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STRAIGHTFORWARDNESS OF GAME FORMS
WITH LOTTERIES AS OUTCOMES

by

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ABSTRACT: Where alternatives, players, and strategies for each player are finitely many, a game form assigns a lottery over alternatives to each configuration of players' strategies. It is straightforward iff it guarantees that each player, whatever his utilities, will have a dominant strategy. It is unilateral iff only one player can influence the outcome, and duple iff it restricts the final outcome to a fixed pair of alternatives. Any straightforward game form, it is shown, is, on a domain which gives each player a dominant strategy for each utility scale, a probability mixture of game forms, each unilateral or duple.
1. INTRODUCTION

A game form is a system which makes an outcome depend on individual actions of some kind, called strategies. It is thus a "game" in the sense of von Neumann and Morgenstern (1947).

A game form is determinate if the dependence of the outcome on individual strategies involves no element of chance. A player's strategy is dominant for him with respect to a weak ordering of the alternatives iff no matter what anyone else does, the strategy secures an outcome at least as high in that weak ordering as is any other lottery he can secure given the strategies of others. A determinate game form is straightforward iff each player, for each weak ordering of the alternatives, has a strategy which is dominant with respect to that weak ordering. The only straightforward determinate game forms are trivial: a straightforward determinate game form either restricts the attainable outcomes in advance to no more than two, or makes one player a dictator among attainable outcomes. (This is shown in Gibbard, 1973).

This paper deals with game forms of a more general kind: systems which make an outcome depend on individual strategies in a way that may involve chance. The systems to be considered have finitely many alternatives, finitely many players, and finitely many strategies for each player. A game form in this expanded sense assigns to each configuration of individual strategies, or strategy profiles, a lottery among alternatives.
Whether a game form makes a given strategy dominant now depends on a player's preferences among lotteries: a strategy is dominant for a player with respect to a weak ordering of all lotteries over alternatives if no matter what anyone else does, the strategy secures a lottery at least as high in that weak ordering as is any other lottery which he could secure given the strategies of others. A weak ordering of lotteries is coherent iff it satisfies standard conditions, such as those of von Neumann and Morgenstern (1947, p. 26), which entail that it is an ordering by expected utility on some cardinal scale; only those weak orderings of lotteries which are coherent will concern us here. A game form is straightforward iff each player, for each coherent weak ordering of all lotteries over the alternatives, has a strategy which is dominant with respect to that weak ordering.

Any straightforward game form, it will be shown, is, on a slightly restricted domain of individual strategies, a probability mixture of game forms each of which either accords a single player a monopoly of influence, or restricts the final outcome to a fixed pair of alternatives. Game forms of the first kind will be called unilateral, and of the second kind, duplo.

The contents of this theorem are perhaps best elucidated through its application to game forms of a special kind, called "decision schemes". A decision scheme is a game form in which players vote weak orderings of the alternatives. The strategy
set of each player, then, consists of all weak orderings of the alternatives; as a game form, a decision scheme assigns probabilities to the alternatives on the basis of the way people vote. The theorem in this paper is a generalization of a theorem about decision schemes (in Gibbard, 1976). We can think of the weak ordering a player votes as truly or falsely representing his preferences among the alternatives, and ask what kinds of decision schemes, if any, logically guarantee that no player will ever benefit from misrepresenting his preferences. A decision scheme will be called strategy-proof iff it logically guarantees that honest voting is always a dominant strategy. Now whether a strategy is dominant depends on a voter's preferences among various lotteries the scheme may yield. Honest voting consists in writing down the weak ordering of alternatives as sure things which, in the obvious sense, fits one's weak ordering of all lotteries over alternatives. A decision scheme is strategy-proof, then, iff for any voter and for any coherent ordering $P^*$ of all lotteries over alternatives, voting the weak ordering of the alternatives which fits $P^*$ is a dominant strategy with respect to $P^*$. (A strategy-proof decision scheme is thus straightforward).

There indeed are strategy-proof decision schemes: one is a random dictatorship, a second is pairwise majority rule over a random pair, and a third is a system which chooses randomly between the first two. (These are discussed in Gibbard, 1976). The theorem proved in that paper is that all strategy-proof decision schemes resemble the ones that
were just described. In the senses given earlier to the terms 'unilateral' and 'duple', a random dictatorship is a probability mixture of decision schemes which are unilateral; pairwise majority rule over a random pair is a probability mixture of decision schemes which are duple, and a random choice between these two systems yields a probability mixture of decision schemes of the two kinds. The earlier theorem, roughly put, is that any strategy-proof decision scheme is a probability mixture of decision schemes, each of which is either unilateral or duple. This rough statement, though, needs an important qualification: the statement holds in general only for a restricted domain, on which voters do not express indifference between alternatives. A correct statement of the theorem is this: If a decision scheme is strategy-proof, then on a domain obtained by restricting voters' ballots to strong orderings, the scheme is a probability mixture of decision schemes, each of which is either unilateral or duple.\(^3\)

This result is generalized by the theorem to be proved in this paper. A set of strategies is adequate for a player iff for any coherent weak ordering of all lotteries over the alternatives, the set includes a strategy which is dominant for the player with respect to that weak ordering. In a weakened form, the theorem in this paper is as follows: if a game form \(g\) is straightforward, then each player \(i\) can be assigned an adequate subset \(S_i\) of his strategy set in such a way that when the strategies of each player \(i\) are
restricted to the set $S_f$, $g$ is a probability mixture of game forms each of which is unilateral or dupl. Now take a strategy-proof decision scheme $d$. Since $d$ is strategy-proof, it is straightforward. It can be shown, moreover, that the set of all strong orderings of the alternatives constitutes a set of strategies which is adequate. The theorem on decision schemes says that when the strategy set for each player is restricted to strong orderings of the alternatives, $d$ is a probability mixture of unilateral and dupl decision schemes.

The theorem in this paper also specifies an assignment of a strategy subset $S_i$ to each player $i$ that will do the required job. A coherent weak ordering of lotteries can be represented by a cardinal utility scale, which is an assignment of a real number to each alternative. Such an assignment can be regarded as a vector in a space with a dimension for each alternative. For each player and strategy, let its domain of dominance be the set of all utility vectors with respect to which it is dominant. We can ask whether the domain of dominance of a strategy has interior points; if it does, call the strategy versatile. The theorem in this paper in its full form is that where $g$ is a straightforward game form, (i) for each player, the set of versatile strategies is adequate, and (ii) with each player restricted to versatile strategies, $g$ is a probability mixture of game forms, each of which is either unilateral or dupl.
Now if a decision scheme is strategy-proof, then every strong ordering is versatile. For its domain of dominance, by the definition of strategy-proofness, includes every utility scale which it fits: where \( \mathbf{P} \) orders the alternatives \( x_M x_{M-1} \ldots x_2 x_1 \), its domain of dominance includes every utility scale \( U \) such that

\[ U(x_M) \geq U(x_{M-1}) \geq \ldots \geq U(x_2) \geq U(x_1). \]

This set of utility scales has interior points; an example is the scale \( U^* \) such that \( U^*(x_i) = i \) for each \( 1 \leq i \leq m \).

From this fact and the theorem in this paper, then, the theorem on strategy-proof decision schemes follows: A strategy-proof decision scheme is, when the strategy set of each player is restricted to strong orderings, a probability mixture of unilateral and duple decision schemes.

The theorem on straightforward game forms has two corollaries which may help to clarify its import; the corollaries tell how requirements beyond straightforwardness yield an even more restricted class of game forms. One property we might desire in a game form is that if everyone likes the same alternative best, they should be able jointly to bring about its selection. For each alternative, then, there should be a strategy profile which results in that alternative as a sure thing. A straightforward game form which satisfies this requirement, the first corollary says, is, with each player restricted to his versatile strategies, a probability mixture of game forms which are dictatorial, where a game form is
dictatorial iff there is a player who, for each alternative, has a strategy which independently of what anyone else does produces that alternative as a sure thing. 4

A further property that might be demanded of a game form is this. Assign each player a weak ordering \( P_1 \) of all lotteries over the alternatives. A lottery is Pareto optimal \textit{ex ante} with respect to that assignment iff there is no other lottery which ranks higher in everyone's ordering. 5 We might require of a game form that for each assignment of a coherent weak ordering of all lotteries to each player, the game form yield, for some strategy profile, a lottery which is Pareto optimal \textit{ex ante} with respect to that assignment. A straightforward game form that satisfies this requirement is itself dictatorial; that is the second corollary. These corollaries are formally stated in Section 3.

The nonstraightforwardness of certain voting procedures was studied by Fagruharson (1969), who introduced the term 'straightforward' in a somewhat different sense from that used here. Zeckhauser (1969, 1973) considers systems of voting with lotteries as outcomes, and studies a property closely related to straightforwardness (1973). Gibbard (1973) discusses game forms with no element of chance, and proves that any straightforward game form of this kind is either dictatorial or duple. An equivalent theorem about non-manipulability of voting schemes is proved independently by Gibbard (1973) and by Satterthwaite (1975). The theorem by Gibbard (1976) on strategy-proof decision schemes has already been
discussed here. The proof of the theorem in this paper follows the proof of the earlier theorem at a higher level of generality. Other work related to the topic of this paper is reviewed in that earlier paper (p. 6).
Let $V$ be a finite non-empty set. A lottery $\mu$ over $V$ is a real-valued function whose domain is $V$, such that for each $x \in V$, $\mu(x) \geq 0$, and $\sum_{x \in V} \mu(x) = 1$. An n-person profile set $\mathcal{A}$ is a Cartesian product $S_1 \times \ldots \times S_n$, where each $S_i$ is a finite non-empty set. A game-form is a function $g$ such that for some positive integer $n$, the domain of $g$ is an n-person profile set $\mathcal{A}$, and for some finite non-empty set $V$, the values of $g$ are lotteries over $V$. $n$ is called the number of players of $g$. $V$ is called the alternative set of $g$, and members of $V$ are called alternatives of $g$. Members of $\mathcal{A}$ are called strategy-profiles of $g$, and where $\mathcal{A} = S_1 \times \ldots \times S_n$, a member $s_k$ of $S_k$ is called a strategy for player $k$ under $g$. Strategy profiles are written in boldface (or on the typewriter with squiggly underlining) on the pattern $\mathbf{s} = (s_1, \ldots, s_n)$. $s_k \mathbf{t}_{-k}$ is the strategy profile $\mathbf{u}$ such that $u_k = s_k$ and for all $i \neq k$, $u_i = t_i$. $\mathbf{t}_{-k}$ is called an environment for $k$. $g(x, \mathbf{s})$ is the probability $g(\mathbf{u})$ assigns to $x$.

A utility scale $U$ over $V$ is a real-valued function whose domain is $V$. Thus both utility scales over $V$ and lotteries over $V$ are vectors in the space $\mathbb{R}^V$ of all functions from $V$ into the reals. Where $U$ is a utility scale over $V$ and $\mu$ is a lottery over $V$, the utility of $\mu$ on scale $U$ is the inner product $U \cdot \mu = \sum_{x \in V} U(x) \mu(x)$. Let $g$ be a game form with domain $S_1 \times \ldots \times S_n$ and alternative set $V$. Strategy $s_k \in S_k$ is $U$-dominant for $k$ iff
for every strategy profile \( \frac{1}{n} \) of \( g \), \( U \cdot g(s_k \frac{1}{n} s_k) \geq U \cdot g(\frac{1}{n}) \). \( g \) is straightforward iff for every \( k \) and \( U \), there is a strategy \( s_k \in S_k \) which is \( U \)-dominant for \( k \). Game form \( g \) is unilateral iff there is a \( k \) such that for all \( s \) and \( t \) in its domain, if \( s_k = t_k \) then \( g(s) = g(t) \). \( g \) is duple iff there are alternatives \( x \) and \( y \) such that for all \( z \in \{x, y\} \) and for all \( g \) in its domain, \( g(z, z) = 0 \). \( g \) is a probability mixture of game forms \( g_1, \ldots, g_d \) on domain \( \tau \) iff there is a lottery \( \tau \) over \( \{g_1, \ldots, g_d\} \) such that, where for each positive integer \( i \leq d \), \( \tau_i = \tau(g_i) \), for every \( g \in \tau \) and for every \( x \), \( g(x, z) = \tau_1 g_1(x, z) + \ldots + \tau_d g_d(x, z) \).

Let \( U_k(s_k) \), the domain of dominance for \( k \) of strategy \( s_k \in S_k \), be the set of all \( U \) such that \( s_k \) is \( U \)-dominant for \( k \).

**Theorem.** Let \( g \) be a straightforward game form with domain \( \mathcal{L} = S_1 \times \ldots \times S_n \) and alternative set \( V \). For each player \( i \), let \( T_i^\ast \) be the set of every \( s_i \in S_i \) whose domain of dominance has a non-empty interior in \( H^i \).

Then (i) for any \( U \) and \( i \), there is an \( s_i \in T_i^\ast \) such that \( s_i \) is \( U \)-dominant for \( i \), and (ii) on the domain \( \tau^\ast = T_1^\ast \times \ldots \times T_n^\ast \), \( g \) is a probability mixture of game forms each of which is either unilateral or duple.
Notation: Let $m$ be $M$, the number of alternatives. Variables will range as follows:

$x, y, z$: Alternatives in $V$.

$i, j, k$: Positive integers $\leq n$, called players.

$s_i, t_i, u_i, v_i$: Strategies for $i$, which are members of $S_i$.

$U$: Utility scales over $V$.

Superscripts, primes, and the like do not affect the range of variables. Sections of the proof are numbered for reference.

1. Let $G_k(s_k, t_k, y_{-k})$, the domain in which $s_k$ is as good as $t_k$ in environment $y_{-k}$, be

$$\{U : U \cdot g(s_k y_{-k}) \geq U \cdot g(t_k y_{-k})\}.$$ 

Where $g(s_k y_{-k}) \neq g(t_k y_{-k})$, this is the closed half-space

$$\{U : U \cdot [g(s_k y_{-k}) - g(t_k y_{-k})] \geq 0\}.$$ 

Let $Y_k(s_k y_{-k})$, the domain of maximality of $s_k$ in environment $y_{-k}$ be

$$\bigcap \{U_k(s_k, t_k, y_{-k}) \mid t_k \in S_k\},$$

the intersection of the domains in which $s_k$ is as good as $t_k$ in $y_{-k}$ for all $t_k$ which are members of $S_k$. $D_k(s_k)$, the domain of dominance of $s_k$, is then

$$\bigcap \{Y_k(s_k, y_{-k}) \mid y_{-k}\}.$$

$D_k(s_k)$ is thus a finite intersection of closed half-spaces; it is therefore closed and convex, and its boundaries are surfaces of dimension $m - 1$. To say that $g$ is straightforward is to say that for every $U$ and $k$, there is an $s_k \in S_k$ such that $U \in D_k(s_k)$, so that the sets $D_k(s_k)$ cover $\mathbb{R}^V$. 

2. The set $T_k^*$ of members of $S_k$ whose domains of dominance have interiors covers $R^V$. Proof: Let $H_1, \ldots, H_d$ be closed sets which cover $R^V$, and let sets $H_1, \ldots, H_p$ have interior points and $H_{p+1}, \ldots, H_d$ not. Then $H^* = H_{p+1} \cup \ldots \cup H_d$ has no interior points, and thus any point in $H^*$ is in the closure of $R^V \setminus H^*$, which is a subset of $H_1 \cup \ldots \cup H_p$. Since these sets are closed, one of them contains each boundary point of their union. Since the sets $D_k(s_k)$ with $s_k \in S_k$ are closed and cover $R^V$, the assertion follows.

3. Strategies $s_k$ and $t_k$ are equivalent for $k$ iff for every $y_k$, $g(s_k, y_k) = g(t_k, y_k)$; we write this $s_k \approx_k t_k$. Assertion: If $D_k(s_k)$ and $D_k(t_k)$ have an interior point in common, then $s_k \approx_k t_k$. Proof: If $g(s_k, y_k) \neq g(t_k, y_k)$, then $G_k(s_k, t_k, y_k)$ and $G_k(t_k, s_k, y_k)$ have no interior points in common. Since $D_k(s_k) \subseteq G_k(s_k, t_k, y_k)$ and $D_k(t_k) \subseteq G_k(t_k, s_k, y_k)$, $D_k(s_k)$ and $D_k(t_k)$ have no interior points in common.

4. The relation $\approx_k$ divides $T_k^*$ into equivalence classes. For each $k$, let $T_k$ consist of one and only one member of each of these equivalence classes. Let $T = T_1 \times \ldots \times T_n$. The sets $T_1, \ldots, T_n$ have these characteristics: $T_k \subseteq T_k^* \subseteq S_k$. $R^V$ is covered by sets $D_k(s_k)$ such that $s_k \in T_k$. For each $s_k \in T_k$, $D_k(s_k)$ has interior points. For any two members $s_k, t_k$ of $T_k$, $D_k(s_k)$ and $D_k(t_k)$ have pairwise disjoint interiors.
5. Where \( s_k, t_k \in T_k \) and \( D_k(s_k) \) and \( D_k(t_k) \) intersect in a surface of dimension \( m-1 \), we say that \( s_k \) and \( t_k \) are adjacent. Let \( A_k \) be the set of all adjacent pairs of strategies in \( T_k \). Where \( \{s_k, t_k\} \in A_k \) or \( s_k = t_k \), the effect \( \varepsilon_k(s_k, t_k, y_{-k}) \) of \( k \)'s switching from \( s_k \) to \( t_k \) in environment \( y_{-k} \) is the vector \( g(t_k y_{-k}) - g(s_k y_{-k}) \). We now show that for adjacent strategies, the direction of the effect of a switch is independent of environment.

6. Let \( \{s_k, t_k\} \in A_k \), and let \( \varepsilon_k(s_k, t_k, y_{-k}) \neq 0 \). Then \( \varepsilon_k(s_k, t_k, y_{-k}) \) is orthogonal to the boundary \( D_k(s_k) \cap D_k(t_k) \), and points in the sense from \( D_k(s_k) \) to \( D_k(t_k) \). Proof: Since \( D_k(s_k) \subseteq G_k(s_k, t_k, y_{-k}) \) and \( D_k(t_k) \subseteq G_k(t_k, s_k, y_{-k}) \), where \( z = G_k(s_k, t_k, y_{-k}) \cap G_k(t_k, s_k, y_{-k}) \), we have \( D_k(s_k) \cap D_k(t_k) \subseteq z \). Now \( z \) is the set

\[
\{U : U \cdot g(s_k y_{-k}) = U \cdot g(t_k y_{-k})\};
\]

thus for \( U \in z \), \( U \cdot [g(t_k y_{-k}) - g(s_k y_{-k})] = 0 \), and \( z \) is orthogonal to \( \varepsilon_k(s_k, t_k, y_{-k}) \). Let \( U_t \) lie in the interior of \( D_k(t_k) \) and \( U_s \) lie in the interior of \( D_k(s_k) \), with \( U_t - U_s \) orthogonal to \( z \). Then \( U_t \cdot g(t_k y_{-k}) > U_t \cdot g(s_k y_{-k}) \) and \( U_s \cdot g(t_k y_{-k}) < U_s \cdot g(s_k y_{-k}) \). Therefore

\[
(U_t - U_s) \cdot [g(t_k y_{-k}) - g(s_k y_{-k})] > 0,
\]

and \( \varepsilon_k(s_k, t_k, y_{-k}) \) has the sense of \( U_t - U_s \).
7. Any lottery lies in the simplex $S^{m-1}$ of $\mathbb{R}^V$ such that $\Sigma_X\varphi(x) = 1$ and $\varphi(x) \geq 0$ for each $x$. This simplex is of dimension $m-1$. A direction $\varphi$ of $S^{m-1}$ is a vector of length one such that $\Sigma_X\varphi(x) = 0$. We have shown that for each pair $(s_k, t_k) \in A_k$, there is a direction $\vartheta_k(s_k, t_k)$ which has this property: for every $\varphi_k$, there is a real $\alpha > 0$ such that $\epsilon_k(s_k, t_k, \varphi_k) = \alpha \vartheta_k(s_k, t_k)$.

8. Now for a fixed environment $\varphi_k$, consider the set of lotteries $\varphi_k(s_k, t_k)$ that $k$ can attain with various strategies $s_k \in \mathcal{T}_k$. They form the vertices of a convex polyhedron $H_k(\varphi_k)$ in $S^{m-1}$ (where more than one strategy may coincide at a vertex). For if $\varphi_k(s_k, t_k)$ is a convex combination of other distinct $\varphi_k(t_k, u_k, \varphi_k)$ and $\varphi_k(u_k, t_k, \varphi_k)$, it can be maximal only on $G_k(t_k, u_k, \varphi_k) \cap G_k(u_k, t_k, \varphi_k)$, and hence its domain of maximality has no interior. A course from vertex $p$ to vertex $q$ of a polyhedron $H$ is a sequence $p_0, \ldots, p_n$ such that $p_0 = p$, $p_n = q$, and each line segment $p_i, p_{i+1}$ is an edge of $H$. A line segment $pq$ lies in direction $\varphi$ iff for some real $\alpha$, $p - q = \alpha \varphi$. The following lemma about convex polyhedra is proved in the Appendix to the proof.

**Lemma 1.** Let $H$ be a convex polyhedron, and where $\psi$ and $\varphi$ are directions, let $p$ and $q$ be vectors such that $p$ is the sole $\psi$-maximal vertex of $H$ and $q$ is a $\varphi$-minimal vertex of $H$. Then there is a course $p_0, \ldots, p_n$ on $H$ from $q$ to $p$ such that (i) for each $i$ such that $1 \leq i \leq n$, $\varphi^T(p_i - p_{i-1}) > 0$, and (ii) if $\psi \varphi \leq 0$, then none of the edges $p_{i-1}, p_i$ lie in direction $\varphi$. 
9. For any \( k \), \( s_k \), and \( \varphi \), either (i) for every \( Y_k \),
\( \epsilon(\nu_k Y_k) \) is connected to every \( \varphi \)-minimal vertex of \( \Gamma_k(Y_k) \)
by a course none of whose segments lie in direction \( \varphi \), or
(ii) for every \( Y_k \), \( \epsilon(\nu_k Y_k) \) is connected to every \( \varphi \)-
maximal vertex of \( \Gamma_k(Y_k) \) by a course none of whose segments
lie in direction \( \varphi \). \textbf{Proof:} Since \( \lambda_k(s_k) \) has interior
points, there is a direction \( \nu \) in the interior of \( \lambda_k(s_k) \)
such that \( \nu \cdot \varphi \neq 0 \). For any \( Y_k \), \( \epsilon(\nu_k Y_k) \) is a uniquely
\( \nu \)-maximal vertex of \( \Gamma_k(Y_k) \). By Lemma 1, if \( \nu \cdot \varphi < 0 \),
then \( \epsilon(\nu_k Y_k) \) is connected to every \( \varphi \)-minimal vertex of
\( \Gamma_k(Y_k) \) by a course none of whose segments lie in direction \( \varphi \).
If \( \nu \cdot \varphi > 0 \), then \( \nu \cdot (-\varphi) < 0 \), and \( \epsilon(\nu_k Y_k) \) is connected
to every \((-\varphi)\)-minimal vertex by such a course.

10. For each \( i \) and \( \varphi \), we now let \( T_i^{-\varphi} \) be the set
of \( \nu_k \in \lambda_k \) which satisfy (i) of \( 9 \), and let \( T_i^{\varphi} = \lambda_k \setminus T_i^{-\varphi} \).
\( T_i^{-\varphi} \) may be empty; if it has members, then they satisfy (ii)
of \( 9 \). It follows that for any two members \( s_k \) and \( t_k \) of
\( T_i^{\varphi} \) and for any \( Y_k \), \( \epsilon(\nu_k Y_k) \) is a course from \( \epsilon(s_k Y_k) \) to
\( \epsilon(t_k Y_k) \) on \( \Gamma_k(Y_k) \) none of whose segments lie in direction \( \varphi \).
The same holds for any two members of \( T_i^{-\varphi} \). If \( T_i^{\varphi} \) is
non-empty, we say that \( T_i \) is \( \varphi \)-separable. Clearly \( T_i \) is
\((-\varphi)\)-separable iff \( T_i \) is \( \varphi \)-separable.

11. where \( \varphi = \theta_k(s_k, t_k) \), let \( v_i \in T_i^{\varphi} \) iff \( w_i \in T_i^{-\varphi} \)
for all \( i \neq k \). Then \( \epsilon(s_k, t_k, Y_k) = \epsilon(s_k, t_k, Y_k) \).
\textbf{Proof:} We first prove the assertion for the case where \( Y_k \)
and \( Y_k \) differ only by a single person \( i \)'s switching \( v_j \)
to \( w_j \), where \( j \neq k \), \( \{v_j, w_j\} \in \Lambda_j \), and \( \theta_j(v_j, w_j) \neq -\theta_k(s_k, t_k) \).
Consider four lotteries: \( P_1 = \mathcal{G}(s_1 t_1 \bar{x}_{j-k}) \), \( P_2 = \mathcal{G}(s_1 t_2 \bar{x}_{j-k}) \), \( P_3 = \mathcal{G}(t_1^* t_3 \bar{x}_{j-k}) \), and \( P_4 = \mathcal{G}(s_3 t_4 \bar{x}_{j-k}) \). \( P_1 P_2 P_3 P_4 \) form a quadrilateral which, by \( \mathcal{E} \), has opposite sides parallel.

Hence opposite sides are equal: \( j \)'s switch does not alter the effect of \( k \)'s switch.

Now let \( v_j \in T_{j} \) iff \( w_j \in T_{j} \). Then \( v_j \) can be obtained from \( s_j \) by a sequence of switches, none of which is in direction \( \pm \varphi \). Hence none alters the effect of \( k \)'s switch in direction \( \varphi \). The same argument applies to every other \( i \neq k \), and the assertion of this section is proved.

\[ 12. \] For the remainder of the proof we adopt these notations. Let \( \varphi^* \) be an arbitrary fixed strategy profile.

If \( T_{i}^{\varphi^*} \) is empty, let \( u_{i}^{\varphi^*} \) be an arbitrary fixed member of \( T_{i} \). If \( T_{i}^{\varphi^*} \) is non-empty, then let \( u_{i}^{\varphi^*} \) and \( u_{1}^{\varphi} \) be such that for all pairs \( \{s_j, t_i\} \in \mathcal{A}_{1} \) and \( \theta_i(s_j, t_i) = \varphi \), \( |\mathcal{G}(t_{1}^{*} \bar{x}_{j-k}) - \mathcal{G}(s_{j} t_{1}^{*})| \) is minimal when \( s_j = u_{j}^{\varphi} \) and \( t_i = u_{i}^{\varphi^*} \). We stipulate further that if \( \varphi = -\varphi \), then \( u_{i}^{\varphi} = u_{i}^{-\varphi} \) and \( u_{i}^{\varphi^*} = u_{i}^{\varphi^*} \). An alignment \( \omega \) is a pair of directions \( \{\varphi, -\varphi\} \). The variable \( \omega \) will range over alignments such that for some \( k, s_{k}, \) and \( t_{k} \) with \( \{s_{k}, t_{k}\} \in \mathcal{A}_{k} \), \( \theta_k(s_{k}, t_{k}) \in \omega \). Clearly the class of such alignments is finite. \( T \) is \( \omega \)-separable iff for some \( i \) and some \( \varphi \in \omega \), \( T_i \) is \( \varphi \)-separable. For any \( s_{i} \) and \( \omega \) with \( \varphi \in \omega \), let \( s_{i}^{\omega} = u_{i}^{\varphi} \) iff \( s_i \in T_{i}^{\varphi} \) and let \( s_{i} = u_{i}^{\varphi^*} \) iff \( s_i \in T_{i}^{\varphi^*} \).
13. If \( \{s_k, t_k\} \in \Lambda_k \) and \( \theta_k(s_k, t_k) = \varphi \in \omega \), then for any two environments \( \omega_{-k} \) and \( \omega_{-k} \),

\[
\epsilon_k(s_k, t_k, \omega_{-k}) - \epsilon_k(s_k, t_k, \omega_{-k}) = \epsilon_k(s_k, t_k, \omega_{-k}) - \epsilon_k(s_k, t_k, \omega_{-k}).
\]

Proof: We know from 11 that \( \epsilon_k(s_k, t_k, \omega_{-k}) = \epsilon_k(s_k, t_k, \omega_{-k}) \) and \( \epsilon_k(s_k, t_k, \omega_{-k}) = \epsilon_k(s_k, t_k, \omega_{-k}) \). We need only show, then, that

\[
\epsilon_k(s_k, t_k, \omega_{-k}) - \epsilon_k(s_k, t_k, \omega_{-k}) = \epsilon_k(s_k, t_k, \omega_{-k}) - \epsilon_k(s_k, t_k, \omega_{-k}). \tag{1}
\]

We have

\[
\epsilon_k(s_k, t_k, \omega_{-k}) = g(t_k, \omega_{-k}) - g(s_k, \omega_{-k});
\]

\[
\epsilon_k(s_k, t_k, \omega_{-k}) = g(t_k, \omega_{-k}) - g(s_k, \omega_{-k}).
\]

Go now from \( s_k \) to \( s_k \) by switches in directions other than \( \varphi \), from \( s_k \) to \( t_k \), and from \( t_k \) to \( t_k \) by switches in directions other than \( \varphi \). \( \omega_{-k} \) and \( \omega_{-k} \) differ from each other only by switches in the directions \( \varphi \); thus the switches in the sequence from \( s_k \) to \( s_k \) have equal effects in both environments. Therefore

\[
g(s_k, \omega_{-k}) - g(s_k, \omega_{-k}) = g(s_k, \omega_{-k}) - g(s_k, \omega_{-k}). \tag{2}
\]

For the same reason,

\[
g(t_k, \omega_{-k}) - g(t_k, \omega_{-k}) = g(t_k, \omega_{-k}) - g(t_k, \omega_{-k}). \tag{3}
\]

Adding (2) and (3), we get
\[ \begin{align*}
\Delta(t_k, s_k, t_{-k}) - g(t_k, s_k, t_{-k}) - g(s_k, t_k, t_{-k}) = \\
g(t_k, s_k, t_{-k}) - g(s_k, t_k, t_{-k}) - g(t_k, s_k, t_{-k})
\end{align*} \]

which is (1).

14. Thus where \( \theta_k(s_k, t_k) \in \omega \), \( \epsilon_k(s_k, t_k, t_{-k}) = \epsilon_k(s_k, t_k, t_{-k}) \) is a constant which is independent of the environment \( t_{-k} \); it depends only on \( s_k \) and \( t_k \). Call this function \( \delta_k(s_k, t_k) \); it is defined for all adjacent pairs of strategies for \( k \). (On the other hand, the function \( \epsilon_k(s_k, t_k, t_{-k}) \) depends only on whether \( s_k \), \( t_k \), and the various \( v_i \)'s for \( i \neq k \) are in \( T_k^{-\varphi} \) or \( T_k^{+\varphi} \) sets. Call this function \( \tau_k(s_k, t_k, t_{-k}) \); it is defined, then, as \( \epsilon_k(s_k, t_k, t_{-k}) \) where \( \theta_k(s_k, t_k) \in \omega \). Since for the \( \gamma \) introduced earlier, \( u_k^{-\varphi} \) and \( u_k^{+\varphi} \), if both are defined, were chosen so as to make \( g_k(u_k^{-\varphi}, u_k^{+\varphi}, t_{-k}) \) minimal for \( \theta_k(u_k^{-\varphi}, u_k^{+\varphi}) = \varphi \), whenever \( \theta_k(s_k, t_k) = \varphi \), we have \( g_k(s_k, t_k, t_{-k}) \leq g_k(u_k^{-\varphi}, u_k^{+\varphi}, t_{-k}) \). Thus if \( t_k \in T_k^{-\varphi} \), then \( g_k(s_k, t_k, t_{-k}) \leq g_k(s_k, t_k, t_{-k}) \), and since both effects are in the \( \theta_k(s_k, t_k) \) direction and their difference is \( \delta_k(s_k, t_k) \), it follows that \( \delta_k(s_k, t_k) \), if non-zero, is a vector in the \( \theta_k(s_k, t_k) \) direction. In short, then, both \( \delta_k(s_k, t_k) \) and \( \tau_k(s_k, t_k, t_{-k}) \) are vectors which, if non-zero, are in the \( \theta_k(s_k, t_k) \) direction, and their sum is \( \epsilon_k(s_k, t_k, t_{-k}) \).

15. Let a path from \( s_k \) to \( t_k \) be a sequence \( s_k^0, \ldots, s_k^\lambda \) of strategies in \( T_k \) such that \( s_k^0 = s_k \), \( s_k^\lambda = t_k \), and \( \{s_{k-1}, s_k^i \} \in \Lambda_k \) for each \( i \) with \( 0 \leq i \leq \lambda \).
A circuit is a path from a strategy to itself. For any circuit \( s_k^0, \ldots, s_k^\lambda \) from \( s_k^0 \) to itself,

\[
\sum_{t=1}^{\lambda} \delta_k(s_k^{t-1}, s_k^t) = 0. \tag{4}
\]

**Proof:** We have at each step in the circuit

\[
\delta_k(s_k^{t-1}, s_k^t) = \varphi_k(s_k^{t-1}, s_k^t, 1) - \lambda_k(s_k^{t-1}, s_k^t, 1).
\]

But where \( \varphi(t) = \theta_k(s_k^{t-1}, s_k^t) \in \omega(t) \), we have

\[
\lambda_k(s_k^{t-1}, s_k^t, 1) = g(s_k^{t-1}(t-1), t(t), 1) - g(s_k^{t-1}(t), t(t), 1).
\]

Therefore

\[
\sum_{t=1}^{\lambda} \lambda_k(s_k^{t-1}, s_k^t, 1) = \sum_{t=1}^{\lambda} [g(s_k^{t-1}(t), t(t), 1) - g(s_k^{t-1}(t), t(t), 1)].
\]

But for \( \varphi(t) \notin \omega \), we have \( s_k^{t-1}(t) = s_k \). For since \( s_k^{t-1} \)

and \( s_k^t \) are adjacent and \( \theta_k(s_k^{t-1}, s_k^t) \in \omega \), where \( \varphi \notin \omega \),

either \( s_k^{t-1} \) and \( s_k^t \) are both in \( T_1^{\omega} \), or \( s_k^{t-1} \) and \( s_k^t \) are both in \( T_1^{\bar{\omega}} \). In the first case, \( s_k^{t-1} = s_k^t \), and

in the second, \( s_k^{t-1} = s_k^t = s_k^{t-1} \). Thus

\[
\sum_{t=1}^{\lambda} [g(s_k^{t-1}, t(t), 1) - g(s_k^{t-1}, t(t), 1)] = \sum_{t=1}^{\lambda} [g(s_k^{t-1}, t(t), 1) - g(s_k^{t-1}, t(t), 1)].
\]

Therefore \( \sum_{t=1}^{\lambda} \lambda_k(s_k^{t-1}, s_k^t, 1) = 0 \). Since also,

\[
\sum_{t=1}^{\lambda} \varphi_k(s_k^{t-1}, s_k^t, 1) = 0, \tag{4}
\]

holds.
16. Since a circuit of $\delta_k$'s sums to zero, and from the way $\delta_k$ is defined, $\delta_k(s_k, t_k) = -\delta_k(t_k, s_k)$, it follows that a sum of $\delta_k$'s is path independent, in the sense that for any two paths $s_0, \ldots, s^t_k$ and $t_0, \ldots, t^t_k$ from $s_k$ to $t_k$,

$$\sum_{t=1}^t \delta_k(s^t_{t-k} - s^t_k) = \sum_{t=1}^t \delta_k(t^t_{t-k} - t^t_k).$$

Call this quantity $f(t_k, s_k)$. For any orientation $\omega$ with $T$ $\omega$-separable, let $h(\omega, s_k) = g(s^\omega_k) - g(\omega)$.

17. Differences in values of $g$ decompose in this way:

$$g(z) - g(z') = \sum_k f_k(t_k, s_k) + \sum_{\omega \in \omega(t)} h(\omega, s_k).$$

**Proof:** Let $z_0, \ldots, z^t_k$ be a sequence with $z_0 = z$, $z^t_k = z'$, and with $z^t_k$ differing from $z^t_{k-1}$ only in that $k(\iota)$ switches in a direction $\in \omega(t)$. We have

$$g(z^t_k) - g(z^t_{k-1}) = \delta_{k(\iota)}(s^t_{k(\iota)}, t^t_{k(\iota)}, z^t_k - z^t_{k-1}).$$

$$= \sum_{\omega \in \omega(t)} f_k(\omega, s^t_k) + \sum_{\omega \in \omega(t)} h(\omega, s^t_k).$$

For $1 \neq \omega(k)$, $f_k(\omega, z^t_k) = f_k(\omega, z^t_{k-1})$, and for $\omega \neq \omega(t)$, $h(\omega, z^t_k) = h(\omega, z^t_{k-1})$. Therefore $g(z^t_k) - g(z^t_{k-1})$ equals

$$\sum_k f_k(z^t_k, z') - \sum_k f_k(z^t_{k-1}, z') + \sum_{\omega \in \omega(t)} h(\omega, z^t_k) - \sum_{\omega \in \omega(t)} h(\omega, z^t_{k-1}),$$

and $g(z') - g(z)$ equals:

$$\sum_k f_k(z^t_k, z') - \sum_k f_k(z^t_{k-1}, z') + \sum_{\omega \in \omega(t)} h(\omega, z^t_k) - \sum_{\omega \in \omega(t)} h(\omega, z^t_{k-1}).$$
18. If for fixed $x$, $g(x, z)$ is minimal for $z = z^X$, then where $x$ and $t$ are fixed, for each $k$, $f_k^*_k(x, z, t_k)$ is minimal for $z = z^X$, and for each $\omega$, $h_\omega^*_k(x, z, t_k)$ is minimal for $z = z^X$. Proof: Where $\varphi \in \omega$, all values of $h_\omega^*_k(x, z, t_k)$ are of the form $\alpha\varphi$, where $\alpha$ is a real number. If $\varphi(x) > 0$, then for fixed $z$, $h_\omega^*_k(x, z, t_k)$ is minimal when $s_k^1 \in T^{-\varphi}_1$ for all $i$, where $\eta_k^X$ is the unit vector in the $x$ direction, $\eta_k^X \varphi > 0$, and hence by Lemma 1 and the definition of $T^{-\varphi}_1$, $s_k^X \in T^{-\varphi}_1$. Therefore $h_\omega^*_k(x, z, t_k)$ is minimal for $z = z^X$. If $\varphi(x) < 0$, repeat the argument using $\psi = -\varphi$. If $\varphi(x) = 0$, then $h_\omega^*_k(x, z, t_k) = \alpha \varphi(x) = 0$, and so any value of $z$ minimizes it.

Now go from $s_k^X$ to $s_k^t$ in a path which makes $\eta_k^X \varphi(s_k^X, t_k)$ non-decreasing; the existence of such a path is guaranteed by (i) of Lemma 1. Since switches in $f_k^*_k(s_k^X, t_k)$ are always in the same direction as switches in $g$ (by 11), these switches are all non-$x$-decreasing, and $f_k^*_k(x, s_k^X, t_k) \geq f_k^*_k(x, s_k^X, t_k)$.

19. A measure over $V$ is a function $\tau$ whose domain is $V$ and whose values are non-negative real numbers; the weight of a measure $\tau$ is $\sum \tau(x)$. A partial game form on $\tau = T_1 \times \ldots \times T_n$ and $V$ is a function whose domain is $\tau$ and whose values, for some $\alpha \geq 0$, are measures over $V$ of weight $\alpha$. Let $f$ be a partial game form on $\tau$ and $V$. $f$ is alternative-eliminating iff for each $x \in V$ there is an $z \in \tau$ such that $f(x, z) = 0$. $f$ is linear iff there
are a real-valued function $\alpha$ with $0 \leq \alpha(s) \leq 1$ and strategy profiles $t_1$ and $t_2^* \in \mathcal{T}$ such that for all $s \in \mathcal{T}$,

$$f(s) = [1 - \alpha(s)]f(t_1) + \alpha(s)f(t_2^*).$$

20. Now for all $s$ and $x$, let $h_\omega(x_1, s, g) = h_\omega^*(x_1, g, s, g^*).$

Then by (18), $h_\omega(x_1, s, g) \geq 0$ for all $x$ and $g$. Moreover, from the way $h_\omega$ is defined, $h_\omega(x_1, s, g) = g(x_1, g^* - g(x_1, g^*)$, and

$$\sum x h_\omega(x_1, s, g) = \sum x [g(x_1, g^*) - g(x_1, g^*)] = 1 - \sum x g(x_1, g^*)$$

which is independent of $\omega$. Thus $h_\omega$ is a partial game form.

Since for all $x$, $h_\omega(x_1, g^*) = h_\omega^*(x_1, g^*, g^*) = 0$, $h_\omega$ is alternative-eliminating. $h_\omega$ is linear, since for any $g$ and $t_1$,

$$h_\omega(g) = h_\omega(g, t_1) + h_\omega(t_1),$$

and for any $i$, $a_1$, and $t_1$ with $\{s_1, t_1\} \in A_1$, either $s_1 = t_1$ or $\theta_1(s_1, t_1) \in \omega$.

21. Similarly, let $f_k(x_1, s, g) = f_k^*(x_1, s, g^*)$ for each $x$.

From (18), it follows that $f_k^*(x_1, s, g) \geq 0$ for all $x$ and $g$.

By path-independence,

$$f_k^*(x_1, s, g^*) = f_k^*(x_1, g^*, g^*) + f_k^*(x_1, g^*, g^*).$$

Hence $\sum x f_k^*(x_1, s, g^*) = \sum x f_k^*(x_1, g^*, g^*) + \sum x f_k^*(x_1, g^*, g^*)$. Since

$$\sum x f_k^*(x_1, g^*, g^*) = 0,$$

this is $\sum x f_k^*(x_1, g^*, g^*)$, which is independent of $g$. Thus each $f_k$ is a partial game form. $f_k$ is unilateral, since it is characterized in terms of $f_k^*$, which in turn is characterized in terms of $f_k^*$, whose values are environment-independent.
22. Let \( f_0(x, g) = g(x, g)^X \) for all \( x \). Then
\[
f_0(x, g) \geq 0
g(x, g) = g(x, g)^X = \Sigma_k f_k(x, g, g)^X + \Sigma_{\omega} h_{\omega}(x, g),
\]
and \( g(x, g)^X = f_0(x, g) \), we have
\[
g(x, g) = f_0(x, g) + \Sigma_k f_k(x, g) + \Sigma_{\omega} h_{\omega}(x, g).
\]
Thus on \( \mathcal{T} \), \( g \) is a finite sum of partial game forms, each of which is either unilateral or both linear and alternative-eliminating.

23. Any linear alternative-eliminating partial game form is a sum of duple partial game forms. \textbf{Proof:} Let \( \varphi^- \) and \( \varphi^+ \) be the extreme values of \( h_{\omega}(g) \). Since \( h_{\omega} \) is linear, there is a real-valued function \( \alpha(g) \) such that for all \( g \),
\[
0 \leq \alpha(g) \leq 1 \quad \text{and} \quad h_{\omega}(g) = [1 - \alpha(g)] \varphi^- + \alpha(g) \varphi^+,
\]
and since in addition \( h_{\omega} \) is alternative-eliminating, for each \( x \), either \( \varphi^-(x) = 0 \) or \( \varphi^+(x) = 0 \).

Where the alternatives in \( V \) are \( x_1, \ldots, x_m \), let \( \sigma_0^+ = \Sigma_{t=1}^m \varphi^+(x_t) \) and let \( \sigma_0^- = \Sigma_{t=1}^m \varphi^-(x_t) \); let \( \sigma_0^+ = 0 = \sigma_0^- \). Let \( \sigma_1, \ldots, \sigma_{\kappa} \) be the values of \( \sigma_0^+ \) and \( \sigma_0^- \) in order of magnitude. Then for each \( \kappa \), the interval \( (\sigma_{\kappa-1}^+, \sigma_{\kappa}^+) \)
consists of zero or more intervals of the form \( (\sigma_{\ell-1}^+, \sigma_{\ell}^+) \). where \( (\sigma_{\ell-1}^+, \sigma_{\ell}^+) \subseteq (\sigma_{\kappa-1}^+, \sigma_{\kappa}^+) \), we shall say \( \kappa(i) = \kappa \).
We have
\[ \Sigma \left[ \sigma_i - \sigma_{i-1} \mid i : \kappa(i) = \kappa \right] = \sigma^+ - \sigma^+_{-1} = \varphi(x_\kappa). \]  
(5)
Likewise for each \( \lambda \), the interval \( (\sigma^\lambda_{i-1}, \sigma^\lambda_i) \) consists of zero or more intervals of the form \( (\sigma^\lambda_{i-1}, \sigma^\lambda_i) \). Let \( \lambda(\iota) = \lambda \)
if \( (\sigma^\lambda_{i-1}, \sigma^\lambda_i) \subseteq (\sigma^\lambda_{i-1}, \sigma^\lambda_i) \); then
\[ \Sigma \left[ \sigma_i - \sigma_{i-1} \mid i : \lambda(i) = \lambda \right] = \varphi^-(x_\lambda). \]  
(6)

For each \( i : 0 \leq i \leq l \), define a duple partial game form \( d_4 \) as follows.
\[ d_4(x_\kappa(i), y, z) = \alpha(z)(\sigma_i - \sigma_{i-1}). \]
\[ d_4(x_\lambda(i), y, z) = [1 - \alpha(z)](\sigma_i - \sigma_{i-1}). \]
\[ d_4(y, z, 0) = 0 \quad \text{for} \quad y \notin \{x_\kappa(i), x_\lambda(i)\}. \]

This is indeed a partial game form. Its values are non-negative, since \( \alpha(z) \), \( 1 - \alpha(z) \), and \( \sigma_i - \sigma_{i-1} \) are always non-negative. The sum of its values is \( \sigma_i - \sigma_{i-1} \), which is constant for all \( z \). Now for any \( x_\kappa \), \( \Sigma_{\iota=1}^l d_4(x_\kappa, z) \) is
\[ \alpha(z)\Sigma \left[ \sigma_i - \sigma_{i-1} \mid i : \kappa = \kappa(i) \right] + [1 - \alpha(z)] \Sigma \left[ \sigma_i - \sigma_{i-1} \mid i : \kappa = \lambda(i) \right] = \alpha(z)\varphi^+(x_\kappa) + [1 - \alpha(z)]\varphi^-(x_\kappa), \]
with the last step following from (5) and (6). Thus
\[ \Sigma_{\iota=1}^l d_4(z) = \alpha(z)\varphi^+ + [1 - \alpha(z)]\varphi = h_\omega(z). \]
24. **Proof of Theorem:** We have seen that on $\mathcal{T}$, $g$ is a sum of partial forms, each of which is either unilateral or duplo. Partial forms of weight zero do not contribute to this sum. Let $g = c_1 + \ldots + c_l$, where these partial forms have positive weights $\alpha_1, \ldots, \alpha_l$. Then let $c_i = (1/\alpha_i)c_i$ for each $i$, $1 \leq i \leq l$; then

$$g = \alpha_1 c_1 + \ldots + \alpha_l c_l,$$

(7)

where each $c_i$ is unilateral or duplo. We may expand the domain of each $c_i$ to $\mathcal{T}'$ by treating equivalent strategies alike: if $g \in \mathcal{T}'$, take the $g' \in \mathcal{T}$ such that $s_1 < s_1', \ldots, s_n < s_n'$, and let $c_i(g) = c_i(g')$ for each $i$. Then since $g(g) = g(g')$, (7) still holds. That proves the Theorem.

**Appendix to the proof:** **Lemma 1.** Let $H$ be a convex polyhedron, and where $\psi$ and $\varphi$ are directions, let $p$ and $q$ be vertices such that $p$ is the sole $\psi$-maximal vertex of $H$ and $q$ is a $\varphi$-minimal vertex of $H$. Then there is a course $p_0, \ldots, p_\nu$ on $H$ from $q$ to $p$ such that (i) for each $i$, $1 \leq i \leq \nu$, $\varphi(p_i - p_{i-1}) \geq 0$, and (ii) if $\psi(p) < 0$, then none of the edges $p_{i-1}p_i$ lie in direction $\varphi$.

**Proof:** The lemma clearly holds for two-dimensional polygons. Let $H$ be of dimension $i$, and suppose the Lemma holds for polyhedra of all lesser dimensions. Here $\mathbf{R}^\psi$ is a two-space containing directions $\psi$ and $\varphi$, let $\pi$ be the orthogonal projection onto $\mathbf{R}^\psi$. Then $\pi H$ is a convex polygon.

For any direction $\psi'$, a side of $\pi H$ which consists of all and only the $\psi'$-maximal points of $\pi H$ is the $\pi$-projection.
of the face of $H$ (which may be an edge) consisting of all and only $v'$-maximal points of $H$. Each vertex of $\pi H$ is the $\pi$-projection of at least one vertex of $H$.

Let $q' = \pi q$. In the polygon $\pi H$, there is a course $q_0', \ldots, q'_\mu$ satisfying (i) and (ii), where $q'_{\mu} = \pi r'$, $q'_0$ is $\varphi'$-minimal, and either $\pi q = q'_0$ or $q'_{\mu}q_{\mu} = \pi q_{\mu}$ is a segment of the $\varphi'$-minimal side of $\pi H$. In the latter case, by inductive hypothesis (i), where $q'_0$ is a vertex of $H$ and $q'_{\mu} = \pi q_{\mu}$, there is a course from $q'_0$ to $q$ on the $\varphi'$-minimal face $F_0$ of $H$. The reverse path from $q$ to $q'_0$ satisfies (i) and (ii) because $\varphi \perp F_0$.

Now for each $1 \leq i \leq \mu$, let $q_i$ be a vertex of $H$ such that $\pi q_i = q'_i$. For $1 \leq i < \mu$, let $F_i$ be the face of $H$ consisting of all points of $H$ which project onto side $q_{i-1}'q'_i$. Let $\varphi_i$ be the direction in $E \varphi F$ in which $F_i$ (and hence $\pi F_i$) is extreme, and let $\lambda_i$ be the direction from $q_{i-1}'$ to $q'_i$. Then $q_{i-1}'$ is a $\varphi_i$-minimal vertex of $F_i$ and $q_i$ is another vertex of $F_i$.

By inductive hypothesis (i), there is a path $q_{i0}'', \ldots, q_i\lambda_i(\mu)$ from $q_{i-1}'$ to $q_i$ in $F_i$, such that where $F_{i+1} = q_{i+1} - q_{i+1}'$, $\varphi_i \cdot \lambda_i \geq 0$ for each $1 \leq i \leq \lambda_i(\mu)$. Since $\varphi_{i+1} \perp F_{i+1}$, $\varphi_{i+1} \cdot r = 0$. Since $\varphi_i$ and $\lambda_i$ are orthogonal directions in $E \varphi F$ and $\varphi$, lies in $E \varphi F$, we have that $\varphi = (\varphi_i \cdot \varphi)\lambda_i + (\lambda_i \cdot \varphi)\varphi_i$.

Hence

$$
\varphi \cdot \lambda_{i+1} = [(\lambda_{i+1} \cdot \varphi)\lambda_i + (\varphi \cdot \varphi)\lambda_i] \cdot \lambda_{i+1}
$$

$$
= (\lambda_{i+1} \cdot \varphi)(\lambda_i \cdot \lambda_{i+1}) + (\varphi \cdot \varphi)(\lambda_i \cdot \lambda_{i+1}).
$$
Since $\chi \cdot \varphi \geq 0$, $\chi \cdot r_{\pm} \geq 0$, and $\varphi \cdot r_{\pm} = 0$, it follows that $\varphi \cdot r_{\pm} \geq 0$. Thus combining the courses from $q$ to $q_0$ and from each $q_{k-1}$ to $q_k$, we get a course on $H$ from $q$ to $p$ satisfying (i). If $\varphi \cdot \varphi < 0$, then no side of $\pi H$ in the $\pi$-projection of this course is oriented in direction $\varphi$, and hence no edge in this path on $H$ from $q$ to $p$ is oriented in direction $\varphi$. Thus (ii) is satisfied.
3. COROLLARIES

Player $k$ is **dictator** under $g$ iff for every alternative $x$ there is a strategy $s^X_k$ for $k$ such that for every environment $y^X_k$, $g(x, s^X_k y^X_k) = 1$. Game form $g$ is **dictatorial** iff some player is dictator under $g$.

**Condition of Attainability:** For every $x \in V$ there is an $n \in J$ such that $g(x, s) = 1$.

**Corollary 1.** Let $|J| \geq 3$. If $g$ is straightforward and satisfies **Attainability**, then on the domain $T^*$ of the theorem, $g$ is a probability mixture of dictatorial game forms.

**Proof:** Consider a player $k$, and let $B^X_k$ be the set of all utility scales that rank $x$ first. Since $B^X_k$ has a non-empty interior and the sets $D_k(s_k)$ cover $B^X_k$, some $D_k(s_k)$ with interior points must intersect $B^X_k$. Let $t^X_k$ be such that $U^X_k(t^X_k)$ has interior points and intersects $B^X_k$, and let $U_k \in U^X_k(t^X_k)$, for each $k$. Let $g^0$ be such that $g(x, g^0) = 1$, and let $g^n = (g^0)^{n-1}$ for each $i$, so that $g^n = n$. Then if $g(x, g^{i-1}) = 1$, then $g(x, g^i) = 1$, since $x$ is top in $U_i$ and $t^X_k$ is $U_i$-dominant for $i$. Hence $g(x, t^X_k) = 1$. We have shown that if $g$ satisfies **Attainability**, then $g|T^*$, $g$ restricted to $T^*$, satisfies attainability.

Now $g|T^*$ is a probability mixture of unilateral and duple game forms. There are no duple game forms in this mixture, for if, say, a $yz$-duple scheme were in the mixture and $x \in \{y, z\}$, then we could never have $g(x, g^i) = 1$. Hence
is a probability mixture of unilateral game forms \( f_1, \ldots, f_n \). Since for each \( x \), \( g(x, i^X) = 1 \), we must have
\[ f_i(x, i^X) = 1 \]; hence each \( f_i \) is dictatorial. That proves the Corollary.

Where \( U_1, \ldots, U_n \) are utility scales over \( V \), a lottery \( \rho \) over \( V \) is Pareto optimal ex ante with respect to \( U_1, \ldots, U_n \) iff there is no lottery \( \sigma \) over \( V \) such that
\[ U_i(\sigma) \geq U_i(\rho) \] for all \( i \).

Condition of ex Ante Acceptability: For every \( U_1, \ldots, U_n \), there is an \( \sigma \) such that \( g(\sigma) \) is Pareto optimal ex ante with respect to \( U_1, \ldots, U_n \).

Corollary 2. If \( g \) is straightforward and satisfies Ex Ante Acceptability and \( |M| \geq 3 \), then \( g \) is dictatorial.

The proof is similar to that of Corollary 2 in Gibbard (1976).
NOTES

1. A decision scheme as the term is used in this paper is called an "unrestricted decision scheme" in Gibbard (1976).

2. The random dictatorship is discussed both in Zeckhauser (1973) and in Gibbard (1973).

3. In Gibbard (1976), this is stated informally on p. 10. The theorem as formally stated in that paper deals with schemes which are defined only for ballots which consist of strong orderings. The theorem as stated here is trivially a consequence of the theorem as stated in the earlier paper.

4. Roughly this corollary was suggested by Hugo Sonnenschein for the theorem on strategy-proofness of decision schemes.

5. This property is studied by Zeckhauser (1973); his Theorem V (p. 945) is similar to the second corollary in this paper. Zeckhauser, however, imposes a non-dictatorship condition which is stronger than the one used in this paper, and hence the corollary here is independent of Zeckhauser's theorem. (Cf. Gibbard, 1976, p. 7).
REFERENCES


