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Stochastic Control of Competition Through Prices

by

S. D. Deshmukh* and Wayne Winston+

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* Graduate School of Management, Northwestern University
Evanston, Illinois 60201

+ Graduate School of Business, Indiana University,
Bloomington, Indiana 47401

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ABSTRACT

We assume that the price of a product set by a firm affects its immediate profit rate as well as the probabilistic rate of arrival of new firms into the industry. Therefore the firm's optimal dynamic pricing strategy must balance the increased current profits from setting a high price against the expected dilution of future profits due to additional competition. We provide a continuous-time Markov decision model and characterize the structure of the optimal control strategy and its sensitivity to the problem parameters. We also indicate the relationship of our problem with the queue control literature.

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Consider an industry consisting of a number of competing firms that produce and sell a single product. Each firm's profit depends upon the price of the product as well as the firm's market share, the former is controllable while the latter is determined by the competition. Moreover, the competition may be changing stochastically due to new firms entering the industry and old ones leaving, depending upon the product price. Thus, setting too high a price increases the established firm's immediate short run profits but also attracts new competitors into the industry, and induces the existing ones to stay on, thereby adversely affecting the firm's long term profit position. On the other hand, setting too low a price, although discouraging new entrants and driving out the existing competitors, may be too unprofitable even in the long run. The established firm wishes to determine an optimal pricing strategy so as to maximize its total expected discounted profit over the entire planning horizon.

Economists have studied problems of this type in the literature on imperfect competition under the general heading of "limit pricing." Bhagwati [2] and Modigliani [14] have provided surveys and syntheses of various economic theories of oligopolistic pricing proposed for taking into account the existing as well as future competition. Gaskins [5] has posed a deterministic optimal control model for taking into account the effect of price on entry and has provided a comparative static and dynamic analysis of the optimal price trajectory. Kamien and Schwartz [7] have incorporated the uncertainty in the time of the entry into an optimal control model via the hazard rate function of price and have analyzed the properties of the optimal pre-entry price using the Pontryagin's maximum principle. In a discrete time dynamic model, Baron [1] has represented the uncertainty in the entry process by a probability distribution of the number of entrants in each period and has investigated the effects of potential entry, barriers to entry and the attitude toward risk on the price and profitability of established firms. Deshmukh and Chikte [4] have proposed an infinite horizon continuous time Markov decision model to study the limit of the sequence of optimal prices to be charged as the number of firms in the industry tends to infinity.

In this paper we consider a more general finite and infinite horizon continuous time Markov decision model with discounting and characterize the structure of the optimal pricing policy, using some of the modern techniques (e.g. by Lippman [9] and Whitt [18]) that have been recently devised for the analysis. Although the basic motivation for the problem treated here arises from an economics or marketing context, the model formulation, the questions asked and the methodology employed are those that are typically found in the operations research and applied probability literature. In this respect, the paper is in the spirit of the articles by Leeman [8], Yechiali [19], Low [12] and Lippman and Stidham [11] all of which arise from an economics motivation. This provides a new application of the methodology of optimization in queueing systems (see, for example, an excellent synthesis and survey of this area by Stidham and Prabhu [17]), thereby illustrating the relevance and importance of operations research concepts and methods in economic analysis.

In section 1 we allow the firms to be of different sizes, drawn from an infinite population, and characterize the finite and infinite horizon optimal pricing strategy of the dominant firm. It is shown that the optimal price has a certain "monotonicity" property and is independent of the size of the dominant firm. In the rest of the paper we assume, for simplicity, that all firms are identical (as in the existing models). In section 2 we consider firms entering the industry from a finite population and show that, under certain assumptions, when the industry is at least half saturated, the optimal price each firm should charge is nondecreasing in the saturation level. The third section is devoted to the sensitivity analysis; it is shown that the optimal price is nondecreasing in the discount rate and nonincreasing in the duration of the planning horizon. In the final section we allow for the firms to leave the industry (unlike in the existing models) and formulate the optimal pricing problem in the queue control framework. Although a further analysis of the model presents formidable theoretical and analytical difficulties, we suggest possibilities for future research.

1. Heterogeneous Infinite Population

Consider an industry consisting of n_i firms of size i, $i=1,2,\ldots,I$, each producing a single product at cost c per unit. Suppose that, as in Gaskins [5], the market price p per unit of the product is controlled by a particular dominant firm (the price leader) of size L. If the industry price is p, the rate of total demand for the product will be denoted by D(p), so that the total industry profit rate may be written as

$$\bar{\Pi}(p) = (p-c)\ D(p).$$

As is typical in economic theory (e.g. see [7],[6],[4]), we will assume that $\bar{\Pi}(\cdot)$ attains its maximum at $\bar{p} < \infty$ and is differentiable and concave on $[c,\bar{p}]$. (Specific examples of demand functions D(p) satisfying this assumption include (i) $a\ e^{-bp}$ with $a,\ b > 0$, yielding $\bar{p} = c + 1/b$, (ii) p^{-d} with $d > 1$, yielding $\bar{p} = cd/(d-1)$ and (iii) $a-bp$ with $a,\ b > 0$, yielding $\bar{p} = (a+bc)/2$.) The continuous time interest rate will be denoted by $\alpha > 0$. We will assume that the total industry demand is shared among the competing firms proportional to the size of each firm, the larger the firm the better being its marketing and distribution capabilities. Consequently, if the industry consists of n_i firms of size i, $i=1,\ldots,I$, and if the price leader of size L selects a price p, the price leader's short run profit rate (i.e. that until the appearance of a new competitor) will be $L\bar{\Pi}(p)/n$, where $n = \sum_{i=1}^{I} in_i$ denotes the total size of the industry. Thus, $[c,\bar{p}]$ is the set of prices of interest and $L\bar{\Pi}(p)/n$ is increasing in $p\in[c,\bar{p}]$ with $L\bar{\Pi}(c)/n = 0$.

If the prevailing price is p, then suppose that new firms enter the industry according to the Poisson process whose rate depends upon the price p. Such an assumption is reasonable if there is a large number of potential entrants whose behavior is independent of one another and stationary in time. The higher the price $p \in [c, \bar{p}]$ the higher is the profit rate of the potential entrant and hence the more attractive the entry becomes. Specifically, suppose that firms of size k arrive according to independent Poisson processes with parameters $\bar{\lambda}_k(p)$, k=1,2,...I, where for each k, $\bar{\lambda}_k(\cdot)$ is a non-decreasing and twice differentiable function on $[c, \bar{p}]$. We assume, as in the other models mentioned before, that, since each firm makes nonnegative profit, there is no incentive for a firm to leave the industry. As a new firm enters the industry, the price leader accepts this fact, chooses a new price and the total industry demand is shared as described above; thus the existing firms are assumed to enjoy no goodwill.

The obvious pricing problem facing the dominant firm is to balance the advantages of setting a high price in terms of increased short run profit rate $L\Pi(p)/n$ and its disadvantages in terms of reduced long run profits from hastening additional future competition at increased entry rates, $\bar{\lambda}_k(p)$, k=1,...,I. Let V(n) denote the maximum expected discounted infinite horizon profit of the dominant firm starting in the industry of total size n and following the optimal pricing strategy. Then it is well known that V(n) is a unique solution of optimality equation

$$V(n) = \underset{p \in [c,\bar{p}]}{\text{Max}} \left\{ [L\bar{\Pi}(p)/n + \sum_{k=1}^{I} \bar{\lambda}_k(p)\; V(n+k)]/[\alpha + \sum_{k=1}^{I} \bar{\lambda}_k(p)] \right\} ,$$

$$n=L,\ L+1, \ldots . \qquad (1.1)$$

and that there exists an optimal stationary pricing strategy which, whenever the industry size is n, specifies a price $P_L^*(n)$ attaining the maximum of the right hand side of this functional equation. Instead of analyzing properties of $V(\cdot)$ and $P_L^*(\cdot)$ using (1.1), which turns out to be a formidable approach, we employ a recent technique devised by Lippman [9] (for optimally controlling exponential queueing systems) and the results of Whitt [18] (in approximating continuous time problems by discrete time ones).

Following Lippman [9], define

$$\Lambda = \sum_{k=1}^{I} \bar{\lambda}_k(\bar{p}) = \underset{p \in [c,\bar{p}]}{\text{Max}} \sum_{k=1}^{I} \bar{\lambda}_k(p) < \infty \qquad (1.2)$$

and consider the continuous time Markov decision process in which the intertransition times are exponentially distributed with parameter Λ, regardless of the state (industry size) n or the action (price) p_n selected. Now, with the additional fictitious decision epochs, define $V_m(n)$ as the optimal expected discounted profit of the dominant firm over the next m stages (transitions), starting in state n. Then the finite stage functional equations become

$$V_m(n) = \operatorname*{Max}_{p\in[c,\overline{p}]} \left\{ L\overline{\Pi}(p)/[n(\alpha+\Lambda)] + \sum_{k=1}^{I} \overline{\lambda}_k(p)/(\alpha+\Lambda)V_{m-1}(n+k) + [\Lambda - \sum_{k=1}^{I} \overline{\lambda}_k(p)]/(\alpha+\Lambda)V_{m-1}(n) \right\},$$

$$m=1,2,\ldots \qquad n=L,L+1,\ldots. \qquad (1.3)$$

where

$$V_0(n) = 0.$$

Now letting

$$\beta = \Lambda/(\alpha+\Lambda), \Pi(p) = \overline{\Pi}(p)/(\alpha+\Lambda), \ \lambda_k(p) = \overline{\lambda}_k(p)/\Lambda,$$

and $\lambda_0(p) = 1 - \sum_{k=1}^{I} \lambda_k(p)$ yields the following equivalent discrete time Markov decision process functional equations.

$$V_m(n) = \operatorname*{Max}_{p\in[c,\overline{p}]} \left\{ L\Pi(p)/n + \beta \sum_{k=0}^{I} \lambda_k(p) \ V_{m-1}(n+k) \right\}, \ m=2,3,\ldots \qquad (1.4)$$

where

$$V_1(n) = L\Pi(\overline{p})/n.$$

The optimal strategy $p_m(n)$, when in state n and m more stages remain, chooses a price attaining the maximum in (1.4). Also in the infinite stage problem we have, as usual,

$$V(n) = \lim_{m\to\infty} V_m(n) \quad \text{and}$$

$$V(n) = \operatorname*{Max}_{p\in[c,\overline{p}]} \left\{ L\Pi(p)/n + \beta \sum_{k=0}^{I} \lambda_k(p) \ V(n+k) \right\} \qquad (1.5)$$

Thus, properties of $V(\cdot)$ can be deduced from the corresponding properties of $V_m(\cdot)$. With this in mind, we now derive some important properties of $V_m(\cdot)$.

Lemma 1: The function $V_m(n)$ is nonnegative and decreasing in $n \geq L$ for all integers $m \geq 0$.

Proof: Clearly, $V_1(n) = L\Pi(\bar{p})/n$ has the required property, so that from (1.4) the lemma follows by a straightforward induction argument.

Q.E.D.

Lemma 2: The function $V_m(n)$ is convex in n for each m, i.e.

$$0 \leq V_m(n) - V_m(n+1) \leq V_m(n-1) - V_m(n)$$

for all $n \geq L$ and $m \geq 1$.

Proof: Now $V_1(n) = L\Pi(\bar{p})/n$ is convex in n. Suppose that $V_m(n)$ is convex in n and let $p_{m+1}(n)$ be the optimal price in state n with (m+1) stages remaining. Then, from (1.4)

$$V_{m+1}(n-1) - V_{m+1}(n) \geq L\Pi(p_{m+1}(n))/[n(n-1)] + \beta \sum_{k=0}^{I} \lambda_k(p_{m+1}(n)) \left[V_m(n-1+k) - V_m(n+k)\right]$$

while

$$V_{m+1}(n) - V_{m+1}(n+1) \leq L\Pi(p_{m+1}(n))/[n(n+1)] + \beta \sum_{k=0}^{I} \lambda_k(p_{m+1}(n)) \left[V_m(n+k) - V_m(n+k+1)\right].$$

Hence, subtracting and using the induction hypothesis yields the result for m+1.

Q.E.D.

Lemma 3: For $n \geq L$ and all $m \geq 1$,

$$V_m(n) - V_m(n+1) \leq L\Pi(\bar{p})(1-\beta^m)/[n(n+1)(1-\beta)].$$

Proof: If $m = 1$, $V_1(n) - V_1(n+1) = L\Pi(\bar{p})/[n(n+1)]$, so that the result holds with equality. Suppose that the lemma is true for m. Then

$$V_{m+1}(n) = L\Pi(p_{m+1}(n))/n + \beta \sum_{k=0}^{I} \lambda_k(p_{m+1}(n))\, V_m(n+k)$$

and

$$V_{m+1}(n+1) \geq L\Pi(p_{m+1}(n))/(n+1) + \beta \sum_{k=0}^{I} \lambda_k(p_{m+1}(n))\, V_m(n+k+1)$$

so that, subtracting and using the induction hypothesis,

$$\begin{aligned} V_{m+1}(n) - V_{m+1}(n+1) &\leq L\Pi(p_{m+1}(n))/[n(n+1)] + \beta \sum_{k=0}^{I} \lambda_k(p_{m+1}(n)) \left[V_m(n+k) - V_m(n+k+1)\right] \\ &\leq L\Pi(\bar{p})/[n(n+1)] + \beta \cdot \sum_{k=0}^{I} \lambda_k(p_{m+1}(n)\, L\Pi(\bar{p}) (1-\beta^m)/\left[(n+k)(n+k+1)(1-\beta)\right] \\ &\leq L\Pi(\bar{p})[1+\beta(1-\beta^m)/(1-\beta)]/[n(n+1)] \\ &= L\Pi(\bar{p})(1-\beta^{m+1})/\left[n(n+1)(1-\beta)\right], \end{aligned}$$

where the third inequality follows because

$$\left[(n+k)(n+k+1)\right]^{-1} \leq \left[n(n+1)\right]^{-1} \text{ and } \sum_{k=0}^{I} \lambda_k(p) = 1.$$

Q.E.D.

Now consider the infinite horizon problem. Since the action space $[c,\bar{p}]$ is compact and the functions $\Pi(p)$ and $\lambda_k(p)$ are continuous, we may apply the results of Maitra [13] to conclude that there exists an optimal stationary policy and the optimal value function $V(\cdot)$ satisfies the optimality equation (1.6) below. Furthermore, $V(n) = \lim_{m\to\infty} V_m(n)$ uniformly in n, so that $V(\cdot)$ inherits the properties of $V_m(\cdot)$ shown in Lemmas 1, 2 and 3. Thus, we may summarize as

<u>Proposition 1</u>: The optimal infinite horizon expected discounted profit $V(\cdot)$ of the dominant firm uniquely satisfies

$$V(n) = \max_{p\in[c,\bar{p}]} \left\{ L\Pi(p)/n + \beta \sum_{k=0}^{I} \lambda_k(p)\ V(n+k) \right\} \quad n=L,L+1,\ldots \qquad (1.6)$$

and has the following properties .

$$0 \le V(n) - V(n+1) \le V(n-1) - V(n), \quad n=L+1,\ L+2,\ldots \qquad (1.7)$$

$$0 \le V(n) - V(n+1) \le L\Pi(\bar{p})/[(1-\beta)n(n+1)], \quad n=L\ L+1,\ldots \qquad (1.8)$$

When the industry size is $n \ge L$ the dominant firm's infinite horizon optimal stationary strategy $P_L^* : \{L,L+1,\ldots\} \to [c,\bar{p}]$ chooses a price $P_L^*(n)$ attaining the maximum in (1.6).

The next proposition shows that the optimal pricing strategy P_L^* is independent of the size L of the dominant firm. Hence, we may denote P_L^* by $P^* = P_1^*$.

Proposition 2: For any $L = 1,2,...,I$,

$$P_L^*(n) = P_1^*(n), \qquad n=1,2,\ldots.$$

Proof: Denote by $V_m(L,n)$ the optimal expected discounted m-stage profit of the dominant firm of size L, starting in a industry of size n. We show by induction that at every stage m, the optimal price in state n is independent of L and that $V_m(L,n) = LV_m(1,n)$. Both of these statements are clearly true for $m = 1$. Suppose that they are true for m. Then

$$V_{m+1}(L,n) = \underset{p\in[c,\bar{p}]}{\text{Max}} \left\{ L\Pi(p)/n + \beta \sum_{k=0}^{I} \lambda_k(p)\ V_m(L,n+k) \right\}$$

$$= L \underset{p\in[c,\bar{p}]}{\text{Max}} \left\{ \Pi(p)/n + \beta \sum_{k=0}^{I} \lambda_p(p)\ V_m(1,n+k) \right\} ,$$

using the induction hypothesis. But this implies that the optimal price in state (m+1) is independent of L and that $V_{m+1}(L,n) = LV_{m+1}(1,n)$, as desired. Letting $m \to \infty$ we get the desired result for the infinite horizon model.

Q.E.D.

Since we have assumed that the total industry demand (and hence profit) is shared among its members so that a firm's share is proportional to its size, there is no real competition among the firms already in the industry; the competition is only between the firms in the industry and the future ones to enter. Also, by Proposition 2, the same optimal pricing strategy would be followed by any one of the members of the group that may be selected to be the price leader. Therefore, it seems clear that the optimal pricing strategy for the price leader (and, hence, for any firm in the current group) is the same as that which optimizes the long run expected discounted profit of the entire group of firms <u>currently</u> in the industry. To see that this is true, let $V'(n)$ be the total maximum expected discounted profit of the cartel of size n, so that the corresponding functional equation for the cartel's pricing problem becomes

$$V'(n) = \max_{p \in [c,\bar{p}]} \left\{ \Pi(p) + \beta \sum_{k=0}^{I} \lambda_k(p)\, nV'(n+k)/(n+k) \right\}$$

$$n = L, L+1, \ldots .$$

However, with $V'(n)=nV(n)/L$ the above expression can be seen to be equivalent to (1.6). Thus, the price leader's problem is the same as that of the group considered as a cartel. Real world examples of such cartels are found among the oil producers, container manufacturers and electrical equipment manufacturers, to name a few.

We are now ready to characterize the structure of the optimal pricing strategy P^*.

<u>Proposition 3</u>: There exists a non-decreasing function $n^* : [c,\bar{p}] \to \{1,2,\ldots\}$ which is independent of L and is given by

$$n^*(p_0) = \left[[\beta\ I(I+1)b\ \Pi(\bar{p})]/[2\ \Pi'(p_0)(1-\beta)] \right] + 1, \quad p_0 \in [c,p], \qquad (1.9)$$

where $b = \sup_{\substack{0 \leq k \leq I \\ c \leq p \leq \bar{p}}} \lambda_k'(p)$ and $[\ \ ]$ is the usual integer notation, such that for any $p_0 \in [c,\bar{p}]$, $P^*(n) \geq p_0$ whenever $n \geq n^*(p_0)$.

__Proof:__ Denote by $V(L,n)$ the optimal value function of the dominant firm of size L, $n \geq L$ being the industry size. Then from the optimality equation (1.6), it suffices to show that, for all $n \geq n^*(p_0)$, all $L \in \{1,2,\ldots,I\}$ and $p \in [c,p_0]$,

$$L\Pi(p_0)/n + \beta \sum_{k=0}^{I} \lambda_k(p_0)V(L,n+k)$$

$$\geq \quad L\Pi(p)/n + \beta \sum_{k=0}^{I} \lambda_k(p)V(L,n+k)$$

i.e.

$$L[\Pi(p_0)-\Pi(p)]/n \geq \beta \sum_{k=0}^{I} [\lambda_k(p)-\lambda_k(p_0)]\ V(L,n+k) \tag{1.10}$$

$$= \beta \sum_{k=0}^{I} a_k(p)\ V(L,n+k)$$

$$= T(p,n), \text{ say,}$$

where $\qquad a_k(p) = \lambda_k(p) - \lambda_k(p_0), \quad p \in [c,p_0].$

Noting that $\sum_{k=0}^{I} a_k(p) = 0$, we can use Abel's method of partial summation to rewrite the last expression as

$$T(p,n) = \beta\Big[a_0(p)\ \Delta\ V(L,n) + (a_0(p) + a_1(p))\Delta\ V(L,n+1)$$

$$\quad \ldots\ldots + \sum_{k=0}^{I-1} a_k(p)\ \Delta\ V(L,n+I-1)\Big]$$

where $\Delta V(L,n) = V(L,n) - V(L,n+1)$,

which is non-negative and non-increasing in n by (1.7). Hence,

$$T(p,n) \leq \beta\Big[|a_0(p)|\Delta V(L,n) + |a_0(p) + a_1(p)|\Delta V(L,n+1)$$

$$\quad . \; . \; . \; . \; . \; . + |\sum_{k=0}^{I-1} a_k(p)|\Delta V(L,n+I-1)\Big]$$

$$\leq \beta \sum_{i=0}^{I-1} |\sum_{k=0}^{i} a_k(p)|\Delta V(L,n).$$

Define $$\sup_{\substack{k \in \{0,1,\ldots,I\} \\ p \in [c,\overline{p}]}} |\lambda_k'(p)| = b < \infty.$$

Then by (1.8), the definition of b and the fundamental theorem of calculus, we have

$$T(p,n) \leq \beta(p_0-p) \; L\Pi(\overline{p})(\sum_{k=1}^{I} kb)/[n(n+1)(1-\beta)] \tag{1.11}$$

$$= U(p,n), \text{ say, } \; p \in [c,p_0].$$

Thus, from (1.10) and (1.11) it suffices to show that, for n sufficiently large,

$$L[\Pi(p_0)-\Pi(p)]/n \geq U(p,n), \quad p \in [c,p_0].$$

Now, by concavity of Π,

$$L[\Pi(p_0)-\Pi(p)]/n \geq L(p_0-p)\ \Pi'(p_0)/n, \quad p \in [c,p_0].$$

Therefore, (1.10) will hold for $n \geq n^*(p_0)$, thereby implying that $P^*(n) \geq p_0$, whenever $n \geq n^*(p_0)$, where $n^*(p_0)$ is given in (1.9). Since Π is concave, $\Pi'(p_0)$ is nonincreasing in p_0 and hence, from (1.9), $n^*(p_0)$ is nondecreasing and independent of L.

Q.E.D.

Thus, given any price p_0 of interest, there exists a critical industry size $n^*(p_0)$ given by (1.9) such that it is optimal to charge at least that high a price once the industry size exceeds the critical number; moreover, the critical size is nondecreasing in the given price and is the same for all firms. This property of the optimal pricing strategy may be called "right-monotonicity"; it would coincide with the usual monotonicity of P^*, if $n^*(p_0)$ given by (1.9) were the smallest such bound. (In Proposition 4 below, we prove, within a certain range, the usual monotonicity of P^* in a special case.) In a small size industry, it may be worth foregoing immediate profits in hope of discouraging new entrants by charging a price less than p_0. However beyond a certain point $n^*(p_0)$, the marginal reduction in the dominant firm's profit due to an additional competitor is not large enough to worry about retarding a new entry, so that the price charged should at least be p_0. Given p_0, the structure of the optimal strategy given in the above proposition is similar to the full service level policies studied by Sobel [16].

From (1.6), (1.9) and the fact that $\Pi'(p_0)$ decreases to $\Pi'(\bar{p}) = 0$ as p_0 increases to $\bar{p}$, we have, as in Deshmukh and Chikte [4],

<u>Corollary</u>. $\lim_{n\to\infty} P^*(n) = \bar{p}$

$$\lim_{n\to\infty} V(n) = 0$$

As the industry size tends to infinity, the effect of an entry becomes truly negligible, so that it is optimal to charge the myopic (immediate profit maximizing) price. However, since the total demand is shared among the firms, the total profit of any firm also becomes negligible.

In the rest of the paper, for simplicity, we take $I = L = 1$ and denote $\lambda_1(\cdot)$ by $\lambda(\cdot)$.

2. <u>Homogeneous Finite Population</u>

Let $N < \infty$ be the total population size, so that when the industry consists of $n < N$ firms and the prevailing price is p, new firms enter the industry according to the Poisson process with rate $\lambda(p)$; when the industry consists of N firms, no more firms can enter. The special case of $N = 2$ has been considered by Kamien and Schwartz [7], while with $N = \infty$ Deshmukh and Chikte [4] have studied the limiting behavior of the $P^*(n)$ as $n \to \infty$. As in these models, we will assume throughout this section that $\lambda(\cdot)$ is increasing and convex on $[c,\bar{p}]$. The functional equation (1.4) now becomes

$$V_m(n) = \max_{p\in[c,\bar{p}]} \{G_m(n,p)\} \tag{2.1}$$

where

$$G_m(n,p) = \Pi(p)/n + \beta[V_{m-1}(n) - \lambda(p)\ \Delta\ V_{m-1}(n)]$$

$$n=1,2,\ldots,N$$

$$m=1,2,\ldots \tag{2.2}$$

with

$$\Delta\ V_{m-1}(n) = V_{m-1}(n) - V_{m-1}(n+1), \quad m=1,2,\ldots \tag{2.3}$$

Clearly, the optimal price $p_m(N) = \overline{p}$ and

$$V_m(N) = \Pi(\overline{p})/N + \beta\ V_{m-1}(N), \qquad m=1,2,\ldots \tag{2.4}$$

In order to characterize $p_m(n)$ for $1 \leq n < N$, we need

Lemma 4: If $N > n \geq (N+1)/2$, then

$$0 < (n+1)\ \Delta\ V_m(n+k+1) \leq n\ \Delta\ V_m(n+k), \quad m=1,2,\ldots \tag{2.5}$$

for all $k \geq 0$ satisfying $n + k \leq N - 2$.

Proof: With $m = 1$, (2.5) becomes

$$0 < (n+1)\ \Pi(\overline{p})/[(n+k+1)(n+k+2)] \leq n\Pi(\overline{p})/[(n+k)(n+k+1)] \tag{2.6}$$

or

$$0 < (n+1)/(n+k+2) \leq n/(n+k)$$

which holds provided $k \leq n$. However, this is true for our constraints on n and k that imply $n \geq (N+1)/2 \geq (N-1)/2 \geq k$. Suppose (2.5) holds for m and verify it for (m+1). We consider two cases.

Case 1: $n + k + 2 \leq N - 1$

Abbreviate $p_{m+1}(n+k+1)$ by p^* and note from (2.1) that

$$(n+1)\ V_{m+1}(n+k+1) = (n+1)\left\{\Pi(p^*)/(n+k+1) + \beta\left[V_m(n+k+1) - \lambda(p^*)\cdot \Delta V_m(n+k+1)\right]\right\}$$

while

$$(n+1)\ V_{m+1}(n+k+2) \geq (n+1)\left\{\Pi(p^*)/(n+k+2) + \beta\left[V_m(n+k+2) - \lambda(p^*)\ \Delta V_m(n+k+2)\right]\right\}.$$

so that

$$(n+1)\ \Delta V_{m+1}(n+k+1) \leq (n+1)\left\{\Pi(p^*)/[(n+k+1)(n+k+2)] + \beta\left[(1-\lambda(p^*))\Delta V_m(n+k+1) + \lambda(p^*)\ \Delta V_m(n+k+2)\right]\right\} \quad (2.7)$$

Similarly, since

$$n\ V_{m+1}(n+k) \geq n\left\{\Pi(p^*)/(n+k) + \beta\left[V_m(n+k) - \lambda(p^*)\Delta V_m(n+k)\right]\right\},$$

we have

$$n\ \Delta V_{m+1}(n+k) \geq n\left\{\Pi(p^*)/[n+k)(n+k+1)] + \beta\left[(1-\lambda(p^*))\ \Delta V_m(n+k) + \lambda(p^*)\ \Delta V_m(n+k+1)\right]\right\}$$

$$\geq (n+1)\left\{\Pi(p^*)/[(n+k+1)(n+k+2)] + \beta\left[(1-\lambda(p^*))\ \Delta V_m(n+k+1) + \lambda(p^*)\ \Delta V_m(n+k+2)\right]\right\} \quad (2.8)$$

by the induction hypothesis and (2.6) with $\overline{p}$ replaced by p^*. Hence, from (2.7) and (2.8)

$$n \, \Delta \, V_{m+1}(n+k) \geq (n+1) \, \Delta \, V_{m+1}(n+k+1),$$

completing the induction argument for Case 1.

Case 2: $n + k + 2 = N$

In this case we must prove that

$$(n+1) \, \Delta \, V_{m+1}(N-1) \leq n \, \Delta \, V_{m+1}(N-2). \tag{2.9}$$

Again, denoting $p_{m+1}(N-1)$ by p^*, we have, as in (2.7),

$$(n+1) \, \Delta \, V_{m+1}(N-1) \leq (n+1) \, \Pi(p^*)/[N(N-1)] + \beta(n+1)[1-\lambda(p^*)] \, \Delta \, V_m(N-1) \tag{2.10}$$

and, as in (2.8),

$$n \, \Delta \, V_{m+1}(N-2) \geq n \, \Pi(p^*)/[(N-1)(N-2)] + \beta \, n(1-\lambda(p^*)) \, \Delta V_m(N-2) + \beta \lambda(p^*) n \Delta \, V_m(N-1) \tag{2.11}$$

Since $(n+1) \, \Delta \, V_m(N-1) \leq n \, \Delta \, V_m(N-2)$ and $\Delta \, V_m(N-1) \geq 0$, (2.10) and (2.11) imply (2.9).

Q.E.D.

Let $p_m(n)$ be the smallest price in $[c,\bar{p}]$ attaining the maximum in (2.1), i.e. $V_m(n) = G_m(n,p_m(n))$.

Proposition 4: For $n \geq (N+1)/2$, $p_m(n)$ is a nondecreasing function of n, for all $m \geq 1$.

<u>Proof:</u> Since $\Pi(.)$ is concave and $\lambda(.)$ is convex, $G_m(n,.)$, given by (2.2), is concave on $[c,\bar{p}]$. If $p_m(n) \in (c,\bar{p})$ maximizes $G_m(n,p)$, then $G'(n,p_m(n))=0$

i.e. $\Pi'(p_m(n))/[\lambda'(p_m(n))] = \beta n \, \Delta V_m(n)$ (2.12)

By Lemma 4, $n \geq (N+1)/2$, implies

$$(n+1) \, \Delta V_m(n+1) \leq n \, \Delta V_m(n) \tag{2.13}$$

Since $\Pi'(p)/\lambda'(p)$ is nonincreasing in $p \in [c,\bar{p}]$, (2,12) and (2.13) imply that $p_m(n+1) \geq p_m(n)$, $n \geq (N+1)/2$, $m \geq 1$.

If $p_m(n) = c$, then clearly $p_m(n+1) \in [c,\bar{p}]$ implies $p_m(n+1) \geq p_m(n)$. Finally, if $p_m(n) = \bar{p}$,

$$\Pi'(\bar{p})/\lambda'(\bar{p}) \geq \beta n \, \Delta V_m(n)$$
$$\geq \beta(n+1) \, \Delta V_m(n+1), \; n \geq (N+1)/2,$$

which implies that $G_m'(n+1,\bar{p}) \geq 0$, i.e. $p_m(n+1) = \bar{p}$

Q.E.D.

Thus, we have a curious result that the optimal pricing strategy is monotone nondecreasing (in the usual sense) once the industry becomes at least half saturated. We do not know if the monotonicity holds when the satuation level is below this critical point.

3. <u>The Sensitivity Analysis</u>

In this section we consider the dependence of the optimal pricing strategy on the interest rate and the length of the planning horizon when the industry consists of identical firms drawn from an infinite population. Our first result is that the optimal price is nondecreasing in the interest rate α, as to be expected, since a higher α means that short term profits (until as entry) become more valuable, in comparison with the long term profits (after the entry).

<u>Proposition 5</u>: The optimal pricing strategy $P^*(\cdot)$ is non-decreasing in α.

<u>Proof</u>: In the m stage process with discount factor $\beta = \Lambda/(\alpha+\Lambda)$, let $V_m(n,\beta)$ be the optimal expected discounted reward starting in an initial state $n \geq 1$. Thus,

$$V_m(n,\beta) = \underset{p\in[c,\bar{p}]}{\text{Max}} \left\{ \bar{\Pi}(p)\beta/(\Lambda n) + \beta\left[\lambda(p)\ V_{m-1}(n+1,\beta) + (1-\lambda(p))\ V_{m-1}(n,\beta)\right]\right\} \quad . \tag{3.1}$$

with $V_0(n,\beta) = 0$.

To show that the optimal price at stage m is non-increasing in β it suffices to show that, for $\beta_2 \geq \beta_1$ we have

$$V_{m-1}(n,\beta_2) - V_{m-1}(n+1,\beta_2) \geq V_{m-1}(n,\beta_1) - V_{m-1}(n+1,\beta_1). \tag{3.2}$$

Again we use induction and observe that (3.2) holds for $m = 2$, since $V_1(n,\beta) = \bar{\Pi}(\bar{p})\beta/(n\Lambda)$. Suppose that (3.2) holds for $m = k + 1$. Now write $p_1 = p_{k+1}\ (n+1,\beta_2)$ and $p_2 = p_{k+1}(n,\beta_1)$. Then

$$V_{k+1}(n,\beta_2) - V_{k+1}(n+1,\beta_2)$$

$$\geq \overline{\Pi}(p_2)\beta_2/(\Lambda n) - \overline{\Pi}(p_1)\beta_2/[\Lambda(n+1)] + \beta_2 \Big\{ \lambda(p_2)\ V_k(n+1,\beta_2)$$

$$+ [1-\lambda(p_2)]\ V_k(n,\beta_2) - \lambda(p_1)\ V_k(n+2,\beta_2)$$

$$- [1-\lambda(p_1)]\ V_k(n+1,\beta_2)\Big\} \tag{3.3}$$

and

$$V_{k+1}(n,\beta_1) - V_{k+1}(n+1,\beta_1)$$

$$\leq \overline{\Pi}(p_2)\beta_1/(\Lambda n) - \overline{\Pi}(p_1)\beta_1/[\Lambda(n+1)] + \beta_1 \Big\{ \lambda(p_2)\ V_k(n+1,\beta_1)$$

$$+ [1-\lambda(p_2)]\ V_k(n,\beta_1) - \lambda(p_1)\ V_k(n+2,\beta_1)$$

$$- [1-\lambda(p_1)]\ V_k(n+1,\beta_1)\Big\} \quad . \tag{3.4}$$

Hence, from (3.3) and (3.4), we will be able to show that (3.2) holds for $m = k + 2$ if

$$\lambda(p_2)\ V_k(n+1,\beta_2) + [1-\lambda(p_2)]\ V_k(n,\beta_2)$$

$$- \lambda(p_1)\ V_k(n+2,\beta_2) - [1-\lambda(p_1)]\ V_k(n+1,\beta_2)$$

$$\geq \lambda(p_2)\ V_k(n+1,\beta_1) + [1-\lambda(p_2)]\ V_k(n,\beta_1)$$

$$- \lambda(p_1)\ V_k(n+2,\beta_1) - [1-\lambda(p_1)]\ V_k(n+1,\beta_1)$$

i.e. if

$$[1-\lambda(p_2)]\ [V_k(n,\beta_2) - V_k(n+1,\beta_2)]$$

$$+ \quad \lambda(p_1)\ [V_k(n+1,\beta_2) - V_k(n+2,\beta_2)]$$

$$\geq [1-\lambda(p_2)]\ [V_k(n,\beta_1) - V_k(n+1,\beta_1)]$$

$$+ \quad \lambda(p_1)\ [V_k(n+1,\beta_1) - V_k(n+2,\beta_1)].$$

But this is immediate from the induction hypothesis, so that (3.2) holds for all m. Letting $m \to \infty$ and considering the corresponding functional equation, the result follows.

Q.E.D.

It seems clear that a firm facing a shorter horizon should be less influenced by the threat of new entrants, and will therefore charge a higher price. This is proved in the following result.

Proposition 6: For $m \geq 2$, $p_m(n) \geq p_{m-1}(n)$, $n=1,2,\ldots.$

Proof: It suffices to show that the following inequality holds for each m.

$$V_{m-2}(n)-V_{m-2}(n+1) \leq V_{m-1}(n) - V_{m-1}(n+1) \qquad (3.5)$$

We prove (3.5) by induction. For $m = 2$, it reduces to $\Pi(\bar{p})/n - \Pi(\bar{p})/(n+1) \geq 0$, which is true. Suppose that (3.5) holds for $m = k$. Let $p_1 = p_{k-1}(n)$ and $p_2 = p_k(n+1)$.

Then

$$V_k(n) - V_k(n+1) \geq \Pi(p_1)/n - \Pi(p_2)/(n+1) + \beta\Big\{\lambda(p_1)\ V_{k-1}(n+1) +$$

$$[1-\lambda(p_1)]\ V_{k-1}(n) - \lambda(p_2)\ V_{k-1}(n+2)$$

$$- [1-\lambda(p_2)]\ V_{k-1}(n+1)\Big\} \qquad (3.6)$$

and

$$V_{k-1}(n) - V_{k-1}(n+1) \leq \Pi(p_1)/n - \Pi(p_2)/(n+1)$$

$$+ \beta\Big\{\lambda(p_1)\ V_{k-2}(n+1) +$$

$$[1-\lambda(p_1)]\ V_{k-2}(n) - \lambda(p_2)\ V_{k-2}(n+2)$$

$$- [1-\lambda(p_2)]\ V_{k-2}(n+1)\Big\} \qquad (3.7)$$

Together (3.6) and (3.7) imply that (3.5) will hold with $m = k + 1$ if

$$[1-\lambda(p_1)]\ \Delta V_{k-1}(n) + \lambda(p_2)\ \Delta V_{k-1}(n+1)$$

$$\geq [1-\lambda(p_1)]\ \Delta V_{k-2}(n) + \lambda(p_2)\ \Delta V_{k-2}(n+1)$$

where $\Delta V_k(n) = V_k(n) - V_k(n+1)$.

The last inequality, however, is a direct consequence of the induction hypothesis, completing the proof.

Q.E.D.

Suppose that the dominant firm wishes to maximize the expected discounted profit earned over a (continuous) time horizon of duration t. If we let $p_t(n)$ be the optimal price when time t remains and the industry size is n, then by combining our results with a modification of Theorem 10.5 of Whitt [18] we may prove the following result.

Proposition 7: Given any $p_0 < \bar{p}$, there exists an integer $n_t^*(p_0)$, which is nondecreasing in p_0, such that $p_t(n) \geq p_0$ if $n \geq n_t^*(p_0)$, for all $t \in [0,\infty)$. Also $p_t(n)$ is nondecreasing in α and t.

Thus far we have assumed that, since each firm makes non-negative profit (by Lemma 1), there is no incentive for a firm to leave the industry. We conclude the paper by indicating the extension of the model to the case where the established firms may exist stochastically from the industry, depending upon the price prevailing.

4. Stochastic Entry and Exits: An M/M/∞ Queue Control Problem

Consider homogeneous firms, which are drawn into the industry from an infinite population and which share the total industry profit equally. Suppose that, if the prevailing price is p, new firms enter the industry according to the Poisson process with rate $\lambda(p)$, while an established firm remains in the industry for a duration which is exponentially distributed with parameter $\mu(p)$. It is meaningful to assume that $\lambda(\cdot)$ is nondecreasing and $\mu(\cdot)$ is nonincreasing, both being continuous on $[c,\bar{p}]$. Thus, setting a higher price increases current profits but also attracts more new firms and induces the existing ones to stay on, thereby increasing future competition and decreasing future profits. Conversely, setting a lower price decreases current profits but also discourages new competition and drives the existing firms out of the industry faster, thereby decreasing future competition and increasing future profits.

The problem of selecting an optimal pricing strategy, as a function of the number $n \geq 2$ of firms in the industry may then be looked upon as that of optimally controlling an $M/M/\infty$ queue, in which a single control variable $p \in [c,\bar{p}]$ affects the arrival rate $\lambda(p)$ as well as the "service" rate $n\mu(p)$ along with the reward rate $\bar{\Pi}(p)/n$. Such a model may be considered as a combination of the model in Crabill [3], (for controlling the service rate in an M/M/1 queue) and the one by Low [12] (for controlling the arrival rate in an M/M/S system). Thus, a seemingly unrelated problem of economic interest can be formulated in the queue control framework. For this problem the finite stage functional equations become

$$V_m(n) = \underset{p \in [c,\bar{p}]}{\text{Max}} \left\{ \Pi(p)/n + \beta \left[V_{m-1}(n) - \lambda(p)\, \Delta V_{m-1}(n) + n\mu(p)\, \Delta V_{m-1}(n-1) \right] \right\} \quad n=2,3,\ldots, \qquad (4.1)$$

where

$$\Delta V_{m-1}(n) = V_{m-1}(n) - V_{m-1}(n+1).$$

However, it does not seem possible to analyze (4.1) further in order to characterize the structure of the optimal pricing strategy $p_m(n)$ as in the previous sections.

Furthermore, we would expect that the finite stage functional equations (4.1) can be extended to obtain the usual infinite horizon functional equation

$$rV(n) = \underset{p\in[c,\overline{p}]}{\text{Max}} \left\{ \Pi(p)/n - \lambda(p)\ \Delta V(n) + n\mu(p)\ \Delta V(n-1) \right\} \qquad n=2,3,..., \tag{4.2}$$

with $\Delta V(n) = V(n) - V(n+1)$, and that the optimal expected discounted reward $V(\cdot)$ is a unique bounded solution of (4.2). However, in this problem, the number of transitions taking place in any time interval is not bounded uniformly in n, since the transition rate is $\lambda(p) + n\mu(p)$, $p \in [c,\overline{p}]$, n=2,3,... Hence the maximum operator implied in (4.2) does not possess the required contraction property and, consequently, (4.2) may not have a unique solution and there may not exist an optimal stationary strategy. On the other hand, because the death rate is linearly increasing in n, one would expect that the number of firms in the industry does not grow indefinitely, so that the usual results still hold. Toward this end, we may first consider the bounded problem, $\mathcal{P}^N$ where the industry size is forced to be less than N, so that the existence results hold and then let $N \to \infty$ and show that the results continue to hold in the limit. However, we have not been able to carry out this approach successfully. Moreover, even assuming that everything works on theoretical grounds we have not been able to establish the form of the infinite horizon optimal pricing policy $P^*(n)$. Other interesting questions would be concerned with finding the probability distribution of the number of firms in the industry or the amount of time each firm spends in the industry under optimal pricing strategy.

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REFERENCES

[1] Baron, D.P., "Limit Pricing, Potential Entry, and Barriers to Entry," American Economic Review, 63 (1973), pp.666-674.

[2] Bhagwati, N.J., "Oligopoly Theory, Entry Prevention and Growth," Oxford Economic Papers, Vol. 22, No. 3 (November 1970), pp.297-310.

[3] Crabill, T., "Optimal Control of a Service Facility with Variable Exponential Service Times and Constant Arrival Rate," Management Science, 18 (1972) pp.560-566.

[4] Deshmukh, S.D., and S.D. Chikte, "Dynamic Pricing with Stochastic Entry," Review of Economic Studies, XLIII, No. 1 (1976), pp.91-97.

[5] Gaskins, D.W. Jr., "Dynamic Limit Pricing: Optimal Pricing under Threat of Entry," Journal of Economic Theory, 3 (1971), pp.306-322.

[6] Howard, R.A., Dynamic Programming and Markov Processes, MIT Press, 1960.

[7] Kamien, M.I. and N.L. Schwartz, "Limit Pricing and Uncertain Entry," Econometrica, 39, No. 3 (May 1971), pp.441-454.

[8] Leeman, W., "The Reduction of Queues Through the Use of Price," Operations Research, 12 (1964) pp.783-785.

[9] Lippman, S.A., "Applying a New Device in the Optimization of Exponential Queueing Systems," Operations Research, 23, 4 (July-August 1975), pp.687-710.

[10] Lippman, S.A., "Countable State Continuous Time Dynamic Programming with Structure," Operations Research, Vol. 24, No.3 (May-June 1976) pp.477-490.

[11] Lippman, S.A., and S. Stidham, "Individual vs. Social Optimization in Exponential Congestion System," Operations Research, Vol. 25, No.3 (March-April 1977), pp. 233-247.

[12] Low, D., "Optimal Dynamic Pricing Policies for an M/M/S Queue," Operationa Research, 22 (1974), pp.545-561.

[13] Maitra, A., "Discounted Dynamic Programming on Compact Metric Spacis," Sankhya, Series A, Vol. 30, part 2, (1968), pp.211-216.

[14] Modigliani, F., "New Developments on the Oligopoly Front," Journal of Political Economy, 46 (1958), pp.215-232.

[15] Ross, S.M., Applied Probability Models with Optimization Applications, Holden Day, 1970.

[16] Sobel, M. J., "Optimality of Full Service Policies," presented at the ORSA-TIMS Joint National Meeting in Las Vegas (November 1975).

[17] Stidham, S. and N. U. Prabhu, "Optimal Control of Queueing Systems," in Mathematical Methods of Queueing Theory, (ed.A.B. Clarke) Springer-Verlag, Berlin 1973, pp.263-294.

[18] Whitt, W., "Continuity of Markov Processes and Programs," Technical report, School of Organization and Management, Yale University, 1975.

[19] Yechiali, U., "On Optimal Balking Rules and Toll Charges in the GI/M/1 Queueing Process," Operations Research 19, (1971), pp.349-370.