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"The Global Asymptotic Stability of Optimal Control:
A Survey of Recent Results"

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on this paper. I wish to thank the National Science Foundation for research
support. Needless to say, all of the above are absolved from all errors and
shortcomings in this paper.
The global asymptotic stability of optimal control: A survey of recent results

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1. Introduction

The purpose of this paper is (1) to hasten the assimilation of some recent results obtained on the global asymptotic stability of optimal control into general economic knowledge, (2) to indicate some possible areas of application of these results, (3) to relate the recent results to standard engineering literature, and (4) to indicate new avenues of research in this area.

It is clear that it is necessary to remain within the space limitation, this survey must be selective. Furthermore, the emphasis will be on basic ideas and not technical details. Not only will this save space, but also it will lave the basic structure of the ideas. Details will be referenced where possible.

In order to describe the results contained in this paper it is useful to state the problem of concern without further ado. Consider the following optimal control problem:

$$\max_{u(t)} \int e^{-U(x(t), u(t))} dt$$

(1)

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The function expressing the multiple of interest.

A similar approach is taken for the case of the equation for the vector of interest. The equation is expressed as a function of the multiple of interest.

The function is defined as:

\[ f(x) = a \cdot x + b \]

where:
- \( a \) is the slope of the function
- \( b \) is the y-intercept

The domain of the function is all real numbers.

The range of the function is all real numbers.

The function is continuous and differentiable for all real numbers.

The function is increasing for all real numbers.

The function is decreasing for all real numbers.

The function is concave up for all real numbers.

The function is concave down for all real numbers.

The function is even for all real numbers.

The function is odd for all real numbers.

The function has no local extrema.

The function has a global maximum at \( x = 0 \).

The function has a global minimum at \( x = 0 \).

The function has no points of inflection.

The function is never equal to zero.

The function is always positive.

The function is always negative.

The function is always increasing.

The function is always decreasing.

The function is never decreasing.

The function is never increasing.

The function is never constant.

The function is never discontinuous.

The function is always continuous.

The function is always differentiable.

The function is always integrable.

The function is always bounded.

The function is always unbounded.

The function is always defined.

The function is always undefined.

The function is always finite.

The function is always infinite.

The function is always real.

The function is always complex.

The function is always positive real.

The function is always negative real.

The function is always positive imaginary.

The function is always negative imaginary.

The function is always real-valued.

The function is always complex-valued.

The function is always rational-valued.

The function is always irrational-valued.

The function is always algebraic-valued.

The function is always transcendental-valued.

The function is always continuous and differentiable with respect to all real numbers.

The function is always continuous and differentiable with respect to all complex numbers.

The function is always continuous and differentiable with respect to all algebraic numbers.

The function is always continuous and differentiable with respect to all transcendental numbers.

The function is always continuous and differentiable with respect to all positive real numbers.

The function is always continuous and differentiable with respect to all negative real numbers.

The function is always continuous and differentiable with respect to all positive imaginary numbers.

The function is always continuous and differentiable with respect to all negative imaginary numbers.

The function is always continuous and differentiable with respect to all rational numbers.

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The function is always continuous and differentiable with respect to all real numbers.
in occupational positions.

The effect of the depression on the economy is also reflected in the terms of trade. If the price of goods increases, the terms of trade will improve, as the country will gain more from the exchange of goods with other countries. If the price of goods decreases, the terms of trade will deteriorate, as the country will lose more from the exchange of goods with other countries. The terms of trade are determined by the relative prices of goods exported and goods imported.

The government should strive to maintain a balanced terms of trade to ensure economic stability and growth. This can be achieved through various measures such as trade policies, currency exchange rates, and fiscal policies. A strong currency can help to boost exports and attract foreign investments, while a weak currency can help to reduce imports and increase exports. A stable currency exchange rate can also help to maintain price stability and promote investment.

The government should also focus on promoting economic diversification and reducing dependence on a single industry or export commodity. This can help to mitigate the impact of any fluctuations in the terms of trade and ensure sustained economic growth. Additionally, the government should invest in education and skills development to enhance the competitiveness of the country's workforce and increase productivity, which can help to improve the terms of trade in the long run.

In conclusion, the terms of trade play a crucial role in determining the economic well-being of a country. By maintaining a balanced terms of trade, the government can promote economic stability and growth, and ensure that the country benefits from its trade activities.
...
Theorem 1. Assume that

\[ \exists \alpha > 0 : \nabla_a \psi(x, \lambda) = 0 \quad \text{for all } x, \lambda \in C \]

Then \( \psi \) is convex on \( C \). If \( C \) is open and \( x \in C \), then convexity is equivalent to

\[ w(x) \leq \lambda \quad \text{for all } x, \lambda \in C \]

where \( w(x) = \nabla_a \psi(x, \lambda) \). A function \( C \to R \) is concave if \( -h \) is convex.


Lemma 2. Assume that

\[ \exists \alpha > 0 : \nabla_a \psi(x, \lambda) = 0 \quad \text{for all } x, \lambda \in C \]

Then \( \psi \) is convex on \( C \). If \( C \) is open and \( x \in C \), then convexity is equivalent to

\[ w(x) \leq \lambda \quad \text{for all } x, \lambda \in C \]

where \( w(x) = \nabla_a \psi(x, \lambda) \). A function \( C \to R \) is concave if \( -h \) is convex.


Corollary 3. Assume that

\[ \exists \alpha > 0 : \nabla_a \psi(x, \lambda) = 0 \quad \text{for all } x, \lambda \in C \]

Then \( \psi \) is convex on \( C \). If \( C \) is open and \( x \in C \), then convexity is equivalent to

\[ w(x) \leq \lambda \quad \text{for all } x, \lambda \in C \]

where \( w(x) = \nabla_a \psi(x, \lambda) \). A function \( C \to R \) is concave if \( -h \) is convex.


Theorem 4. Assume that

\[ \exists \alpha > 0 : \nabla_a \psi(x, \lambda) = 0 \quad \text{for all } x, \lambda \in C \]

Then \( \psi \) is convex on \( C \). If \( C \) is open and \( x \in C \), then convexity is equivalent to

\[ w(x) \leq \lambda \quad \text{for all } x, \lambda \in C \]

where \( w(x) = \nabla_a \psi(x, \lambda) \). A function \( C \to R \) is concave if \( -h \) is convex.

\[ 0 < (x - x_0 - b)(x - x_0 - b)H_0(x - x_0 - b - b) \]

\[ f(x) = -((x - x_0 - b)(x - x_0 - b)H_0(x - x_0 - b - b)) \]

\[ (x - x_0 - b)(x - x_0 - b)H_0(x - x_0 - b - b) \]

\[ (x - x_0 - b)(x - x_0 - b)H_0(x - x_0 - b - b) \]

\[ \text{Theorem 2:} \]

\[ \forall x, y \in \mathbb{R} : f (x) = -((x - x_0 - b)(x - x_0 - b)H_0(x - x_0 - b - b)) \]

\[ \text{Theorem 3:} \]

\[ \forall x, y \in \mathbb{R} : f (x) = -((x - x_0 - b)(x - x_0 - b)H_0(x - x_0 - b - b)) \]
of the operators are suggested for G-V: 

\[
\begin{align*}
\text{Operator} & \quad \text{Definition} \\
\land & \quad \text{Logical AND} \\
\lor & \quad \text{Logical OR} \\
\neg & \quad \text{Logical NOT} \\
\to & \quad \text{Implication} \\
\iff & \quad \text{Equivalence} \\
\forall & \quad \text{Universal Quantification} \\
\exists & \quad \text{Existential Quantification} \\
\end{align*}
\]

Several extensions to the operators are possible, such as the use of parentheses to group expressions, or the introduction of additional logical connectives.

For example, consider the expression:

\[
(x \land y) \lor z
\]

This expression can be read as "(x and y) or z".
The ODE is given by the linear differential equation:

\[ \frac{d}{dt}y(t) + \lambda y(t) = f(t) \]

where \( y(t) \) is the dependent variable, \( \lambda \) is a constant, and \( f(t) \) is a function of time.

The solution to this ODE is given by:

\[ y(t) = e^{-\lambda t} \int e^{\lambda \tau} f(\tau) d\tau + C \]

where \( C \) is the constant of integration.

To solve the initial value problem, we substitute the initial condition \( y(t_0) = y_0 \):

\[ y(t_0) = e^{-\lambda t_0} \int_0^{t_0} e^{\lambda \tau} f(\tau) d\tau + C = y_0 \]

Solving for \( C \):

\[ C = y_0 - e^{-\lambda t_0} \int_0^{t_0} e^{\lambda \tau} f(\tau) d\tau \]

The general solution then becomes:

\[ y(t) = e^{-\lambda t} \int e^{\lambda \tau} f(\tau) d\tau + y_0 e^{-\lambda t_0} \int_0^{t_0} e^{\lambda \tau} f(\tau) d\tau \]

This solution describes the behavior of the system over time, given the initial conditions and the input function.
where the matrix $(\mathbf{A})$ is a square matrix of size $m \times m$. The sum of the diagonal elements of $\mathbf{A}$ is denoted by $\text{tr}(\mathbf{A})$.

The trace of a matrix is the sum of its diagonal elements. In other words, if $\mathbf{A}$ is a square matrix, then

$$\text{tr}(\mathbf{A}) = \sum_{i=1}^{m} A_{ii}$$

where $A_{ii}$ is the element in the $i$th row and $i$th column of the matrix $\mathbf{A}$.
Theorem 1 (Cantor, 1874).

Assume (F, Y, M, C).

Let $S$ be a set of all elements of $F$. Let $O$ be the set of all elements of $F$. Let $C$ be the set of all elements of $F$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $S$ is a subset of $O$. Let $C$ be the set of all elements of $F$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $C$ is a subset of $M$. Let $O$ be the set of all elements of $F$. Let $C$ be the set of all elements of $F$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $O$ is a subset of $C$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $M$ is a subset of $Y$. Let $C$ be the set of all elements of $F$. Let $O$ be the set of all elements of $F$. Then $C$ is a subset of $O$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $M$ is a subset of $Y$. Let $O$ be the set of all elements of $F$. Let $C$ be the set of all elements of $F$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $O$ is a subset of $C$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $M$ is a subset of $Y$.

Theorem 2. If $O$, $C$, $M$, $Y$ are disjoint, then $S$ is a subset of $O$. Let $C$ be the set of all elements of $F$. Let $O$ be the set of all elements of $F$. Then $C$ is a subset of $O$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $M$ is a subset of $Y$. Let $O$ be the set of all elements of $F$. Let $C$ be the set of all elements of $F$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $O$ is a subset of $C$. Let $M$ be the set of all elements of $F$. Let $Y$ be the set of all elements of $F$. Then $M$ is a subset of $Y$.
The output where we can see a new expression is when
\[ \frac{\partial}{\partial x} \left( \Phi(x) \right) = 0 \]
and that is equal to
\[ \frac{\partial}{\partial x} \left( \Phi(x) \right) = 0 \]

Replace:
\[ \frac{\partial}{\partial x} \left( \Phi(x) \right) = 0 \]

The expression is equal to:
\[ \frac{\partial}{\partial x} \left( \Phi(x) \right) = 0 \]

Some other QF's results are discussed only very briefly due to lack of

2. Other QF's results

The two main results of this paper are:
\[ 0 > \Phi(x) = \Phi(x-1) \]
and the expression is:
\[ 0 > \Phi(x) = \Phi(x-1) \]

Replace:
\[ 0 > \Phi(x) = \Phi(x-1) \]

The expression is:
\[ 0 > \Phi(x) = \Phi(x-1) \]
The magnetic fields are described by the equation \( \mathbf{B} = \nabla \times \mathbf{A} \), where \( \mathbf{A} \) is the vector potential. This equation is derived from Maxwell's equations and is fundamental in electromagnetism.

The vector potential \( \mathbf{A} \) is related to the magnetic field \( \mathbf{B} \) by the equation \( \mathbf{B} = \nabla \times \mathbf{A} \). This relationship is crucial in understanding the behavior of magnetic fields in various applications, such as in the design of electrical devices and in the study of natural phenomena like lightning and Earth's magnetic field.

The magnetic field \( \mathbf{B} \) is also related to the electric field \( \mathbf{E} \) through the equation \( \mathbf{B} = \nabla \times \mathbf{E} \), which describes how changes in the electric field give rise to magnetic field lines.

These equations form the basis for many applications in physics and engineering, including the design of transformers, generators, and motors, as well as the study of electromagnetic waves and their interactions with matter.

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \]
The correct answer is (D).