ell \neq k$ such that $(\overline{g},\overline{x}^i) \in P_i(g,x^i) \cap X_i$ $\forall$ $i \in B_k$.

(ii) Let $B = \{B_1,\ldots,B_K\}$ be a jurisdiction structure. An allocation (g,z,x,y) is called a <u>Pareto optimum</u> relative to B if it it Pareto optimum for each $B_k \in B$.

Since jurisdiction k controls only the production of its public goods (g^k, z^k) and the taxes it levies on its residents (and thus $\{x^{i}\}$, $i \in B_k$), we required $\bar{y} = y$ and for $\ell \neq k$, $\bar{z}^\ell = z^\ell$, $\bar{g}^\ell = g^\ell$. As the preference ordering P_i is not necessarily either transitive or complete we cannot have a stronger notion of Pareto optimality. However, if the preference ordering of each $i \in N$ can be represented by a utility function u^i, we can define:

<u>Definition 7:</u> (i) An attainable allocation (g,z,x,y) is a <u>strong Pareto optimum for B_k</u> if there exists no other attainable allocation $(\bar{g},\bar{z},\bar{x},\bar{y})$ with $\bar{y} = y$, $\bar{z}^\ell = z^\ell$, $\bar{g}^\ell = g^\ell$ $\forall$ $\ell \neq k$ such that $u^i(\bar{g},\bar{x}^i) \geq u^i(g,x^i)$ $\forall$ $i \in B_k$ and $u^i(\bar{g},\bar{x}^i) > u^i(g,x^i)$ for some $i \in B_k$.

(ii) Let $B = \{B_1,\ldots,B_K\}$ be a jurisdiction structure. An allocation (g,z,x,y) is a <u>strong Pareto optimum</u> relative to B if it is a strong Pareto optimum for each $B_k \in B$.

By "(strong) Pareto optimum" we shall mean Pareto optimum and strong Pareto optimum if each individual's preferences can be represented by a utility function defined over his consumption set. From Definition 5, using ($\mathcal{g}$.5) and Assumption (a.3) every p.c.e. allocation is a (strong) Pareto optimum relative to B.

<u>Assumptions</u>:

For each agent $i \in N$,

(a.1) X_i is a closed and convex subset of $R_+^{qK} \times R_+^\ell$ with $(g,0) \in X_i$ for all $g \in R_+^{qK}$. 4/

(a.2) P_i has an open graph in $X_i \times R_+^{qK+\ell}$, and (a.3) $P_i(g,x^i)$ is convex and $(g,x^i) \in \text{Bdry}\,\{P_i(g,x^i) \cap X_i\}$

for each $(g,x^i) \in X_i$. (I.e., local non-satiation).

For each firm $j = 1,\ldots,J$ [and for each jurisdiction $k = 1,,,.K$],

(b.1) $Y_j[G_k]$ is a closed and convex set of $R^\ell [R^q_+ \times R^\ell_-]$,

(b.2) $\mathcal{Y}_j(g)[\mathcal{S}_k(g)]$ is a closed convex set containing 0 for each

$g \in R^{qK}_+$. 5/

(c) The attainable set A is bounded.

[Note that (a.1) and (b.2) imply that $0 \in A$, hence A is nonempty.]

The Main Result:

Theorem: Let B be a partition of N, and $\mathcal{E} = (G_k, \mathcal{S}_k, X_i, w^i, P_i, Y_j, \mathcal{Y}_j, B)$ be an economy which satisfies assumptions (a), (b) and (c). Then, for any tax scheme α, there exists a p.c.q.e. relative to $(\mathcal{E},\alpha)$.

Proof: We apply the theorem of the existence of a quasi equilibrium in abstract economies [4] to our economy $\mathcal{E}$. Except for the local governments, and the fact that we consider quasi-equilibrium rather than an equilibrium, the application is closely related to that in [6].

By (c), there exists a compact and convex set $D \subset R_+^{qK} \times R_+^{\ell K} \times R_+^{\ell n} \times R^{\ell J}$ which contains A in its interior. By interecting the respective projections of D with the sets: X_i, G_k and Y_j we may assume, using (a.1) and (b.1), that these sets are compact and convex.

For each $d = (g,z,x,y) \in D$ and $p \in \Delta$ define:

For all $i \in N$,

$$\mathcal{X}_i(d) = \{\tilde{x}^i \in R_+^\ell \mid (g,\tilde{x}^i) \in X_i\}$$

$$\mathcal{A}_i(d,p) = \{\tilde{x}^i \in R_+^\ell \mid \tilde{x}^i \in \mathcal{X}_i(d) \text{ and }$$
$$p\tilde{x}^i \leq pw^i + \text{Max}\,[0, \sum_{j=1}^{J} \theta_{ij} py^j] - \alpha_i(d,p)\}$$

$$\hat{P}_i(d) = \{\tilde{x}^i \in R_+^\ell \mid (g,\tilde{x}^i) \in P_i(g,x^i) \cap X_i\}$$

For all $j = 1,\ldots,J$,

$$\mathcal{A}_j(d) = \mathcal{Y}_j(g)$$

$$\hat{P}_j(d,p) = \{\tilde{y}^j \in \mathcal{Y}_j(g) \mid p\tilde{y}^j > py^j\}$$

For all $k = 1,\ldots,K$,

$$\mathcal{A}_k(d,p) = \{(\tilde{g}^k,\tilde{z}^k) \in R_+^{q+\ell} \mid (\tilde{g}^k, -\tilde{z}^k) \in \mathcal{G}_k(g) \text{ and }$$
$$p\tilde{z}^k \leq \sum_{i \in B_k} pw^i + \text{Max}\,[0, \sum_{i \in B_k} \sum_{j=1}^{J} \theta_{ij} py^j]\}$$

$$\hat{P}_k(d,p) = \{(\tilde{g}^k,\tilde{z}^k) \in R_+^{q+\ell} \mid \exists \{\bar{x}^i\}_{i \in B_k} \cdot \ni \cdot (\tilde{g},\bar{x}^i) \in P_i(g,x^i) \cap X_i$$
$$\forall\ i \in B_k,\ (\tilde{g}^\ell = g^\ell \quad \ell \neq k), \text{ and}$$

$$p\tilde{z}^k < \sum_{i\in B_k} (pw^i - p\bar{x}^i) + \sum_{i\in B_k} \sum_{j=1}^{J} \theta_{ij} py^j\}$$

And,

$$\mathcal{P}_o(d) \equiv \Delta$$

$$\hat{P}_o(d,p) = \{\tilde{p} \in \Delta \,|\, \tilde{p}(\sum_{k=1}^{K} z^k + \sum_{i\in N} x^i - \sum_{j=1}^{J} y^j - \sum_{i\in N} w^i) >$$

$$p(\sum_{k=1}^{K} z^k + \sum_{i\in N} x^i - \sum_{j=1}^{J} y^j - \sum_{i\in N} w^i)\}.$$

<u>Lemma 1</u>: Under assumptions (a), (b) and (c), there exists a $d^* = (g^*, z^*, x^*, y^*) \in D$ and a $p^* \in \Delta$ such that:

For all $i \in N$,

(L.1) $x^{*i} \in \mathcal{P}_i(d^*, p^*)$

(L.2) $\hat{P}_i(d^*) \cap \mathcal{P}_i(d^*, p^*) = \phi$ and/or $\alpha_i(d^*, p^*) = v^i(d^*, p^*)$

For all $j = 1, \ldots, J$

(L.3) $y^{*j} \in \mathcal{P}_j(d^*)$

(L.4) $\hat{P}_j(d^*, p^*) \cap \mathcal{P}_j(d^*) = \phi$.

For all $k = 1, \ldots, K$

(L.5) $(g^{*k}, z^{*k}) \in \mathcal{P}_k(d^*, p^*)$

(L.6) $\hat{P}_k(d^*, p^*) \cap \mathcal{P}_k(d^*, p^*) = \phi$ and/or $\sum_{i\in B_k} p^* w^i + \sum_{i\in B_k} \sum_{j=1}^{J} \theta_{ij} p^* y^{*j} = 0.$

And

(L.7) $p^* \in \mathcal{Q}_o(d^*,p^*)$

and

(L.8) $\mathcal{Q}_o(d^*,p^*) \cap \hat{P}_o(d^*,p^*) = \emptyset$.

We shall present the proof of this lemma after completing the proof of the theorem. We shall now show that (L.1) - (L.8) imply that (d^*,p^*) is a p.c.q.e. relative to $(\mathcal{E},\alpha)$.

By (L.3) and (L.4), ($\mathcal{E}$.1) holds. Using assumption (b.2), we have $p^*y^{*j} \geq 0$ for all $j = 1,\ldots,J$, hence by (L.1), ($\mathcal{E}$.2) is satisfied. ($\mathcal{E}$.3) is implied directly by (L.2). By (L.5) and the definition of a tax scheme ($\mathcal{E}$.4) holds. Using (L.1) and summing over all $i \in N$, we get:

$$p^*\left(\sum_{i\in N} x^{*i} + \sum_{k=1}^{K} z^{*k} - \sum_{j=1}^{J} y^{*j} - \sum_{i\in N} w^i\right) \leq 0$$

which together with (L.8), (L.1), (L.3) and (L.5) imply that d^* is an attainable allocation (i.e., $d^* \in A$). It is left to be shown that ($\mathcal{E}$.5) holds as well.

Suppose,

$$(*) \qquad \sum_{i\in B_k} p^*w^i + \sum_{i\in B_k} \sum_{j=1}^{J} \theta_{ij} p^*y^{*j} > 0 \quad .$$

By (L.6) if $(\bar{g},\bar{x}^i) \in P_i(g^*,x^{*i}) \cap X_i \quad \forall\ i \in B_k(\bar{g}^\ell = g^{*\ell}\ \forall\ \ell \neq k)$, and $(\bar{g}^k, -\bar{z}^k) \in \mathcal{Y}_k(g^*)$, then:

$$(**)\quad p^{*}\bar{z}^{k} \geq \sum_{i\in B_k} (p^{*}w^{i} - p^{*}\bar{x}^{i}) + \sum_{i\in B_k} \sum_{j=1}^{J} \theta_{ij} p^{*} y^{*j}$$

To establish ($\mathcal{G}$.5) we have to show that equality in (**) cannot prevail. By (b.2), for all $0 \leq \lambda \leq 1$ $(\lambda \bar{g}^{k}, \lambda \bar{z}^{k}) \in \mathcal{T}_k(d^{*}, p^{*})$. By (a.1) and (a.2), for $\lambda < 1$ large enough, $(\bar{g}^{1},\ldots,\lambda \bar{g}^{k},\ldots,\bar{g}^{K},\lambda \bar{x}^{i}) \in P_i(g^{*},x^{*i}) \cap X_i$ $\forall$ $i \in B_k$. Using again (L.6), together with (*):

$$p^{*}\lambda \bar{z}^{k} \geq \sum_{i\in B_k} (p^{*}w^{i} - p^{*}\lambda \bar{x}^{i}) + \sum_{i\in B_k} \sum_{j=1}^{J} \theta_{ij} p^{*} y^{*j}$$

$$> \lambda \left[\sum_{i\in B_k} (p^{*}w^{i} - p^{*}\bar{x}^{i}) + \sum_{i\in B_k} \sum_{j=1}^{J} \theta_{ij} p^{*} y^{*j} \right]$$

Hence,

$$p^{*}\bar{z}^{k} > \sum_{i\in B_k} (p^{*}w^{i} - p^{*}\bar{x}^{i}) + \sum_{i\in B_k} \sum_{j=1}^{J} \theta_{ij} p^{*} y^{*j} .$$

Q.E.D.

By the standard technique of adding plausible assumptions every p.c.q.e. is a p.c.e.; however, the proof of existence of such equilibria cannot be derived directly but rather via the existence of quasi equilibria. In all the above three examples, (α.1) - (α.3), it is possible (at least a-priori) that $\alpha_i = pw^{i}$. Moreover, even if we assume that an individual must have a positive after tax income, (so that he may be in the interior of his consumption set), there can be no direct proof for the existence of a proportional tax system (example (α.1)) since, though the proportion must be less than 1, it can be arbitrarily close to 1. On the other hand, there exists no $0 < \epsilon < 1$, such that the solution can be restricted to lie in $[0,\epsilon]$.

Corollary: Let $\mathcal{E}$ be an economy which satisfies assumptions (a), (b), (c) and either:

(d.1) monotonicity of preferences: For all $i \in N$ and $(g,x^i) \in X_i$, $P_i(g,x^i) \supset P_i(g,x^i) + R_+^{qK+\ell}$

or 6/

(d.2) For all $k = 1,\ldots,K$, $\sum_{i \in B_k} w^i >> 0$.

Then, every p.c.q.e. allocation, for any tax scheme, is a (strong) Pareto optimum.

Proof of Lemma 1:

Lemma 1 is a direct application of the Theorem in [4]. The following lemma guarantees the continuity of $\mathcal{X}_i, i \in N$:

Lemma 2: Let $H = \{(a,b)\}$ be a compact convex subset of R^{s+t} such that H^s is a polytope. Then, the correspondence $h: R^s \twoheadrightarrow R^t$, defined by:

$h(a) = \{b \in R^t \mid (a,b) \in H\}$ is a continuous correspondence in H^s.

(H^s denotes the projection of H on R^s).

Proof:

(i) Since H is closed, $h(a)$ is upper semi continuous .

(ii) Let $a^k \to \bar{a}, \bar{b} \in h(\bar{a})$. We have to find a sequence b^k such that $b^k \to \bar{b}, b^k \in h(a^k)$ for all k. Let $R(\bar{a},a^k)$ denote the ray originating at $\bar{a}$ and passing through a^k, and let r^k be the furthest point on this ray which belongs to H^s, i.e., for all $r \in R(\bar{a},a^k)$: either $r \in [\bar{a},r^k]$ or else $r \notin H^s$. (Since H^s is compact r^k is well-defined.) Clearly, $a^k \in [\bar{a},r^k]$, thus $a^k = \lambda_o^k \bar{a} + (1-\lambda_o^k)\, r^k$, $0 \leq \lambda_o^k \leq 1$, and as H^s is a polytope. $a^k \to \bar{a}$ if and only if $\lambda_o^k \to 1$. (If $r^k = \bar{a}$, then $a^k = \bar{a}$ and we choose $\lambda_o^k = 1$. In this case, the proof of (ii) is trivial.) Let $\{\alpha^1, \ldots, \alpha^m\}$ be the extreme points of H^s, hence there exist non-negative numbers $\{\delta_i^k\}$ $i = 1,\ldots,m$, $\sum_{i=1}^m \delta_i^k = i$, such that $r^k = \sum_{i=1}^m \delta_i^k \alpha^i$. (These numbers need not be unique.)

Defining $\lambda_i^k = (1 - \lambda_o^k)\delta_i^k$, $i = 1,\ldots,m$, we have:

$$a^k = \lambda_o^k \bar{a} + \sum_{i=1}^m \lambda_i^k \alpha^i .$$

$\alpha^i \in H^s$ implies $(\alpha^i,\beta^i) \in H, i = 1,\ldots,m$. By the convexity of H,

$x^k \equiv \lambda_o^k(\bar{a},\bar{b}) + \sum_{i=1}^m \lambda_i^k(\alpha^i,\beta^i) \in H$. Define, $b^k \equiv \lambda_o^k \bar{b} + \sum_{i=1}^m \lambda_i^k \beta^i$

then, $x^k = (a^k,b^k)$ and hence $b^k \in h(\alpha^k)$. As $a^k \to \bar{a}$, $\lambda_o^k \to 1$ and therefore $b^k \to \bar{b}$.

Q.E.D.

To conclude that $\chi^i(d)$ is continuous, we choose the compact set D (which contains the attainable set in its interior) to be a cube, and using assumption (a.1), namely that for all $g \in D, (g,0) \in X_i$, H^s corresponds to the qK-dimensional cube.

Remark 1:*/ The condition that H^s is a polytope is necessary. Let H^s be the unit sphere in R^2, $\bar{a} = (1,0)$, $\bar{b} = 1$, and $H = \text{conv}\{(H^s,0),(1,0,1)\} \subset R^3$. Define $a^k = ((1 - \frac{1}{k^2})^{\frac{1}{2}}, \frac{1}{k})$ $k = 1,2,\ldots$. Clearly, $a^k \to \bar{a}$, $a^k \in H^s$. However, $h(a^k) = 0 \;\forall\, k$, therefore $b^k \equiv 0 \neq \bar{b}$.

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*/ This example was suggested to me by E. Kalai.

Denote: $T = \{k = 1,\ldots,K;\ j = 1,\ldots,J;\ i \in N;\ 0\}$. By (a.1) and the definition of a tax scheme, $\mathcal{O}_i(d,p) \neq \emptyset$ $i \in N$. Assumption (b.2) implies that $\mathcal{O}_k(d,p)$ and $\mathcal{O}_j(d,p)$ are nonempty. Thus $\mathcal{O}_t(d,p) \neq \emptyset$ $\forall (d,p) \in D \times \Delta$ and $\forall\ t \in T$. Clearly, $\mathcal{O}_t(d,p)$ and $\hat{P}_t(d,p)$ are convex for all $(d,p) \in D \times \Delta$ and $\forall\ t \in T$. Since $\mathcal{O}_j(d,p)$ and $\mathcal{O}_o(d,p)$ are continuous correspondences, in order to apply the result in [4] it is left to be shown that:

For all $i \in N$ $[k=1,\ldots,K]$, $\mathcal{O}_i(d,p)$ $[\mathcal{O}_k(d,p)]$ is a continuous correspondence for all $(d,p) \in D \times \Delta$ with $v^i(d,p) > \alpha_i(d,p)$ $[\sum_{i\in B_k} pw^i + \text{Max}\,[0, \sum_{i\in B_k}\sum_{j=1}^{J} \theta_{ij}py^j] > 0]$ and is upper semi-continuous for all $(d,p) \in D \times \Delta$ with $v^i(d,p) = \alpha_i(d,p)$ $[\sum_{i\in B_k} pw^i + \text{Max}[0, \sum_{i\in B_k}\sum_{j=1}^{J} \theta_{ij}py^j] = 0]$.

This is an immediate corollary of

<u>Lemma 3:</u> Let D be a compact subset of R^t_+ and $\phi: D \rightarrow\!\!\!\!\rightarrow R^s_+$ be a continuous correspondence such that for all $d \in D$, $\phi(d)$ is nonempty compact and convex. Denote: $S = \{1,\ldots,s\}$, $\hat{S} \subset S$, and $\Pi = \{\pi \in R^s_+ \mid \sum_{i\in\hat{S}} \pi_i = 1,\ \pi_i = 0\ \forall\ i \notin \hat{S}\}$. Let f be a continuous function $f: D \times \Pi \rightarrow R_+$ such that $f(d,\pi) > 0$ implies that $\exists\ \tilde{c} \in \phi(d)$ with $\pi\tilde{c} < f(d,\pi)$. Then the correspondence $\mathcal{O}: D \times \Pi \rightarrow\!\!\!\!\rightarrow R^s_+$ given by: $\mathcal{O}(d,\pi) = \{c \in R^s_+ \mid c \in \phi(d) \text{ and } \pi c \leq f(d,\pi)\}$

is upper semi continuous in $D \times \Pi$ and is lower semi-continuous for all $(d,p) \in D \times \Pi$ with $f(d,\pi) > 0$.

Proof:

(i) Upper semi continuity: Let $(d^k,\pi^k) \to (\bar{d},\bar{\pi}), c^k \in \mathcal{A}(d^k,\pi^k)$, and $c^k \to \bar{c}$. I.e., $\pi^k c^k \leq f(d^k,\pi^k)$ and $c^k \in \phi(d^k)$, $k = 1,2,\ldots$. Since D is closed and ϕ is upper semi continuous, $\bar{c} \in \phi(\bar{d})$, and since f is continuous,

$$\bar{\pi}\,\bar{c} \leq f(\bar{d},\bar{\pi}). \text{ Hence } \bar{c} \in \mathcal{A}(\bar{d},\bar{\pi}).$$

(ii) Lower semi continuity: Let $(d^k,\pi^k) \to (\bar{d},\bar{\pi}), \bar{c} \in \mathcal{A}(\bar{d},\bar{\pi})$.

We have to find a sequence c^k such that

$$c^k \to \bar{c} \text{ and } c^k \in \mathcal{A}(d^k,\pi^k) \text{ for all } k = 1,2,\ldots \text{ .}$$

$\bar{c} \in \mathcal{A}(\bar{d},\bar{\pi}) \Rightarrow \bar{c} \in \phi(\bar{d})$ and $\pi\bar{c} \leq f(\bar{d},\bar{\pi})$. Distinguish the two cases:

I. $\bar{\pi}\,\bar{c} < f(\bar{d},\bar{\pi})$:

Since ϕ is lower semi continuous, $\exists$ $c^k \to \bar{c}$ with $c^k \in \phi(d^k)$. As $c^k \to \bar{c}$ and $\bar{\pi}\,\bar{c} < f(\bar{d},\bar{\pi})$, for k large enough, $\pi^k c^k < f(d^k,\pi^k)$. Hence, $c^k \in \mathcal{A}(d^k,\pi^k)$.

II. $\bar{\pi}\,\bar{c} = f(\bar{d},\bar{\pi}) > 0.$

By assumption, there exists $\tilde{c} \in \phi(\bar{d})$ with:

(1) $\bar{\pi}\,\tilde{c} < f(\bar{d},\bar{\pi})$ and thus $\bar{\pi}\,\tilde{c} < \bar{\pi}\,\bar{c}$.

Since ϕ is lower semi continuous, there exist:

(a) $\{c^k\}$ s.t. $c^k \to \bar{c}$ and $c^k \in \phi(d^k)$ and,

(b) $\{\tilde{c}^k\}$ s.t. $\tilde{c}^k \to \tilde{c}$, $\tilde{c}^k \in \phi(d^k)$.

By (1),(a) and (b), for k large enough: $(k \geq \bar{k})$

(2) $\pi^k \tilde{c}^k < f(d^k,\pi^k)$ and $\pi^k \tilde{c}^k < \pi^k c^k$.

For $k \leq \bar{k}$, define $\hat{c}^k = \tilde{c}^k$. By (2) and (b) $\hat{c}^k \in \sigma(d^k,\pi^k)$.
For $k \geq \bar{k}$, define $\hat{c}^k = \lambda^k c^k + (1-\lambda^k)\tilde{c}^k$, where

$$\lambda^k = \underset{0 \leq \lambda \leq 1}{\text{Max}} \; [\lambda \,|\, \lambda \pi^k c^k + (1-\lambda)\pi^k \tilde{c}^k \leq f(d^k,\pi^k)].$$

(By (2), λ^k exists and is unique).

By (a), (b) and the convexity of $\phi(d^k)$, $\hat{c}^k \in \phi(d^k)$. Moreoever, since

$\bar{\pi}\,\bar{c} = f(\bar{d},\bar{\pi})$ and f is continuous, $\lambda^k \to 1$. By (a), therefore $\hat{c}^k \to \bar{c}$. Clearly, $\hat{c}^k \in \sigma(d^k,\pi^k)$ for all k.

Q.E.D.

FOOTNOTES

1/ In particular:

For all $i \in N$, $X_i = R_+^{q+\ell}$ and $w^i \gg 0$,

The preference ordering can be represented by a strictly concave utility function which is strictly monotonic increasing ,

The (unique) technology set is a convex cone containing the origin and is "round" (i.e., "the supportung plane through any boundary point is unique up to a constant of proportionality").

2/ The usual definition of local public goods is: "goods which cannot readily be supplied and priced on a variable unit-of-service basis, but which are best provided communally at the same level to all members of the association, in such a way that nonmembers are excluded from enjoying them altogether." (M. McGuire, "Group Segregation and Optimal Jursidictions," Journal of Political Economy, 82, (1974), 112-132.) This is the definition used by Richter. We, however, allow for the case where nonmembers cannot be excluded from consumption . The "locality" of these goods is, therefore, defined only via the production (and financing) and not necessarily via their consumption, as well. In other words, in this paper spillovers of local public goods are allowed.

3/ In particular:

For all $i \in N$, $X_i = R_+^{q+\ell}$ (no spillovers are allowed),

The preference ordering can be represented by a continuous utility function which is strictly monotonic increasing in each coordinate ,

The (unique) technology set is a convex cone containing the origin, and public goods do not enter into production of private goods,

For all $k = 1,\ldots,K$, $\sum_{i \in B_k} w^i \gg 0$.

4/ Note that $X^i = R_+^{qK+\ell}$ implies (a.1).

5/ Assumptions (b.1) and (b.2) are implied by:

b.2') $Y_j[G_k]$ is a closed convex set of $R^{qK} \times R^{\ell}$

$[R^{qK} \times R_+^q \times R_-^{\ell}]$, with $(g,0) \in Y_j\ [G_k]$ for all $g \in R_+^{qK}$.

For a proof see lemma 2.

6/ Foley as well as Richter assume both (d.1) and (d.2) (see footnotes 1 and 3 above).