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THE CONICAL DUALITY AND COMPLEMENTARITY OF PRICE AND QUANTITY FOR MULTICOMMODITY SPATIAL AND TEMPORAL NETWORK ALLOCATION PROBLEMS

by

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The Conical Duality and Complementarity of Price and Quantity for Multicommodity Spatial and Temporal Network Allocation Problems*

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Abstract. Consider a graph G with each node i representing the set of "producers" and/or "consumers" at a specific spatial as well as temporal location. Each link k is directed so that it represents a specific storage and/or transportation facility for transfering certain "commodities" from a given node i to another given node j. Each commodity r is produced and/or consumed by certain nodes.

Suppose that the excess quantity of commodity r produced by node i is a variable q_{ir} (which is positive when node i produces more than it consumes); and suppose that the quantity of commodity r transfered via link k (in the direction of link k) is a variable $q^{kr} \geq 0$. Conservation of each commodity r at each node i then requires that the quantity vector q (whose components are the q_{ir} and the q^{kr}) be in the cone

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$$Q = \{q \mid q^{kr} \geq 0 \text{ and } q_{ir} + \sum_{[i]} q^{kr} = \sum_{(i)} q^{kr}\},$$

where [i] denotes the set of all links k directed into node i and (i) denotes the set of all links k directed out from node i.

Suppose that the unit price of commodity r for node i is a variable p_{ir}; and suppose that the unit price of transfering commodity r via link k is a variable p^{kr}. Price stability for each commodity r then requires that the price vector p (whose components are the p_{ir} and the p^{kr}) be in the cone

$$P = \{p \mid p_{ir} + p^{kr} \geq p_{jr} \text{ for each } k \in (i) \cap [j]\}.$$

The main result given here is that P and Q are a pair of dual convex polyhedral cones, whose corresponding (conical) "complementarity conditions"

$$p \in P \qquad \text{and} \qquad q \in Q,$$

$$0 = \langle p, q \rangle$$

can be used to characterize the solution sets for various important network allocation problems. The main implications of this result are that generalized geometric programming and generalized complementarity theory, along with convex analysis, monotone mapping theory, and generalized fixed point theory, can now be exploited in a much deeper study of such problems than has previously been possible.

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1. Introduction. According to Samuelson [34], the first network allocation problem was considered at least as early as 1838 by A.A. Cournot [8] -- who also supplied at that time the seemingly first explicit statement that competitive market price is determined by the intersection of supply and demand curves. Moreover, subsequent investigations of the so-called "communication of markets" problem were carried out by many of the great theoretical economists of the early 20th century, including Cunyngham (1904), Barone (1908), Pigou (1904), and H. Schultz (1935) -- as described by Taussig [43] and Viner [44]. In fact, their investigations culminated in 1951 with the work of Enke [12], who gave the problem a more general formulation along with a "solution by electric analogue".

Stimulated by that solution and the "variational (or extremum) principles" for "simple passive electric networks" discovered around 1850 by the great theoretical physicists Maxwell and Kirchoff, Samuelson [34] provided in 1952 a variational principle for the Enke problem and showed that the "transportation problem" first studied during the 1940's by Hitchcock [18], Kantorovich [20], and Koopmans [22] could be viewed as a special case. Subsequent work by Reiter [28], Beckmann and Marschak [3], Lefeber [24], and Stevens [38] also helped to clarify the relations between the "spatial equilibrium problems" of Enke and the "linear programming" theory developed in 1951 by Dantzig [9]. Moreover, in 1957, Samuelson [35] observed that spatial and temporal network allocation problems are, in principle, no more difficult than purely spatial network allocation problems.

It seems that "duality" was first introduced into such problems in 1963 by Smith [36] -- who evidently received his primary stimulation from the 1958 treatise of Arrow, Hurwicz, and Uzawa [1], which contains expositions of both the 1951 linear programming "duality theory" of Gale, Kuhn, and Tucker [16] and the 1951 nonlinear programming "Lagrangian theory" of Kuhn and Tucker [23]. For independent and subsequent developments having to do with related network allocation problems, see both the treatise of Ford and Fulkerson [15] and the treatise of Berge and Ghouila-Houri [4], as well as the more recent treatises by Hu [19], Spivey and Thrall [37], and Rockafellar [31].

It seems that "multicommodity" network allocation problems were first considered in 1964 by Takayama and Judge [39,40]. In collaboration with some of their colleagues and co-workers, they subsequently made numerous applications, including rather extensive numerical calculations carried out via the "quadratic programming" algorithms that had previously been developed in 1958 and 1959 by Barankin and Dorfman [2] and Wolfe [46]. For detailed descriptions of all such work, including a more thorough historical account of the developments having to do with multicommodity spatial and temporal network allocation problems, see the relatively recent treatises by Takayama and Judge [41, 42].

The purpose of this paper is to uncover a new result that is to serve as the key to a much deeper study of multicommodity network allocation problems than has previously been possible. Actually, the main result given here can be viewed primarily as a (multicommodity conical) extension of a fundamental result obtained in 1923 by Weyl [45] -- who discovered during his study of (single-commodity) electric and hydraulic networks that the vector space (or cone) of all "current vectors" satisfying Kirchoff's "nodal conservation law" is the orthogonal complement (or dual) of the vector space (or cone) of all "potential vectors" satisfying Kirchoff's "circuit conservation law". In fact, Weyl's result served as the key to a much deeper study of electric and hydraulic networks initiated in 1947 by Duffin [10] and further pursued by Bott and Duffin [6,7], Birkhoff and Diaz [5], Minty [25,26], Berge and Ghouila-Houri [4], and Rockafellar [29,31].

The main result of this paper is actually the key to exploiting, within the context of network allocation problems, both the generalized "geometric programming" of Peterson [27] and the generalized "complementarity theory" of Habetler and Price [17], Karamardian [21], and Saigal [32,33] -- mathematical tools that also rely heavily on the "convex analysis" of Fenchel [13,14] and Rockafellar [30], as well as the generalized "monotone mapping theory" discussed by Rockafellar [30] and the generalized "fixed point theory" discussed by Eilenberg and Montgomery [11].

2. The model. The allocation problems to be studied here can be conveniently represented by a "directed graph" (consisting of "nodes" and "directed links") on which there is "multicommodity flow".

The total number of nodes is m, and the nodes are chosen and enumerated so that

> node i, for $i = 1,2,\ldots,m$, represents the set of "producers" and/or "consumers" at a specific spatial as well as temporal location.

The total number of links is n, and the links are chosen and enumerated so that

> link k, for $k = 1,2,\ldots,n$, represents a specific storage and/or transportation facility for transfering certain "commodities" from some node i to another

(different) node j (which means in particular that there are no links connecting a given node to itself).

Moreover,

each link k is directed to coincide with the direction of a possible transfer of commodities (which means in particular that there are at least two links connecting those nodes between which there is a possible transfer of commodities in both directions).

The total number of commodities is w, and the commodities are chosen and enumerated so that

commodity r, for $r = 1,2,...,w$, represents either a "raw material", an "intermediate product", or a "finished product", each of which can be produced and/or consumed by certain nodes.

We also assume that

the unit price of commodity r for node i is a variable p_{ir}; and the unit price for transfering commodity r via link k is a variable p^{kr}.

We further assume that

> the excess quantity of commodity r produced by node i is a variable q_{ir} (which is positive when node i produces more of commodity r than it consumes); and the quantity of commodity r transfered via link k (in the direction of link k) is a variable $q^{kr} \geq 0$.

Now, for a given link k connecting, say, node i to node j, if the unit purchase - price p_{ir} of a given commodity r for node i plus the unit transfer-price p^{kr} for commodity r on link k were less than the unit selling-price p_{jr} of commodity r for node j, some "entrepreneurs" would obviously begin to purchase as much of commodity r as possible from node i and resell it to node j -- an economically unstable situation. On the other hand, if the quantity q^{kr} of commodity r being transfered via link k were strictly positive and if p_{ir} plus p^{kr} were strictly greater than p_{jr}, some entrepreneurs would obviously begin to lower q^{kr} to zero -- another economically unstable situation. To provide a purely mathematical expression of the resulting economic stability conditions, suppose that

> the symbol (i) denotes the set of all links k directed out from node i, while the symbol [i] denotes the set of all links k directed into node i.

Then, the network is in a state of "economic stability" only if

$$p_{ir} + p^{kr} \geq p_{jr}, \text{ with equality holding if } q^{kr} > 0, \qquad (1)$$

$$\text{for each } k \in (i) \cap [j], \text{ for } 1 \leq i \leq m, \text{ for } 1 \leq j \leq m, \text{ and for } 1 \leq r \leq w.$$

Naturally, the network is in a state of "economic equilibrium" only if certain other conditions are also satisfied. In particular, we have already mentioned the feasibility condition

$$q^{kr} \geq 0 \text{ for } 1 \leq k \leq n \text{ and for } 1 \leq r \leq w; \tag{2}$$

and conservation of each commodity r at each node i clearly requires that

$$q_{ir} + \sum_{[i]} q^{kr} = \sum_{(i)} q^{kr} \text{ for } 1 \leq i \leq m \text{ and for } 1 \leq r \leq w. \tag{3}$$

There is still another condition that relates prices to quantities via supply and demand mappings, but that condition is superfluous to the main results of this paper and hence will not be introduced until section 4.

3. <u>The main result</u>. Notationally, we suppose that

> the symbol p_i denotes the vector of unit prices of the various commodities r for node i (and hence has components p_{ir} for $r = 1,2,\ldots,w$).

Similarly, we suppose that

> the symbol q_i denotes the vector of excess quantities of the various commodities r produced by node i (and hence has components q_{ir} for $r = 1,2,\ldots,w$).

We also suppose that

> the symbol p^k denotes the vector of unit

prices for transfering the various commodities r via link k (and hence has components p^{kr} for $r=1,2,\ldots,w$).

Similarly, we suppose that

the symbol q^k denotes the vector of quantities of the various commodities r transfered via link k (and hence has components q^{kr} for $r=1,2,\ldots,w$).

Finally, we suppose that

the symbol p denotes the vector $(p_1,\ldots,p_m, p^1,\ldots,p^n)$, each of whose components is itself a vector.

Likewise, we suppose that

the symbol q denotes the vector $(q_1,\ldots,q_m,q^1,\ldots,q^n)$, each of whose components is itself a vector.

Needless to say, both p and q also have scalar components; in fact, they clearly have the same number of scalar components, and their inner product $\langle p,q\rangle$ can be expressed in terms of those scalar components as

$$\langle p,q\rangle = \sum_{r=1}^{w} \sum_{i=1}^{m} p_{ir}q_{ir} + \sum_{r=1}^{w} \sum_{k=1}^{n} p^{kr}q^{kr}.$$

The economic equilibrium conditions (1) through (3) can be rephrased in a mathematically more tractable form that involves both the set

$$\mathbf{P} \stackrel{\Delta}{=} \{p \mid p_{ir} + p^{kr} \geq p_{jr} \text{ for each } k \in (i) \cap [j],$$
$$\text{for } 1 \leq i \leq m, \text{ for } 1 \leq j \leq m, \text{ and for } 1 \leq r \leq w\}$$

and the set

$$Q \triangleq \{q \mid q^{kr} \geq 0 \text{ for } 1 \leq k \leq n \text{ and for } 1 \leq r \leq w;$$

$$\text{and } q_{ir} + \sum_{[i]} q^{kr} = \sum_{(i)} q^{kr} \text{ for } 1 \leq i \leq m \text{ and for } 1 \leq r \leq w\}.$$

In fact, some important properties of P and Q along with that more tractable form constitute the main result of this paper -- as crystallized in the following proposition.

PROPOSITION. The sets P and Q are dual convex polyhedral cones: that is, P and Q are convex polyhedral cones for which

$$P = \{p \mid 0 \leq \langle p,q \rangle \text{ for each } q \in Q\}$$

and

$$Q = \{q \mid 0 \leq \langle p,q \rangle \text{ for each } p \in P\}.$$

Moreover, a given vector (p,q) satisfies the economic equilibrium conditions (1) through (3) if and only if it satisfies the (conical) complementarity conditions

(I) $p \in P$ and $q \in Q$

(II) $0 = \langle p,q \rangle$.

The following two theorems, which are of some interest in their own right, help to prove the preceding proposition.

Theorem 1. If $0 \leq \langle p,q \rangle$ for each $q \in Q$, then $p \in P$.

Proof. Given arbitrary but fixed indices i', j', k', r' such that $k' \in (i') \cap [j']$, note that the definition of Q along with the parenthetical part of the second displayed statement in section 2 implies that the vector $\bar{q}$ with components

$$\bar{q}_{ir} \stackrel{\Delta}{=} \begin{cases} 1 \text{ for } i = i' \text{ and } r = r' \\ -1 \text{ for } i = j' \text{ and } r = r' \\ 0 \text{ otherwise} \end{cases}$$

and

$$\bar{q}^{kr} \stackrel{\Delta}{=} \begin{cases} 1 \text{ for } k = k' \text{ and } r = r' \\ 0 \text{ otherwise} \end{cases}$$

is in Q. Consequently, the obviously valid equation

$$\langle p, \bar{q} \rangle = p_{i'r'} + p^{k'r'} - p_{j'r'}$$

implies that

$$0 \le p_{i'r'} + p^{k'r'} - p_{j'r'} \text{ when } 0 \le \langle p, q \rangle \text{ for each } q \in Q.$$

This completes our proof of Theorem 1.

<u>Theorem</u> 2. <u>If</u> $p \in P$ <u>and</u> $q \in Q$, <u>then</u>

$$0 \le \langle p, q \rangle,$$

<u>with equality holding if and only if</u>

<u>either</u> $p_{ir} + p^{kr} = p_{jr}$ <u>or</u> $q^{kr} = 0$ for each $k \in (i) \cap [j]$,

<u>for</u> $1 \le i \le m$, <u>for</u> $1 \le j \le m$, <u>and for</u> $1 \le r \le w$.

Proof. Given $p \in P$ and $q \in Q$, note from the definitions of P and Q that for an arbitrary $k \in (i) \cap [j]$

$$0 \le p_{ir} q^{kr} - p_{jr} q^{kr} + p^{kr} q^{kr},$$

with equality holding if and only if

$$\text{either } p_{ir} + p^{kr} = p_{jr} \text{ or } q^{kr} = 0.$$

Summing these inequalities over all $k \in (i) \cap [j]$, we see that

$$0 \le p_{ir} \sum_{(i) \cap [j]} q^{kr} - p_{jr} \sum_{(i) \cap [j]} q^{kr} + \sum_{(i) \cap [j]} p^{kr} q^{kr},$$

with equality holding if and only if

$$\text{either } p_{ir} + p^{kr} = p_{jr} \text{ or } q^{kr} = 0 \text{ for each } k \in (i) \cap [j].$$

Now, letting

$$\langle i \rangle \overset{\Delta}{=} \{ j \mid (i) \cap [j] \ne \emptyset \} \text{ for } 1 \le i \le m$$

and summing the preceding inequalities over all $j \in \langle i \rangle$, we see from elementary graph-theoretic considerations that

$$0 \le p_{ir} \sum_{(i)} q^{kr} - \sum_{\langle i \rangle} p_{jr} \Big(\sum_{(i) \cap [j]} q^{kr} \Big) + \sum_{(i)} p^{kr} q^{kr},$$

with equality holding if and only if

$$\text{either } p_{ir} + p^{kr} = p_{jr} \text{ or } q^{kr} = 0 \text{ for each } k \in (i) \cap [j].$$

Consequently, using the definition of Q, we infer that

$$0 \le p_{ir} [q_{ir} + \sum_{[i]} q^{kr}] - \sum_{\langle i \rangle} p_{jr} \Big(\sum_{(i) \cap [j]} q^{kr} \Big) + \sum_{(i)} p^{kr} q^{kr},$$

with equality holding if and only if

$$\text{either } p_{ir} + p^{kr} = p_{jr} \text{ or } q^{kr} = 0 \text{ for each } k \in (i) \cap [j].$$

Summing these inequalities over all i, we now see from elementary graph-theoretic considerations that

$$0 \le \sum_{i=1}^{m} p_{ir}q_{ir} + \sum_{i=1}^{m} p_{ir}\Big[\sum_{[i]} q^{kr}\Big] - \sum_{i=1}^{m}\Big[\sum_{\langle i\rangle} p_{jr}\Big(\sum_{(i)\cap[j]} q^{kr}\Big)\Big] + \sum_{k=1}^{n} p^{kr}q^{kr},$$

with equality holding if and only if

$$\text{either } p_{ir} + p^{kr} = p_{jr} \text{ or } q^{kr} = 0 \text{ for each } k \in (i) \cap [j],$$

$$\text{for } 1 \le i \le m, \text{ and for } 1 \le j \le m.$$

Elementary graph-theoretic considerations also show that the second and third summations over all i cancel one another. Consequently, summing these inequalities over all r, we now conclude that

$$0 \le \sum_{r=1}^{w} \sum_{i=1}^{m} p_{ir}q_{ir} + \sum_{r=1}^{w} \sum_{k=1}^{n} p^{kr}q^{kr},$$

with equality holding if and only if

$$\text{either } p_{ir} + p^{kr} = p_{jr} \text{ or } q^{kr} = 0 \text{ for each } k \in (i) \cap [j],$$

$$\text{for } 1 \le i \le m, \text{ for } 1 \le j \le m, \text{ and for } 1 \le r \le w.$$

This completes our proof of Theorem 2.

Proof of the proposition. To establish the convex polyhedral conicality of P and Q, simply note that P and Q are defined by linear homogeneous inequalities and equations. To prove the representation formula for P, simply use both

Theorem 1 and Theorem 2. To prove the representation formula for Q, simply use the representation formula for P along with "conical duality theory" (i.e. Theorem 14.1 on page 121 of [30]). Finally, to establish the equivalence of the conditions (1) through (3) and the conditions (I) and (II), simply use Theorem 2. q.e.d.

4. The main implications. As previously mentioned, conditions (1) through (3) must be augmented by another condition involving supply and demand mappings to completely characterize the states of "economic equilibrium" for the model being considered. To obtain that condition, we assume that

> with each node i there is associated a generally multivalued (point-to-set) mapping γ_i with domain Γ_i, whose functional value $\gamma_i(p_i)$ depends only on the vector p_i of unit prices p_{ir} for node i and the various commodities r.

We also assume that

> with each link k there is associated a generally multivalued (point-to-set) mapping γ^k with domain Γ^k, whose functional value $\gamma^k(p^k)$ depends only on the vector p^k of unit prices p^{kr} for transfering via link k the various commodities r.

Economically, each mapping $\gamma_i:\Gamma_i$ is actually a given (generally multivalued) "supply mapping" $\sigma_i:\Sigma_i$ minus (in the algebraic sense) a given (generally multivalued) "demand mapping" $\delta_i:\Delta_i$; so for a specific unit price vector $p_i \in \Gamma_i \overset{\Delta}{=} \Sigma_i \cap \Delta_i$, the only possible excess quantity vectors $q_i \in \gamma_i(p_i) \overset{\Delta}{=} \{s_i - d_i \mid s_i \in \sigma_i(p_i) \text{ and } d_i \in \delta_i(p_i)\}$. On the other hand, each function $\gamma^k:\Gamma^k$ is simply a given (generally multivalued) "supply mapping"; so for a specific unit price vector $p^k \in \Gamma^k$, the only possible transfer quantity vectors $q^k \in \gamma^k(p^k)$.

In particular then, the other condition needed to completely characterize the states of economic equilibrium for the model being considered is

$$q_i \in \gamma_i(p_i) \text{ for } 1 \le i \le m \quad \text{and} \quad q^k \in \gamma^k(p^k) \text{ for } 1 \le k \le n. \tag{4}$$

In fact, the model being considered is said to be in a state of <u>economic equilibrium</u> if conditions (1) through (4) are satisfied.

The following corollary to the proposition of section 3 is to play a key role in the author's future study of economic equilibria, as well as his future study of other types of network equilibria.

<u>Corollary</u>. <u>The economic equilibrium conditions</u> (1) <u>through</u> (4) <u>are equivalent to the network equilibrium conditions</u>

(I) $\quad p \in P \quad$ <u>and</u> $\quad q \in Q$,

(II) $\quad 0 = \langle p, q \rangle$,

(III) $\quad q \in \gamma(p)$,

<u>where</u> γ <u>is, of course, the generally multivalued (point-to-set) mapping with domain</u>

$$\Gamma \overset{\Delta}{=} \Gamma_1 \times \cdots \times \Gamma_m \times \Gamma^1 \times \cdots \times \Gamma^n$$

<u>and function values</u>

$$\gamma(p) \stackrel{\Delta}{=} \gamma_1(p_1) \times \cdots \times \gamma_m(p_m) \times \gamma^1(p^1) \times \cdots \times \gamma^n(p^n).$$

Conditions (I) through (III) are termed the <u>network equilibrium conditions</u> because we shall soon see that they can also be used to characterize other types of network equilibria -- simply by choosing $\gamma:\Gamma$ differently while maintaining the same definition for P and Q. In fact, a common attribute of all network allocation problems (known to the author) is the conical duality and complementarity of the network equilibrium conditions (I) and (II) -- as expressed by the proposition in section 3.

As an example of another type of network equilibria, we now consider a network allocation problem that arises from (centralized) network optimization. Prior to doing so, we must delete the only two assumptions given in this section, namely, the two assumptions that essentially postulate the existence of the (generally multivalued) mapping $\gamma:\Gamma$ and hence lead to the network equilibrium condition (III). In place of those two assumptions we assume that

> each node i requires (for consumption or otherwise) a specified quantity d_{ir} of each commodity r.

We also assume that

> with each node i there is associated a function g_i with domain C_i, whose functional value $g_i(q_i)$ depends only on the vector q_i of excess quantities q_{ir} of the various commodities r produced by node i.

Finally, we assume that

with each link k there is associated a function g^k with domain C^k, whose functional value $g^k(q^k)$ depends only on the vector q^k of quantities q^{kr} of the various commodities r transfered via link k.

Economically, each function $g_i:C_i$ is actually a cost function $c_i(\cdot + d_i):C_i$; that is, the production of a specific quantity vector $q_i + d_i \in C_i + \{d_i\}$ entails a production cost $g_i(q_i) = c_i(q_i + d_i)$. On the other hand, each function $g^k:C^k$ is just another cost function; that is, the transfer of a specific quantity vector $q^k \in C^k$ entails a transfer cost $g^k(q^k)$.

In particular then, the total production and transfer cost

$$g(q) \stackrel{\Delta}{=} g_1(q_1) + \ldots + g_m(q_m) + g^1(q^1) + \ldots + g^n(q^n)$$

is to be minimized subject to the feasibility conditions

$$q \in C \stackrel{\Delta}{=} C_1 \times \cdots \times C_m \times C^1 \times \cdots \times C^n$$

and

$$q \in Q.$$

Due to the conical nature of Q, the preceding cost minimization problem can be viewed most effectively as a generalized geometric programming problem [27]. In fact, when the mappings γ_i and γ^k are taken to be the "subgradient mappings" ∂g_i and ∂g^k respectively, the conical duality between P and Q together with the theory described in [27] shows that the resulting network equilibrium conditions are just the corresponding geometric programming "extremality conditions". Given then that the cost function g:C has certain properties [27], the network equilibrium conditions (I) through (III) have a solution set

$$E^* = S^* \times T^*,$$

where S^* is the optimal solution set for the preceding cost minimization problem, while T^* is the optimal solution set for its geometric programming "dual problem" [27]. In the dual problem, the "conjugate transform" [13, 14, 30] of $g{:}C$ is to be minimized subject to certain feasible conditions, including the cone condition $p \in P$. In fact, its optimal solution set T^* consists of "equilibrium price" vectors (or "shadow price" vectors) that can at times be used to solve the "primal problem" (i.e. the cost minimization problem) through "decomposition" [27] and "decentralized planning". However, the most important fact is the validity of the preceding displayed equation, which shows how to recover the desired optimal solution set S^* from the solution set E^* for the network equilibrium conditions (I) through (III).

Actually, the network equilibrium conditions (I) through (III) that arise in the context of predicting economic equilibria can frequently be viewed as the extremality conditions for an appropriate pair of "dual geometric programming problems". In such cases, the appropriate dual problems constitute a pair of "dual (or complementary) variational principles", and the objective function g for the problem involving q can be viewed as a quasi-cost function (or, in the terminology of Samuelson [34], as the negative of a "quasi-welfare function").

Such dual variational principles [27] are known to exist only when there is a "closed convex function" $g{:}C$ whose subgradient mapping ∂g is identical to the given mapping γ. However, according to Corollary 31.5.2 on page

340 of [30], such a function g:C exists only if γ is a "maximal monotone mapping." On the other hand, Corollary 37.5.2 on page 396 of [30] indicates that γ can be a maximal monotone mapping without such a function g:C existing.

In summary then, there are at least three different levels of generality at which the preceding network allocation problems should be studied. In increasing order of generality they are: (1) the case where $\gamma = \partial g$ for some closed convex function g:C, (2) the case where γ is a maximal monotone mapping (or perhaps just a monotone mapping), (3) the case where γ has no such monoticity properties.

For case (1), dual variational principles exist, and the powerful theories of geometric programming [27] and convex analysis [30] can be exploited. For case (2), the less powerful theories having to do with monotone mappings [30], complementarity [17,21,32,33] and fixed points [11] can be utilized. Finally, for case (3), complementarity theory and fixed point theory might still be usable.

5. Future work. The author intends to explore in great detail: (I) existence and uniqueness theorems as well as sensitivity analyses and computational algorithms for each of the three different levels of generality (discussed in the preceding section), (II) the relations between production functions and supply mappings, as well as the relations between utility functions and demand mappings -- with a view toward characterizing the three different levels of generality in terms of appropriate properties of production and utility functions, (III) the qualitative and quantitative effects of various governmental policies on economic equilibria.

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