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THE MANIPULATION OF SOCIAL CHOICE MECHANISMS THAT DO NOT LEAVE "TOO MUCH" TO CHANCE

by

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SUMMARY

In this paper we study the possibility of constructing satisfactory social choice mechanisms whose outcomes are determined by a combination of voting and chance. The following theorem is obtained: if a social choice mechanism does not leave "too much" to chance and satisfies a unanimity condition, then it is either uniformly manipulable or dictatorial. The result contributes to the program suggested by Gibbard [2] for the study of the extent to which social choice mechanisms in which chance plays a role can be freed from strategic manipulation.[1]

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1. INTRODUCTION

A voting scheme is a mechanism that assigns a single social outcome to each set of individual preferences. Gibbard [2] and Satterthwaite [6] independently considered the possibility that such schemes may be subject to strategic manipulation. They proved that all voting schemes are either manipulable or dictatorial (provided the range of the scheme contains more than two alternatives). This result makes explicit a central difficulty in constructing satisfactory mechanisms for social choice. However, it is marred by the restrictive requirement that the voting schemes always determine a single social alternative.

In order to motivate our analysis, consider the case in which a committee composed of three individuals I_1, I_2 and I_3 is facing three possible alternative courses of action, a, b, and c. Assume the preferences of these individuals are:[2]

$$I_1 : abc \qquad I_2 : (ab)c \qquad I_3 : bac.$$

A voting scheme would have to determine one and only one of the alternatives a, b and c as the outcome. While under the above set of individual preferences, it is natural to assume that c would not be chosen, it is not clear that there are sufficient grounds to decide whether a or b should be the selected outcome. Mechanisms based on equalitarian principles would indeed have to give the same treatment to both alternatives. We could say that there exists a "natural tie" between a and b. For another example, consider the case where the individuals' preferences lead to the voting paradox:

$$I_1 : abc \qquad I_2 : bca \qquad I_3 : cab$$

Once again there exists a "natural tie", now between a, b and c. Yet a voting scheme of the Gibbard-Satterthwaite type would have to select one and only one of these alternatives as the social outcome.

The above cases are but two examples of situations where individual preferences are "symmetric" and yet the single-valuedness of voting schemes forces the alternatives to be treated in an "assymmetric" manner. One way to correct this anomaly is to consider choice mechanisms that allow for ties among alternatives, i.e., which for some configurations of indiviual preferences select sets of alternatives rather than single alternatives. We can then assume that one of the alternatives from this set will be chosen as the social outcome by means of a random device.[3] We call such mechanisms selection methods.

Once we allow for a mechanism to select sets of alternatives and to pick one of them at random it is no longer true that all nondictatorial mechanisms are manipulable. For example, consider the mechanism that picks any of the alternatives at random, regardless of the individuals' preferences. Clearly such a mechanism is non-manipulable, but also extremely unattractive, since chance plays too large a role. A somewhat more attractive rule has been described by Gibbard [2, 3] and Zeckhauser [8]. For simplicity, assume that the individuals' preferences are strict. Each individual fills a ballot with his top alternative. One of the ballots is then drawn at random to determine the outcome. In that case also, it is in the individuals' interest to always vote according to their preferences. But again, chance plays a large role in the determination of the social outcome. Indeed, the final outcome depends on chance except when all individuals unanimously agree on some alternative as best.

Gibbard and Satterthwaite's conclusion that all voting schemes must be either manipulable or dictatorial does not therefore generalize to the larger class of mechanisms that we have called selection methods. The two examples above show that there exist some selection methods which are neither manipulable nor dictatorial. These examples are flawed, however, by the fact that chance plays an excessive role in determining the social outcome. Just think of a legislature that would systematically resort to flipping coins in order to pass laws! We take the view that an attractive balance in the use of chance is met by selection methods that allow chance to play a role in only very special circumstances, and yield single alternatives for most sets of individual preferences. This motivates our interest in the class of positive responsive selection methods. A selection method is positive responsive if whenever several alternatives are selected, any improvement of one of them in any individual's preference order would make this alternative the only selected one. Selection methods allow for sets of alternatives to be treated equally and thus may operate symmetrically over symmetrical preferences. However, positive responsive selection methods do not leave much to chance, since ties among alternatives are easily broken.

When is a selection method manipulable? Any notion of manipulability for selection methods must be based on assumptions about the individuals' preferences over sets of alternatives. These, in turn, depend on the individuals' attitude toward risk and on the random device that is used to choose outcomes from selected sets. One natural approach is to consider a selection method manipulable if there exists some preference profile and some attitude toward risk for which an individual would find it profitable to misrepresent his preferences.[4] In this chapter we take a polar viewpoint and concentrate on those situations for which it would be to some individual's advantage to misrepresent his preferences over alternatives regardless of his particular attitudes toward risk. We endow individuals with preferences over sets of alternatives. An individual's preferences over sets of alternatives are then said to be consistent with his preferences over alternatives if whenever he prefers alternative x to alternative y, he prefers the set ⟨x,y⟩ to the singleton ⟨y⟩ and ⟨x⟩ to ⟨x,y⟩. A selection method is uniformly manipulable if there exists a profile at which an individual can misrepresent his preferences to his advantage provided that his preferences over sets and over alternatives are consistent. Uniformly manipulable selection methods are indeed very vulnerable since for some profile it is in <u>any</u> consistent individual's interest to manipulate them, regardless of the individual's further attitude toward risk.

The main result in this chapter shows that if a positive responsive selection method satisfies a unanimity condition it is either dictatorial or uniformly manipulable (when there are more than three alternatives). Thus, the problems in arriving at satisfactory mechanisms are not substantially diminished by just letting chance play a role in the mechanism. If the role of chance is a limited one, our theorem shows that the same dilemma between manipulability and dictatoriality arises than for voting schemes. Therefore, the only selection methods that can be both nondictatorial and nonmanipulable are those in which chance plays an extensive role, and this in itself would make them rather unattractive.

2. NOTATION AND DEFINITIONS

Alternatives

Let X be a finite set.[5] The elements of X are denoted by x, y, z, u, v, w, ... and are called alternatives. Let $\mathscr{X}$ be the set of all nonempty subsets of X. Elements of $\mathscr{X}$ are denoted by Z, Y, ...

Individuals

Let I be the initial segment of the integers with cardinality N; i.e., $I = \langle 1,2,\ldots,N\rangle$. Elements of I are called individuals.

Preferences

Let $\mathcal{R}$ be the set of orderings on X.[6] Elements of $\mathcal{R}$ are denoted by R and are called preferences (over alternatives).

Let $\bar{\mathcal{R}}$ be the set of strong orderings on X.[7] Elements of $\bar{\mathcal{R}}$ are called strict preferences (over alternatives). The strict preference relation $R \in \bar{\mathcal{R}}$ associated with $R \in \mathcal{R}$ is defined by

$$x\bar{R}y \longleftrightarrow \sim (yRx)$$

The indifference relation $\tilde{R}$ associated with $R \in \mathcal{R}$ is defined by

$$x\tilde{R}y \longleftrightarrow (xRy \wedge yRx)$$

Choice Sets

Given $R \in \mathcal{R}$, $y \in \mathcal{X}$ the choice set $C(R,Y)$ is defined by

$$C(R,Y) = \{y \in Y | (\forall z \in Y) yRz\}$$

Preference Profiles

Let $\mathcal{S}$ be the set of functions from I to $\mathcal{R}$. Elements of $\mathcal{S}$ are denoted by s, s',..., and are called preference profiles.[8] Given $s \in \mathcal{S}$, $i \in I$, we denote by $\bar{s}(i)$, $\tilde{s}(i)$, respectively, the strict preference and the indifference relation associated with s(i). Let $\bar{\mathcal{S}}$ be the set of functions from I to $\bar{\mathcal{R}}$. Elements of $\bar{\mathcal{S}}$ are called strict preference profiles.

Definition 1

A selection method M is a function from $\mathcal{S}$ to $\mathcal{X}$.

A selection method is, thus, a function that assigns a set of alternatives to each possible preference profile.[9] These sets of alternatives are called selection sets. As we noted in the introduction the leading interpretation that we give to selection methods assumes that one of its elements will be chosen to be the outcome by means of a random device.[10]

3. CONSISTENT INDIVIDUALS

We have assumed that individuals have preferences on alternatives and formalized this idea by endowing each of them with an ordering on X. We are now going to introduce the notion that individuals might regard some sets of alternatives as more desirable than others depending on their preferences on alternatives. We define individuals to be consistent when their preferences over sets of alternatives satisfy a certain minimal condition.[11]

Let $\mathcal{R}^*$ be the set of reflexive and transitive binary relations on $\mathcal{X}$. Elements of $\mathcal{R}^*$ are denoted by R^* and are called preferences over $\mathcal{X}$ (or preferences over sets).[12]

The strict preference $\bar{R}^*$ on $\mathcal{X}$ associated with $R^* \in \mathcal{R}^*$ is defined by

$$Y\bar{R}^*Z \longleftrightarrow [YR^*Z \wedge \sim(ZR^*Y)]$$

Elements of the Cartesian product $\mathcal{R} \times \mathcal{R}^*$ are denoted by (R,R^*) and are called extended preferences.

Definition 2

An extended preference $(R,R^*) \in \mathcal{R} \times \mathcal{R}^*$ is consistent iff

$$(\forall x,y \in X)[x\bar{R}y \rightarrow (\langle x\rangle \bar{R}^* \langle x,y\rangle \wedge \langle x,y\rangle \bar{R}^* \langle y\rangle)]$$

where $\overline{R}$, $\overline{R}^*$ are the strict preference over sets and over alternatives associated with R and R* respectively.

Consistent extended preferences (R,R*), thus, rank one-element sets by R* in the same order as they rank the corresponding alternatives by R. Moreover, if an alternative x is preferred to another alternative y by R, the two-element set $\langle xy \rangle$ is ranked between the one-element sets $\langle x \rangle$ and $\langle y \rangle$ by R*.

Definition 3

Let $\mathcal{S}^*$ be the set of functions from I to $\mathcal{R} \times \mathcal{R}^*$. Elements of $\mathcal{S}^*$ are denoted by (s,s*) and called extended preference profiles. Given $(s,s^*) \in \mathcal{S}^*$, $i \in I$, we denote by $\overline{s}^*(i)$, $\tilde{s}^*(i)$, respectively, the strict preference and the indifference relation associated with s(i).

Individual i is consistent at (s,s*) iff (s(i),s*(i)) is consistent.

It is important to note, in closing this section, that individuals endowed with any of the usual criteria of behavior under risk will be consistent, regardless of the specific random mechanism to be used in the cases where more than one alternative is selected. This is the case, among others, for all individuals whose preferences over sets of alternatives can be rationalized by a Von Neumann-Morgenstern utility function.

4. A POSSIBILITY THEOREM FOR SELECTION METHODS

Consider the case of a committee of three individuals facing three alternatives. We describe in this section a specific selection method for that committee and show that it satisfies a number of conditions which, in the light of our preceding discussion, are desirable. We formalize these condtions[13] and ask the question whether it is possible to find selection methods that satisfy all of them simultaneously for cases other than the particular one where $N = 3$ and $|X| = 3$. We conclude the section by stating our main theorem, which answers this question in the negative.

Let $N = 3$, $|X| = 3$. Given $s \in \bar{\mathcal{S}}$, define the binary relation m_s on X so that xm_sy iff x has a majority over y at preference profile s. Let $W_s = \{x | \nexists y \in X$ for which ym_sx and $\sim(xm_sy)\}$. Define the selection method $\overline{M}$ on $\bar{\mathcal{S}}$ so that, for all $s \in \bar{\mathcal{S}}$,

$$\overline{M}(s) = W_s \text{ if } W_s \neq \emptyset, \text{ and } \overline{M}(s) = \langle x,y,z \rangle \text{ if } W_s = \emptyset.$$

$\overline{M}$, as defined, is just the simple majority rule except that, whenever a cycle arises all elements in the cycle are selected.

The rule $\overline{M}$ has a number of attractive features. In the first place, it has no dictator. By this we mean that no individual has the power to always impose its best preferred alternative as the selection method's outcome.

To substantiate this assertion it suffices to note that whenever two committee members agree on an alternative as best this alternative is selected regardless of the third member's preferences. Formally, we define a dictator as follows:

Definition 4

An individual $i \in I$ is a dictator for a selection method M iff[14]

$$(\forall x \in X)(C(s(i),X) = \langle x \rangle) \rightarrow M(s) = \langle x \rangle$$

If there is a dictator for M, we say that M is dictatorial. A related concept, that we use later on , is that of a weak dictator.

Definition 5

An individual $i \in I$ is a weak dictator for a selection method M iff

$$(\forall s \in \mathcal{S})C(s(i),X) \subset M(s)$$

Going back now to the properties of our selection method $\overline{M}$, notice that it is unanimous. By this we mean that it never selects an alternative x if there is some other alternative y that all individuals prefer to x. In formal terms, $\overline{M}$ satisfies the following requirement.

Definition 6

A selection method is unanimous iff

$$(\forall x,y \in X)[(\forall i)(x\overline{s}(i)y) \rightarrow \sim(y \in M(s))]$$

Third, $\overline{M}$ is positive responsive. By this we mean the following. Consider any preference profile s for which the selection set contains more than one alternative. Consider now a second profile s' at which some alternative x among the selected ones at s has improved upon some of the remaining alternatives in some committeeman's preferences. Suppose that all other alternatives stand in the same relationship among each other in s and s'. Then, x is the unique alternative selected by $\overline{M}$ at s'. In the light of our introductory remarks this is indeed a very satisfactory feature of $\overline{M}$. By not ruling out the possibility that several alternatives be selected, $\overline{M}$ may operate symmetrically over symmetric preferences. Yet the mechanism's responsiveness guarantees that multivalued selections are not a permanent feature of the mechanisms's operation: any appropriate change in the relative position of alternatives will yield to a new situation for which only one alternative is selected. Therefore, chance enters the mechanism only in a limited number of cases. This property, thus, eliminates one of the possible sources of criticism to mechanisms that combine voting with chance: that by leaving too much to chance they may become too arbitrary in the way they arrive at decisions on often important issues. We now formalize that property by the following definition.

Definition 7

A selection method M is positive responsive iff

$$(\forall x \in X)\{(\forall w,z \neq x)(\forall i \in I)[ws(i)z \longleftrightarrow ws'(i)z] \wedge (\forall z \neq x)$$
$$(\forall i \in I)[(x\tilde{s}(i)z \rightarrow xs'(i)z) \wedge (x\overline{s}(i)z \rightarrow x\overline{s}'(i)z] \wedge$$
$$s \neq s' \wedge x \in M(s)\} \rightarrow \langle x \rangle = M(s')$$

Finally, consider the possibilities that $\overline{M}$ leaves open to manipulation. For profiles in $\bar{\mathcal{S}}$ at which one alternative is a majority winner, the only possible changes that individuals can induce on the selected alternatives by misrepresenting their preference would yield outcomes that are strictly worse than those resulting from declaring their actual preferences. The only possibilities for manipulation of $\overline{M}$ arise in the cases where there is no majority winner, i.e., for profiles of the form:

$$I_1 : xyz \qquad I_2 : yzx \qquad I_3 : zxy.$$

Let a particular preference profile of this form be denoted by s. According to our rule, $\overline{M}(s) = \langle x,y,z \rangle$. By declaring yxz to be his preferences, I_1 could change the preference profile to an s' for which $\overline{M}(s') = \langle y \rangle$. Similarly, I_2 could misrepresent his preferences to zyx and change the preference profile to an s" for which $\overline{M}(s") = \langle z \rangle$, and I_3 could give rise to a profile s''' for which $M(s''') = \langle x \rangle$. Would this mean that $\overline{M}$ is manipulable? In a sense yes, since there may be individuals who prefer $\langle y \rangle$ to $\langle x,y,z \rangle$, say, when their preferences over alternatives are of the form xyz. Such would be the case, for example, for maximiners and for some expected utility maximizers. However, other individuals with the same ranking of alternatives, xyz, may prefer $\langle x,y,z \rangle$ to $\langle y \rangle$. Clearly, $\overline{M}$ would be vulnerable in a stronger sense if it was open to manipulations that all individuals with given preferences over alternatives would find profitable <u>regardless</u> of their attitudes toward risk. This remark motivates the following definition.

<u>Definition 8</u>

A selection method M is <u>uniformly manipulable</u> iff $\exists s,s' \in \mathcal{S}$, $\exists i \in I$ such that $(\forall j \neq i)\ s(j) = s'(j)$, and $M(s')\overline{s}^*(i)M(s)$ for <u>all</u> $s^*(i)$ consistent with $s(i)$. We then say that M is uniformly manipulable by i at profile s.

To explain the meaning of this definition, let us first concentrate on a specific $\hat{s}^*(i)$ consistent with $s(i)$. Individual i, endowed with extended preferences $(s(i),\hat{s}^*(i))$ could, by misrepresenting his preferences to $s'(i)$, change the preference profile from s to s'. By so doing he could secure an outcome that he strictly prefers to the one that would result if he was to reveal his true preferences s(i). That fact in itself can already be thought of as a threat of manipulation, since i may try to take advantage of the possibility to improve himself by misrepresenting his preferences. Still, our definition of uniform manipulability involves the further requirement that the threat should be advantageous to <u>all</u> consistent individuals (s(i),s*(i)). By imposing this requirement we concentrate only on those situations where an unambiguous threat arises, i.e., one that it would be in the interest of <u>any</u> consistent individual with prefereneces s(i) to carry through, regardless of his attitude toward risk. Even if a selection method is not uniformly manipulable, some individuals may still find it profitable to misrepresent their preferences depending on their attitude toward risk. On the other hand, any selection method which is uniformly manipulable is extremely vulnerable.

We can now summarize our discussion of the properties of $\overline{M}$ by stating that, for the case where $N = 3$, $|X| = 3$, there exists a selection method on $\overline{\mathcal{F}}$ that is unanimous, positive responsive, has no dictator and is not uniformly manipulable.

Could this result be extended to the case where $|X| > 3$? The following theorem states that such an extension is not possible and constitutes the main result of the paper.

Theorem 1

Let $|X| > 3$. A unanimous and positive responsive selection method M is either dictatorial or uniformly manipulable.

5. A THEOREM FOR GROUP DECISION RULES

We now present a theorem for group decision rules that is central to our proof of Theorem 1. This theorem is due to Mas-Colell and Sonnenschein [4].
Let $\mathscr{B}$ be the set of complete and reflexive binary relations on X. Let $B \in \mathscr{B}$. The strict relation $\bar{B}$ associated with B is defined by

$$x\bar{B}y \longleftrightarrow \sim (yBx)$$

A relation $B \in \mathscr{B}$ is quasi-transitive iff

$$(\forall x,y,z \in X) \; \{(x\bar{B}y \wedge y\bar{B}z) \rightarrow x\bar{B}z\}$$

Definition 9

A group decision rule on $\bar{\mathscr{F}}$ is a function from $\bar{\mathscr{F}}$ to $\mathscr{B}$.

Definition 10

A group decision rule on $\bar{\mathscr{F}}$, F is quasi-transitive iff F(s) is quasi-transitive for all $s \in \bar{\mathscr{F}}$.

Definition 11

A group decision rule on $\bar{\mathcal{J}}$, F is unanimous iff

$$(\forall x,y \in X)\ \{(s \in \bar{\mathcal{J}} \wedge (\forall i)x\bar{s}(i)y) \rightarrow x\bar{B}y\},\text{ where } F(s) = B.$$

Definition 12

A group decision rule on $\bar{\mathcal{J}}$, F is binary iff

$$(\forall x,y \in X)[s \in \bar{\mathcal{J}} \wedge s' \in \bar{\mathcal{J}} \wedge (\forall i)(xs(i)y \longleftrightarrow xs'(i)y)] \rightarrow (xBy \longleftrightarrow xB'y)$$

where $F(s) = B$ and $F(s') = B'$.

Definition 13

A group decision rule on $\bar{\mathcal{J}}$, F has a weak dictator iff $\exists i \in I$ such that

$$(\forall s \in \bar{\mathcal{J}})(\forall x,y \in X)(x\bar{s}(i)y \rightarrow xBy),\text{ where } B = F(s).$$

Theorem 2

Let $|X| > 2$. Let F be a group decision rule on $\bar{\mathcal{J}}$. If F is quasi-transitive, unanimous and binary, then it has a weak dictator.

The proof of this theorem is due to Mas-Colell and Sonnenschein [5]. Although the theorem is valid for the general case where F is defined on $\mathcal{J}$, its proof only involves preference profiles in $\bar{\mathcal{J}}$ and thus applies to our restricted version.

6. PROOF OF THE MAIN THEOREM

As a preliminary to the proof of Theorem 1, we define the following property of selection methods and prove a lemma on its relationship to positive responsiveness.

Definition 14

A selection method M is nonnegative responsive iff

$$(\forall x \in X)\{(\forall w,z \neq x)(\forall i \in I)[ws(i)z \Longleftrightarrow ws'(i)z] \wedge (\forall z \neq x)$$
$$(\forall i \in I)[(z\tilde{s}(i)x \rightarrow zs'(i)x) \wedge (z\overline{s}(i)x \rightarrow z\overline{s}'(i)x)] \wedge x \notin M(s)\}$$
$$\rightarrow x \notin M(s')$$

Lemma 1

If a selection method M is positive responsive, it is nonnegative responsive.

Proof of Lemma 1

Suppose M was not nonnegative responsive. Then $\exists s,s' \in \mathcal{J}$ satisfying the conditions of the definition of nonnegative responsiveness and such that $x \in M(s')$. Since $x \notin M(s)$, it must be that $s \neq s'$. But then all the requirements are met for positive responsiveness to imply that $M(s) = \langle x \rangle$, a contradiction.

The proof of Theorem 1 proceeds as follows. We assume that our selection method M is not uniformly manipulable. We then use M to generate a group decision rule F on $\bar{\mathcal{S}}$, and prove that F is quasi-transitive, binary and unanimous. Thus, by Theorem 2 , F has a weak dictator (on $\bar{\mathcal{S}}$). Finally, we show that if F has a weak dictator (on $\bar{\mathcal{S}}$), M must have a dictator (on $\mathcal{S}$). The proof is organized in eight steps.

STEP 1

Given $s \in \mathcal{S}$, $A \in \mathcal{X}$, let s_A be the profile defined by

$$(\forall i)(\forall w,z \in A) \qquad w s_A(i) z \Longleftrightarrow w s(i) z$$

$$(\forall i)(\forall w,z \notin A) \qquad w s_A(i) z \Longleftrightarrow w s(i) z$$

$$(\forall i)(\forall w \notin A, \forall z \in A) \qquad z \bar{s}_A(i) w$$

Informally, s_A is obtained from s by advancing all elements of A to the top and keeping the rankings of alternatives within A and within X-A unchanged.

For each $s \in \bar{\mathcal{S}}$, let B_s be defined by

$$x B_s y \Longleftrightarrow x \in M(s_{\langle xy \rangle})$$

Since M is unanimous, B_s is a complete and reflexive binary relation for all $s \in \bar{\mathcal{S}}$. Let F be the group decision rule on $\bar{\mathcal{S}}$ defined by: $(\forall s \in \bar{\mathcal{S}}) F(s) = B_s$.

STEP 2. (F is binary)

Proof.

Assume it was not, i.e., $\exists s, s' \in \bar{\mathcal{S}}$, $x, y \in X$ such that

$$(\forall i)\ xs(i)y \Longleftrightarrow xs'(i)y$$

and yet $x \in M(s_{\langle xy\rangle})$, $x \notin M(s'_{\langle xy\rangle})$.
Let s^k be defined as

$$s^k = (s_{\langle xy\rangle}(1),\ldots,s_{\langle xy\rangle}(k-1), s'_{\langle xy\rangle}(k),\ldots,s'_{\langle xy\rangle}(N))$$
$$\text{for } k \in \{1,\ldots,N+1\}$$

Notice that, since M is unanimous, for all $k \in \{1,\ldots,N+1\}$, either

$$F(s^k) = \langle x\rangle \text{ or } F(s^k) = \langle y\rangle \text{ or } F(s^k) = \langle xy\rangle$$

There must exist s^j, s^{j+1} such that

$$x \notin M(s^j) = M(s_{\langle xy\rangle}(1),\ldots,s_{\langle xy\rangle}(j-1), s'_{\langle xy\rangle}(j)$$
$$s'_{\langle xy\rangle}(j+1),\ldots,s'_{\langle xy\rangle}(N)) = \langle y\rangle$$

and

$$x \in M(s^{j+1}) = M(s_{\langle xy\rangle}(1),\ldots,s_{\langle xy\rangle}(j-1),s_{\langle xy\rangle}(j),$$
$$s'_{\langle xy\rangle}(j+1),\ldots,s'_{\langle xy\rangle}(N))$$

If $xs(j)y$, then $xs'_{\langle xy\rangle}(j)y$. Therefore, for all $s^*(j)$ consistent with $s'_{\langle xy\rangle}(j)$ it must be that $M(s^{j+1})\overline{s}^*(j)M(s^j)$, and M will be uniformly manipulable by j at s^j. Similarly, if $ys(j)x$, it must be that $ys_{\langle xy\rangle}(j)x$, and then M is uniformly manipulable by j at s^{j+1}. Since denying our conclusion would, in both cases, contradict our hypothesis, F must be binary.

STEP 3. (F is unanimous)

Proof.

Let $s \in \bar{\mathcal{S}}$ be such that, for some $x,z \in X$,

$$(\forall i \in I)xs(i)z.$$

We must prove that $x\overline{B}_s y$.

Notice that, by construction,

$$(1) \quad [(\forall i \in I)xs(i)z] \rightarrow [(\forall i \in I)xs_{\langle xz\rangle}(i)z]$$

Since M is unanimous

(2) $[(\forall i \in I) x s_{\langle xz \rangle}(i) z] \rightarrow M(s_{\langle xz \rangle}) = \langle x \rangle$

Therefore, combining (1) and (2) we get

$$[(\forall i \in I) x s(i) z] \rightarrow x \bar{B}_s z,$$

which is an expression for unanimity.

STEP 4. (F is quasi-transitive)

Proof.

Assume F was not quasi-transitive. There would exist, then, $s \in \bar{\mathcal{S}}$, $x, y, z \in X$ such that

$$\langle x \rangle = M(s_{\langle xy \rangle}), \langle y \rangle = M(s_{\langle yz \rangle}) \text{ and } z \in M(s_{\langle xz \rangle}).$$

Let $A = M(s_{\langle xyz \rangle})$.

Since M is unanimous, it must be that either

1) $A = \langle x \rangle$ 2) $A = \langle y \rangle$ 3) $A = \langle z \rangle$ 4) $A = \langle x, z \rangle$

5) $A = \langle y, z \rangle$ 6) $A = \langle x, y \rangle$ or 7) $A = \langle x, y, z \rangle$.

Case 1. ($A = \langle x \rangle$). Let $\hat{s} = s_{\langle xyz \rangle}$, $\hat{\hat{s}} = (s_{\langle xyz \rangle})_{\langle xz \rangle}$.

Let $s^k = (\hat{\hat{s}}(1), \ldots, \hat{\hat{s}}(k-1), \hat{s}(k), \ldots, \hat{s}(N))$. Since M is unanimous and nonnegative responsive, for all $k \in \{1, \ldots, N+1\}$ either $M(s^k) = \langle x \rangle$ or $M(s^k) = \langle xz \rangle$ or $M(s^k) = \langle z \rangle$.

Suppose $\exists s^k$ such that $z \in M(s^k)$. Then $\exists s^j, s^{j+1}$ such that

$$\langle x \rangle = M(s^j) = M(\hat{\hat{s}}(1),\ldots,\hat{\hat{s}}(j-1),\hat{s}(j),\hat{s}(j+1),\ldots,\hat{s}(N))$$

and $z \in M(s^{j+1}) = M(\hat{\hat{s}}(1),\ldots,\hat{\hat{s}}(j-1),\hat{\hat{s}}(j),\hat{s}(j+1),\ldots,\hat{s}(N))$.

If $x\hat{s}(j)z$, then $x\hat{\hat{s}}(j)z$ and for all $s^*(j)$ consistent with $\hat{\hat{s}}(j)$ it will be that $M(s^j)\overline{s}^*(j)M(s^{j+1})$. Thus, in this case, M would be uniformly manipulable at s^{j+1} by individual j. Similarly, if $z\hat{s}(j)x$, M would be uniformly manipulable at s^j by individual j. Since, in both cases, we would arrive at a contradiction to our assumption that M is not uniformly manipulable, it must be that $\nexists s^h$ such that $z \in M(s^h)$. Therefore, for all $k \in \langle 1,\ldots,N+1 \rangle$, $M(s^k) = \langle x \rangle$.

In particular, $M(\hat{\hat{s}}) = \langle x \rangle$.

But, by assumption, $z \in M(s_{\langle xz \rangle})$. Since, by construction,

$$(\forall i \in I)\ xs_{\langle xz \rangle}(i)z \longleftrightarrow x\hat{\hat{s}}(i)z,$$

the binarity of F, as established in Step 2, would be contradicted.

<u>Case 2</u>. $(A = \langle y \rangle)$. A proof similar to that of Case 1 would apply, letting $\hat{s} = s_{\langle xyz \rangle}$ and $\hat{\hat{s}} = (s_{\langle xyz \rangle})_{\langle xy \rangle}$.

<u>Case 3</u>. $(A = \langle z \rangle)$. A proof similar to that of Case 1 would apply, letting $\hat{s} = s_{\langle xyz \rangle}$ and $\hat{\hat{s}} = (s_{\langle xyz \rangle})_{\langle yz \rangle}$.

<u>Case 4</u>. $(A = \langle x,z\rangle)$. Since M is unanimous, it must be that, for some i, $xs(i)z$.

<u>Case 4a</u>. $(xs(i)ys(i)z)$. Then $xs_{\langle xyz\rangle}(i)ys_{\langle xyz\rangle}(i)z$. Let s' be defined so that

$$(\forall j \neq i)\ \ s'(j) = s_{\langle xyz\rangle}(j)$$

$$(\forall\langle w,v\rangle \neq \langle y,z\rangle)(vs'(i)w \Longleftrightarrow vs_{\langle xyz\rangle}(i)w) \text{ and } z\overline{s}'(i)y.$$

Informally, s' is obtained from $s_{\langle xyz\rangle}$ by advancing z over y in individual i's ranking. Since M is positive responsive, $M(s') = \langle z\rangle$. Yet, at s', $xs'(i)z$ and, therefore, for all $s^*(i)$ consistent with $s'(i)$ it must be that $M(s_{\langle xyz\rangle})\overline{s}^*(i)M(s')$. Thus, M would be uniformly manipulable, contrary to our assumption.

<u>Case 4b</u>. $(xs(i)zs(i)y)$. Then $xs_{\langle xyz\rangle}(i)zs_{\langle xyz\rangle}(i)y$. Let s' be defined so that

$$(\forall j \neq i)\ \ s'(j) = s_{\langle xyz\rangle}(j)$$

$$(\forall\langle w,v\rangle \neq \langle x,z\rangle)(vs_{\langle xyz\rangle}(i)w \Longleftrightarrow vs'(i)w) \text{ and } z\overline{s}'(i)x.$$

By construction,

$$s' = s'_{\langle xyz \rangle}$$

$$s'_{\langle yz \rangle} = (s_{\langle xyz \rangle})_{\langle yz \rangle}$$

$$\text{and } (\forall i \in I) \ ys'(i)z \Longleftrightarrow ys'_{\langle yz \rangle}(i)z$$

Informally, s' is obtained from $s_{\langle xyz \rangle}$ by advancing z over x in individual i's ranking.

Since M is positive responsive, $M(s') = \langle z \rangle$.

Let $s^k = (s'_{\langle yz \rangle}(1), \ldots, s'_{\langle yz \rangle}(k-1), s'(k), \ldots, s'(N))$ for $k \in \{1, \ldots, N+1\}$.

Since M is unanimous and nonnegative responsive, for all $k \in \{1, \ldots, N+1\}$ either $M(s^k) = \langle z \rangle$ or $M(s^k) = \langle y \rangle$ or $M(s^k) = \langle y, z \rangle$. Suppose $\exists s^h$ such that $y \in M(s^h)$. Then $\exists j$ such that

$$\langle z \rangle = M(s^j) = M(s'_{\langle yz \rangle}(1), \ldots, s'_{\langle yz \rangle}(j-1), s'(j), s'(j+1), \ldots, s'(N)), \text{ and}$$

$$y \in M(s^{j+1}) = M(s'_{\langle yz \rangle}(1), \ldots, s'_{\langle yz \rangle}(j-1), s'_{\langle yz \rangle}(j), s'(j+1), \ldots, s'(N)).$$

If $ys'(j)z$, $M(s^{j+1})\overline{s}^*(j)M(s^j)$ for all $s^*(j)$ consistent with $s'(j)$, and M is uniformly manipulable at s^j by individual j. Similarly, if $zs'(j)y$, then $zs'_{\langle zy \rangle}(j)y$, and M is uniformly manipulable at s^{j+1} by individual j. In both cases, M would be uniformly manipulable, contrary to our assumption. Therefore, it must be that

$$M(s'_{\langle yz\rangle}) = \langle z\rangle = M[(s_{\langle xyz\rangle})_{\langle yz\rangle}]$$

But, by assumption, $\langle y\rangle = M(s_{\langle yz\rangle})$, and this contradicts the binarity of F.

Case 4c. $(ys(i)xs(i)z)$. The same proof that in Case 4b applies, establishing that, also in this case, M is uniformly manipulable.

Cases 5 and 6. $(A = \langle y,z\rangle)$ and $(A = \langle x,y\rangle)$. Proofs similar to the one for Case 4 would apply.

Case 7. $(A = \langle x,y,z\rangle)$. Since M is unanimous, there must exist at least two alternatives in $\langle x,y,z\rangle$ which are ranked in third place by some individual at $s_{\langle xyz\rangle}$. For suppose, otherwise, that for all individuals in I the same alternative in $\langle x,y,z\rangle$, say x, was ranked third at $s_{\langle xyz\rangle}$. Then, by construction

$$(\forall i \in I)\ (ys_{\langle xyz\rangle}(i)x \text{ and } zs_{\langle xyz\rangle}(i)x)$$

and any of the two would be sufficient, by M's unanimity to establish that $x \notin M(s_{\langle xyz\rangle}) = A$, contrary to our assumption.

Assume, then, that x and y were two of the alternatives that are ranked third by some $i \in I$ in $s_{\langle xyz \rangle}$.[15] There will exist $m,n \in I$ such that

$$zs_{\langle xyz \rangle}(m)x \wedge ys_{\langle xyz \rangle}(m)x$$

$$\text{and } zs_{\langle xyz \rangle}(n)y \wedge xs_{\langle xyz \rangle}(n)y$$

Let $w \in X$ be such that $w \notin \langle x,y,z \rangle$ and
$(\forall u \in X-\langle x,y,z,w \rangle) ws_{\langle xyz \rangle}(m)u$.
Let $v \in X$ be such that $v \notin \langle x,y,z \rangle$ and
$(\forall u \in X-\langle x,y,z,v \rangle) vs_{\langle xyz \rangle}(n)u$.
In words, w and v are the alternatives that m and n rank in fourth place at $s_{\langle xyz \rangle}$.
Let $s' \in \bar{\mathcal{F}}$ be such that

$$(\forall i \neq m) \; s_{\langle xyz \rangle}(i) = s'(i)$$

$$(\forall \langle o,p \rangle \neq \langle x,w \rangle) \; os_{\langle xyz \rangle}(m)p \longleftrightarrow os'(m)p, \text{ and } ws'(m)x$$

Let $s'' \in \bar{\mathcal{F}}$ be such that

$$(\forall i \neq n) \; s_{\langle xyz \rangle}(i) = s''(i)$$

$$(\forall \langle o,p \rangle \neq \langle y,v \rangle) os_{\langle xyz \rangle}(n)p \longleftrightarrow os''(n)p, \text{ and } vs''(n)y$$

Let s''' be such that

$$(\forall i \notin \langle n,m\rangle)\ s'''(i) = s_{\langle xyz\rangle}(i) = s'(i) = s''(i)$$

$$s'''(m) = s'(m)$$

$$s'''(n) = s''(n)$$

Informally, s' is obtained from $s_{\langle xyz\rangle}$ by advancing w over x in individual m's ranking; s" is obtained from $s_{\langle xyz\rangle}$ by advancing v over y in individual n's ranking; s''' is the profile at which both changes above have been performed.

Notice that, by M's unanimity, $M(s') \subset \langle x,y,z\rangle$ and $M(s'') \subset \langle xyz\rangle$. Suppose $x \in M(s')$. Since M is positive responsive, this would imply $\langle x\rangle = M(s)$, contrary to our assumption. Thus, $M(s') \subset \langle y,z\rangle$. By a similar argument, it must be that $M(s'') \subset \langle x,z\rangle$.

<u>Case 7a</u>. $(M(s') = \langle y\rangle)$. Let then $\hat{s}$ be defined so that

$$(\forall i \neq m)\ \hat{s}(i) = s'_{\langle xyz\rangle}(i) \text{ and } \hat{s}(m) = s'(m)$$

Notice that, $(\forall i \in I)\ y\hat{s}(i)z \longleftrightarrow ys'(i)z$.

Let $s^k = (s'(1),\ldots,s'(k-1),\hat{s}(k),\ldots,\hat{s}(N))$ for $k \in \{1,\ldots N+1\}$. By construction, and since M is unanimous, it must be that either $M(s^k) = \langle y\rangle$ or $M(s^k) = \langle yz\rangle$ or $M(s^k) = \langle z\rangle$ for all $k \in \{1,\ldots,N+1\}$.

Suppose $\exists s^h$ such that $z \in M(s^h)$. Then, there must exist $j \in I$ such that

$$z \in M(s^j) = M(s'(1),\ldots,s'(j-1),\hat{s}(j),\hat{s}(j+1),\ldots,\hat{s}(N))$$

and $\langle y \rangle = M(s^{j+1}) = M(s'(1),\ldots,s'(j-1),s'(j),\hat{s}(j+1),\ldots,\hat{s}(N))$.

Then, if $y\hat{s}(j)z$, M is uniformly manipulable by j at s^j. And, if $z\hat{s}(j)y$, then $zs'(j)y$, and M is uniformly manipulable by j at s^{j+1}.

Thus, it must be that $M(\hat{s}) = \langle y \rangle$.

Since F must be binary, $M[(s_{\langle xyz \rangle})_{\langle xy \rangle}] = M(s_{\langle xy \rangle}) = \langle x \rangle$.

Notice that $(s_{\langle xyz \rangle})_{\langle xy \rangle}$ and $\hat{s}$ only differ by m's preference, and that

$$y(s_{\langle xyz \rangle})_{\langle xy \rangle}x.$$

Therefore, M would be uniformly manipulable by m at $(s_{\langle xyz \rangle})_{\langle xy \rangle}$, a contradiction.

Case 7b. $(M(s'') = \langle x \rangle)$. A similar argument to that developed for Case 7a would contradict our assumption that M is not uniformly manipulable.

Case 7c. $(M(s') \neq \langle y \rangle$ and $M(s') \neq \langle x \rangle)$. Since, as noted above, $M(s') \subset \langle y,z \rangle$ and $M(s'') \subset \langle x,z \rangle$, it must be that $z \in M(s')$ and $z \in M(s'')$. We claim that, in this case, $M(s''') = \langle z \rangle$. To prove it, first note that, by unanimity, it must be that $M(s''') \subset \langle x,y,z\rangle$. Suppose $x \in M(s''')$. Then, by positive responsiveness, it would be that $\langle x\rangle = M(s'')$, contrary to our assumption that $z \in M(s'')$. Similarly, if $y \in M(s''')$, then $\langle y\rangle = M(s')$, a contradiction to $z \in M(s')$. Thus, it must be that, as we claimed, $\langle z\rangle = M(s'')$.

An argument similar to that of Case 7a would now conclude that M is uniformly manipulable.

We have now exhausted all the possibilities for the value of $A = M(s_{\langle xyz\rangle})$ and, for all cases, reached a contradiction to our hypothesis from our denial of F's quasi-transitivity. It must, therefore, be that F is quasi-transitive.

STEP 5. (All weak dictators for F (on $\bar{\mathcal{S}}$) are weak dictators for M (on $\bar{\mathcal{S}}$).)

Proof. Suppose not. Then $\exists s \in \bar{\mathcal{S}}$, $i \in I$, $x \in X$ such that

1) i is a weak dictator for F on $\bar{\mathcal{S}}$
2) $\langle x\rangle = C(s(i),X)$
3) $x \notin M(s)$.

Suppose, without loss of generality, that $i = N$.

Let $z \in M(s)$. Define s' as follows:

$$(\forall j \neq N)\ [(\forall y,w \neq z)(ys(j)w \longleftrightarrow ys'(j)w) \wedge (\forall y \neq z)(zs'(j)y)]$$

$$(\forall y \in X)\ xs'(N)y \wedge (\forall w \notin \langle x,z\rangle)\, zs'(N)w \wedge (\forall w,v \notin \langle x,z\rangle)$$

$$(ws'(N)v \longleftrightarrow ws(N)v)$$

Informally, s' is obtained from s by advancing z to first place in the ranking of all individuals other than N, and to second place (after x), in the ranking of individual N. If $s \neq s'$, since M is positive responsive and $z \in M(s)$, $\langle z \rangle = M(s')$.

If $s = s'$, since $x \notin M(s)$ and, by construction and unanimity, $M(s') \subset \langle x,z \rangle$, it must also be that $M(s') = \langle z \rangle$. Let $s^k = (s'_{\langle xz \rangle}(1),\ldots,s'_{\langle xz \rangle}(k-1), s'(k),\ldots,s'(N))$ for $k \in \{1,\ldots,N+1\}$.

Notice that, by unanimity, for all $k \in \{1,\ldots,N+1\}$ it must be that either $M(s^k) = \langle x \rangle$ or $M(s^k) = \langle z \rangle$ or $M(s^k) = \langle x,z \rangle$.

Clearly, $s^1 = s'$ and $s^{N+1} = s'_{\langle xz \rangle}$. Notice that $s^{N+1} = s^N$. Therefore, since $\langle z \rangle = M(s^1)$ and $x \in M(s^{N+1})$, there must exist $j \in \{1,\ldots,N-1\}$ such that

$$\langle z \rangle = M(s^j) = M(s'_{\langle xz \rangle}(1),\ldots,s'_{\langle xz \rangle}(j-1), s'(j), s'(j+1),\ldots,s'(N))$$

and $x \in M(s^{j+1}) = M(s'_{\langle xz \rangle}(1),\ldots,s'_{\langle xz \rangle}(j-1), s'_{\langle xz \rangle}(j), s'(j+1),\ldots,s'(N))$

Since, by construction z is the most preferred alternative for all $i \in \{1,\ldots,N-1\}$, $zs'_{\langle xz \rangle}(i)x$, we would have that $M(s^j)\overline{s}*(j)M(s^{j+1})$ for all $s*(j)$ consistent with $s'_{\langle xz \rangle}(i)$, and, therefore, M would be uniformly manipulable by j at

s^{j+1}, contrary to our assumption. Thus, if N is a weak dictator for F on $\bar{\mathcal{F}}$, it is also a weak dictator for M on $\bar{\mathcal{F}}$.

STEP 6. (If there is a weak dictator for M on $\bar{\mathcal{F}}$, it is a dictator for M on $\bar{\mathcal{F}}$.)

Proof. Let $m \in I$ be a weak dictator and suppose it is not a dictator. Then, $\exists s \in \bar{\mathcal{F}}$, $x \in X$ such that $x \in M(s)$ and $x \notin C(s(m),X) = \langle y \rangle$. Since m is a weak dictator, $y \in M(s)$.

<u>Case 1</u>. ($\exists z \neq y$ such that $zs(m)x$).

Define s' to be the preference profile at which

$$(\forall j \neq m) \; s(j) = s'(j)$$

and $ys'(m)x \wedge (\forall w \notin \langle y,x \rangle) xs'(m)w \wedge (\forall w,v \neq x) ws(m)v \longleftrightarrow ws'(m)v$.

Informally, s' is obtained from s by advancing x to second place, after y, in individual m's ranking.

Since M is positive responsive, $M(s') = \langle x \rangle$.

Still, by construction, $C(s'(m),X) = \langle y \rangle \not\subset M(s')$, and this contradicts our assumption that m is a weak dictator.

<u>Case 2</u>. $((\forall w \notin \langle y,x \rangle) xs(m)w$, and $\exists i \in I$, $z \in X$ such that $zs(i)x)$

Let then s' be defined so that

$(\forall j \neq i)\ s(j) = s'(j)$

$(\forall w \neq x)\ xs'(i)w \wedge (\forall w,v \neq x)(ws(i)v \Longleftrightarrow ws'(i)v)$

Informally, s' is obtained from s by advancing x to second place (after y) in individual i's ranking.
Since M is positive responsive, $M(s') = \langle x \rangle$. Yet, $C(s'(m),X) = \langle y \rangle \not\subset M(s')$, and this again contradicts our assumption that m is a weak dictator.

Case 3. $((\forall w \notin \langle x,y \rangle) xs(m)w$, $(\forall i \neq m)(\forall w \neq x)xs(i)w$, and $\exists z \notin \langle x,y \rangle$, $j \in I$ for which $zs(j)y$.)
Given M's unanimity and our assumptions for the case, $M(s) = \langle x,y \rangle$.
Define s' to be such that

$(\forall i \neq j)\ s(i) = s'(i)$

$$(\forall w \neq x)\ xs'(j)w \wedge (\forall w \notin \langle x,y \rangle) ys'(j)w \wedge (\forall w,v \notin \langle x,y \rangle)(ws(j)v \Longleftrightarrow ws'(j)v)$$

Informally, s' is obtained from s by advancing y to second place in individual j's ranking.
Since M is positive responsive, $M(s') = \langle y \rangle$.
Yet, by construction, $\langle x \rangle = C(s'(j),X)$. Therefore, for all $s^*(j)$ consistent with $s'(j)$, $M(s) = \langle xy \rangle \dot{\overline{s}}{}^*(j) \langle y \rangle = M(s')$.
This means that M would be uniformly manipulable, by j,

at s' in contradiction to our hypothesis.

Case 4. $((\forall w \notin \langle x,y\rangle) x s(m) w$, $(\forall i \neq m)(\forall w \neq x) x s(i) w$, and $(\forall i \in I)(\forall w \in \langle x,y\rangle) y s(i) w)$

Given M's unanimity and our assumptions, $M(s) = \langle x,y\rangle$.

Let s' be defined so that

$$(\forall i \neq m)\ s(i) = s'(i)$$

$$(\forall w,v \neq x)\ v s(m) w \Longleftrightarrow v s'(m) w \wedge (\forall w \neq x) w s'(m) x.$$

Informally, s' is obtained from s by making x to be the last alternative in individual m's ranking.

Since M is unanimous, it has to be that either $M(s') = \langle x\rangle$ or $M(s') = \langle y\rangle$ or $M(s') = \langle x,y\rangle$.

We claim that $x \notin M(s')$. Assume, a contrario, that $x \in M(s')$. Then, by positive responsiveness, $M(s) = \langle x\rangle$, contrary to the fact that $M(s) = \langle x,y\rangle$.

The two above remarks, therefore, imply that $\langle y\rangle = M(s')$. Since $y s(m) x$, it would be that, for all $s^*(m)$ consistent with $s(m)$, $M(s')\overline{s}^*(m)M(s)$. Therefore, M would be uniformly manipulable by m at s, a contradiction.

The four cases above exhaust all the possible configurations of s. In all cases, denial of our conclusion led to a contradiction of our hypothesis. Therefore, the proof of this step is now complete.

STEP 7. (If there is a dictator for M on $\bar{\mathcal{S}}$, there is a dictator for M on $\mathcal{S}$.)

Proof. Suppose, without loss of generality, that N is the dictator on $\bar{\mathcal{S}}$. Assume it was not a dictator on $\mathcal{S}$. Then, there would exist $s \in \mathcal{S}$, $y \in X$ such that

$$C(s(N),X) = \langle y \rangle \text{ and } M(s) \neq \langle y \rangle.$$

Let $x \in M(s)$, $x \neq y$. Define s' as follows.

$$(\forall i \neq N)\ [(\forall w \neq x)\ x\bar{s}'(i)w \wedge (\forall w,v \neq x)(ws(i)v \longleftrightarrow ws'(i)v)]$$

$$y\bar{s}'(N)x \wedge (\forall w \notin \langle x,y \rangle)\ x\bar{s}'(N)w \wedge (\forall w,v \neq x)$$

$$(ws'(m)v \longleftrightarrow ws(m)v).$$

Informally, s' is obtained by advancing x to first place in the rankings of all individuals but N, and to second place in the ranking of N.

If $s = s'$, $x \in M(s')$. If $s \neq s'$, by positive responsiveness, $\langle x \rangle = M(s')$. Therefore, in all cases, $x \in M(s')$. By unanimity and since x is preferred by all individuals to all alternatives but y, it must be that either $M(s') = \langle x \rangle$ or $M(s') = \langle x,y \rangle$.

Let T be an arbitrary strong ordering on X, and define s^T as follows:

$(\forall i \in I)(\forall w,v \in X)\ w\bar{s}^T(i)v \longleftrightarrow [w\bar{s}'(i)v \text{ or } (ws'(i)v \wedge wTv)]$

Informally, s^T is obtained by breaking whatever ties appear in the individual preferences at s' by means of the strong ordering T.

By construction $s^T \in \mathcal{S}$. Also by construction, since

$$(\forall i \neq N)\ x\bar{s}'(i)y$$

$$\text{and} \qquad y\bar{s}(N)x,$$

we have that

$$(\forall i \neq N)\ xs^T(i)y$$

$$\text{and} \qquad ys^T(N)x.$$

Since $s^T \in \bar{\mathcal{S}}$ and M is a dictator on $\bar{\mathcal{S}}$, $M(s^T) = \langle y \rangle$.
Still, we have that $x \in M(s')$.
Let $s^k = (s^T(1),\ldots,s^T(k-1)s'(k),\ldots,s'(N))$ for $k \in \langle 1,\ldots,N+1 \rangle$.
Notice that, by M's unanimity, for all $k \in \langle 1,\ldots,N+1 \rangle$ it must be that $M(s^k) = \langle x \rangle$ or $M(s^k) = \langle y \rangle$ or $M(s^k) = \langle x,y \rangle$.
There must exist $j \in \langle 1,\ldots,N \rangle$ such that

$$x \in M(s^j) = M(s^T(1),\ldots,s^T(j-1),s'(j),s'(j+1),\ldots,s'(N))$$

and $x \notin M(s^{j+1}) = M(s^T(1),\ldots,s^T(j-1),s^T(j),s'(j+1),\ldots,s'(N))$

$$= \langle y \rangle$$

If $j \neq N$, then $xs^T(j)y$, and M is uniformly manipulable by j at s^{j+1}. If $j = N$, then $y\bar{s}'(j)x$, and M is uniformly manipulable by N at $s^j = s^N$. In both cases, by denying the conclusion for this step, we would contradict our hypothesis on M. Therefore, it must be that the dictator on $\bar{\mathcal{S}}$ is also a dictator on $\mathcal{S}$.

STEP 8.

Now, to complete the proof of Theorem 1, notice that Steps 2, 3 and 4, along with Theorem 2 imply that F has a weak dictator on $\bar{\mathcal{S}}$. Thus, by Steps 5, 6 and 7, M must have a dictator (on $\mathcal{S}$).

7. NOTES AND EXTENSIONS

In this section we investigate whether it would have been possible to relax the assumptions of our main theorem or to strengthen its conclusion.
The following examples show that such relaxation or strengthening are not possible.

Example 1. There exists a selection method that is positive responsive, nondictatorial and not uniformly manipulable, for any number of alternatives.

Let M_1 be defined by

$$(\forall s \in \mathcal{S})\ M_1(s) = \langle x \rangle,$$

where x is any arbitrary element of X.
The reader may check that M_1 indeed satisfies the above mentioned properties. Notice, however, that M_1 is not unanimous.

Example 2. There exists a selection method that is unanimous, nondictatorial and not uniformly manipulable, for any number of alternatives.

Let M_2 be defined by

$$(\forall s \in \mathcal{S})\ M_2(s) = \{x \in X \mid (\forall y \in X)\ \exists i \ni xs(i)y\}.$$

It is left to the reader to check that M_2 satisfies the above mentioned properties. Notice, however, that M_2 is not positive responsive.

Example 3. There exists a selection method that is unanimous, positive responsive and not uniformly manipulable for any number of alternatives.

Let T be an arbitrary strong ordering on X, and i be any arbitrary element of I. Define M_3 by

$$(\forall s \in \mathcal{S})\ M_3(s) = C(T, C(s(i), X)).$$

The reader may check that M_3 satisfies the above mentioned properties. Notice, however, that i is a dictator for M. Finally we want to inquire whether our definition of a dictator could have been stronger. A mild strengthening of our definition would be the following:

Definition 15

A selection method M has a total dictator iff $\exists i \in I$ such that

$$(\forall s \in \mathcal{S})\ M(s) \subset C(s(i), X).$$

This stronger form of a dictator is not implied, however, by the assumptions in our Theorem 1, as shown by the following example.

Example 4. There exists a unanimous and positive responsive selection method that is not uniformly manipulable and has no total dictator (for any number of alternatives).

Let T be an arbitrary strong ordering on X. Let x be an arbitrary element of X.

Let $\hat{s}$ be the preference profile for which

$$C(\hat{s}(N), X) = X - \langle x \rangle \wedge (\forall j \neq N)[C(s(j), X) = \langle x \rangle \wedge C(s(j), X - \langle x \rangle) = X - \langle x \rangle]$$

Let $\tilde{\mathcal{P}}$ be the set of profiles defined by

$$\tilde{\mathcal{P}} = \{s \mid C(s(N),X) = X\}$$

Let $\hat{\mathcal{P}}$ be the set of profiles defined by

$$\hat{\mathcal{P}} = \mathcal{P} - (\hat{s} \cup \tilde{\mathcal{P}})$$

For any $s \in \mathcal{P}$, let

$$C_s^0 = C(s(N),X)$$

$$C_s^i = C(s(i-1),C_s^{i-1}) \quad \text{for } i \in \langle 1,\ldots,N-1\rangle$$

$$C_s^N = C(T,C_s^{N-1})$$

Let C_s^k be such that $C_s^{k-j} = X$ for $j \in \langle 1,\ldots,k\rangle$ and $C_s^k \neq X$. Now, define M_4 as follows:

$$M_4(s) = C(T,C_s^k) \quad \text{whenever } s \in \hat{\mathcal{P}}$$

$$M_4(s) = X \quad \text{whenever } s = \hat{s}$$

$$M_4(s) = \langle x\rangle \quad \text{whenever } s \in \tilde{\mathcal{P}}$$

It is left to the reader to check that M_4 is unanimous, positive responsive and not uniformly manipulable. Notice, however, that $M_4(\hat{s}) = X \not\subset X - \langle x\rangle = C(\hat{s}(N),X)$.

Hence, M_4 has no total dictator. However, N is a dictator for M_4.

A slight strengthening of our nonmanipulability condition would be enough to avoid this very anomalous case where a dictator is not a total dictator for selection methods satisfying the remaining conditions of our theorem.

Definition 16

A preference relation over sets $R^* \in \mathscr{R}^*$ is consistent* with a preference over alternatives $R \in \mathscr{R}$ iff it is consistent and

$$(\forall Y,Z \in \mathscr{X})\{[Y \subset C(R,X) \wedge Z \not\subset C(R,X)] \rightarrow Y\overline{R}^*Z\},$$

where $\overline{R}^*$ is the strict preference relation over sets associated with R^*.

Definition 17

A selection method M is uniformly manipulable* iff $\exists s,s' \in \mathscr{S}$, $i \in I$ such that

$$(\forall j \neq i)\ s(j) = s'(j)$$

and $M(s')\overline{s}^*(i)M(s)$ for <u>all</u> $s^*(i)$ that are consistent* with $s(i)$.

Theorem 3

Let $|X| > 3$. A unanimous and positive responsive selection method M is either uniformly manipulable* or has a total dictator.

Proof. If M is not uniformly manipulable*, it is not uniformly manipulable. Thus, by Theorem 1, it has a dictator. Suppose the dictator, i, was not a total dictator. Then $\exists s \in \mathcal{S}$, $x \in X$ such that $x \notin C(s(i),X)$ and $x \in M(s)$.

Therefore, $M(s) \not\subset C(s(i),X)$.

Let $y \in C(s(i),X)$ and define s' so that

$$(\forall j \neq i)\ s(j) = s'(j) \text{ and } (\forall z \neq y)\ y\overline{s}'(i)z$$

Then, $C(s'(i),X) = \langle y \rangle$ and $M(s') = \langle y \rangle$, since i is a dictator. Thus $M(s') \subset C(s(i),X)$. Therefore, for any $s^*(i)$ which is consistent* with s(i) it will be that

$$M(s')\overline{s}^*(i)M(s).$$

This would imply that M is uniformly manipulable.*

8. A COMPARISON WITH GIBBARD'S RESULTS

We will conclude this chapter by commenting on a related result obtained independently and at very nearly the same time by A. Gibbard [2].

Decision schemes assign a prospect (i.e., a probability distribution over alternatives) to each preference profile. A decision scheme is manipulable if there exists a profile at which an individual whose preferences conform to some Von Neumann-Morgenstern utility function would find it profitable to misrepresent his preferences over alternatives. Gibbard proves that, on the set of strict profiles, the only decision schemes that are not manipulable are probability mixtures of schemes which either (1) restrict the possible outcomes to a fixed pair of alternatives, or (2) make a single individual the sole determiner of the probabilities the various alternatives get. This theorem provides a full characterization of those decision schemes that are not manipulable.

Since we can interpret selection methods as decision schemes, by appending a random mechanism to each subset of alternatives, Gibbard's theorem is very closely related to the subject of this chapter, and it is worthwhile to digress and compare his results to ours.

Gibbard does not "a priori" restrict in any way the kind of decision schemes under his consideration. We concentrate on the class of positive responsive selection methods, owing to our special interest in those mechanisms for which chance plays a limited role. For this narrower class of functions, however, the results we have presented are stronger. Dictatorial selection methods correspond to only a small subclass of those decision schemes that Gibbard characterizes as nonmanipulable. Moreover, for positive responsive selection methods, the alternative to being dictatorial is to be uniformly manipulable, which is a stronger form of vulnerability to strategy than being manipulable in Gibbard's sense.

Both results closely follow, with different emphasis, the program suggested by Gibbard himself [2, p. 593] in the following words: "Exactly what is required for a (decision) scheme to be attractive I cannot specify . . . A system which allowed only occasional ties to be broken by chance might be quite attractive. Work needs to be done on . . . decision schemes. It would be good to identify properties which would make a . . . decision scheme attractive and to have theorems on the manipulability of classes of . . . decision schemes which, by various criteria, are attractive."

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FOOTNOTES

1. After the results of this chapter were obtained I became aware of a contribution by A. Gibbard to the same area [2]. Gibbard's approach and result are substantially different than the ones communicated here. The relationship between the two is discussed in the last section of this chapter.

2. In this Introduction, expressions of the form I_i:xyz mean that individual I_i prefers x to y, x to z and y to z. Expressions of the form I_j:(xy)z mean that I_j is indifferent between x and y and prefers them both to z.

3. This is the usual interpretation attached to sets of socially indifferent alternatives in the social choice literature. We shall use it at several points to provide heuristic support for some of our definitions. However, our results do not depend on this or any particular interpretation attached to sets of selected outcomes.

4. This is the approach taken by Gibbard [3] and by Zeckhauser [8].

5. We assume that X is finite for convenience of exposition. The reader may check that our theorems are also valid when the number of alternatives is infinite.

6. An ordering on X is a binary relation that satisfies the following properties:
   (a) Reflexivity: $(\forall x \in X)\ xRx$
   (b) Completeness: $(\forall x,y \in X)\ [x \neq y \rightarrow (xRy \vee yRx)]$
   (c) Transitivity: $(\forall x,y,z \in X)\ [(xRy \wedge yRz) \rightarrow xRz]$

7. A strong ordering on X is a complete and transitive binary relation that satisfies the condition of Assymmetry: $(\forall x,y \in X)\ [xRy \rightarrow \sim(yRx)]$

8. Viewing I as the set of individuals in a society, a preference profile $s \in \mathcal{S}$ specifies the preferences on alternatives held by each of society's members.

9. Notice that built into our definition is the hypothesis that M operates over the universal domain. In the course of our proofs we find it convenient to refer to functions from I to $\mathcal{S}$ (the set of strict preference profiles) which we then call selection methods on $\mathcal{S}$.

10. An alternative formalization of mechanisms that combine voting with chance is given by decision schemes. Decision schemes assign a probability distribution on the set of alternatives to each preference profile. Our results could be reformulated in terms of decision schemes. We prefer, however, to leave the meaning of selection sets open to other possible interpretations.

11. Our consistency requirement is strongly reminiscent of Luce and Raiffa's Condition 6 (monotonicity) for the existence of a Von Neumann-Morgenstern utility function conforming to an individual's preferences [4, p. 28]. Professor Fishburn called my attention to his paper on "Even Chance Lotteries in Social Choice Theory" [1], where a number of conditions on individual preferences over sets of alternatives are described. Fishburn's conditions are stronger, but in the same spirit, than our consistency requirement.

12. Notice that $R^* \in \mathcal{R}^*$ need not be complete.

13. We formalize these conditions with reference to preference profiles in $\mathcal{S}$. On occasion, however, we find it convenient to refer to the case where the same conditions hold for strict preference profiles only (i.e., for $s \in \mathcal{S}$). When this is the case, we state explicitly that the property holds on $\mathcal{S}$.

14. Notice that a dictator can impose his choice whenever he prefers one alternative to all others. For the case where he is indifferent among several best alternatives, see Section 7.

15. The reader may check that this arbitrary selection of a pair in $\langle x,y,z \rangle$ implies no loss of generality in our proof.