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AN IMPOSSIBILITY THEOREM FOR VOTING WITH A DIFFERENT INTERPRETATION

by

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ABSTRACT

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Consider a group which must select one alternative from a set of three or more alternatives. Members each cast a ballot which the voting procedure counts. For a given alternative x, let two ballot profiles C and D have the property that if a member ranks alternative x above alternative y within C, then he also ranks x above that y within D. Strong nonnegative response requires that if the voting procedure selects x when the profile is C, then it must also select x when the profile is D. We prove that no nondictatorial voting procedure exists which satisfies both strong nonnegative response and Pareto optimality. The proof depends on showing that strategy-proofness and nonnegative response are equivalent.

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1. Introduction

Consider a group which must, through voting, select a single alternative from a set of feasible alternatives. Specifically, define a group to be a set N whose $|N|$ elements are the group's members. They select one element of the set X, the feasible set of alternatives, by each casting a ballot and then using a voting procedure to count the ballots. A ballot B_i is a strict ordering of the elements within S, the universal set of alternatives, e.g. $B_i = (x\ y\ z)$ where $S = \{x,y,z\}$ and x is ranked highest, y second highest, and z lowest. Indifference is not allowed. A voting procedure is a singlevalued function $v(B_1,\ldots,B_n|X)$ which evaluates the profile of ballots and selects one element of X, $X \subseteq S$, as the group's chosen alternative. For example, if $S = \{x,y,z\}$, if $X = \{x,y\}$, and if the voting procedure is based on majority rule, then $v[B_1 = (x\ y\ z),\ B_2 = (z\ x\ y),\ B_3 = (y\ x\ z)|X = \{x,y\}] = x$.

A classic question within social choice theory is whether satisfactory voting procedures exist. The usual answer has been negative: those reasonable sets of criteria which different investigators have proposed for the evaluation of voting procedures have generally led to impossibility theorems. The oldest and most celebrated of these theorems is that of Arrow [1].

The criteria which Arrow posited are that a voting procedure should be nondictatorial, should give the group sovereignty to pick whatever feasible alternative it prefers, should not be negatively responsive to the preferences members express on their ballots, and should give choices that are independent of the preferences members express for infeasible alternatives. In addition, as a fifth criterion, Arrow argued for group rationality: as the feasible set varies the group's choices should be consistent with a complete and transitive social ordering in exactly the same manner that a rational individual's personal choices are consistent with that ordering which describes his preferences.[1] Arrow showed that if individual preferences are free to vary and S contains at least three alternatives, then these five conditions are inconsistent.[2]

Gibbard [4] and Satterthwaite [7] have more recently derived a second impossibility theorem which is based on different criteria. They asked if a voting procedure can satisfy simultaneously strategy-proofness, citizens' sovereignty, and nondictatoriality and showed that it is impossible whenever preferences are unrestricted and the universal set S contains at least three elements. A voting procedure is strategy-proof if each group member never can have an opportunity to manipulate the group's choice of an alternative by misrepresenting on his ballot what his preferences truly are. For a common example of misrepresentation, consider a plurality rule election between two major party candidates and an independent candidate. A registered voter may decide to rank his second choice, a mojor party candidate, first on his ballot instead of ranking his first choice, the independent, first because he wants to avoid "throwing away" his ballot.

The question which Arrow asked and the question which Gibbard and Satterthwaite asked are substantively different from each other. Arrow's five conditions may be interpreted as asking if a group's decisions are analogous to a rational individual's decisions. Gibbard and Satterthwaite's strategy-proofness criterion focuses on incentives, not rationality. Our question in this paper fits into a third category of criteria: does the voting procedure count the ballots in a reasonable manner?

Specifically, consider two related ballot profiles $B = (B_1,\ldots,B_n)$ and $B' = (B_1',\ldots,B_n')$. Suppose that $v(B_1,\ldots,B_n|X) = x$ for some feasible set X. Suppose also, that for all members $i \in N$ and all alternatives $y \in S$, if alternative x is ranked above alternative y on ballot B_i, then on ballot B_i' alternative x is also ranked above alternative y. In other words, the switch from profile B to profile B′ precludes any alternative y, which was ranked below alternative x on ballot B_i, from jumping above alternative x on ballot B_i'. Given these ballot profiles B and B′, a reasonable requirement to place on the manner in which v counts the ballots is that $v(B_1',\ldots,B_n'|X)=x$. After all, $v(B_1,\ldots,B_n|X) = x$ and in the switch from profile B to profile B′ alternative x has retained or improved its relative position with respect to every other alternative. We call this requirement, which will be defined more formally in Section 3, strong nonnegative response.

An example where $S = \{w,x,y,z\}$ and $|N| = 5$ easily shows that both plurality rule and the Borda count fail to satisfy this requirement. Let profiles B and B′ be:

$$B_1 = (x\ w\ z\ y)\ , \quad B'_1 = (x\ y\ z\ w),$$
$$B_2 = (x\ w\ z\ y)\ , \quad B'_2 = (x\ z\ y\ w),$$
$$B_3 = (y\ z\ x\ w)\ , \quad B'_3 = (y\ z\ x\ w),$$
$$B_4 = (z\ y\ x\ w)\ , \quad B'_4 = (y\ z\ x\ w),$$
$$B_5 = (w\ y\ x\ z)\ , \quad B'_5 = (y\ x\ w\ z).$$

With respect to alternative x profiles B and B′ satisfy the requirements of strong nonnegative response. Note that the condition's requirements on B and B′ relate only to those pairs of alternatives that include x, not to all possible pairs of alternatives. Consider plurality rule (v_P) first. Profile B gives x two first place votes compared to one first place vote each for the other elements of S. Therefore $v_P(B|S) = x$. But, contrary to the requirement of strong nonnegative response, $v_P(B'|S) = y$. The story repeats itself for the Borda count (v_B).[3] For profile B alternative x receives nine points while the other alternatives each receive seven points. For profile B′, alternative y receives twelve points while alternative w, x, and z respectively receive one, ten, and seven points. Therefore $v_B(B|S) = x$ and $v_B(B'|S) = y$.

These examples illustrate this paper's main result: if preferences are unrestricted and S contains at least three alternatives, then no voting procedure exists which satisfies strong nonnegative response, citizens' sovereignty, and nondictatoriality. Thus an impossibility result also obtains for this third category of criteria. The differences between this impossibility theorem and the impossibility theorems of Arrow and of Gibbard and Satterthwaite, however, are primarily difference of interpretation, not of logical structure. We prove this paper's impossibility theorem by showing that strong nonnegative response is formally equivalent to strategy-proofness. Therefore the impossibility theorem for strong nonnegative response is a restatement of the strategy-proofness theorem. Moreover, in the paper's final section we summarize and add to the work of Gibbard [4], Satterthwaite [7], and Blin and Satterthwaite [2] and show the close relationship which Arrow's theorem has to the theorems on strategy-proofness and strong nonnegative response.

2. Preferences, Sincere Strategies, and Insincere Strategies

Each group member $i \in N$ has preferences P_i over the universal set of alternatives S. Preferences, like a ballot, are a complete, asymmetric, and transitive ordering of S. A member's preferences P_i describe what he truly desires. For example, $P_i = (x\ y\ z)$ denotes that individual i most prefers that the group's choice be x, next prefers that it be y, and least prefers that it be z. An alternative notation for the preference ordering $P_i = (x\ y\ z)$ is $x\ P_i\ y$, $x\ P_i\ z$, and $y\ P_i\ z$ where $x\ P_i\ y$ means individual i prefers x to y. Similarly an alternative notation for the ballot $B_i = (x\ y\ z)$ is $x\ B_i\ y$, etc.
Beyond completeness, asymmetry, and transitivity we place no restrictions such as single-peakedness, on either admissible preferences or admissible ballots. Any strict ordering is admissible. Indifference, however, is excluded as inadmissible.[4]

A group member's choice of ballot B_i need not be identical to his true preferences P_i. Any attempt through direct regulation to make him reveal his true preferences is certain to fail because his preferences are purely internal to him and are consequently unverifiable by an outside observer. If a member selects a ballot B_i which is identical to his preferences P_i, then $B_i \equiv P_i$ is called his sincere strategy. If, however, he selects a ballot B_i which is different than his preferences, then $B_i \neq P_i$ is called his insincere strategy. The n-tuple of sincere strategies $B \equiv (B_1,\ldots,B_n) \equiv (P_1,\ldots,P_n) \equiv P$ is called the sincere strategy profile.

3. Equivalence of Strong Nonnegative Response and Strategy-proofness

In this section we show that, despite their different interpretation, strong nonnegative response and strategy-proofness are equivalent conditions. The formal definitions of the two conditions are as follows:

Strong Nonnegative Response (SNNR). For any $x \in S$, let C and D be any two ballot profiles such that, for all $s \in S$ and all $i \in N$, $x\ B_i\ s$ implies $x\ C_i\ s$. A voting procedure satisfies SNNR if and only if $v(D|W) = x$ for all feasible sets $W \subset S$ such that $v(C|W) = x$.

Strategy-proofness (SP). A voting procedure satisfies SP if and only if no sincere strategy profile $C \equiv P$ exists such that, for some feasible set $W \subset S$, for some member $i \in N$, and for some insincere strategy C_i',

$$v(C/C_i'|W)\ P_i\ v(C/C_i|W) \tag{3.1}$$

where

$$C/C_i' = (C_1,\dots,C_{i-1},C_i',C_{i+1},\dots,C_n) = P/C_i' \tag{3.2}$$

and

$$C/C_i = (C_1,\dots,C_{i-1},C_i,C_{i+1},\dots,C_n) = P/P_i = P \tag{3.3}$$

Strong nonnegative response requires that if some ballot profile results in x being the group's choice, then a second ballot profile must also result in x being the group's choice provided that within the second profile each member's ballot ranks x above every alternative y above which x was ranked on that member's ballot within the first profile. Strategy-proofness requires that no group member has an incentive to employ an insincere strategy. Specifically notice in (3.1) that $C_i \equiv P_i$ because C_i is the sincere strategy of member i; therefore (3.1) may be rewritten as $v(C/C_i'|W)\ C_i\, v(C/C_i|W)$.

Theorem 1. A voting procedure satisfies SP if and only if it satisfies SNNR.

Proof. First we prove that SP implies SNNR. Suppose that SP does not imply SNNR. Therefore a voting procedure $v(B|X)$ exists which satisfies SP, but not SNNR. This means that a set $W \subset S$, distinct alternatives $x, z \in S$, and profiles B and C must exist such that, for all $i \in N$ and all $y \in S$,

$$x\ B_i\ y \Rightarrow x\ C_i\ y, \tag{3.4}$$

$v(B|W) = x$, and $v(C|W) = z$. Consider the sequence:

$$\begin{aligned}
&v(B_1, B_2, \ldots, B_n|W) = x,\\
&v(C_1, B_2, \ldots, B_n|W)\\
&\qquad \cdots\\
&v(C_1, \ldots, C_{j-1}, B_j, B_{j+1}, \ldots, B_n|W) = v(D/B_j|W),\\
&v(C_1, \ldots, C_{j-1}, C_j, B_{j+1}, \ldots, B_n|W) = v(D/C_j|W),\\
&\qquad \cdots\\
&v(C_1, \ldots, C_{n-1}, B_n|W),\\
&v(C_1, \ldots, C_{n-1}, C_n|W) = z.
\end{aligned} \tag{3.5}$$

Since $v(B|W) = x$ and $v(C|W) = z$ a switching point must exist: a $j \in N$ exists such that $v(D/B_j|W) = x$ and $v(D/C_j|W) = u$ (u might equal z). Two possibilities exist for B_j -- either $x\ B_j\ u$ or $u\ B_j\ x$ -- and both lead to contradictions. If $x\ B_j\ u$, then (3.4) implies that $x\ C_j\ u$. Therefore $v(D/B_j|W)\ C_j\ v(D/C_j|W)$ which is a contradiction of SP. If $u\ B_j\ x$, then $v(D/C_j|W)\ B_j\ v(D/B_j|W)$ which is also a contradiction of SP. Therefore SNNR implies SP.

We now show that SNNR implies SP. Suppose that SNNR does not imply SP. Therefore a voting procedure $v(B|X)$ must exist which satisfies SNNR but not SP. Consequently a distinct pair $x, y \in S$, a sincere strategy $B_j \leq P_j$, a profile B, a member $j \in N$, and an insincere strategy B_j' must exist such that

$$v(B/B_j'|W) = y, \tag{3.6}$$

$$v(B/B_j|W) = x, \tag{3.7}$$

and

$$y\ B_j\ x. \tag{3.8}$$

Partition S into three exhaustive and disjoint subsets:

$$W^+ = \{z \in S | z\ B_j\ x\} \tag{3.9}$$

$$X = \{z \in S | (xB_j\ z\ \&\ z\ B_j'\ x) \text{ or } z = x\} \tag{3.10}$$

$$W^- = \{z \in S | x\ B_j\ z\ \&\ x\ B_j'\ z\}. \tag{3.11}$$

Construct a ballot Q_j from ballots B_j and B_j' as follows:

$$[s \in W^{+} \ \& \ (t \in X \text{ or } t \in W^{-})] \Rightarrow s \ Q_j \ t, \tag{3.12}$$

$$(s \in X \ \& \ t \in W^{-}) \Rightarrow s \ Q_j \ t, \tag{3.13}$$

$$s,\ t \in W^{+} \Rightarrow (s \ Q_j \ t \Leftrightarrow s \ B_j \ t), \tag{3.14}$$

$$s,\ t \in X \Rightarrow (s \ Q_j \ t \Leftrightarrow s \ B_j' \ t), \tag{3.15}$$

and

$$s,\ t \in W^{-} \Rightarrow (s \ Q_j \ t \Leftrightarrow sB_j \ t). \tag{3.16}$$

The effect of this construction is to order the three sets W^{+}, X, and W^{-} in descending order: $Q_j = (W^{+} \ X \ W^{-})$. The individual alternatives within W^{+} are ordered as ballot B_j ordered them, the alternatives within X are ordered as ballot B_j' ordered them, and the alternatives within W^{-} are ordered as ballot B_j ordered them. Notice that (3.10) and (3.14) imply that if $s \in X$ and $s \neq x$, then $s \ Q_j \ x$.

Denote by w the group choice which the ballot profile B/Q_j generates: $v(B/Q_j|W) = w$. Since W^{+}, X, and W^{-} partition S, three possibilities exist for w: $w \in W^{+}$, $w \in X$, or $w \in X^{-}$. We consider each of these possibilities in turn and show that application of SNNR, which v is assumed to satisfy, leads to a contradiction of our assumptions (3.6), (3.7), or (3.8). Therefore SNNR implies SP.

If $w \in W^{+}$, then for all $z \in S$, (3.12) and (3.14) imply that

$$w \ Q_j \ z \Rightarrow w \ B_j \ z. \tag{3.17}$$

Moreover $v(B/Q_j|W) = w$ and the only difference between ballot profile B/Q_j and B/B_j is the ballot of member j. Therefore SNNR is applicable to the switch from profile B/Q_j to B/B_j. It implies that $v(B/B_j|W) = w$ because $v(B/Q_j|W)=w$. But our assumption (3.7) states that $v(B/B_j|W)=x$. Since $x \notin W^+$ we have our first contradiction.

If $w \in X$, then for all $z \in S$, (3.10), (3.11), (3.13), and (3.15) imply that

$$w\ Q_j\ z \Rightarrow w\ B_j'\ z. \tag{3.18}$$

The same argument, _mutatis mutandis_, as we used in analyzing the first possibility implies that SNNR is applicable to the switch from profile B/Q_j to profile B/B_j'. Therefore, $v(B/B_j'|W) = w$ because $v(B/Q_j|W) = w$. Assumption (3.6), however, is that $v(B/B_j'|W) = y$. Moreover (3.6) when coupled with (3.9) implies that $y \in W^+$. Therefore, a contradiction exists because $w \in X$ and y must equal w.

The one remaining possibility is $w \in W^-$. If $w \in W^-$, then for all $z \in S$, (3.12), (3.13), and (3.16) imply that

$$w\ Q_j\ z \Rightarrow w\ B_j\ z. \tag{3.19}$$

The same argument, _mutatis mutandis_, implies that SNNR is applicable to the switch from profile B/Q_j to profile B/B_j. Therefore $v(B/B_j|W) = w$ because $v(B/Q_j|W) = w$. But assumption (3.7) states that $v(B/B_j|W) = x$. Since $x \notin W^-$ we have our third contradiction.||

4. Impossibility Theorem for Strong Nonnegative Response and Strategy-proofness

In order to state and prove the impossibility theorem for strong nonnegative response and strategy-proofness we must formally define three additional conditions. Citizens' sovereignty requires that the group can actually choose any alternative within the feasible set X by casting an appropriate ballot profile. Pareto optimality requires that if the group's members on their ballots unanimously prefer one alternative x to another alternative y, then the group's choice is not y, the dominated alternative. A dictatorial voting procedure is a voting procedure which vests all decision making power into one individual, the dictator. Formally these three conditions are:

Citizens' Sovereignty (CS). A voting procedure satisfies CS if and only if, for every feasible set $W \subset S$ and every alternative $x \in W$, a ballot profile B exists such $v(B|W) = x$.

Pareto Optimality (PO). A voting procedure satisfies PO if and only if, for any feasible set $W \subset S$, any pair $x, y \in W$, and any ballot profile B, $v(B|W) \neq y$ whenever, for all $i \in N$, $x \; B_i \; y$.

Dictatoriality (D). A voting procedure is dictatorial if and only if, for some feasible set $W \subset S$, a member $i \in N$ exists such that, for all ballot profiles B and all alternatives $x \in W$, either $v(B|W) = x$ or $v(B|W) \; B_i \; x$.

The main theorem is: if $|S| \geq 3$, then only dictatorial voting procedures satisfy conditions SP, SNNR, CS, and PO.[5] The theorem's form is that of a possibility theorem. But, because a dictatorial voting procedure is the antithesis of democratic decision making, its substance is that of an impossiblity theorem.

Lemma. The following four pairs of conditions are equivalent:

a. SNNR and CS,

b. SNNR and PO,

c. SP and CS,

d. SP and PO.

Theorem 2. If $|S| \geq 3$, then every voting procedure which satisfies any of the four pairs of conditions in the lemma is dictatorial.

Proof. We prove the lemma and theorem together. First we show that (a)-(d) are equivalent sets of conditions. From Theorem 1 we know that

$$\text{SNNR} \iff \text{SP}. \tag{4.1}$$

Consequently if we show that

$$\text{SNNR \& PO} \iff \text{SNNR \& CS}, \tag{4.2}$$

then substitution of SP for SNNR proves the full equivalence of (a)-(d). We show (4.2) by first noting an immediate consequence of the definitions of PO and CS is that PO implies CS. Therefore

$$\text{SNNR \& PO} \implies \text{SNNR \& CS}.$$

Condition CS, however, does not imply PO directly.

Nevertheless PO is implied by CS and SNNR together. This can be shown by supposing that a voting procedure $v(B|X)$ exists which satisfies CS and SNNR, but not PO. Therefore a set $W \subseteq S$, a pair $x, y \in W$, and a profile B exist such that $v(B|W) = y$ and, for all $i \in N$, $x\ B_i\ y$. Construct a ballot profile C from the profile B by moving alternative x to the top of each ballot within C and leaving the ordering of all other elements unchanged from B. More formally, construct each ballot C_i such that:

(i), for all $i \in N$ and all $s \in S-\{x\}$, $x\ C_i\ s$ and

(ii), for all $i \in N$ and all pairs $s, t \in S-\{x\}$, $s\ C_i\ t$ if and only if $s\ B_i\ t$.

Recall that, for all $i \in N$, $x\ B_i\ y$. Therefore, by property (ii),

$$y\ B_i\ s \Rightarrow y\ C_i\ s \tag{4.4}$$

for all $i \in N$ and all $s \in S$. Consequently SNNR is applicable: $v(C|W) = y$ because $v(B|W) = y$.

The voting procedure v satisfies CS. Therefore a profile D exists such that $v(D|W) = x$. Clearly, for all $i \in N$ and all $s \in S$,

$$x\ D_i\ s \Rightarrow x\ C_i\ s$$

because, for all $i \in N$ and $t \in S-\{x\}$, $x\ C_i\ t$. SNNR therefore is applicable: $v(C|W) = x$ because $v(D|W) = x$. This conclusion contradicts our earlier conclusion that $v(C|W) = y$. This means that SNNR and CS imply PO. Therefore, in conjunction with (4.3), we have the desired result:

$$\text{SNNR \& CS} \Leftrightarrow \text{SNNR \& PO}. \qquad (4.5)$$

Thus the lemma is proved.

Theorem 2 follows directly from the equivalence among (a)-(d) and a theorem of Gibbard (4) and Satterthwaite [7]: if $|S| \geq 3$, then every voting procedure which satisfies SP and CS is dictatorial. ||

5. The Relationship of Strong Nonnegative Response and Strategy-proofness to Arrow's Conditions

The conditions which Arrow [1] used in the construction of his impossibility theorem were rationality, nonnegative response, independence of irrelevant alternatives, CS, and D.[6] The definition of the rationality condition itself depends on the concept of a social welfare function. A <u>social welfare function</u> is a function which associates with each ballot profile a unique, complete, asymmetric, and transitive ordering of the alternatives within S, i.e. $u(B) = B_N$ where B is a ballot profile and B_N, the social ordering of S, is a strict ordering on S. A voting procedure is rational if and only if as the feasible set X varies the choices which the voting procedure generates are consistent with some underlying social welfare function in the same manner that an individual's choices are consistent with his personal preferences. Formally:

> Rationality (R). A voting procedure $v(B|X)$ is rational if and only if a social welfare function $u(B)$ exists such that, for all ballot profiles B, all feasible sets $W \subseteq S$, and all alternatives $y \in W$, either $v(B|W) = y$ or $v(B|W)\ B_N\ y$ where $B_N = u(B)$.

Independence of irrelevant alternatives states that the only information which should affect the group's choice of an element within the feasible set is how the members order the elements of the feasible set X on their ballots. The manner in which the members order those elements of S that are infeasible should have no effect on the group's choice. Nonnegative response is a less demanding version of strong nonnegative response. It states that if x is the group's choice for a ballot profile C and if the only difference between ballot profies C and D is that x has moved up on some ballots relative to some other alternatives, then x should continue to be the group's choice for ballot profile D. Nonnegative response differs from SNNR by holding the ordering of all alternatives other than x constant relative to each other. SNNR allows the shuffling of alternatives other than x as long as no jumps above x occur.

Independence of Irrelevant Alternatives (IIA). For any feasible set $W \subseteq S$, let C and D be any two ballot profiles such that, for all $i \in N$ and all s, $t \in W$, $s\ C_i\ t$ if and only if $s\ D_i\ t$. A voting procedure satisfies IIA if and only if $v(C|W) = v(D|W)$ for all such feasible sets W and all such ballot profiles C and D.

Nonnegative Response (NNR). For any $x \in S$, let C and D be any two ballot profiles C and D such that (i), for all $i \in N$ and all $s \in S$, $x\ C_i\ s$ implies $x\ D_i\ s$ and (ii), for all $i \in N$ and all $s, t \in S - \{x\}$, $s\ C_i\ t$ if and only if $s\ D_i\ t$. A voting procedure satisfies NNR if and only if $v(D|W) = x$ for all feasible sets $W \subseteq S$ such that $v(C|W) = x$.

Arrow's theorem is that if $|S| \geq 3$, then every voting procedure which satisfies R, IIA, NNR, CS is dictatorial. The similarity with Theorem 2 is manifest: the conditions R, IIA, NNR, and CS imply a dictator as do the conditions SNNR, SP, and CS. The theorems, however, are not equivalent because the sets of conditions are not equivalent. The actual relationship among the conditions is this:

$$\text{R, IIA, NNR} \Rightarrow \text{SP} \Leftrightarrow \text{SNNR} \Rightarrow \text{IIA, NNR} \tag{5.1}$$

Blin and Satterthwaite [2, Theorem 2] showed that R, IIA, and NNR imply SP. SP, however, does not imply R, IIA, and NNR; based on an example of Karni and Schmeidler [6], Blin and Satterthwaite [2,Section 8] have constructed a counter example where a voting procedure satisfies SP but not R, IIA, and NNR. Theorem 1 of this paper establishes the equilvalence between SNNR and SP. Inspection of the definitions shows that SNNR implies NNR. The proof that SP implies IIA parallels our demonstration within the proof of Theorem 1 that SP implies SNNR. Finally IIA and NNR do not imply SP or NNR; for example, the type β Borda count voting procedure described by Blin and Satterthwaite [2,Section 4] does not satisfy SP or SNNR but does satisfy IIA and NNR.

The implication of (5.1) is that Arrow's theorem is an immediate consequence of Theorem 2. Since SP and CS imply the existence of a dictator and since R, IIA, NNR, and CS imply satisfaction of SP and CS, it follows that R, IIA, NNR, and CS must imply the existence of a dictator. Theorem 2, however, can not be proved directly from Arrow's theorem because of the lack of equivalence between the two sets of conditions.

Footnotes

1. Arrow [1] developed his theorem in terms of social welfare functions, not voting procedures. Nevertheless his result, as shown in Blin and Satterthwaite [2], may be reinterpreted within the context of voting procedures.

2. Many variations on Arrow's theorem have been constructed; for a review of them see Sen's book [8].

3. The Borda count, which is named after its eighteenth century French inventor, selects a winning alternative by assigning each alternative ($|S|$-k-1) points for each ballot in which it is ranked k positions from the top. The points for each alternative are summed and the winner is that alternative, from among the alternatives contained in the feasible set X, which received the most points. If two alternatives receive the same number of points, then individual one's ballot, B_1, is used to break the tie.

4. The strong assumption that members are not indifferent among alternatives is justifiable because we are proving impossibility results within this paper. If no satisfactory voting procedure exists when only strict orders are admissible as preferences and stated preferences, then certainly no satisfactory voting procedure exists which is satisfactory when weak orders, as well as strict orders, are admissible. See Gibbard [5,Section III] for a more detailed exposition of this argument.

5. This theorem does not remain valid if the set of admissible preference profiles and ballot profiles are restricted severely enough, e.g. to be single-peaked. See Blin and Satterthwaite [3] for a discussion of this.

6. As stated in footnote 1, Arrow [1] originally stated his theorem solely in terms of social welfare functions. His theorem, however, may be restated in terms of voting procedures by introducing,as we do below, the concept of a rational voting procedure. See Blin and Satterthwaite [2] for a complete development of this particular interpretation of Arrow's theorem.

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