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EQUITABILITY IN MULTI-AGENT DYNAMIC SYSTEMS

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Abstract

A desirable feature of a socio-economic system is that it be forgiving of the past mistakes and misfortunes of an individual, and always present to him the opportunity to rise and compete, if he will but apply himself. In this paper we pose a mathematical definition of equitability — a property intended to capture the afforementioned forgiveness feature — and we examine the structure it imposes on several types of Markovian systems. It is found that equitability does not imply that the system be insensitive to the agents' desires (i.e., centrally run), nor does it imply that one agent's actions must not influence the welfare of the others.

EQUITABILITY IN DYNAMIC SYSTEMS

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I. Introduction

The economic well-being of an individual at some time in the future can often be anticipated accurately through knowledge of his current economic status. A man with half a million dollars in blue chip securities now and no large debts will probably be well-off for the rest of his days, while an uneducated migrant farm laborer having nothing now will likely continue with nothing. Wealth usually enables an individual to take advantage of special opportunities such as sudden bargains at the grocery store or in the stock market, and it enables one to economize through bulk purchases. Further, a wealthy individual can generally acquire useful information, control factors relating to his future wealth, and plan his economic activities well in advance of their execution. Thus individual wealth tends to preserve and even enhance itself in a process not unlike that ascribed by Galbraith [1] to the growth of giant corporations.

Poverty, on the other hand, breeds poverty. A poor person cannot save for the future because all must be spent now for his survival. He takes advantage of few bargains, as his expenditures each day are closely dictated by the requirement that he live. Information about economic opportunities, if it is obscure or expensive, cannot be sought out and purchased by the poor; their poverty perpetuates itself.

Socialistic economic systems deal with the difficulty by making it improbable that an individual's state will deteriorate beyond a certain point, but there is the danger that such systems may do little to make it possible for an individual of small means to rise to the higher levels of economic well-being: a potential down-and-outer may be trapped in the down state, and permitted neither to rise nor go out. Communistic systems offer another type of relief: because property is largely communally held, individuals tend to be relatively no better nor worse off than others around them.

There exists the possibility that a system can be devised in which the industrious will be rewarded, and in which poverty will not be eternally binding, nor wealth automatically perpetuated. We shall term such a system "equitable", and offer a more precise definition in the next section. Given a central planner able to surmount the considerable information processing difficulties, one can envision an equitable system in which individuals are constantly monitored, carefully exposed to opportunities, and subsequently rewarded. Of interest in the present paper is the possibility that an equitable, decentralized system can be constructed. The decentralization places much of the information processing burden on individuals, and it permits them wider lattitude in making decisions.

In pragmatic terms, it seems that a decentralized equitable system would feature programs to educate and employ the poverty stricken, and to supply outright subsistence support in a manner which prepares and encourages an individual to raise himself beyond need of it. Further, heavy estate and property (as opposed to income) taxes would serve to keep individuals from remaining wealthy without their renewed effort.

In the present paper our task is to take the first timid step toward making equitability a precise concept. We shall define equitability for a simple abstract dynamic system, and explore what implications equitability has on the structure of the system. It is our explicit purpose to determine if there exist equitable structures which are not wholly centralized, and which are not merely a collection of non-interacting, single-agent sub-structures.

II. Definitions and Notation

We shall analyze initially a discrete time, two-agent, four-state system with regard to its equitability, leaving to the appendix the outline of the m-agent, n^m-state analysis. The set of possible states of the system is $\overline{X} = \{X_1, X_2, X_3, X_4\}$, while the state of the system at time n is denoted x^n.

The agents, A_1 and A_2, have preference orderings over the states as follows:

A_1 prefers X_1 to X_2 and is indifferent between X_1 and X_3;
X_3 to X_4 X_2 and X_4

A_2 prefers X_1 to X_3 and is indifferent between X_1 and X_2.
X_2 to X_4 X_3 and X_4

These preferences are illustrated in Figure 1.

At each time n, each agent makes a binary choice which influences the future states of the system. In general, his choice will depend upon the current system state, the time n, and an exogenous random event.

Thus the i^{th} agent's choice is represented as

$$u_i^n(x^n) = \begin{cases} U_1 \text{ with probability } q_i^n(x^n) \\ U_0 \text{ with probability } 1-q_i^n(x^n), \ i=1,2. \end{cases}$$

In response to the decisions of the agents at time n, the system moves probabilistically to a new state by time n+1. The transition probabilities are the entries of one of four matrices $\Pi^1, \Pi^2, \Pi^3, \Pi^4$ characterizing the system:

If (A_1, A_2) choose (U_1, U_1), the transition matrix is Π^1

(U_o, U_1), " " " " Π^2

(U_1, U_o), " " " " Π^3

(U_o, U_o), " " " " Π^4 .

Each Π^i is a 4x4 matrix of probabilities. The jkth entry of Π^i is π^i_{jk}, the probability that if the system is in state X_k and if Π^i is selected by the agents, that the next state of the system will be X_j. Hence the elements of each column of Π^i sum to 1, and each element is nonnegative and no greater than 1. The association between the agents' actions and the Π^i matrix selected is depicted in Figure 2, while the significance of the Π^i matrix entries is suggested in Figure 3.

The probability that at time n=0 the system is in the state X_i is denoted p^o_i, and P^o is defined as

$$P^o = \begin{pmatrix} p^o_1 \\ p^o_2 \\ p^o_3 \\ p^o_4 \end{pmatrix} .$$

Given the initial vector P^o, we let P^n be the vector of probabilities with which the system is in its possible states at time n. As p_i^n, the i^{th} component of P^n, depends on the functions q_j^n, we should perhaps indicate the dependence thus: $p_i^n(q_1^o,\ldots,q_1^{n-1};\ q_2^o,\ldots,q_2^{n-1})$. Instead, we shall use the briefer $p_i^n(q_1;q_2)$ to call attention to the dependence of p_i^n on the q_i^n.

As time passes, the system moves from state to state. In state X_1, both agents are well-off, while in state X_4 neither agent is in a state he prefers. In states X_2 and X_3, one agent is satisfied while the other is not. We define equitability in terms of the relative frequencies with which the agents are able to direct the system to states they prefer. According to our definition, Agent A_1 should be able, if the system is equitable, to force the system to reside in states X_1 or X_3 (his preferred states) at least as often as it resides in states X_1 or X_2 (A_2's preferred states). It should be possible to do this regardless of whether A_2 is making decisions selfishly, maliciously or ludicrously. Similarly, if the system is equitable, A_2 should be able to direct the system to assume his preferred states at least as often as it assumes those preferred by A_1. Specifically, we define equitability of the two-agent four-state system as follows:

Definition: The system is equitable iff there exist sequences of functions $(\hat{q}_i^n)_{n=o}^{\infty}$, i=1,2, such that $\hat{q}_i^n$: $\underline{X}\to[0,1]$ and for any $\varepsilon > 0$, any P^o, and any functions $(q_i^n)_{n=o}^{\infty}$, i=1,2, mapping $\underline{X}$ into $[0,1]$, both of the following inequalities hold for n sufficiently large:

$$\frac{1}{n}\sum_{m=0}^{n-1}\left[p_1^m(\hat{q}_1;q_2)+p_3^m(\hat{q}_1;q_2)\right] \geq \frac{1}{n}\sum_{m=0}^{n-1}\left[p_1^m(\hat{q}_1;q_2)+p_2^m(\hat{q}_1;q_2)\right] - \varepsilon$$

$$\frac{1}{n}\sum_{m=0}^{n-1}\left[p_1^m(q_1;\hat{q}_2)+p_2^m(q_1;\hat{q}_2)\right] \geq \frac{1}{n}\sum_{m=0}^{n-1}\left[p_1^m(q_1;\hat{q}_2)+p_3^m(q_1;\hat{q}_2)\right] - \varepsilon.$$

The $\hat{q}_i^n$ are called equitable probability functions.
The following theorem is an obvious variation of the definition, and it provides a more compact set of conditions for equitability.

Theorem I. The system is equitable iff there exist sequences of functions $(\hat{q}_i^n)_{n=0}^{\infty}$, $i=1,2$, such that $\hat{q}_i^n$: $\bar{X} \to [0,1]$ and such that for any $\varepsilon > 0$, any P^o, and any functions $(q_i^n)_{n=0}^{\infty}$, $i=1,2$, mapping $\bar{X}$ into $[0,1]$, both of the following inequalities hold for n sufficiently large:

$$\frac{1}{n}\sum_{m=0}^{n-1}\left[p_3^m(\hat{q}_1;q_2) - p_2^m(\hat{q}_1;q_2)\right] \geq -\varepsilon$$

$$\frac{1}{n}\sum_{m=0}^{n-1}\left[p_2^m(q_1;\hat{q}_2) - p_3^m(q_1;\hat{q}_2)\right] \geq -\varepsilon.$$

A necessary condition for equitability follows at once:
Corollary I. If the system is equitable, then for any P^o, the equitable probability functions satisfy

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}\left[p_2^m(\hat{q}_1;\hat{q}_2) - p_3^m(\hat{q}_1;\hat{q}_2)\right] = 0.$$

III. The Autonomous Case

A special case of the two-agent four-state system is that in which the Π^i are identical to each other. In this, the autonomous case, the system evolves independently of the decisions of the agents: it is rather like a centrally planned economy in which one is free to decide as he wishes as long as his actions conform with the dictates of the central planner.

In the autonomous case, p_i^n does not depend on $q_1^o, \ldots, q_1^{n-1}$; $q_2^o, \ldots, q_2^{n-1}$ and Theorem I takes the form:

Corollary II. The autonomous system is equitable iff for any p^o,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=o}^{n-1} (p_2^m - p_3^m) = 0.$$

Hence for an autonomous system, the necessary condition of Corollary I is also sufficient.

For many Π matrices which might characterize an autonomous system, p^n will approach a limit as n becomes large, and this limit will be independent of p^o.

Definition: A transition matrix Π is p^o-independent iff

$$L = \lim_{n\to\infty} \mathop{\times\!\!\times}_{m=o}^{n} \Pi$$

exists, and the columns of L are identical to each other.

An autonomous system with a P^o-independent matrix is called a P^o-independent autonomous system. We use ℓ to denote the typical column of the matrix L associated with such a system.

Lemma I. If Π is P^o-independent, then

i) $$L = \lim_{n\to\infty} \frac{1}{n} \sum_{m=o}^{n-1} \prod_{k=o}^{m} \Pi$$

ii) $\lim\limits_{n\to\infty} P^n = \ell$, independent of P^o.

The condition of Corollary II thus becomes:

Corollary III. If the autonomous system is P^o-independent, it is equitable iff $\ell_2 = \ell_3$.

To determine if $\ell_2 = \ell_3$ for a given P^o-independent Π matrix one may use Proposition I, in which the following variables are cited:

$$D_2 = -\begin{vmatrix} \pi_{21} & \pi_{23} & \pi_{24} \\ \pi_{31} & \pi_{33}-1 & \pi_{34} \\ \pi_{41} & \pi_{43} & \pi_{44}-1 \end{vmatrix}, \qquad D_3 = \begin{vmatrix} \pi_{21} & \pi_{22}-1 & \pi_{24} \\ \pi_{31} & \pi_{32} & \pi_{34} \\ \pi_{41} & \pi_{42} & \pi_{44}-1 \end{vmatrix}$$

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \pi_{21} & \pi_{22}-1 & \pi_{23} & \pi_{24} \\ \pi_{31} & \pi_{32} & \pi_{33}-1 & \pi_{34} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44}-1 \end{vmatrix}$$

Proposition I. $D \leq D_2 + D_3 \leq 0$. If Π is P^o-independent then $D < 0$ and $\ell_i = D_i/D$, $i=2,3$.

For autonomous systems, the property of P^o-independence neither implies nor precludes equitability, as is demonstrated in

IV. The Constant Strategy Case

Relaxing the restriction on the previous section that the Π^i be identical, we impose the looser restriction that the strategies, by law or special agreement, cannot vary with time. Thus

$$q_i^n(x^n) = q_i(x^n), \qquad n = 0,1,\ldots,; \qquad i = 1,2.$$

The expression for p^n in this situation is

$$p^n(q_1;q_2) = \prod_{m=0}^{n-1} \Pi(q_1;q_2)\ P^0$$

where

$$\Pi(q_1;q_2) = \Pi^1K^1 + \Pi^2K^2 + \Pi^3K^3 + \Pi^4K^4$$

$$K^1 = Q^1Q^2 \quad ; \quad K^2 = (I-Q^1)Q^2$$

$$K^3 = Q^1(I-Q^2) \quad ; \quad K^4 = (I-Q^1)\ (I-Q^2)$$

$$Q^1 = \begin{pmatrix} q_{11} & 0 & 0 & 0 \\ 0 & q_{12} & 0 & 0 \\ 0 & 0 & q_{13} & 0 \\ 0 & 0 & 0 & q_{14} \end{pmatrix}, \quad Q^2 = \begin{pmatrix} q_{21} & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 \\ 0 & 0 & q_{23} & 0 \\ 0 & 0 & 0 & q_{24} \end{pmatrix}$$

I = 4x4 identity matrix; $q_{ij} = q_i(X_j)$, $i=1,2$; $j=1,2,3,4$.

The assumption that the q_i^n do not depend on n assures that the relative frequency with which the system occupies any given state approaches a limit as n increases without bound.

Proposition II. For any functions $q_i : \bar{X} \to [0,1]$, $i=1,2$, and any P^0, the limits

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} p_j^m(q_1;q_2), \qquad j = 1,2,3,4,$$

exist. (Proof sketch in appendix.)

The values of the limits depend, in general, both on the q_i and on P^0.

In many instances $\Pi(q_1;q_2)$ is P^o-independent regardless of what specific q_1, q_2 functions are employed. A system with such Π^i is called P^o-independent. The following proposition gives conditions necessary to ensure that a system be P^o-independent.

Proposition III. If $\Pi(q_1;q_2)$ is P^o-independent for all q_i: $\bar{X} \to [0,1]$, $i=1,2$, then each Π^j, $j=1,2,3,4$, is P^o-independent. Moreover, each transition matrix Π whose k^{th} column is the k^{th} column of one of the Π^j, $k=1,2,3,4$, is P^o-independent.

If a system is P^o-independent, we let $\ell(q_1;q_2)$ denote the typical column vector of

$$L(q_1;q_2) = \sum_{n=o}^{\infty} \Pi(q_1;q_2),$$

and $\ell_i(q_1;q_2)$ the i^{th} component of $\ell(q_1;q_2)$. Theorem I then becomes:

Corollary IV. A P^o-independent system is equitable for constant strategies iff there exist functions $\hat{q}_i$: $\bar{X} \to [0,1]$, $i=1,2$, such that for any q_i: $\bar{X} \to [0,1]$, $i=1,2$,

$$\ell_3(\hat{q}_1;q_2) - \ell_2(\hat{q}_1;q_2) \geq 0 \geq \ell_2(q_1;\hat{q}_2) - \ell_3(q_1;\hat{q}_2).$$

Defining $D_2(q_1;q_2)$, $D_3(q_1;q_2)$ and $D(q_1;q_2)$ in terms of $\Pi(q_1;q_2)$ in a manner analogous to that in Section III, we can modify Corollary IV to yield a computationally more useful result. First we establish some properties of D_2, D_3 and D:

Lemma II. $D(q_1;q_2) \leq D_2(q_1;q_2) + D_3(q_1;q_2) \leq 0$ for all q_i: $\bar{X} \to [0,1]$. Furthermore, D, D_2 and D_3 are each 8-affine+ in the variables q_{ij}, $i=1,2$; $j=1,2,3,4$.

Defining $\Delta = D_2 - D_3$, we modify Corollary IV to give necessary and sufficient conditions in terms of a saddle point of the 8-affine function Δ:

+A function of n variables is n-affine if it is affine in each variable when the others are held constant.

Corollary V. A P^o-independent system is equitable for constant strategies iff there exist functions $\hat{q}_i:\underline{\overline{X}}\to[0,1]$, $i=1,2$, such that for any q_i: $\underline{\overline{X}} \to [0,1]$, $i=1,2$,

$$\Delta(\hat{q}_1;q_2) \le 0 \le \Delta(q_1;\hat{q}_2).$$

An example of a P^o-independent system which is equitable for constant strategies is:

$$\Pi^1 = \begin{pmatrix} .25 & .25 & .25 & 0 \\ .25 & .5 & 0 & 0 \\ .25 & 0 & .5 & .5 \\ .25 & .25 & .25 & .5 \end{pmatrix} \qquad \Pi^2 = \begin{pmatrix} 0 & .25 & 0 & .25 \\ 1 & .25 & .5 & 0 \\ 0 & .25 & 0 & .25 \\ 0 & .25 & .5 & .5 \end{pmatrix}$$

$$\Pi^3 = \begin{pmatrix} 0 & .5 & .25 & 0 \\ 0 & .5 & .25 & .5 \\ .5 & 0 & .25 & .25 \\ .5 & 0 & .25 & .25 \end{pmatrix} \qquad \Pi^4 = \begin{pmatrix} 1 & 0 & 0 & .25 \\ 0 & 0 & .5 & .25 \\ 0 & .5 & .25 & .25 \\ 0 & .5 & .25 & .25 \end{pmatrix}$$

State	x_1	x_2	x_3	x_4
Value of $\hat{q}_1$	1	0	1	0
Value of $\hat{q}_2$	1	1	0	0

$$\Pi(\hat{q}_1;\hat{q}_2) = \begin{pmatrix} .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \\ .25 & .25 & .25 & .25 \end{pmatrix}$$

$$\ell_2(\hat{q}_1;\hat{q}_2) = \ell_3(\hat{q}_1;\hat{q}_2) = .25.$$

It is of interest to note that the agents may wish to abandon the use of $\hat{q}_1$ and $\hat{q}_2$ in this example in favor of the strategies $\overline{q}_1$ and $\overline{q}_2$ defined thus:

State	x_1	x_2	x_3	x_4
Value of $\overline{q}_1$	0	0	1	0
Value of $\overline{q}_2$	0	1	0	0

Using $\hat{q}_1$, $\hat{q}_2$, the agents cause

$$\Pr\{x^n = X_1 \text{ or } X_3\} = .5; \quad \Pr\{x^n = X_1 \text{ or } X_2\} = .5$$

for $n \geq 1$. Using $\bar{q}_1$, $\bar{q}_2$, however, results in

$$\Pr\{x^n = X_1 \text{ or } X_3\} \geq .5; \quad \Pr\{x^n = X_1 \text{ or } X_2\} \geq .5$$

for $n \geq 1$, and further,

$$\lim_{n\to\infty} \Pr\{x^n = X_1\} = 1.$$

However, the $\bar{q}_1$, $\bar{q}_2$ leave each player vulnerable to an opposing constant strategy for which he would be at a disadvantage.

V. The General Case

At present we have but one proposition to assert about this, the most difficult case to analyze. It is a useful result, however, because it provides a fairly simple condition sufficient to ensure equitability. To facilitate the statement of the proposition, we define $\Pi(q_1^n;q_2^n)$ as before:

$$\Pi(q_1^n;q_2^n) = \sum_{i=1}^{4} \Pi^i K^i$$

where the K^i are as given in the preceding section. In case q_1 takes only the values 0 or 1 as it ranges over $\underline{\bar{X}}$, we define the function $q_1{}^{(j)}$ to be the same as q_1, except that the value of $q_1(X_j)$ is taken as $1 - q_1(X_j)$. For example,

$$\text{if } q_1(x) = \begin{cases} 1, \text{ if } x = X_1 \\ 0, \text{ if } x = X_2 \\ 1, \text{ if } x = X_3 \\ 1, \text{ if } x = X_4 \end{cases}, \quad \text{then } q_1{}^{(3)}(x) = \begin{cases} 1, \text{ if } x = X_1 \\ 0, \text{ if } x = X_2 \\ 0, \text{ if } x = X_3 \\ 1, \text{ if } x = X_4 . \end{cases}$$

The notation $q_2{}^{(j)}$ is similarly defined, for $j=1,2,3,4$. By

$$\left[\Pi(q_1;q_2)\right]_{ij}$$

is meant the ij^{th} element of the matrix in brackets.

Proposition IV. If there exist functions $\hat{q}_1$ and $\hat{q}_2$ from $\underline{X}$ into $\{0,1\}$ satisfying

$$\left[\Pi(\hat{q}_1;\hat{q}_2)\right]_{2j} = \left[\Pi(\hat{q}_1;\hat{q}_2)\right]_{3j}$$

$$\left[\Pi(\hat{q}_1^{(j)};\hat{q}_2)\right]_{2j} \geq \left[\Pi(\hat{q}_1^{(j)};\hat{q}_2)\right]_{3j}$$

$$\left[\Pi(\hat{q}_1;\hat{q}_2^{(j)})\right]_{2j} \leq \left[\Pi(\hat{q}_1;\hat{q}_2^{(j)})\right]_{3j}$$

for $j=1,2,3,4$, then the system is equitable. Furthermore, $\hat{q}_1$ and $\hat{q}_2$ are constant strategies satisfying the conditions of Theorem I. (Proof in appendix.)

The example of the preceding section satisfies the hypothesis of Proposition IV, and hence is not only equitable for constant strategies as asserted there, but equitable.

VI. Discussion

The motivation for studying system equitability comes from the observation that as the earth becomes more populous and our mutual interdependencies multiply, there appear to be increasingly many opportunities for individuals or groups to offend others, intentionally or unintentionally, and to do so with impunity. In bygone days offenses tended to be obvious and brutal, examples being slavery or vigilante law enforcement. Today's offenses are more subtle: many firms pollute the air or water, yet do not pay sufficiently for the priviledge; schools and residential areas practice, by some default-type mechanism in many cases, forms of racial or religious segregation; poor people are unable to break out of their poverty within a reasonable time because the socio-economic system is biased against them. It seems desirable to develop a technical vocabulary and some general relationships concerning these effects, in hopes that thereby recognition and treatment of discriminatory situations will be facilitated.

In developing such a vocabulary and theory, one runs immediately into a stumbling block: the problem of interpersonal utility comparisons. Our treatment is founded on the somewhat heavy-handed assumption that on at least some coarse scale, agents' utilities are comperable: that for example two agents' preference intensities are equal on the matter of having an estate of $100,000-$200,000 as opposed to one of $10,000 - $20,000. How these intensities compare on the matter of hearing Bach as opposed to Bernstein is a consideration too fine to be relevant to our model.

There are many theoretical developments which appear relevant to the study of systems which possess discriminatory features. The game theory of von-Neumann and Morgenstern [5] focuses on static systems involving two or more agents, each influencing the welfare of the other. The concept of a fair static game, as discussed by von Neumann and Morgenstern, can be extended to differential games (Isaacs [3]) in an obvious way, but a fair differential game need not possess the forgiveness of an equitable system: it may place at a permanent severe disadvantage an agent who errs early. The concept of Pareto-optimality extends also to differential games (Ho [2]), but a system offering incentive for the use of Pareto-optimal strategies will, in general, still penalize permanently any participant who is delinquent in effecting his best strategy. In the multi-agent dynamic system literature generally, little attention has been given to analyzing systems with regard to the ease with which they permit an agent to recover from a disadvantageous state. Much of the literature to date has been war-game motivated and has focused on determining strategies for forcing other agents into disadvantageous states. An ideal socio-economic system would, on the contrary, feature inexhaustible opportunities for the recovery of any agent coming into hard times, provided, of course, that he properly exert himself.

Future technical work in the development of a satisfactory concept of equitability must incorporate some measure of how soon an agent can, by properly applying himself, recover from a disadvantageous state. Eventually, models of differing economic systems should be examined with regard to their equitability, just as they are currently studied with regard to their informational efficiency and growth potential.

Herein we have suggested a definition of equitability applicable to a simple, highly artificial system. But we have shown that systems exist which satisfy the definition, and that such systems need not be wholly trivial: agents may influence the system with acts of individual discretion, and the effects of one agent's acts may be felt by all agents.

VI. References

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[4] Kemeny, J.G., Snell, J.L., and Knapp, A.W., Denumerable Markov Chains, D. Van Nostrand, Princeton, N.J., 1966.

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VII. Appendix

Proof sketch for Proposition II:

Proposition 4-28, p. 102, and Theorem 6-1, p. 130 of Kemeny, Snell and Knapp [4] lead directly to our Proposition I.

Proof of Proposition IV:

Suppose agent A_1 constantly uses $\hat{q}_1$, while at the m^{th} instant, A_2 employs some arbitrary function q_2^m, m=0,1,...,n-1. Then

$$\begin{aligned} p_3^n - p_2^n &= (0,-1,1,0)P^n \\ &= (0,-1,1,0)\ \Pi(\hat{q}_1;q_2^{n-1})P^{n-1} \quad . \end{aligned}$$

Let

$$\Lambda(\hat{q}_1;q_2^{n-1}) = (0,-1,1,0)\Pi(\hat{q}_1;q_2^{n-1}).$$

We shall show that regardless of q_2^{n-1}, each component of Λ is non-negative, which, since each component of P^{n-1} is non-negative, will show that $p_3^n - p_2^n \geq 0$ independent of the sequence of functions $q_2^o,\dots,q_2^{n-1}$ used by A_2.

The j^{th} component of Λ is

$$\left[\Lambda(\hat{q}_1;q_2^{n-1})\right]_j = \left[\Pi(\hat{q}_1;q_2^{n-1})\right]_{3j} - \left[\Pi(\hat{q}_1;q_2^{n-1})\right]_{2j},$$

which is affine (jointly) in the variables $q_2^{n-1}(X_i)$, i=1,2,3,4. Because of the assumptions

$$\begin{aligned} \left[\Pi(\hat{q}_1;\hat{q}_2{}^{(j)})\right]_{2j} &\leq \left[\Pi(\hat{q}_1;\hat{q}_2{}^{(j)})\right]_{3j} \\ \left[\Pi(\hat{q}_1;\hat{q}_2)\right]_{2j} &= \left[\Pi(\hat{q}_1;\hat{q}_2)\right]_{3j}, \end{aligned}$$

it follows at once that $\Lambda_j \geq 0$, and that $\Lambda(\hat{q}_1;\hat{q}_2) = (0,-1,1,0)$. $\Pi(\hat{q}_1;\hat{q}_2)$ has components consisting only of zeroes. Hence the condition in Theorem I that

$$\frac{1}{n}\sum_{m=0}^{n-1}\left[p_3^m(\hat{q}_1;q_2) - p_2^m(\hat{q}_1;q_2)\right] \geq -\varepsilon$$

is met for any n and for any $\varepsilon \geq 0$.

A similar demonstration shows that the symmetric condition

$$\frac{1}{n}\sum_{m=0}^{n-1}\left[p_2^m(q_1;\hat{q}_2) - p_3^m(q_1;\hat{q}_2)\right] \geq -\varepsilon$$

also holds. Hence the system is equitable.

<u>Development of m-agent, m^n-state case:</u>

AI. Definition and Notation

Consider a discrete-time dynamic system which may reside in any of n^m states $\underline{\overline{X}} = \{X_\alpha\}_{\alpha\varepsilon\mathcal{A}}$, where $\mathcal{A}$ is the set of m-tuples of integers from the set $\{1,\ldots,n\}$. It is postulated that m agents, $A_1,\ldots,A_m$, influence the sequence of states visited by the system. Each agent has a preference ordering on the set $\underline{\overline{X}}$ of states as follows:

A_j strictly prefers X_α to X_β if $\alpha_j > \beta_j$

A_j is indifferent between X_α and X_β if $\alpha_j = \beta_j$

where α_j, β_j are the j^{th} components of the m-tuples α and β, respectively. The state of the system at time t is denoted x^t.

At each time t, each agent, cognizant of x^t, makes a decision, and the decisions jointly determine the probabilities with which the possible states will be visited at time t+1. Each agent A_j selects a probability function $\psi_j^t:\underline{\overline{X}}\times\{1,\ldots,k\}\to[0,1]$ in accordance with which an action $\gamma_j\varepsilon\{1,\ldots,k\}$ is taken. The m-tuple γ of actions serves to identify a transition matrix Π^γ in accordance with which the system moves to its next state. The probability of moving from the given state, say α, to some other specific state, say β, is denoted $\pi^{\gamma}_{\beta\alpha}$.

The probability that at time t=0 the system will be in the state X_α is denoted p^0_α. The probability that at time t the system will be in the state X_β, given the values of the p^0_α, $\alpha\varepsilon\mathcal{A}$, is denoted p^t_β. Generally, p^t_β depends on the functions $\psi^0_j,\ldots,\psi^{t-1}_j$, $j=1,2,\ldots,m$, and we denote this dependence as $p^t_\beta(\psi)$. In what follows, it is convenient to use $p^t_\beta(\psi_{\hat{j}})$ to denote the p^t_β value resulting when agent j uses particular functions $\hat{\psi}^0_j,\ldots,\hat{\psi}^{t-1}_j$ but the functions used by the other agents are unspecified.

An equitable system is defined in a way ensuring that any agent in the system has the capability, regardless of the action of other agents, to force the system to reside in states favorable to him at least as frequently as in states favorable to other agents.

Definition: A system is equitable iff for each agent A_j there exist functions $\hat{\psi}^t_j: X\times\{1,\ldots,k\}\to[0,1]$, $\hat{\psi}^t_j(x^t,\cdot)$ a probability function on $\{1,\ldots,k\}$, such that for any initial probabilities p^0_α, $\alpha\varepsilon\mathcal{A}$, and any $\varepsilon > 0$, the following inequalities hold for all t sufficiently large:

$$\frac{1}{t}\sum_{s=0}^{t-1}\sum_{\alpha:\alpha_j\geq r} p^s_\alpha(\psi_{\hat{j}}) \geq \frac{1}{t}\sum_{s=0}^{t-1}\sum_{\alpha:\alpha_i\geq r} p^s_\alpha(\psi_{\hat{j}}) - \varepsilon,$$

$r=1,2,\ldots,n$; $i=1,2,\ldots,m$; regardless of the functions $\psi^0_\ell,\ldots,\psi^{t-1}_\ell$, $\ell\neq j$, used by agents other than A_j.

Denoting by $\hat{p}^s_\alpha$ the probability of being in state α at time s when, in an equitable system, the functions (strategies) $\hat{\psi}^t_j$ are employed, $j=1,2,\ldots,m$, $t=0,1,\ldots,s-1$, we obtain the following necessary condition:

Proposition AI: If a system is equitable, then the functions $\hat{\psi}_j^t$ are such that for any initial probabilities p_α^o, $\alpha\varepsilon$, and each $r\varepsilon\{1,2,\ldots,n\}$,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=o}^{t-1} \left(\sum_{\alpha:\alpha_j\geq r} \hat{p}_\alpha^s - \sum_{\alpha:\alpha_i\geq r} \hat{p}_\alpha^s \right) = 0,$$

$j=1,2,\ldots,m$; $i=1,2,\ldots,m$.

AII. The Autonomous Case

If Π^γ, the transition matrix jointly selected by the m agents, does not depend on γ, we write $\Pi = \Pi^\gamma$ and say the system is autonomous: its evolution is not influenced by the actions of the agents. In an autonomous system, the conditions of Proposition AI are both necessary and sufficient for equitability.

Definition. An autonomous system is said to be P^o-independent iff

$$L = \prod_{t=o}^{\infty} \Pi$$

exists, and the columns of L are identical.

Denoting by ℓ the typical column of L associated with a P^o-independent autonomous equitable system, and by ℓ_α the α^{th} component of ℓ, we obtain the following necessary and sufficient condition:

Proposition AII. A P^o-independent autonomous system is equitable iff there exists a constant c_r (depending on r) such that

$$\sum_{\alpha:\alpha_j\geq r} \ell_\alpha = c_r,\ r=1,2,\ldots,m.$$

AIII. The Constant Strategy Case

If by law or mutual agreement the agents' strategies ψ_j^t do not vary with t, we write $\psi_j = \psi_j^t$ and say the strategies are constant. (The autonomous case may be regarded as a special constant strategy case.)

Proposition AIII. For any constant strategies ψ_j, $j=1,2,\ldots,m$, the limits

$$\lim_{t\to\infty} \frac{1}{t} \sum_{s=o}^{t-1} \sum_{\alpha:\alpha_j \geq r} p_\alpha^s(\psi) \quad \text{exist, } j=1,2,\ldots,m;$$

$r=1,2,\ldots,n.$

Definition. If for all constant strategies ψ_j, $j=1,2,\ldots,m$, $\lim_{t\to\infty} p_\alpha^t(\psi)$ exists for each $\alpha\varepsilon\mathcal{A}$ and equals (say) $\ell_\alpha(\psi)$, independent of p_α^o, the system is said to be P^o-independent for constant strategies.

Proposition AIV. If a system is, for constant strategies, both equitable and P^o-independent, then for any m-tuple γ of possible joint actions, the autonomous system using transition matrix Π^γ is equitable.

Proposition AV. A system P^o-independent for constant strategies is equitable for constant strategies iff there exist strategies $\hat{\psi}_j$ such that for any arbitrary (constant) strategies ψ_i,

$$\sum_{\alpha:\alpha_j \geq r} \ell_\alpha(\psi_{\hat{j}}) - \sum_{\alpha:\alpha_i \geq r} \ell_\alpha(\psi_{\hat{j}}) \geq 0$$

$j=1,2,\ldots,m;\ i=1,2,\ldots,m;\ r=1,2,\ldots,n.$

When constant strategies are employed, the system motion is governed by a constant transition matrix $\Pi(\psi)$ which is an nm-affine combination of the matrices Π^γ. The elements of $\Pi(\psi)$ are denoted $\pi_{\alpha\beta}(\psi)$. Denoting by $D_\beta(\psi)$ the cofactor of $\pi_{1\beta}(\psi)$, where the subscript 1 indicates $\alpha=\{1,1,\ldots,1\}$, we have the following:

Proposition AVI. A system P^O-independent for constant strategies is equitable for constant strategies iff there exist strategies $\hat{\psi}_j$ such that for any constant strategies ψ_i,

$$\sum_{\beta:\beta_j \geq r} D_\beta(\psi_{\hat{j}}) - \sum_{\beta:\beta_i \geq r} D_\beta(\psi_{\hat{j}}) \leq 0,$$

$i=1,2,\ldots,m;\ j=1,2,\ldots,m;\ r=1,2,\ldots,n.$

AIV. The General Case

The general system is difficult to analyze for equitability. However, two propositions of significance are readily obtained.

Proposition AVII. An equitable system is necessarily equitable for constant strategies.

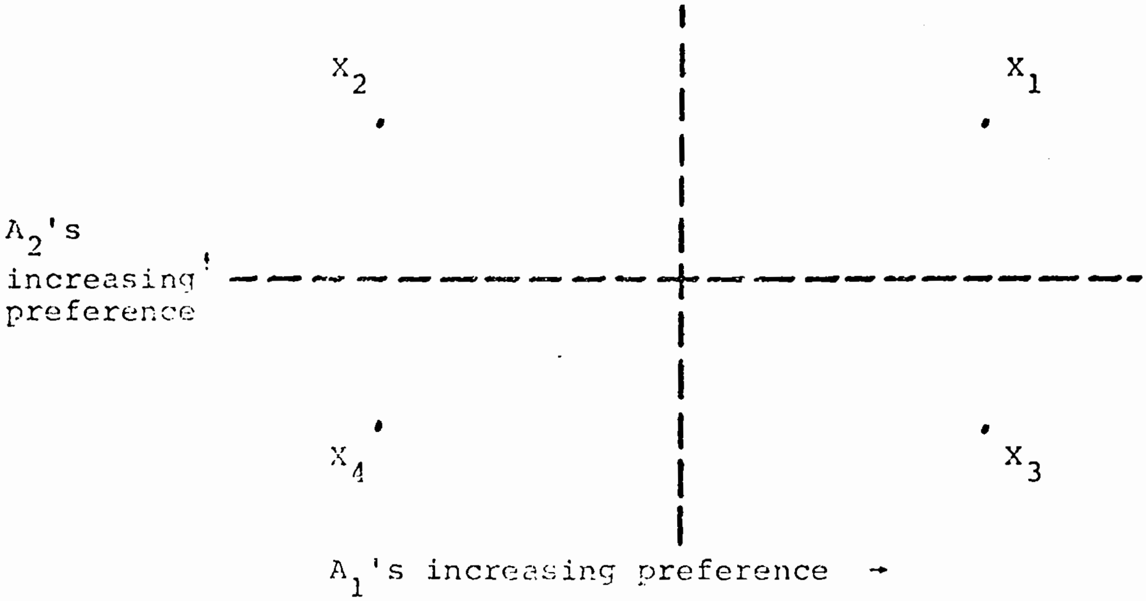
Denote by $\Pi(\psi)$ the transition matrix in effect at the instant t at which the agents choose their strategies $\psi_1^t,\ldots,\psi_m^t$, and by $\pi_{\alpha\beta}(\psi)$ the $\alpha\beta^{th}$ component of $\Pi(\psi)$. If the system is in state β, the j^{th} agent might consider seeking an apparently near-sighted policy $\hat{\psi}_j^t$ satisfying

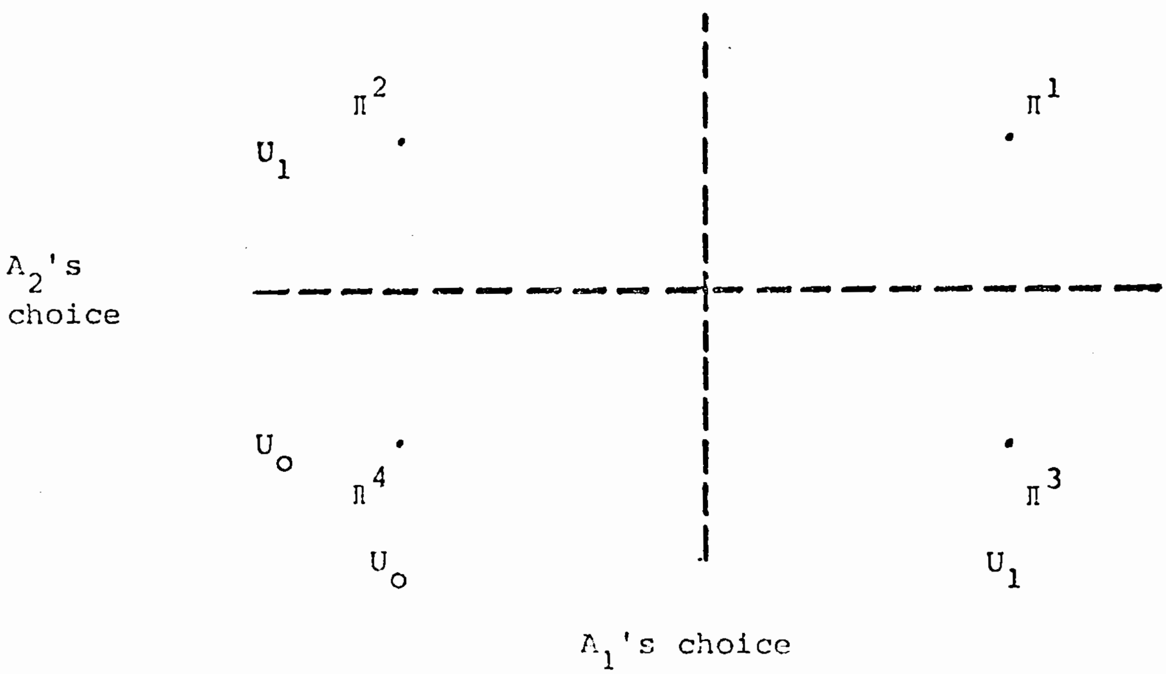
$$\sum_{\alpha:\alpha_j \geq r} \pi_{\alpha\beta}(\psi_{\hat{j}}) - \sum_{\alpha:\alpha_i \geq r} \pi_{\alpha\beta}(\psi_{\hat{j}}) \geq 0, \quad \text{all } \psi_i^t,$$

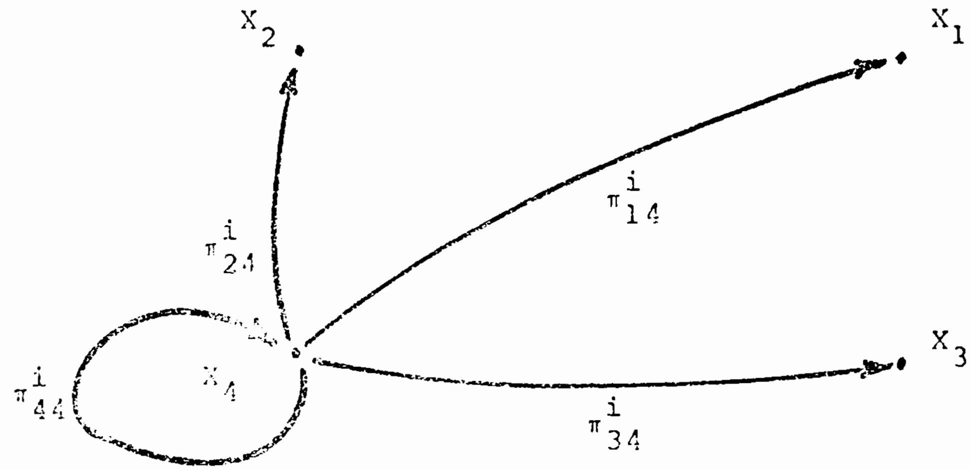
$r=1,2,\ldots,n;\ i=1,2,\ldots,m.$

Proposition AVIII. If for each state β and each time t such a $\hat{\psi}_j^t$ exists for each agent, then the system is equitable and the $\hat{\psi}_j^t$ are equitable strategies. As $\Pi(\psi)$ depends on the ψ_j^t but not explicitly on t, the $\hat{\psi}_j^t$ need not depend on t.

Examples of systems satisfying the hypothesis of Proposition AVIII can be constructed with moderate ease.


X2
X1
A2's
increasing
preference
X4
X3
A1's increasing preference →


π2
π1
U1
A2's
choice
Uo
π4
π3
Uo
U1
A1's choice


X2
X1
π i 24
π i 14
X4
X3
π i 44
π i 34

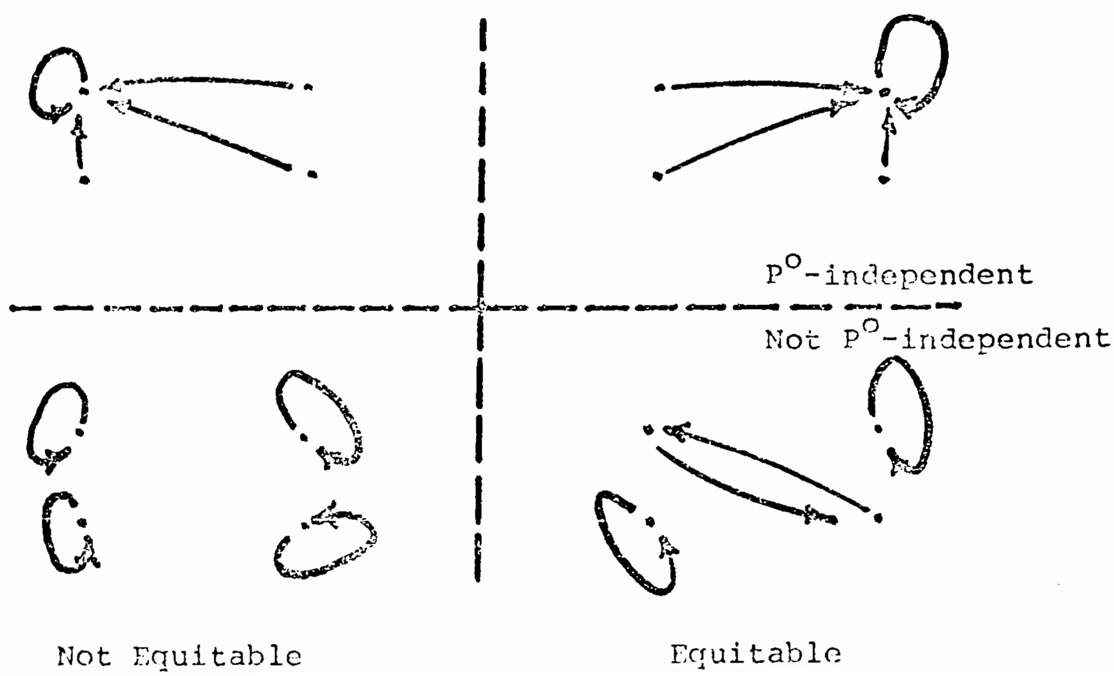

P^O-independent
Not P^O-independent
Not Equitable
Equitable

Figure 1. System states and agent preferences.

Figure 2. Agent actions determine which Π^i will be used. If A_1 chooses U_1 while A_2 chooses U_2, then Π^3 guides the system to its next state.

Figure 3. The entries in Π^i are transition probabilities related as suggested to the states X_j.

Figure 4. Examples of autonomous systems exhibiting that neither of the properties "P^0-independent" and "equitability" implies the other. The four dots in each sector represent states, numbered as in Figure 3. Lines leaving a dot represent non-zero transition probabilites. All such probabilities in the above diagram are 1.