DISCUSSION PAPER NO. 200

"The Global Asymptotic Stability of Optimal Control:
A Survey of Recent Results"

William A. Brock*
University of Chicago
and
Cornell University

February 25, 1976

*I thank R. Lucas, M. Intriligator, and J. Scheinkman for helpful comments on this paper. I wish to thank the National Science Foundation for research support. Needless to say, all of the above are absolved from all errors and shortcomings in this paper.

Discussion Paper #200

"The Global Asymptotic Stability of Optimal Control: A Survey of Recent Results"

by

William A. Brock

THIS IS A REPRINT FROM A JOURNAL.

The global asymptotic stability of optimal control: A survey of recent results

WILLIAM A. BROCK*

University of Chicago and Cornell University

1. *Introduction*

The purpose of this paper is (1) to hasten the assimilation of some recent results obtained on the global asymptotic stability of optimal control into general economic knowledge, (2) to indicate some possible areas of application of these results, (3) to relate the recent results to standard engineering literature, and (4) to indicate new avenues of research in this area.

In order to remain within the space limitation, this survey must be selective. Furthermore, the emphasis will be on basic ideas and not technical details. Not only will this save space, but also it will lay bare the basic structure of the ideas. Details will be referenced where possible.

In order to describe the results contained in this paper it is useful to state the problem of concern without further ado. Consider the following optimal control problem:

$$\max_{v(\cdot)} \int_0^T e^{-\rho t} U[x(t), v(t)]\, dt, \tag{1}$$

*I thank R. Lucas, M. Intriligator, and J. Scheinkman for helpful comments on this paper. I wish to thank the National Science Foundation for research support. Needless to say, all of the above are absolved from all errors and shortcomings.

subject to

$$\dot{x} = T[x(t), v(t)t], \tag{2}$$

$$x(0) = x_0, \quad \text{given } x(t) \in R^n, \quad v(t) \in R^m, \tag{3}$$

$$v(\cdot):[0, T] \to R^m \text{ measurable,}$$

where U is instantaneous utility, t is time, T is planning horizon, $x(t)$ is state vector at time t, $v(t)$ is instrument vector at time t, $T[\cdot]$ is the technology which relates the rate of change of the state vector $\dot{x} \equiv dx/dt$ to the state $x(t)$ and instruments $v(t)$ at time t, $\rho \geqq 0$ is the discount in future utility, and x_0 is the initial position of the state vector at time 0. The objective is to maximize

$$\int_0^T U \, dt,$$

subject to eqs. (2) and (3) over some set $\mathscr{A}$ of instrument functions,

$$v(t):[0, T] \to R^m,$$

which is usually taken to be the set of all measurable $v(\cdot)$ or the set of all piecewise continuous $v(\cdot)$.

Problem (1) was chosen as the vehicle of explanation of the results in this survey because it is described in chapter 2 of the well-known book by Arrow and Kurz (1970). Let us specialize problem (1) somewhat. Set

$$T = \infty, \quad \rho > 0. \tag{4}$$

Put[1]

$$W(x, t_0) = \sup \int_{t_0}^{\infty} e^{-\rho(t-t_0)} U[x(t), v(t)] \, dt \tag{5}$$

s.t.

$$\dot{x} = T[x(t), v(t)], \quad x(t_0) = x, \quad v(\cdot) \in \mathscr{A}.$$

As pointed out by Arrow and Kurz, W is independent of t_0, and, under strict concavity assumptions on $U(\cdot)$ and $T(\cdot)$, the optimal $v(t)$ (denote it by $v^*(t)$) is of the time stationary feedback form: there is a function $h(x)$ such that

$$v^*(t) = h(x^*(t)). \tag{6}$$

[1] Here "sup" denotes supremum. The supremum is taken over all instrument functions $v(\cdot) \in \mathscr{A}$.

Thus optimal paths $x^*(t)$ satisfy

$$\dot{x}^*(t) = T[x^*(t), h(x^*(t))] \equiv F(x^*(t)), \quad x^*(t_0)) = x, \tag{7}$$

which is an autonomous of time set of differential equations.

The basic problem addressed in this article may now be stated.

Basic problem (P). Find sufficient conditions on the utility function $U[\cdot]$, the technology $T[\cdot]$, and the discount ρ such that there exists a steady state of eq. (7), call it x^* such that (a) x^* is locally asymptotically stable (L.A.S.), (b) x^* is globally asymptotically stable (G.A.S.).

Here L.A.S. and G.A.S. are defined as follows.

Definition. *The steady state solution x^* of eq.* (7) *is* L.A.S. *if there is $\epsilon > 0$ such that*

$$|x^* - x_0| < \in \quad \text{implies} \quad x(t|x_0) \to x^*, t \to \infty,$$

where

$$x(t|x_0)$$

is the solution of eq. (7) *with* $x(0) = x_0$. *The steady state solution x^* is* G.A.S. *if for* all $x_0 \in R^n$, $x(t|x_0) \to x^*$, $t \to \infty$.

Here $|y|$ denotes the norm of vector y:

$$|y| = \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}.$$

Before getting into the results, it is useful to discuss why such results are important. There are several areas of applications of stability results. A first area is the neoclassical theory of investment associated with the names of Eisner–Strotz, Lucas, Mortensen, Jorgenson, Treadway, and others. A version of this theory was used by Nadiri and Rosen (1969) in a well-known article on estimating interrelated factor demand functions. The Nadiri–Rosen work culminated in their book (1973) which ended with a plea for useful results on problem (P).

The paper by Mortensen (1973) derives a set of useful empirical restrictions on dynamic interrelated factor demand functions derived from the neoclassical theory of investment *provided that the stability hypothesis is satisfied*. Mortensen's paper can be viewed as "Samuelson's Correspondence Principle Done Right" in the context of the neoclassical theory of investment. Thus, there is no doubt that stability results are of great importance in the neoclassical theory of investment.

A second area of applications of stability results is economic growth theory. Fortunately, this area is well covered in the paper by Cass and Shell (1976), so we will not spend much time on it here. It is, basically, the extension of the well known turnpike theory of McKenzie, Gale, Radner, Samuelson, and others to the case $\rho > 0$.

A third area of applications of the results reported here is the dynamic oligopoly games of Flaherty (1974), J. Friedman (1971), Prescott (1973), and others. These games represent exciting new efforts to "dynamize" the field of industrial organization. Indeed, this area is "wide open" for new researchers.

A fourth area of applications is the optimal regulator problems of engineering (see Anderson and Moore (1971), Kwakernaak and Sivan (1972)), the optimal filtering problem (Kwakernaak and Sivan), and the integral convex cost problem of operations research (see Lee and Markus (1967, ch. 3)). This application is developed by Magill (1975). No doubt there are many more applications, but this should be enough to convince the reader that the stability problem is of basic importance in a number of areas. Let us turn to the results.

2. *Stability results*

It will be useful to write down some specializations of the general problem (5). Put $t_0 = 0$, $x = x_0$, and suppress t_0 in W, henceforth. The neoclassical theory of investment as stated by Mortensen (1973) is

$$W(x_0) = \sup \int_0^\infty e^{-\rho t}(f(x(t), v(t)) - w^T x(t) - g^T v(t))\,dt, \tag{8}$$

s.t.

$$\dot{x} = v(t), \quad x(0) = x_0, \quad v(\cdot):[0,\infty)\to R^n \text{ measurable.}$$

Here $f(x, v)$ is a generalized production function which depends upon the vector of n factors $x(t)$, and the rate of adjustment of the factors $v(t)$. The cost of obtaining factor services in each instant of time is $w^T x(t)$ (w^T denotes the transpose of the vector w) and the cost of adding to the stock of factors (which may be negative) is $g^T v(t)$. It is assumed that a stationary solution $x_0 = x^*$ exists for eq. (8), that the optimal plan exists and is unique for each $x_0 \gg 0$[2] and is interior to any natural boundaries (i.e.

[2] Let $x \in R^n$, $x \geqq 0$, $x > 0$, $x \gg 0$ mean in turn: $x_i \geqq 0$, $i = 1, 2, \dots, n$; $x_i \geqq 0$, $i = 1, 2, \dots, n$, but $x \neq 0$; $x_i > 0$, $i = 1, 2, \dots, n$.

$x_i^*(t) > 0$, for all $t \geqq 0$, $i = 1, 2, \dots, n$), $f \in C^2$ (f is twice continuously differentiable), and that the optimal plan is one with piecewise continuous time derivatives. These assumptions which avoid many tangential technical side issues will be *placed on the general problem* (5) *also, and will be maintained throughout this article.*

The next few pages will attempt to summarize the fundamental work of Magill[3] (1972, 1974, 1975) on the G.A.S. of the linear quadratic approximation around a steady state solution of (5). These results go far beyond the simple checking of eigenvalues that most people associate with a local analysis. Furthermore, the local results of Magill lead naturally to the global results of Cass–Shell, Rockafellar, and Brock–Scheinkman, which are discussed below.

The linear quadratic approximation of eq. (8) at a steady state x^* is

$$W(\xi_0) = \sup \int_0^\infty e^{-\rho t}\{(\xi(t), \eta(t))^T A^*(\xi(t), \eta(t)\,dt, \tag{9}$$

s.t.

$$\dot{\xi} = \eta(t), \qquad \xi(0) = x_0 - x^* = \xi_0,$$
$$\eta(\cdot):[0,\infty)\to R^n \text{ measurable,}$$

where

$$\xi(t) = x(t) - x^*, \qquad \eta(t) = v(t) - v^* = v(t), \tag{10}$$
$$A^* = \begin{bmatrix} f_{xx}^*, & f_{xv}^* \\ f_{vx}^*, & f_{vv}^* \end{bmatrix}.$$

The symbols $f_{xx}^*, f_{xv}^*, f_{vx}^*, f_{vv}^*, f_x^*, f_v^*$ denote

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial v \partial x}, \quad \frac{\partial^2 f}{\partial x \partial v}, \quad \frac{\partial^2 f}{\partial v^2}, \quad \frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial v},$$

all evaluated at $(x^*, v^*) = (x^*, 0)$, respectively.

The linear quadratic approximation at a steady state (x^*, v^*) for the general problem (5) is the following.

Linear quadratic approximation to general problem (Magill (1975, p. 7).

$$W(\xi_0) = \sup \int_0^\infty e^{-\rho t}\left\{(\xi(t), \eta(t))^T \begin{bmatrix} U_{xx}^* & U_{xv}^* \\ U_{vx}^* & U_{vv}^* \end{bmatrix} (\xi(t), \eta(t))\right\}dt, \tag{12}$$

[3] Magill (1972, 1974, 1975) also treats the case of uncertainty. Due to the lack of space only his certainty results will be treated here.

s.t.

$$\dot{\xi} = T^*_x \xi(t) + T^*_v \eta(t),$$
$$\xi(0) = x_0 - x^* = \xi_0,$$
$$\eta(\cdot) : [0, \infty) \to R^m \text{ measurable,}$$

where

$$U^*_{xx}, U^*_{xv}, U^*_{vx}, U^*_{vv}, T^*_x, T^*_v$$

are the appropriate matrices of partial derivatives evaluated at (x^*, v^*). The quadratic approximation (12) can be expected to hold only in the neighborhood of x_0. The validity of the linear quadratic approximation for the *infinite horizon* problem (5) has not been studied yet, as far as I can tell. The finite horizon case is studied by Breakwell, Speyer and Bryson (1963), for example (see also Magill (1975)).

A problem that is extensively studied in the engineering literature and is closely related to eq. (12) is the following.

Time stationary optimal linear regulator problem (OLRP)

$$-W(\xi_0) = \inf \int_0^\infty e^{-\rho t} \left\{ (\xi(t), \eta(t)) \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} (\xi(t), \eta(t)) \right\} dt, \tag{13}$$

s.t.

$$\dot{\xi} = F\xi(t) + G\eta(t),$$
$$\xi(0) = \xi_0,$$
$$\eta(\cdot) : [0, \infty) \to R^m \text{ measurable.}$$

Clearly, by putting

$$Q = -U^*_{xx}, \quad S = -U^*_{xv}, \quad S^T = -U^*_{vx},$$
$$R = -U^*_{vv}, \quad F = T^*_x, \quad G = T^*_v,$$

this is the same problem as eq. (12). The importance of observing that eqs. (12) and (13) are the same problem is that it enables us to carry the extensive set of results derived by engineers on OLRP (see Anderson and Moore (1971) and Kwakernaak and Sivan (1972) for example) to linear quadratic approximations (12) to economic problems.

Such an approach would virtually resolve the local asymptotic stability question for problem (5) if the engineers had spent more time on the case $\rho > 0$ instead of the case $\rho \leqq 0$. Fortunately, the paper by Magill (1975) fills this gap. Results on the OLRP are applicable (provided that the question of sufficient conditions for the validity of the linear quadratic approximation is resolved for problem (5)) to the L.A.S. problem for economic problems with $\rho > 0$ for two reasons.

First, $\xi(t)$, $\eta(t)$ may be replaced by $\xi(t) = e^{(\rho/2)t}\hat{\xi}(t)$, $\eta(t) = e^{(\rho/2)t}\hat{\eta}(t)$ in eq. (13). The constraint in eq. (13) becomes

$$\dot{\hat{\xi}} = (F - (\rho/2)I)\hat{\xi}(t) + G\hat{\eta}(t), \; \hat{\xi}(0) = \xi_0, \tag{14}$$

where I denotes the $n \times n$ identity matrix. This transformation of variables (used by Magill (1975) and Anderson and Moore (1971, p. 53) allows results for the case $\rho = 0$ (the bulk of results on the OLRP) to be carried over directly to the case $\rho > 0$. Second, the OLRP suggests an important class of Lyapunov functions upon which theorems 1–6 below will be based, viz. the minimum[4] from ξ_0 itself (Anderson and Moore (1971, p. 41)). The minimum is a positive definite quadratic form $\xi_0^T P \xi_0$ under general assumptions (see Anderson and Moore (1971)).

Before plunging into statements of formal theorems, let us use the OLRP to explore the determinants of L.A.S. The following is based upon Magill (1975), but brevity demands that many of his results be passed over. Assume that the matrix

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$$

is positive definite[5] in order to reflect the concavity of $U(x, v)$, leading to the negative definiteness of the matrix

$$\begin{bmatrix} U^*_{xx} & U^*_{xv} \\ U^*_{vx} & U^*_{vv} \end{bmatrix}$$

in economic problems.

When is the OLRP eq. (13) unstable? First, put $S = 0$. Then in the one dimensional case, we see that instability is more likely the larger is F, the smaller[6] is $|G|$, the larger is R, the smaller is Q, and the larger is ρ. The

[4] Magill (1972) recognized the importance of the Lyapunov function $V_1 = \xi_0^T P \xi_0$ for the case $\rho > 0$ as well as for the standard engineering case, $\rho \leqq 0$, as early as 1972.

[5] An $n \times n$ matrix A is positive definite if it is symmetric and positive quasi-definite. An $n \times n$ matrix is positive quasi-definite if for all $x \neq 0$, $x \in R^n$ we have

$$x^T A x > 0.$$

An $n \times n$ matrix B is negative quasi-definite (negative definite) if $-B$ is positive quasi-definite (positive definite).

[6] $|G|$ denotes the absolute value of the number G.

intuition behind this is quite compelling for if F is positive the system

$$\dot{\xi} = F\xi(t), \qquad \xi(0) = \xi_0$$

is unstable. If $|G|$, the absolute value of G, is small then a lot of input $\eta(t)$ must be administered in order to have much impact on $\xi(t)$. But inputs $\eta(t)$ cost $\eta^T(t)R\eta(t)$ to administer. If $\eta(0)$ is administered today, then $\xi(t)Q\xi(t)$ will be smaller in the "next instant." But the future is discounted by ρ. To sum up in words: why stabilize a highly unstable system (large F) when control input is ineffective (small $|G|$), when control input is expensive (R is large) when deviation of the state from the origin is not very costly (Q is small) and the future is not worth much (ρ is large).

Now assume that $S \neq 0$. Change units to reduce the problem to the case $S = 0$. Following Anderson and Moore (1971, p. 47), and Magill (1975),

$$\begin{aligned}(\xi, \eta)^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} (\xi, \eta) &= \eta^T R\eta + 2\xi^T S\eta + \xi^T Q\xi \\ &= (\eta + R^{-1}S^T\xi)^T R(\eta + R^{-1}S^T\xi) \\ &\quad + \xi^T(Q - SR^{-1}S^T)\xi. \end{aligned} \tag{15}$$

Note that since the L.A.S. is positive definite, R, Q, and $Q - SR^{-1}S^T$ are all positive definite. Defining

$$\eta_1 = \eta + R^{-1}S^T\xi,$$

the OLRP (13) with $S \neq 0$ becomes

$$\inf \int_0^\infty e^{-\rho t}\{\eta_1^T R\eta_1 + \xi^T(Q - SR^{-1}S^T)\xi\}\, dt \tag{16}$$

s.t.

$$\dot{\xi} = (F - GR^{-1}S^T)\xi + G\eta_1, \qquad \xi(0) = \xi_0.$$

Clearly, eq. (16) is unstable iff eq. (13) is unstable.

Let us use eq. (16) to explore when instability may be likely. Consider the one dimensional case. Without loss of generality we may assume $G \geqq 0$. For if $G \leqq 0$, put $\eta_2 = -\eta_1$ and stability will not be affected. It is clear from eq. (16) that when $S < 0$ a decrease in S is destabilizing. For a decrease in S makes $F - GR^{-1}S^T$ larger and makes

$$Q - SR^{-1}S^T$$

smaller. For $S > 0$ an increase in S makes the "underlying system matrix,"

$$F - GR^{-1}S^T$$

smaller (a stabilizing force), but

$$Q - SR^{-1}S^T$$

becomes smaller (a destabilizing force). Hence, ambiguity is obtained in this case.

It is known that instability in the multidimensional case is related to the amount of "asymmetry" in the underlying system matrix,

$$A = F - GR^{-1}S^T.$$

This is so because, roughly speaking, instability of A makes instability of the optimal path more likely and instability of A is related to its lack of symmetry.

In the multidimensional case when $\rho \leqq 0$, G.A.S. is intuitive from the existence of a finite value to the integral and the positive definiteness of the matrix

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}.$$

Roughly speaking, $(x^*(t), v^*(t))$ must converge to 0, $t \to \infty$, or else the integral will "blow up." Sufficient conditions for G.A.S. of the OLRP are covered in detail in Anderson and Moore (1971, ch. 4) for the case $\rho \leqq 0$. The important paper by Magill (1975) develops a rather complete set of results for *all* ρ.

Let us apply our intuitive understanding of the determinants of G.A.S. gained from the OLRP to the linear quadratic approximation (9) to the neoclassical model of investment (8). Here

$$Q = -f^*_{xx}, \quad R = -f^*_{vv}, \quad S = -f^*_{xv}, \quad G = I, \quad F = 0.$$

By the reasoning above from eq. (16), in the one dimensional case, provided that $A > 0$, instability at x^* is likely when ρ is large, $-f^*_{vv}$ is large, (it is positive by concavity of f in (x, v)), and $-f^*_{xx} + f^*_{xv}f^{*-1}_{vv}f^*_{xv}$ is small (it is positive by concavity of f). Note that when $F = 0$ and $S = 0$, since 0 is not a stable matrix, the underlying system matrix is not stable. If $S = 0$, however, a theorem to be proved below will show that G.A.S. holds anyway.

2.1. *Results for the general nonlinear nonquadratic problem*

In searching for sufficient conditions on U, T for G.A.S. to hold it turns out to be convenient to form the *current* value Hamiltonian (following Arrow and Kurz (1970, p. 47)) for eq. (5)

$$H(q,x,v) = U[x,v] + qT[x,v]. \tag{17}$$

Let $v^*(\cdot)$ be a choice of instruments that maximizes

$$\int_0^\infty e^{-\rho t} U[x(t),v(t)]\,dt,$$

s.t.

$$\dot{x} = T[x(t),v(t)], \qquad x(0) = x_0,$$

over all measurable $v(\cdot)$. Then Arrow and Kurz (1970, p. 48) showed that there exist costate variables $q^*(t)$ (expressed in current value) such that on each interval of continuity of $v^*(t)$,

$$\dot{q}^* = \rho q^*(t) - H_x^0(q^*(t),x^*(t)), \tag{18}$$

$$\dot{x}^* = H_q^0(q^*(t),x^*(t)),\ x^*(0) = x_0, \tag{19}$$

where $v^*(t)$ solves

$$\max_{v \in R^m} H(q^*(t),x^*(t),v) \equiv H^0(q^*(t),x^*(t)).$$

Also (1970, p. 35), if W_x exists,

$$q^*(t) = W_x(x^*(t)),$$

where, the reader will recall, $W(x)$ is the current value state valuation function. Note further that W_{xx} exists almost everywhere and is negative and is negative semi-definite when U and T are concave.[7] This is so because $W(x)$ is concave in this case.

[7]A qualification must be made here. Concavity of U and T implies concavity of $W(x)$ for the problem with inequality constraints,

$$\max \int_0^\infty e^{-\rho t} U[x(t),v(t)]\,dt,$$

s.t.

$$\dot{x} \leqq T[x(t),v(t)], \qquad x(0) = x_0.$$

But the assumptions usually made on economic problems lead to

$$\dot{x} = T[x(t),v(t)]$$

for optimum paths.

We are now in a position to state the G.A.S. results of Cass and Shell (1976), Rockafellar (1976), and Brock and Scheinkman (1975a, 1974a, 1974b). These results are based on the Lyapunov functions

$$V_1 = -(q-q^*)^T(x-x^*) \tag{20}$$

(Cass and Shell (1976), Rockfellar (1976), and Brock and Scheinkman (1974a, 1974b)),[8]

$$V_2 = -\dot{q}^T\dot{x}$$

(Rockafellar (1976) and Brock and Scheinkman (1975a)), where (q^*, x^*) is a steady state of the system eqs. (18) and (19).

2.2. *Results based on the Lyapunov[9] function* $V_1(q,x) = -(q-q^*)^T(x-x^*)$ *and* $V_2 = -\dot{q}^T\dot{x}$

Cass and Shell (forthcoming) formulate a general class of economic dynamics in price–quantity space which includes descriptive growth theory and optimal growth theory. This article is concerned only with their stability analysis. Roughly speaking, they take the time derivative of V_1 along solutions of eqs. (18) and (19) that satisfy

$$\lim_{t\to\infty} q^*(t)x^*(t)\,e^{-\rho t} = 0, \tag{21}$$

[8]It is important to interpret the meaning of V_1 for the OLRP. Here, since the state–costate equations (18) and (19) are linear, $x^* = q^* = 0$. Also the minimum cost, $C(\xi_0) = -W(\xi_0)$, given by eq. (13) is quadratic in ξ_0 and is 0 when $\xi_0 = 0$. Thus, there is a matrix P such that

$$C(\xi_0) = \xi_0^T P \xi_0.$$

Also, P is positive semi-definite when the integrand is convex in (ξ, η). Furthermore, the costate $q^*(t)$ in eqs. (18) and (19) for the OLRP is given by

$$q^*(t) = W_x(\xi^*(t)) = -2P\xi^*(t).$$

Thus,

$$V_1 = -2\xi^{*T}(t)P\xi^*(t),$$

and asking that $\dot{V}_1 > 0$ is just asking that the minimal cost fall as time increases when the minimal cost is calculated at $\xi^*(t)$ for each t. See Anderson and Moore (1971), Kwakernaak and Sivan (1972), and Magill (1975) for a more complete discussion of why V_1 is the basic Lyapunov function in the OLRP literature.

[9]Cass and Shell (1976) and Rockafellar (1976) were the first to recognize that V_1 is of basic importance for G.A.S. analysis in economics. It was used earlier, by Samuelson (1972) to eliminate limit cycles in the case $\rho = 0$.

and interpret the economic meaning of the assumption

$$0 \geqq \dot{V}_1(q^*(t), x^*(t)) = -\{[\rho q^*(t) - H_x^0(q^*(t), x^*(t))]^{\mathrm{T}}(x^*(t) - x^*) + (q^*(t) - q^*)^{\mathrm{T}} H_q^0(q(t), x^*(t))\}. \tag{22}$$

They show that eq. (22) implies

$$V_1(q^*(t), x^*(t)) \geqq 0. \tag{23}$$

Thus, only a slight strengthening of eq. (22) and assumptions sufficient to guarantee that $x^*(t)$ is uniformly continuous on $[0, \infty)$ allow them to prove that the steady state solution x^* is G.A.S. in the set of all solutions of eqs. (18) and (19) that satisfy eq. (21). More precisely:

Theorem 1 (Cass and Shell (1976)). *Assuming (stability assumption for $\rho \geqq 0$): for every $\epsilon > 0$, there is a $\delta > 0$ such that $\|x - x^*\| > \epsilon$ implies*

(S) $$(q - q^*)^{\mathrm{T}} H_q^0(q, x) - (H_x^0)^{\mathrm{T}}(q, x)(x - x^*) + \rho q^{*\mathrm{T}}(x - x^*) > -\rho(q - q^*)^{\mathrm{T}}(x - x^*) + \delta.$$

Then if $(q^*(t), x^*(t))$ solves eqs. (18) and (19) and if eq. (21) holds, then

$$\|x^*(t) - x^*\| \to 0, \qquad t \to \infty.$$

Proof. See Cass and Shell (1976).

It should be noted that (S) is only required to hold on the set of (q, x) such that $(q, x) = (q^*(t), x^*(t))$ for some $t \geqq 0$. Also (S) is the same as

$$-\dot{V}_1(q, x) > \delta.$$

Cass and Shell (1976) also proves the following useful theorem.

Theorem 2. *Assume that*

$$H^0(q, x)$$

is convex in q and concave in x. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-\rho t}(q(t) - q^*)^{\mathrm{T}}(x(t) - x^*)] \geqq 0, \tag{24}$$

for any solution $(q(t), x(t))$ of eqs. (18) and (19).

Proof. See Cass and Shell (1976).

Note that if U, T are concave in (x, v), then it is trivial to show that $H^0(q, x)$ is concave in x. Convexity in q follows from the very definition of H^0:

$$H^0(q, x) = \max_v [U(x, v) + q^{\mathrm{T}} T(x, v)],$$

regardless of whether U, T are concave. The proof uses the definition of convexity and the definition of maximum. See Rockafellar (1976) and its references for a systematic development of properties of the function H^0. Let us turn now to Rockafellar's work.

Rockafellar (1976) studies the case in which $U(x, v)$ is concave in (x, v) and $T(x, v) = v$. He points out, though, that the restriction $(x, \dot{x}) \in X$, X convex may be treated by defining U to be equal to $-\infty$ when off X. Thus, a very general class of problems may be treated by his methods. The paper (and its references) develops, with no differentiability assumptions on $U(x, v)$ – only concavity is assumed – the following ideas, to name a few: (1) a dual problem that the optimal costate $q^*(t)$ must solve; (2) duality theory of the Hamiltonian function $H^0(q, x)$; (3) existence and uniqueness theory for optimum paths; (4) theorems on the differentiability of $W(x)$ under assumptions sufficient for G.A.S. of the stationary solution x^*; (5) relations between $W(x)$ and its analogue for the dual problem; (6) theorems on the monotonicity of the expression

$$V = -(q_1(t) - q_2(t))^{\mathrm{T}}(x_1(t) - x_2(t))$$

for any pair of solutions $(q_1(t), x_1(t))$, $(q_2(t), x_2(t))$ of eqs. (18) and (19) starting from any set of initial conditions; and (7) the notion of (α, β) convexity–concavity for the H^0 function and its relation to G.A.S. of the stationary solution x^*. Due to lack of space, only the main G.A.S. theorem of Rockafellar's will be given here. A definition is needed.

Definition. *Let $h: C \to R$ be a finite function on a convex set $C \leqq R^n$. Then h is α-convex, $\alpha \in R$, if the function*

$$h(x) - \frac{1}{2}\alpha|x|^2$$

is convex on C. If C is open and $h \in C^2$, then α-convexity is equivalent to: for all $x_0 \in C$, for all $w \in R^n$,

$$w^{\mathrm{T}} h_{xx}(x_0) w \geqq \alpha w^{\mathrm{T}} w$$

must hold. Here $h_{xx}(x_0)$ is the matrix of second-order partial derivatives of h evaluated at x_0. A function $g: C \to R$ is β-concave if $-g$ is β-convex.

Theorem 3 (Rockafellar (1976)). *Assume that $H^0(q,x)$ is finite and β-convex–α-concave on $R^n \times R^n$. Also assume that a stationary solution (q^*, x^*) to eqs. (18) and (19) exists and that optimum paths exist from the initial condition x_0. Then the stationary solution x^* is G.A.S. provided that*

(R) $\quad 4\alpha\beta > \rho^2.$

Proof. See Rockafellar (1976).

A more precise statement of theorem 3 is given in Rockafellar (1976). The basic idea of the proof is just to show that (R) implies

$$\dot V_1 < 0$$

along solutions of eqs. (18) and (19) that correspond to optimal paths. It should be noted that Rockafellar works with the more general system,[10]

$$(\dot x(t), -\dot q(t) + \rho q(t)) \in \partial H^0(q(t), x(t)), \tag{26}$$

where

$$\partial H^0(q(t), x(t)) = \{(a, b) \in R^n \times R^n \mid a \text{ is a subgradient of } H^0 \text{ w.r.t. } q(t) \text{ and } b \text{ is a subgradient w.r.t. } x(t)\}.$$

Let us turn now to the results reported in Brock and Scheinkman (1975a, 1974a).

In these papers, the Lyapunov function $V_2 = -\dot q(t)^{\mathrm T}\dot x(t)$ is differentiated along solutions of eqs. (18) and (19) to yield

$$\dot V_2(q(t), x(t)) = -(\dot q(t), \dot x(t))^{\mathrm T} B(q(t), x(t))(\dot q(t), \dot x(t)), \tag{27}$$

where

$$B(q(t), x(t)) = \begin{bmatrix} H^0_{qq} & (\rho/2)I \\ (\rho/2)I & -H^0_{xx} \end{bmatrix}, \tag{28}$$

and where I denotes the $n \times n$ identity matrix and the matrices of partial derivatives

$$H^0_{qq}, \; -H^0_{xx}$$

(which are positive semi-definite since H^0 is convex in q and concave in x) are evaluated at $(q(t), x(t))$. The following sequence of theorems summarize the results in Brock and Scheinkman (1975a, 1974a).

Theorem 4. *Let $(q^*(0), x^*(0))$ be a solution of eqs. (18) and (19). Then*

$$\dot V_2(q^*(t), x^*(t)) < 0 \tag{29}$$

provided that $z^*(t)^{\mathrm T} B(q^*(t), x^*(t))\, z^*(t) > 0$ for all $t \geqq 0$, where

$$z^*(t) = [\rho q^*(t) - H^0_x(q^*(t), x^*(t)), H^0_q(q^*(t), x^*(t))]. \tag{30}$$

Furthermore, if $(q^*(\cdot), x^*(\cdot)t$ is bounded independently of t,[11] of if

$$\limsup_{t\to\infty} -\dot V_2(q^*(t), x^*(t)) < \infty, \tag{31}$$

then there is a stationary solution (q^*, x^*) of (18) and (19) such that

$$(q^*(t), x^*(t)) \to (q^*, x^*),\; t \to \infty. \tag{32}$$

Proof. Inequality (29) is obvious from eq. (27). The second part of the theorem is just a standard application of results on G.A.S. by means of Lyapunov functions. See Brock and Scheinkman (1975a, theorem 2.1) for details.

Theorem 5. *If* (a) *(q^*, x^*) is the unique stationary solution of eqs. (18) and (19);* (b) *For all $(q, x) \neq (q^*, x^*)$,*

$$(q - q^*)^{\mathrm T} H^0_q(q, x) + (x - x^*)^{\mathrm T}(\rho q - H^0_x(q, x)) = 0,$$

implies

$$(q - q^*, x - x^*)^{\mathrm T} B(q, x)(q - q^*, x - x^*) > 0;$$

[10] Let $f: R^n \to R^n$ be a point to set mapping. We say that $x(\cdot)$ is a solution of

$$\dot x(t) \in f(x(t)), \quad x(0) = x_0,$$

if $x(\cdot)$ is absolutely continuous on $[0, \infty)$ and

$$\dot x(t) \in f(x(t)), \quad x(0) = x_0$$

for almost every $t \in [0, \infty)$.

[11] Here, given a function $y(\cdot): [0, \infty) \to R$, lim sup $y(t)$ denotes the largest cluster point of the function values $y(t)$ as $t \to \infty$. Assumption (31) is quite natural for optimal paths because if $W_{xx}(\cdot)$ exists it will be negative semi-definite since $W(\cdot)$ is concave for U, T concave. Thus,

$$\dot q^{*\mathrm T}\dot x^* = \dot x^*(t)^{\mathrm T} W_{xx}(x^*(t))\dot x^*(t) \leqq 0,$$

and eq. (31) holds automatically.

(c) *For all* $w \neq 0$, $w^{\mathrm{T}}B(q^*, x^*)w > 0$. *Then all solutions of* (18) *and* (19) *that are bounded*[12] *for* $t \geqq 0$ *converge to* (q^*, x^*) *as* $t \to \infty$.

Proof. Put $V_1(q,x) = -(q-q^*)^{\mathrm{T}}(x-x^*)$ and use (b) and (c) to show that $\dot{V}_1(q,x) < 0$ for $(q,x) \neq (q^*, x^*)$. The rest is just a standard Lyapunov function stability exercise. See Brock and Scheinkman (1975a, theorem 3.2) for the details.

Note that theorem 5 gives a set of sufficient conditions for the Cass and Shell hypothesis,

$$\dot{V}_1(q,x) < 0,$$

to hold.

Theorems 4 and 5 are, in some sense, complementary since each asks that Q be "positive definite" in directions which are transversal to each other.

Theorem 6. *Assume that* $W_{xx}(\cdot)$ *exists and is negative definite on* $\mathbf{R}^n_+$. *Let* $(q^*(t), x^*(t))$ *be a solution of eqs.* (18) *and* (19) *that corresponds to an optimal path. Assume* (a) $H^0 \in C^2$; (b) (q^*, x^*) *is the unique stationary solution of* (18) *and* (19); (c) x^* *is a locally asymptotically stable solution of the "reduced form" system*

$$\dot{x}(t) = H^0_q(W_x(x(t)), x(t)); \tag{33}$$

(d) $H^0(q,x)$ *is locally* α*-convex-*β*-concave at* (q^*, x^*); *and* (e) $H^0(q,x)$ *is locally* α*-quasi-convex and* $H^0(q,x) - pqx$ *is* β*-quasi-concave along* $(q^*(t), x^*(t))$ *where* $4\alpha\beta > \rho^2$. *Then* (q^*, x^*) *is G.A.S.*

[12]Boundedness of $(q^*(t), x^*(t))$ may be dispensed with provided that one assumes

(T) $\quad \lim_{t\to\infty} q^*(t)x^*(t)\,\mathrm{e}^{-\rho t} = 0,$

and refines the Lyapunov analysis a bit or one assumes that $W_x(\cdot)$ exists. Benveniste and Scheinkman (1975) provide a set of very general conditions on eq. (5) that imply that W_x exists and that (T) holds for optimal paths. For if $W(\cdot)$ is concave and $W_x(\cdot)$ exists, then

$$(q^*(t) - q^*)^{\mathrm{T}}(x^*(t) - x^*) = [W_x(x^*(t)) - W_x(x^*)]^{\mathrm{T}}(x^*(t) - x^*) \leqq 0.$$

The use of $W(\cdot)$ in the last line exposes why the Lyapunov function, $-(q-q^*)^{\mathrm{T}}(x-x^*)$ is "natural."

[13]The unwary reader, after reading Rockafellar (1976) might think that $W(\cdot) \in C^2$ implies G.A.S. But the OLRP gives examples of unstable systems where the value $W(\cdot)$ is C^∞.

Here $f: R^n \to R$, $f \in C^2$ is locally β-quasi-concave at $x_0 \in R^n$ if for all $w \in R^n$, $w \neq 0$, we have

$$w^{\mathrm{T}} f_{xx}(x_0) w \leqq -\beta w^{\mathrm{T}} w,$$

for all w such that

$$w^{\mathrm{T}} f_x(x_0) = 0.$$

$g: R^n \to R$ is locally α-quasi-convex at $x_0 \in R^n$ if $-g$ is locally α-quasi-concave at x_0.

Proof. This is an adaptation of Hartman–Olech's theorem (Hartman (1964, p. 548)) to the system eq. (33) with their $G(\cdot) = -W_{xx}(\cdot)$. See Brock and Scheinkman (1974b).

Theorem 6 allows a weak form of increasing returns to the state variable. For β-quasi-concavity in x of the imputed profit function – the Hamiltonian function $H^0(q,x) - pqx$ – amounts to allowing increasing returns to x provided that the "isoquants" for each fixed q have enough "curvature." We say that theorem 6 allows a *weak* form of increasing returns because we assume $W_{xx}(\cdot)$ is negative definite which implies a form of long run *decreasing* returns. In particular, the state valuation function is concave. Note that concavity of W does not imply concavity of U or T, although vice versa holds.

Theorem 6 is in an unsatisfactory state of affairs at the moment since it requires that $W_{xx}(\cdot)$ exist and be negative definite, but we do not have a useful set of sufficient conditions on U and T for this to happen. Both this question and the question of stability analysis under increasing returns seem to us to be "wide open" and important fields of research. Systematic study of economic dynamics under increasing returns is likely to change our view of how a dynamic economy functions.

Theorems 1–6 are all unified by the fact that they represent results that can be obtained from the Lyapunov functions,

$$V_1 = -(q-q^*)^{\mathrm{T}}(x-x^*) \quad \text{and} \quad V_2 = -\dot{q}\dot{x},$$

and their analogues. These results lead us intuitively to expect that G.A.S. is likely when $H^0(q,x)$ has a "lot of convexity in q" and a "lot of concavity in x" relative to the discount rate ρ. More specifically, the Rockafellar condition,

(R) $\quad 4\alpha\beta > \rho^2,$

or its analogues, are sufficient for G.A.S.

To sharpen our understanding of functions U and T that satisfy the hypotheses of the above G.A.S. theorems, it is useful to look at the OLRP and, what is the same thing, the linear quadratic approximation to problem (5), as given in eq. (12). The Hamiltonians for these problems are (converting the OLRP into a maximization problem) OLRP:

$$H(q,x,v) = x^{\mathrm{T}}(-Q)x + 2x^{\mathrm{T}}(-S)v + v^{\mathrm{T}}(-R)v + q^{\mathrm{T}}[Fx + Gv],^{14} \tag{34}$$

$$H^0(q,x) = x^{\mathrm{T}}[SR^{-1}S^T - Q]x + q^{\mathrm{T}}[F - GR^{-1}S^{\mathrm{T}}]x + \tfrac{1}{4}q^{\mathrm{T}}GR^{-1}G^{\mathrm{T}}q. \tag{35}$$

[14] The derivation of formula (35) follows. First, if A is a matrix, then

(a) $\dfrac{\partial}{\partial x}(x^{\mathrm{T}}Ax) = (A + A^{\mathrm{T}})x,$

(b) $\dfrac{\partial}{\partial y}(x^{\mathrm{T}}Ay) = A^{\mathrm{T}}x,$

must hold. The optimal control v must maximize H. Therefore,

(c) $0 = H_v = -2Rv - 2S^{\mathrm{T}}x + G^{\mathrm{T}}q.$

Hence, letting v^0 denote the optimal v,

(d) $v^0 = -R^{-1}S^{\mathrm{T}}x + \tfrac{1}{2}R^{-1}G^{\mathrm{T}}q.$

Substituting (d) into eq. (34), we get

$$\begin{aligned} H^0 &= -x^{\mathrm{T}}Qx + [-R^{-1}S^{\mathrm{T}}x + \tfrac{1}{2}R^{-1}G^{\mathrm{T}}q]^{\mathrm{T}}(-R)[-R^{-1}S^{\mathrm{T}}x + \tfrac{1}{2}R^{-1}G^{\mathrm{T}}q] \\ &\quad + [G^{\mathrm{T}}q - 2S^{\mathrm{T}}x]^{\mathrm{T}}[-R^{-1}S^{\mathrm{T}}x + \tfrac{1}{2}R^{-1}G^{\mathrm{T}}q] + q^{\mathrm{T}}Fx \\ &= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}). \end{aligned}$$

Now

$$\begin{aligned} (\mathrm{II}) &= [S^{\mathrm{T}}x - \tfrac{1}{2}G^{\mathrm{T}}q]^{\mathrm{T}}[\tfrac{1}{2}R^{-1}G^{\mathrm{T}}q - R^{-1}S^{\mathrm{T}}x] \\ &= -x^{\mathrm{T}}SR^{-1}S^{\mathrm{T}}x + \tfrac{1}{2}x^{\mathrm{T}}SR^{-1}G^{\mathrm{T}}q + \tfrac{1}{2}q^{\mathrm{T}}GR^{-1}S^{\mathrm{T}}x \\ &\quad - \tfrac{1}{4}q^{\mathrm{T}}GR^{-1}G^{\mathrm{T}}q, \\ (\mathrm{III}) &= 2x^{\mathrm{T}}SR^{-1}S^{\mathrm{T}}x - x^{\mathrm{T}}SR^{-1}G^{\mathrm{T}}q - q^{\mathrm{T}}GR^{-1}S^{\mathrm{T}}x + \tfrac{1}{2}q^{\mathrm{T}}GR^{-1}G^{\mathrm{T}}q. \end{aligned}$$

Add (II) to (III) to get

(e) $(\mathrm{II}) + (\mathrm{III}) = x^{\mathrm{T}}SR^{-1}Sx + \tfrac{1}{4}q^{\mathrm{T}}GR^{-1}G^{\mathrm{T}}q - x^{\mathrm{T}}SR^{-1}G^{\mathrm{T}}q.$

Insert (e) into the expression for H^0,

$$\begin{aligned} H^0(q,x) &= (\mathrm{I}) + (\mathrm{IV}) + (\mathrm{II}) + (\mathrm{III}) \\ &= -x^{\mathrm{T}}Qx + q^{\mathrm{T}}Fx + x^{\mathrm{T}}SR^{-1}S^{\mathrm{T}}x \\ &\quad + \tfrac{1}{4}q^{\mathrm{T}}GR^{-1}G^{\mathrm{T}}q - x^{\mathrm{T}}SR^{-1}G^{\mathrm{T}}q \\ &= x^{\mathrm{T}}[SR^{-1}S^{\mathrm{T}} - Q]x + q^{\mathrm{T}}[F - GR^{-1}S^{\mathrm{T}}]x \\ &\quad + \tfrac{1}{4}q^{\mathrm{T}}GR^{-1}G^{\mathrm{T}}q, \end{aligned}$$

which is eq. (35). Equation (37) is proved by putting $Q = -U^*_{xx}$, $S = -U^*_{xv}$, $R = -U^*_{vv}$, $F = T^*_x$, $G = T^*_v$ in eq. (35).

2.3. *Linear quadratic approximation to (P)*

Let

$$H(\eta,\xi,\gamma) = \xi^{\mathrm{T}}(U^*_{xx})\xi + 2\xi^{\mathrm{T}}U^*_{xv}\gamma + \gamma^{\mathrm{T}}(U^*_{vv})\gamma + \eta^{\mathrm{T}}[T^*_x\xi + T^*_v\gamma], \tag{36}$$

$$\begin{aligned} H^0(\eta,\xi) &= \xi^{\mathrm{T}}[U^*_{xx} - U^*_{xv}(U^*_{vv})^{-1}U^{*\mathrm{T}}_{xv}]\xi + \eta^{\mathrm{T}}[T^*_x - T^*_v U^{*-1}_{vv}U^{*\mathrm{T}}_{xv}]\xi \\ &\quad - \tfrac{1}{4}\eta^{\mathrm{T}}[T^*_v U^{-1}_{vv}T^{*\mathrm{T}}_v]\eta. \end{aligned} \tag{37}$$

Substitute the formula

$$v^0 = -R^{-1}S^{\mathrm{T}}x + \tfrac{1}{2}R^{-1}G^{\mathrm{T}}q$$

to obtain the system

$$\begin{aligned} \dot{x} = Fx + Gv^0 &= (F - GR^{-1}S^{\mathrm{T}})x + \tfrac{1}{2}GR^{-1}G^{\mathrm{T}}q \\ &= H^0_q(q,x). \end{aligned} \tag{38}$$

Similarly,

$$\dot{\xi} = T^*_x\xi + T^*_v\eta = (T^*_x - T^*_v U^{*-1}_{vv}U^{\mathrm{T}}_{xv})\xi + T^*_v\eta. \tag{39}$$

Let us use these formulas to build some understanding of the meaning of the G.A.S. tests in theorems 1–5, and to obtain at the same time some of the Magill (1975) G.A.S. results for the OLRP. The Cass and Shell test requires that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1 = -\frac{\mathrm{d}}{\mathrm{d}t}(q - q^*)^{\mathrm{T}}(x - x^*) = -[(\rho q - H^0_x)^{\mathrm{T}}x + q^{\mathrm{T}}H^0_q] < 0, \tag{40}$$

for all q, x. Note the existence of the stationary solution $(q^*, x^*) = (0,0)$ for the OLRP. From eqs. (35) and (40)

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}V_1 &= [\rho q - 2(SR^{-1}S^{\mathrm{T}} - Q)x]^{\mathrm{T}}x + q^{\mathrm{T}}[\tfrac{1}{2}GR^{-1}G^{\mathrm{T}}]q \\ &= (q,x)^{\mathrm{T}}B(q,x), \end{aligned} \tag{41}$$

where

$$B = \begin{bmatrix} H^0_{qq} & (\rho/2)I \\ (\rho/2)I & -H^0_{xx} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}GR^{-1}G^{\mathrm{T}} & (\rho/2)I \\ (\rho/2)I & 2(Q - SR^{-1}S^{\mathrm{T}}) \end{bmatrix}, \tag{42}$$

which is the negative of the Magill (1975) K^{ρ} matrix. Now, it is easy to see that the Rockafellar condition (R) is, basically, the same thing as B positive definite (see Brock and Scheinkman (1975a)). Thus, in the case of the OLRP, all five G.A.S. tests developed in theorems 1–5 amount to the same thing. This will not be true for nonquadratic problems, of course.

When is B positive definite? To get some feel for this put $G = I$, $S = 0$. Then in the one dimensional case, we require

$$(\tfrac{1}{2}R^{-1})(2Q) = R^{-1}Q > \rho^2/4. \tag{43}$$

Inequality eq. (43) holds if Q is large, R is small and ρ is small. This is in accord with our earlier heuristic discussion of the stability of the OLRP. But one source of stability or instability is ignored by eq. (43), and indeed by all of the theorems 1–6. That is the matrix F, which is the very law of motion of the system! We shall say more about F later.

In general, as is easy to see, B will be positive definite when

$$\underset{\sim}{\lambda}[\tfrac{1}{2}GR^{-1}G^{\mathrm{T}}]\underset{\sim}{\lambda}[2(Q - SR^{-1}S^{\mathrm{T}})] = \underset{\sim}{\lambda}[GR^{-1}G^{\mathrm{T}}]\underset{\sim}{\lambda}[Q - SR^{-1}S^{\mathrm{T}}] > \rho^2/4, \tag{44}$$

where $\underset{\sim}{\lambda}(A)$ = the smallest eigenvalue of $(A + A^{\mathrm{T}})/2$. Inequality (44) is the same thing as (R) since α-convexity of $H^0(q, x)$ means α = the smallest eigenvalues of H^0_{qq} and, in this case,

$$H^0_{qq} = \tfrac{1}{2}GR^{-1}G^{\mathrm{T}}.$$

Similarly for β-concavity.

Return now to the role of the matrix F. Any information on F is "wasted" by theorems 1–6. Indeed, it is pointed out by Magill (1975) that a fruitful way to view G.A.S. tests based on the theorems 1–6 is that they give sufficient conditions for G.A.S. no matter *how stable or unstable the matrix F is*. A test needs to be developed that uses information on F, for rough intuition suggests that, for the OLRP, if F has all eigenvalues with negative real parts (i.e. F is a stable matrix), then it seems odd that it would be optimal to destabilize the system.[15] This seems plausible because it costs $v^{\mathrm{T}}Rv$ to administer control, and one would think that in view of the cost $x^{\mathrm{T}}Qx$ of x being away from zero it would be sensible to use v to speed up the movement of x to zero when F is a stable matrix. However there are

[15] Of course, from eq. (16) when $S \neq 0$, then $GR^{-1}S^{\mathrm{T}}$ may be such that

$$F - GR^{-1}S^{\mathrm{T}} \equiv A$$

is unstable even though F is stable. Thus, the "cross-effects" S may act to destabilize a stable law of motion. The conjecture is plausible for $S = 0$, however. Magill (1975) calls A the "underlying system" matrix. A natural and interesting conjecture is: does A stable imply G.A.S. of the optimal solutions to eq. (16). We sketch the construction of a counterexample in the Appendix.

two state variable examples where the underlying system matrix is stable, but it is optimal to administer control to destabilize the system! See the appendix for an outline of how to construct such an example.

Now there is one test for G.A.S. of the OLRP that wastes no information at all. That is to count the eigenvalues of the linear system

$$\text{(L)}\qquad \begin{aligned} \dot{q} &= \rho q - H^0_{xx}x - H^0_{xq}q, \\ \dot{x} &= H^0_{qx}x + H^0_{qq}q, \end{aligned}$$

and check whether half of them have negative real parts. Then, provided that the corresponding eigenvectors in (q, x) space generate a linear space whose projection on x space is all of R^n, G.A.S. holds. The problem (posed by Harl Ryder) of finding a neat set of conditions on (L), making full use of its structure, for half of the eigenvalues of (L) to have negative real parts and for the projection property to hold seems to be open. However, there are Routh–Hurwitz-type tests for $k \leqq n$ of the eigenvalues to have negative real parts, but the problem appears to be in developing a test that is "efficient" in the use of the structure of (L).

Let us call this kind of test the *ideal OLRP test*. This sort of test has not been generalized in an interesting and useful way to nonquadratic problems, however. This is another research problem of great importance.

The tests proposed in theorems 1–6, wasteful relative to the ideal test for the OLRP though they may be, give good results for nonlinear problems and generalize easily to the case of uncertainty. For example, they indicate that G.A.S. follows from just convexo-concavity of H^0 for the case $\rho = 0$, which is the famous no discounting case in optimal growth theory. Brock and Majumdar (1975) develop stochastic analogues of theorems 1–5 and obtain G.A.S. results for a highly nonlinear multisector model under uncertainty. The objective of the work being surveyed in this article is to develop G.A.S. tests on U and T that work for *nonlinear, nonquadratic problems*, and that generalize easily to uncertainty. Turn now to the development of a test that uses information on F.

2.4. *Results based upon the Lyapunov function $V_3 = \dot{x}^{\mathrm{T}}G(q, x)\dot{x}$*

Brock and Scheinkman (1975b) consider the class of Lyapunov functions

$$V_3 = \dot{x}^{\mathrm{T}}G(q, x)\dot{x}, \tag{45}$$

where the matrix $G(q, x)$ is positive definite. Look at

$$\dot{x} = H^0_q(q, x), \qquad x(0) = x_0. \tag{46}$$

Evaluate $\dot V_3$ along solutions of eq. (46). One obtains

$$\begin{aligned}\dot V_3 &= \ddot x^{\mathrm T} G\dot x + \dot x^{\mathrm T} G\ddot x + \dot x^{\mathrm T}\dot G\dot x\\ &= [H^0_{qq}\dot q + H^0_{qx}\dot x]^{\mathrm T} G\dot x + \dot x^{\mathrm T} G[H^0_{qq}\dot q + H^0_{qx}\dot x] + \dot x^{\mathrm T}\dot G\dot x.\end{aligned} \tag{47}$$

Here $\dot G(q,x)$, the trajectory derivative of G, is defined by

$$\begin{aligned}\dot G_{ij} &= \sum_s (\dot G_{ijq_s}\dot q_s + \dot G_{ijx_s}\dot x_s)\\ &= \sum_s [G_{ijq_s}(\rho q_s - H^0_{x_s}) + G_{ijx_s}H^0_{q_s}].\end{aligned} \tag{48}$$

Assuming $W_{xx}(\cdot)$ exists and is negative semi-definite, so that $\dot q^{\mathrm T}\dot x \leqq 0$ along solutions of eqs. (18) and (19) that correspond to optimal paths, eq. (47) suggests choosing

$$G = H^{0^{-1}}_{qq}. \tag{49}$$

Thus,

$$\dot V_3 = 2\dot q^{\mathrm T}\dot x + \dot x^{\mathrm T}\{H^{0^{-1}}_{qq}H^0_{qx} + (H^{0^{-1}}_{qq}H^0_{qx})^{\mathrm T} + (\dot H^{0^{-1}}_{qq})\}\dot x. \tag{50}$$

We may now state the following theorem.

Theorem 7. *Let $q(\cdot), x(\cdot)$ be a solution of eqs.* (18) *and* (19) *such that*

$$\dot q^{\mathrm T}\dot x \leqq 0 \quad on \quad [0,\infty).$$

Assume

$$\dot x^{\mathrm T}\{H^{0^{-1}}_{qq}H^0_{qx} + (H^{0^{-1}}_{qq}H^0_{qx})^{\mathrm T} + (\dot H^0_{qq})^{-1}\}\dot x \leqq 0 \tag{51}$$

along $x(\cdot)$. Also assume that $W_x(\cdot)$ exists.[16] *Then $x(t)$ converges to the largest future invariant set contained in*

$$\{\bar x \mid \dot V_3(W_x(\bar x), \bar x) = 0\}.$$

Proof. This is just a standard application of Lyapunov theory to the function V_3. See Brock and Scheinkman (1975b) for the proof and some extensions of theorem 7. Here X is "future invariant" under $\dot x = f(x)$ if for each $x_0 \in X$ the solution $x(t|x_0)$ starting from x_0 stays in X for $t \geqq 0$. Many times the special structure of the Lyapunov function V and the law of motion f can be used to show that the largest future invariant set contained in $\{x_0 | \dot V(x_0) = 0\}$ is a point. Obviously, theorems 1–5 may be sharpened in this way.

Theorem 7 would be very nice if the term

$$\dot H^{0^{-1}}_{qq}$$

did not appear. This term is a hard one on which to get a grip. However, for quadratic problems and especially for problems where H^0_{qq} is independent of (q,x), the G.A.S. test (51) is useful. Problems where H^0_{qq} is independent of (q,x) arise in neoclassical investment models where the adjustment cost function is quadratic, but the *production function* is not necessarily quadratic. See Brock and Scheinkman (1975b, forthcoming) for a wide class of investment models where eq. (51) is applicable.

Let us apply eq. (51) to the OLRP. Here

$$H^{0^{-1}}_{qq}H^0_{qx} = (\tfrac{1}{2}GR^{-1}G^{\mathrm T})^{-1}(F - GR^{-1}S^{\mathrm T}). \tag{52}$$

To get some "feel" for negative quasi-definiteness of eq. (52) put $G = I$, $S = 0$, then

$$H^{0^{-1}}_{qq}H^0_{qx} = 2RF. \tag{53}$$

If R and F are one dimensional, then since $R > 0$ by convexity of the objective, we see that $F < 0$ implies G.A.S. by theorem 7, regardless of the size of ρ and Q. This is in accord with our intuition that if F is a stable matrix, then the OLRP should be G.A.S. independently of ρ.

None of the G.A.S. tests mentioned above make use of H^0_{xx} in a way that parallels H^0_{qq} in theorem 7. This brings us to one of Magill's (1975, theorem 2 ii c) clever results for the OLRP. We state it for the case of certainty only.

Theorem 8 (Magill (1975)). *Assume that*

$$-H^0_{qq}, H^0_{xx}$$

are negative definite. Furthermore, assume that

$$M^\rho = [\rho I - H^0_{xq}]^{\mathrm T}(-H^0_{xx})^{-1} + (-H^0_{xx})^{-1}[\rho I - H^0_{xq}]$$

is nonpositive definite. Then G.A.S. holds for the OLRP.

Proof. Put

$$V = \eta^{\mathrm T}(-H^0_{xx})^{-1}\eta.$$

[16] This result may be generalized by considering

$$V_5 = \alpha_1\dot q^{\mathrm T}\dot x + \alpha_2\dot x^{\mathrm T}H^{0^{-1}}_{qq}\dot x + \alpha_3\dot q^{\mathrm T}(-H^{0^{-1}}_{xx})\dot q$$

and arguing as above. We leave this to the reader.

Then

$$\dot{\eta} = \rho\eta(t) - H^0_{xx}\xi(t) - H^0_{xq}\eta(t), \qquad \xi(t)^{\mathrm{T}}\eta(t) < 0$$

(Magill (1975, pp. 19–20)) implies

$$\begin{aligned}\dot{V} &= \dot{\eta}^{\mathrm{T}}(-H^0_{xx})^{-1}\eta + \eta^{\mathrm{T}}(-H^0_{xx})^{-1}\dot{\eta}\\ &= [\rho\eta - H^0_{xx}\xi - H^0_{xq}\eta]^{\mathrm{T}}(-H^0_{xx})^{-1}\eta\\ &\qquad + \eta^{\mathrm{T}}(-H^0_{xx})^{-1}[\rho\eta - H^0_{xx}\xi - H^0_{xq}\eta]\\ &= 2\xi^{\mathrm{T}}\eta + \eta^{\mathrm{T}}(\rho I - H^0_{xq})^{\mathrm{T}}(-H^0_{xx})^{-1}\eta + \eta^{\mathrm{T}}(-H^0_{xx})^{-1}(\rho I - H^0_{xq})\eta < 0.\end{aligned}$$

The rest is standard, since $\xi^{\mathrm{T}}\eta = 2\xi^{\mathrm{T}}W_{xx}\xi < 0$ under Magill's hypotheses.

2.5. *Other G.A.S. results*

Some other G.A.S. results are described only very briefly due to lack of space. The Lyapunov function,

$$V_4 = a\dot{q}^{\mathrm{T}}\dot{x} + b\dot{x}^{\mathrm{T}}H^{0^{-1}}_{qq}(q,x)\dot{x},$$

leads to the following question: find sufficient conditions on $H^0(q,x)$ such that $\dot{V}_4 \leqq 0$ holds along $(q(t),x(t))$, $t \in [0,\infty)$, for particular choices of a, $b \in R$. One useful sufficient condition for $\dot{V}_4 \leqq 0$ that emerges from this approach is that

$$\rho\bar{\lambda}[H^{0^{-1}}_{qq}H^0_{qx} + (H^{0^{-1}}_{qq}H^0_{qx})^{\mathrm{T}} + (\dot{H}^{0^{-1}}_{qq})] \leqq 2\underline{\lambda}q[-H^0_{xx}] \tag{54a}$$

holds along the solution $(q(t),x(t))$ to eqs. (18) and (19). Here $\bar{\lambda}(A) =$ largest eigenvalue of $(A + A^{\mathrm{T}})/2$.

Relationship (54a) is derived by differentiating V_4 along solutions to eqs. (18) and (19),

$$\begin{aligned}\dot{V}_4 &= \rho a\dot{q}^{\mathrm{T}}\dot{x} + a\dot{x}^{\mathrm{T}}(-H^0_{xx})\dot{x} + a\dot{q}^{\mathrm{T}}H^0_{qq}\dot{q}\\ &\qquad + b\dot{x}^{\mathrm{T}}[H^{0^{-1}}_{qq}H^0_{qx} + (H^{0^{-1}}_{qq}H^0_{qx})^{\mathrm{T}} + (H^{0^{-1}}_{qq})]\dot{x} + 2b\dot{q}^{\mathrm{T}}\dot{x}\end{aligned}$$

Note that the first line of $\dot{V}_4$ is just

$$(\dot{q},\dot{x})^{\mathrm{T}}B(q,x)(\dot{q},\dot{x}),$$

so if B is positive semi-definite along $(q(t),x(t))$, just put $b = 0$, $a < 0$ to get $\dot{V}_4 < 0$ for all $t \geqq 0$. Similarly if $A \equiv H^{0^{-1}}_{qq}H^0_{qx} + (H^{0^{-1}}_{qq}H^0_{qx})^{\mathrm{T}} + (H^{0^{-1}}_{qq})$ is negative definite along $(q(t),x(t))$ and if we assume that $\dot{q}^{\mathrm{T}}\dot{x} \leqq 0$ along optimal paths, we then get $\dot{V}_4 \leqq 0$ for all $t \geqq 0$ by putting $a = 0$, $b > 0$. The only case where we can get a new theorem, therefore, is when B is not positive semi-definite and A is not negative quasi semi-definite for all $t \geqq 0$. Suppose that $\bar{\lambda}(A) > 0$, $t \geqq 0$. Grouping the terms common to $\dot{q}^{\mathrm{T}}\dot{x}$ in $\dot{V}_4$, choose $a < 0$, $b > 0$ such that $2b + a\rho \geqq 0$. Then

$$\dot{V}_4 \leqq a\dot{x}^{\mathrm{T}}(-H^0_{xx})\dot{x} + b\dot{x}^{\mathrm{T}}A\dot{x} \leqq a\underline{\lambda}(-H^0_{xx})|\dot{x}|^2 + b\bar{\lambda}(A)|\dot{x}|^2,$$

since $\dot{q}^{\mathrm{T}}\dot{x} \leqq 0$. The R.H.S. is nonpositive provided that

$$\bar{\lambda} \leqq (-a)\underline{\lambda},$$

but

$$(-a)\underline{\lambda} \leqq \frac{2b}{\rho}\underline{\lambda}.$$

Therefore,

$$b\bar{\lambda} \leqq \frac{2}{\rho}\underline{\lambda},$$

and this last is eq. (54a)[17]

If the Hamiltonian $H^0(q,x)$ is separable, i.e.

$$H^0(q,x) = F_1(q) + F_2(x), \tag{55}$$

for some pair of functions $F_1(\cdot)$, $F_2(\cdot)$, then Scheinkman (1975) shows that G.A.S. holds under convexo-concavity of H^0 by setting $V_5 = F_1(q)$, and using $\dot{q}^{\mathrm{T}}\dot{x} \leqq 0$ to show that $\dot{V}_5 \leqq 0$ along optimal paths. Scheinkman has also generalized the above result to discrete time. Note that stability does not depend on the size of ρ. Separable Hamiltonians arise in adjustment cost models where the cost of adjustment is solely a function of net investment (see Scheinkman (1975) for details).

Swapan Dasgupta (1975) and Araujo–Scheinkman (1975) have developed notions of dominant diagonal and block dominant diagonal that take into account the "saddle point character" of eqs. (18) and (19), and have obtained interesting G.A.S. results. The Araujo–Scheinkman paper also relates l_∞ continuity of the optimal path in its initial condition (in discrete time) to G.A.S. They show that l_∞ continuity plus L.A.S. of a steady state x^* implies G.A.S. of that steady state! Thus, l_∞ continuity is not an assumption to be taken lightly. They also show the "converse" result that if G.A.S. is true, then (l_∞) differentiability of the optimal paths

[17] Benveniste and Scheinkman (1975) provide a useful set of sufficient conditions on eq. (5) for existence of W_x. They provide conditions only for the case $T(x,v) = v$, but it should not be too difficult to generalize their argument to more general functions T.

with respect to both initial conditions and discount factor must hold. This establishes in particular that both the policy function and the value function are differentiable if G.A.S. holds.

Magill (1972, 1974, 1975) has formulated a linear quadratic approximation to a continuous time stochastic process version of problem (P), and has established the stochastic stability of this approximation. His paper (1972) was the first to point out the "correct" Lyapunov function to use, namely the minimal expected value of the objective as a function of the initial condition x_0. Magill's is the only stability result that I know of for the multisector optimal growth model driven by a continuous time stochastic process. Brock–Majumdar (1975) treats the discrete time case. This area is largely undeveloped and a promising area for future research.

Scheinkman's thesis (1973) shows the important result that G.A.S. is a "continuous" property in $\rho > 0$ at $\rho = 0$. This result was generalized for any $\rho > 0$ in Araujo and Scheinkman (1975). Burmeister and Graham (1973) present the first set of G.A.S. results for multisector models under adaptive expectations. This looks like a promising area for future developments.

Another particularly promising area of research is to apply the program of results surveyed in this paper to noncooperative equilibria generated by N-player differential games, where the objective of player i is to solve

$$\sup \int_0^\infty e^{-\rho_i t} U_i(x_i(t), \hat{x}_i(t), v_i(t), \hat{v}_i(t))\, dt$$

s.t.

$$\dot{x}_i(t) = T_i(x_i(t), \hat{x}_i(t), v_i(t), \hat{v}_i(t)),$$

$$x_i(0) = x_{i0} \text{ given},$$

$$v_i(\cdot): [0, \infty) \to R^m \text{ measurable}.$$

Here $x_i(t) \in R^n$ denotes the vector of state variables under the control of player i; $v_i(t) \in R^m$ denotes the vector of instrument variables under control by i; and $\hat{x}_i(t)$, $\hat{v}_i(t)$ denote the state and instrument variables under control by all players but i. A discussion of the economic basis for these games and some very preliminary results is given in Brock (1975).

Finally, the relationship between L.A.S. and G.A.S. is not very well understood at this point. In particular, suppose that eqs. (5), (6) and (7) had only one rest point x^* and assume it is L.A.S. What additional assumptions are needed on the Hamiltonian to ensure G.A.S.? A non-

linear version of the OLRP suggests that this problem may be difficult. For example, consider the problem

$$\min \int_0^\infty e^{-\rho t}(x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)\, dt$$

s.t.

$$\dot{x} = F(x) + Gu, \qquad x_0 \text{ given}.$$

Note that only $F(x)$ is nonlinear. Now, arguing heuristically, if we let $\|G\| \to 0$, $\|R\| \to \infty$, $\|Q\| \to 0$, $\rho \to \infty$, then the optimal solution of eq. (56), call it $\bar{x}(t|x_0)$, should converge to the solution $x(t|x_0)$ of

$$\dot{x} = F(x), \qquad x(0) = x_0. \tag{57}$$

Here $\|A\|$ denotes a norm of the matrix A. In other words, given any differential equation system (57) we should be able to construct a problem (56) that generates optimum paths that lie arbitrarily close to the solution trajectories of eq. (57). This suggests that any behavior that can be generated by systems of the form (57) can be generated by optimum paths to problems of the form (56). There are many systems $\dot{x} = F(x)$ that possess a unique L.A.S. rest point, but are not G.A.S. An obvious example in the plane is concentric limit cycles surrounding a unique L.A.S. rest point.

In economic applications more information on F is available. We may assume $F(x)$ is concave in x, for example. But, still, a lot of phenomena may be generated by systems of the form $\dot{x} = F(x)$, F concave.

An important research project would be to classify the class of optimal paths generated by problem (5) for all concave U, T. The heuristic argument given above suggests that anything generated by eq. (57) for F concave is possible. Thus, it appears that strong additional hypotheses must be placed on the Hamiltonian, above and beyond convexo-concavity, to get G.A.S. even when the rest point is L.A.S. and unique. It should be pointed out that there is a close relationship between uniqueness of the rest point and G.A.S. Obviously, G.A.S. cannot hold when there is more than one rest point. The uniqueness of rest points is fairly comprehensively studied in Brock (1973) and Brock and Burmeister (1976).

In summary, let us say that a dent has been made in providing economists with a useful set of G.A.S. results for their dynamic models, but much more needs to be done.

Appendix: An example where the underlying system matrix is stable, but the optimal path is unstable

Without loss of generality, we may assume that $S = 0$ in eq. (13). For the one dimensional case, it is fairly easy to show that $F < 0$ implies $\xi(t|\xi_0) \to 0$, $t \to \infty$ for all ξ_0, where $\xi(t|\xi_0)$ denotes the optimum solution of eq. (13) starting from ξ_0. Thus, we must go to the two dimensional case in order to construct a counter-example. Put $\xi = x$, $\eta = q$ to ease the typing.

The Hamilton–Jacobi equation,

$$\rho W(x) = H^0(W_x(x), x), \tag{A.1}$$

generates, letting $W(x_0) = x_0^{\mathrm{T}} W x_0$ be the state valuation function of eq. (13), the matrix equation

$$\rho W = -Q + F^{\mathrm{T}}W + WF + WGR^{-1}G^{\mathrm{T}}W. \tag{A.2}$$

See Magill (1975) for this easy derivation and a discussion of the properties of the quadratic matrix Ricatti equation (A.2).

Now, the system

$$\dot{x} = Fx + Gu^{\circ} \tag{A.3}$$

becomes

$$\dot{x} = Fx + \tfrac{1}{2}GR^{-1}G^{\mathrm{T}}q = [F + GR^{-1}G^{\mathrm{T}}W]x \equiv Ax, \tag{A.4}$$

using

$$q = W_x = 2Wx,$$

and the formula for the optimal control,

$$u^{\circ} = \tfrac{1}{2}R^{-1}G^{\mathrm{T}}q.$$

The task is to construct a matrix F that has all eigenvalues with negative real parts and to construct ρ, Q, R, G so that $F + GR^{-1}G^{\mathrm{T}}W$ is unstable. The easiest way to do this is to divide both sides of (A.2) by ρ, let $\rho \to \infty$, and change Q so that $Q/\rho = \bar{Q}$, where $\bar{Q}$ is a fixed positive definite matrix. Thus for ρ large, W is approximately given by $W = -\bar{Q}$ and

$$A = F - GR^{-1}G^{\mathrm{T}}\bar{Q}.$$

Our task reduces to: construct stable matrix F and two positive definite matrices

$$B \equiv GR^{-1}G^{\mathrm{T}}, \bar{Q}$$

so that A is unstable.

Put

$$B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \quad \bar{Q} = I,$$

so that

$$BQ = B$$

and

$$A = F - B.$$

Just pick stable F and positive diagonal B so that the determinant of $F - B$ is negative:

$$|F - B| = (F_{11} - b_1)(F_{22} - b_2) - F_{12}F_{21} < 0. \tag{A.5}$$

To do this set $F_{11} + F_{22} < 0$,

$$F_{11} < 0, \qquad F_{22} > 0, \qquad F_{11}F_{22} - F_{12}F_{21} > 0.$$

Obviously if b_1 is large enough and b_2 is small enough, then R.H.S. (A.5) is negative. This ends the sketch of the counterexample.

What causes this odd possibility that it may be optimal to destabilize a stable system when more than one dimension is present? To explain, let us call

$$B = GR^{-1}G^{\mathrm{T}}$$

the control gain. It is large when control cost, R, is small and Gu is "effective" in moving x. Now, the discount ρ is high on the future, but the state disequilibrium cost $Q = \rho I$ is large. For $i = 1, 2$, the cost of x_i – disequilibrium is weighted equally by Q. But if b_1 is large and b_2 is small, then control gain is larger for x_1 than for x_2. Therefore, the optimizer administers more control to x_1 relative to x_2. But $-F_{12}F_{21} > 0$ in order that $F_{11} < 0$, $F_{22} > 0$, $|F| > 0$, so that the sign of the impact of an increase of x_2 on $\dot{x}_1$ is opposite to the sign of the impact of an increase of x_1 on $\dot{x}_2$. Thus, the optimizer is lead to destabilize the system.

The economic content of our example is: optimal stabilization policy may be destabilizing when there is a high cost of state disequilibrium, a high discount on the future, and differential control gains or differential state costs.

References

Anderson, B. and J. Moore (1971), *Linear Optimal Control*, Prentice-Hall, Englewood Cliffs, N.J.

Araujo, A. and J. Scheinkman (1975), "Smoothness, Comparative Dynamics, and the Turnpike Property," University of Chicago.
Arrow, K. and M. Kurz (1970), *Public Investment, The Rate of Return, and Optimal Fiscal Policy*, The Johns Hopkins Press, Baltimore.
Benveniste, L. and J. Scheinkman (1975), "Differentiable Value Functions in Concave Dynamic Optimization Problems," Report 7549, Center for Mathematical Studies in Business and Economics, The University of Chicago.
Breakwell, J., J. Speyer and A. Bryson (1963), "Optimization and Control of Nonlinear Systems Using the Second Variation," *SIAM Journal of Control*, 1, 193–223.
Brock, W. (1975), "Differential Games With Active and Passive Variables," University of Chicago.
Brock, W. (1973), "Some Results on the Uniqueness of Steady States in Multisector Models of Optimum Growth When Future Utilities are Discounted," *International Economic Review*, 14, 535–559.
Brock, W. and E. Burmeister (1976), "Regular Economies and Conditions for Uniqueness in Optimal Multi-Sector Economic Models," *International Economic Review*, Feb.
Brock, W. and M. Majumdar (1975), "Asymptotic Stability Results for Multisector Models of Optimal Growth When Future Utilities are Discounted," Cornell University Discussion Paper; University of Chicago Working Paper.
Brock, W. and J. Scheinkman (1975a), "Global Asymptotic Stability of Optimal Control With Applications to the Theory of Economic Growth," *Journal of Economic Theory* Symposium, Feb. 1976: University of Chicago Working Paper. (Revision of Brock and Scheinkman (1974a).)
Brock, W. and J. Scheinkman (1974a), "Global Asymptotic Stability of Optimal Control With With Applications to Dynamic Economic Theory," *Applications of Control Theory to Economic Analysis*, J.D. Pitchford and S.J. Turnovsky (eds.), North-Holland, Amsterdam. Also, Discussion Paper No. 151, Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Il.
Brock, W. and J. Scheinkman (1974a), "Global Asymptotic Stability of Optimal Control With Applications to the Theory of Economic Growth," Report 7426, Center for Mathematical Studies in Business and Economics, University of Chicago.
Brock, W. and J. Scheinkman (1974b), "Some Results on Global Asymptotic Stability of Control Systems," Report 7422, Center for Mathematical Studies in Business and Economics, University of Chicago.
Brock, W. and J. Scheinkman (forthcoming), "On The Long Run Behavior of a Competitive Firm," *1974 Vienna Conference Volume on Equilibrium and Disequilibrium in Economic Theory*.
Burmeister, E. and D. Graham (1973), "Price Expectations and Stability in Descriptive and Optimally Controlled Macro-Economic Models," *I.E.E. Conference Publication No. 101*, Institute of Electrical Engineers, London.
Cass, D. and K. Shell (1976), "The Structure and Stability of Competitive Dynamical Systems," *Journal of Economic Theory*, 12, 31–70.
Dasgupta, S. (1975), "On the Stability of Capital Stock Structure of a Competitive Firm," University of Rochester.
Flaherty, M. (1974), "Industry Structure and R and D," Carnegie-Mellon University Working Paper.
Friedman, J. (1971), "A Noncooperative Equilibrium for Supergames," *Review of Economic Studies*, 38, 1–21.
Hartman, P. (1964), *Ordinary Differential Equations*, John Wiley, New York.
Kwakernaak, H. and R. Sivan (1972), *Linear Optimal Control Systems*, Wiley–Interscience, New York.
Lee, E. and L. Markus (1967), *Foundations of Optimal Control Theory*, John Wiley, New York.

Magill, Michael J.P. (1972), "Capital Accumulation Under Random Disturbances," University of Indiana.
Magill, Michael J.P. (1974), "A Local Analysis of *N*-Sector Capital Accumulation Under Uncertainty," paper presented at European Meeting of the Econometric Society, Grenoble, France.
Magill, Michael J.P. (1975), "Some New Results on the Local Stability of the Process of Capital Accumulation," Department of Economics, University of Indiana, Bloomington, IN. To appear in the *Journal of Economic Theory*.
Mortensen, D. (1973), "Generalized Costs of Adjustment and Dynamic Factor Demand Theory," *Econometrica*, 41, 657–665.
Nadiri, I. and S. Rosen (1969), "Interrelated Factor Demand Functions," *American Economic Review*, 59, 457–471.
Nadiri, I. and S. Rosen (1973), *A Disequilibrium Model of Demand for Factors of Production*, National Bureau of Economic Research, New York.
Prescott, E. (1973), "Market Structure and Monopoly Profits: A Dynamic Theory," *Journal of Economic Theory*, 6, 546–557.
Rockafellar, R.T. (1976), "Saddle Points of Hamiltonian Systems in Convex Lagrange Problems Having a Nonzero Discount Rate," *Journal of Economic Theory*, 12, 71–113.
Samuelson, P. (1972), "The General Saddle Point Property of Optimal Control Motions," *Journal of Economic Theory*, 5.
Scheinkman, J. (1975), "Stability of Separable Hamiltonians and Investment Theory," University of Chicago.
Scheinkman, J. (1976), "On Optimal Steady States of *n*-Sector Growth Models When Utility is Discounted," *Journal of Economic Theory*, 12, 11–30. Also, Report 7357, Center for Mathematical Studies in Business and Economics, University of Chicago.