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Strategy-proofness and Single-peakedness

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1. Introduction

Is the occurrence of a single-peaked preference profile sufficient to guarantee that individuals within a group which uses generalized majority rule to choose among three or more alternative have no incentive to misreveal their true preferences on their ballots? A recent result by Pattanaik [7, Theorem 3] would seem to lend support to this hypothesis. He has shown that if the true preference profiles of the group's members are single-peaked and if the ballots which members are permitted to cast are restricted so as to allow expression only of single-peaked orderings, then every member's dominant strategy is to cast that ballot which faithfully represents his true preferences. In other words, Pattanaik's theorem states that generalized majority rule is strategy-proof whenever <u>both</u> sincere preferences and ballots are <u>a priori</u> restricted to be single-peaked.

Our question in this note therefore concerns the possibility of strengthening Pattanaik's theorem by eliminating the requirement that ballots, as well as preferences, be restricted to be single-peaked. Specifically consider a group which uses generalized majority rule without any restriction, such as single-peakedness, on how members may rank the various alternatives being considered. On a particular issue suppose members' preferences over the alternatives happen to be single-peaked. Is it possible that, despite the single-peakedness of true preferences, some member may have an incentive to cast a ballot which misrepresents his preferences? Our answer to this is affirmative: single-peakedness of preferences alone without a restriction on admissible ballots is insufficient to guarantee strategy-proofness. We show this in three steps: (1) formalization of the concepts of voting procedure, manipulation, and strategy-proofness; (2) definition of a specific variety of generalized majority rule; and (3) construction of an example where an individual has an incentive to manipulate through misrevelation despite the existence of a single-peaked preference profile within the group.

2. Formulation

A group consists of a set $N = \{1,2,\ldots,n\}$ of members whose task is to select a single alternative from a set $S = \{x,y,z,\ldots\}$ of alternatives. The number of individuals $|N|$ in the group is assumed to be odd and the number of alternatives $|S|$ in the alternative set is assumed to be at least three. Each individual $i \in N$ is rational: his preferences $\overline{P}_i$ are a strict order on S, i.e. $\overline{P}_i$ is a complete, asymmetric, and transitive binary relation. In the usual manner $x \overline{P}_i y$ means that individual i prefers that the group select alternative x instead of alternative y. Indifference between alternatives is inadmissible.[1]

A n-tuple $(\overline{P}_1,\ldots,\overline{P}_n)$ of all individuals' preferences is called a preference profile. An individual's preferences $\overline{P}_i$ must be an element of the set of admissible preferences $\overline{\mathcal{P}}$. The set of admissible preference profiles is consequently $\overline{\mathcal{P}}^n$, the n-fold cartesian product of $\overline{\mathcal{P}}$. If no restriction is placed a priori on individual preferences then $\overline{\mathcal{P}} = \pi$ and $\overline{\mathcal{P}}^n = \pi^n$ where π is the collection of all possible strict orderings on S. If some restriction is placed on individual preferences, then $\overline{\mathcal{P}}$ is a proper subset of π.

The group makes its choice among the elements of S by voting. Each individual casts a ballot P_i which is a strict order of the alternatives within S. The resulting ballot profile $P = (P_1,\ldots,P_n)$ is inserted into the voting procedure. A voting procedure is a single valued function v(P) whose argument is the ballot profile P and whose image is a single element of S. This image is defined to be the group's choice. Thus, given a ballot profile of P, $v(P) = x \in S$ is the group's choice. We assume that, for each alternative $x \in S$, an admissible ballot profile P exists such that $v(P) = x$.

The preference ordering P_i which an individual reveals as his ballot may or may not be an accurate statement of his true preferences $\overline{P}_i$. Any attempt through direct regulation to require him to reveal his true preferences is certain to fail because his true preferences are purely internal to him and thus unobservable. If individual i does reveal his true preferences, then $P_i \equiv \overline{P}_i$ and P_i is said to be his sincere strategy. If he misrepresents his true preferences, then $P_i \neq \overline{P}_i$ is an insincere strategy. A ballot profile $P \in P^n$ is called the sincere strategy profile if and only if $P \equiv \overline{P}$. An individual's ballot P_i must be an element of the set of admissible ballots $\mathcal{P}$. If no restriction is placed on ballots, then $\mathcal{P} = \pi$. If ballots are *a priori* restricted, then $\mathcal{P}$ is a proper subset of π.

A voting procedure is strategy-proof if it never gives any individual an incentive to employ an insincere strategy. Formally, individual i can manipulate the voting procedure v(P) at profile $P \in \mathcal{P}^n$ if and only if an insincere strategy $P_i' \in \mathcal{P}$ exists such that

$$v(P/P_i')\ \overline{P}_i\ v(P/P_i)$$

where P_i is his sincere strategy, $P/P_i = P = (P_1, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_n)$, and $P/P_i' = (P_1, \ldots, P_{i-1}, P_i', P_{i+1}, \ldots, P_n)$. In other words, individual i can manipulate v at profile P if and only if he prefers the outcome which he obtains by playing the insincere strategy P_i' to the outcome which he obtains by playing his sincere strategy $P_i = \overline{P}_i$. A voting procedure v(P) is strategy-proof if (a) $\overline{\mathcal{P}}^n \subset \mathcal{P}^n$ and (b) no sincere strategy profile $P \in \overline{\mathcal{P}}^n$ exists at which some individual $i \in N$ can manipulate the outcome. Requirement (b) states that if a voting procedure is strategy-proof, then no situation can occur where an individual has an incentive to employ anything but his sincere strategy. In other words, if a voting procedure is strategy-proof, then every individual's dominant strategy is to reveal his true preferences. Requirement (a) ensures that the sincere strategy of every individual -- no matter what his true preferences $\overline{P}_i \in \overline{\mathcal{P}}$ are --

is always admissible.

Strategy-proofness is difficult to achieve. Gibbard [5] and Satterthwaite [8] have independently proved an impossiblity theorem which states that if S contains at least three elements and neither admissible preferences nor admissible ballots are restricted, then every strategy-proof voting procedure is dictatorial. A dictatorial voting procedure is a voting procedure which vests all power in one individual, the dictator, i.e. individual $i \in N$ is dectator for $v(P)$ if and only if, for all $P = (P_1,\ldots,P_i,\ldots,P_n) \in \mathcal{P}^n$ and all $y \in S$, either $v(P) = y$ or $v(P)\ P_i\ y$. Formally stated, the theorem is:

> Theorem 1. If $|S| \geq 3$ and $\overline{\mathcal{P}}^n = \mathcal{P}^n = \pi^n$, then no strategy-proof voting procedure exists which is not dictatorial.

Proofs of this theorem, in addition to the original ones by Gibbard [5] and Satterthwaite [8], may be found in Schmeidler and Sonnenschein [9] and Blin and Satterthwaite [2].

3. Majority Rule with Borda Completion

Given a ballot profile $P = (P_1,\ldots,P_i,\ldots,P_n) \in \mathcal{P}^n$, majority rule with Borda completion calculates the group's choice as follows.[2] If a Condorcet winner exists, then it is the group's choice. A Condorcet winner is that element of S which defeats every other alternative within S on the basis of simple majority rule. If the voting paradox occurs, then no Condorcet winner exists. In such cases the Borda count is used as a secondary rule to make the group's choice determinate. The Borda count selects a winning alternative by assigning points: each alternative $x \in S$ receives $(|S|-k-1)$ points for each ballot P_i in which it is ranked k positions from the top. The points each alternative receives are summed and the winner is that alternative which receives the most points. Finally, if two alternatives receive the same number of points, then the ballot P_1 of individual one, who may be interpreted as the group's chairman, is used to break the tie. Let $v_M(P)$ denote this voting procedure.

Suppose, for example, that $|N| = 3$, $S = \{w,x,y,z\}$ and

$$\begin{aligned} P_1 &= (w\ x\ y\ z), \\ P_2 &= (x\ z\ y\ w), \\ P_3 &= (y\ w\ x\ z). \end{aligned} \tag{3.1}$$

where $P_1 = (w\ x\ y\ z)$ means that $w\ P_1\ x$, $w\ P_1\ y$, $w\ P_1\ z$, $x\ P_1\ y$, etc. In this case no majority winner exists: w has a majority over x, x has a majority over y, and y has a majority over w. The Borda count assigns five points to w (three from P_1, zero from P_2, and two from P_3), six points to x, five points to y, and two points to z. Therefore $v_M(P_1,P_2,P_3) = x$.

This example also illustrates that Theorem 1 is correct in stating that majority rule with Borda completion is not strategy-proof when admissible preferences and admissible revealed preferences are unrestricted. Suppose $P = (P_1,P_2,P_3)$ as defined by (3.1) represents the sincere strategies of individuals one, two, and three, i.e. $P = \overline{P}$. Individual three, who prefers that the group's choice be either y or w instead of x, can manipulate the group's decision by playing the insincere strategy $P_3' = (w\ y\ x\ z)$. This causes the outcome to become $v_M(\overline{P}_1, \overline{P}_2, \overline{P}_3{}') = w$ because the switch from $\overline{P}_3$ to $\overline{P}_3'$ breaks the cycle among w, x, and y and results in w being the majority winner. Therefore $v_M(\overline{P}/P_3') \ \overline{P}_3 \ v_M(\overline{P}/\overline{P}_3)$ i.e. v_M is not strategy-proof when $\mathcal{P} = \overline{\mathcal{P}} = \pi$.

4. Single-peakedness and Strategy-proofness.

The most intuitively appealing restriction on individuals' preferences is single-peakedness.[3] Formally:

Single-peakedness. The set of admissible preferences $\overline{\mathcal{P}}$ is single-peaked if and only if a strict ordering $Q \in \pi$ exists such that, for all triples x, y, $z \in S$, x Q y Q z implies that if $\overline{P} \in \overline{\mathcal{P}}$, then neither $x \overline{P} z \overline{P} y$ nor $z \overline{P} x \overline{P} y$. Single-peakedness of the admissible ballot set $\mathcal{P}$ is defined in the same manner, mutatis mutandis.

The reasonableness of the single-peakedness condition is supported by the following example: Consider individuals' preferences smong different sized pieces of steak. Let S = {1,2,3,4,5,6,...,32} where 1 represents a one ounce steak, 2 represents a two ounce steak, etc. Let Q order S as 1 Q 2 Q 3 ... 31 Q 32. If my most preferred steak is a seventeen ounce steak, then my preferences among fifteen ounce, nineteen ounce, and a twenty-three ounce steak could easily be any of the following:

$\overline{P}_1 = (15 \ 19 \ 23)$

$\overline{P}_1' = (19 \ 15 \ 23)$

$\overline{P}_1'' = (19 \ 23 \ 15)$

Therefore $\overline{P}_1$, $\overline{P}_1'$, $\overline{P}_1'' \in \mathcal{P}$. Preference ordering $\overline{P}_1^* = (23 \ 15 \ 19)$, however, makes no logical sense given that my most preferred steak is seventeen ounces. Therefore $\overline{P}_1^*$ may reasonably be excluded from the domain of admissible preferences, a conclusion which is consistent with the formal requirements of single-peakedness: 15 Q 19 Q 23 implies that neither $\overline{P}_1^* = (23\ 15\ 19)$ nor $\overline{P}_1^{**} = (15\ 23\ 19)$ are admissible. This example also explains why such preferences are said to be single-peaked. If we list the alternatives along a horizontal axis according to their "objective" order -- their weight in this case -- and indicate levels of preference by points on a vertical axis, then the resulting preference graphs have a single mode: they are single-peaked. Preferences for alternatives other than the most preferred one decrease monotonically from this maximum.

As we stated in the introduction, Pattanaik [7, Theorem 3] has shown that majority rule with Borda completion is strategy-proof if both admissible preferences and admissible ballots are restricted to be single-peaked. Formally:

Theorem 2. If $\mathcal{P}$ and $\overline{\mathcal{P}}$ are both single-peaked and $\mathcal{P} \equiv \overline{\mathcal{P}}$, then majority rule with Borda completion is a strategy-proof voting procedure.

Our question concerns the robustness of this result.

Specifically, does this strategy-proofness of majority rule with Borda completion depend critically on the single-peakedness of both $\overline{\mathcal{P}}$ and $\mathcal{P}$? Or in order to be strategy-proof, is it sufficient that only $\overline{\mathcal{P}}$ be restricted to be single-peaked? Our conclusion is negative. Admissible preferences and admissible ballots must both be single-peaked in order for majority rule with Borda completion to be strategy-proof. We can demonstrate this with a simple example.

Consider the case where admissible preferences are restricted but admissible ballots are not. Specifically suppose that $|N| = 3$, $S = \{w, x, y, z\}$, $\overline{\mathcal{P}}$ is single-peaked with respect to the strict ordering $Q = (w\ x\ y\ z)$, and $\mathcal{P} = \pi$. Let the profile of preferences $\overline{P} \in \overline{\mathcal{P}}^n$ be:

$$\overline{P}_1 = (w\ x\ y\ z)$$
$$\overline{P}_2 = (x\ y\ w\ z)$$
$$\overline{P}_3 = (z\ y\ x\ w).$$

Inspection shows that this profile is in fact single-peaked with respect to Q. If each individual plays his sincere strategy $P_i \equiv \overline{P}_i$, then the majority winner is x, i.e. $v_M(P) = v_M(\overline{P}) = x$.

This outcome, however, is not stable despite the single-peakedness of the sincere strategy profile $P \equiv \overline{P}$. Individual three can manipulate the outcome by changing his ballot from $P_3 = \overline{P}_3 = (z\ y\ x\ w)$ to $P_3' = (y\ z\ w\ x)$. This switch creates the voting paradox among alternatives w, x, and y which, when resolved by application of the Borda count, results in alternative y being chosen, i.e. $v_M(\overline{P}/P_3') = y$. Thus

$$v_M(\overline{P}/P_3')\ \overline{P}_3\ v_M(\overline{P}/\overline{P}_3),$$

i.e. individual three can manipulate v_M at the sincere strategy profile $\overline{P}$. Note, however, that this example does not violate Theorem 2. The ordering P_3', while admissible as his ballot, is not admissible as his preferences because it violates the single-peakedness requirement of $\overline{\mathcal{P}}$. Thus the mere occurrence of a single-peaked preference profile is not sufficient to guarantee strategy-proofness. Single-peakedness of the set of admissible ballots is also required.

This negative conclusion suggests a second, complementary question concerning the robustness of Theorem 2. In order to guarantee strategy-proofness is it sufficient to restrict the admissible ballot set to be single-peaked even while the admissible preference set $\overline{\mathcal{P}}$ is unrestricted? The answer is negative. If $\mathcal{P}$ is single-peaked while $\overline{\mathcal{P}}$ is unrestricted, then $\overline{\mathcal{P}} \not\subset \mathcal{P}$. A requirement of strategy-proofness is that $\overline{\mathcal{P}} \subset \mathcal{P}$; therefore the voting procedure can not be strategy-proof.

This particular argument that restriction of $\mathcal{P}$ is insufficient to guarantee strategy-proofness is not fully satisfactory because it depends critically on the formal requirement that $\overline{\mathcal{P}} \subset \mathcal{P}$. We can make a more satisfactory argument if we use Gibbard's concept [5] of a straightforward voting procedure. A voting procedure is straightforward if and only if each member's optimal choice of a ballot always depends only on his sincere preferences $\overline{P}_i$ and never depends on the other members' choice of ballots. In other words, straightforwardness removes the incentive which group members might otherwise have to react strategically to each others preferences and ballots. Formally, v(P) is <u>straightforward</u> if and only if, for any member $i \in N$ with any preferences $\overline{P}_i \in \overline{\mathcal{P}}$, a ballot $P_i' \in \mathcal{P}$ exists such that, for any ballot profile $P \in \mathcal{P}^n$, either $v(P/P_i') = v(P)$ or $v(P/P_i')\overline{P}_i\ v(P)$. Notice that straightforwardness includes strategy-proofness as a special case and does not require $\overline{\mathcal{P}} \subset \mathcal{P}$.

Given this concept, we can easily construct examples which show that majority rule with Borda completion is not straightforward when $|N| = 3$, $S = \{a,b,c\}$, $\overline{\mathcal{P}} = \pi$, and $\mathcal{P} = \{(a\ b\ c),$ $(b\ a\ c), (b\ c\ a), (c\ b\ a)\}$. Let member three have preferences

$\overline{P}_3 = (a\ c\ b) \in \pi$. Note that $\overline{P}_3 \notin \mathcal{P}$. Suppose first that the two other members cast ballots $P_1 = (a\ b\ c)$ and $P_2 = (c\ b\ a\ )$. Inspection shows that member three's unique optimal ballot is $P_3 = (a\ b\ c)$: $v_M\ (P_1, P_2, P_3) = a$, member three's most preferred alternative. Now consider the case where members one and two do not cast ballots P_1 and P_2, but rather cast ballots $P_1' = (c\ b\ a)$ and $P_2' = (b\ a\ c)$. In this changed situation if member three casts ballot $P_3 = (a\ b\ c)$, then the outcome is $v_M(P_1', P_2', P_3) = b$, his least preferred alternative according to his sincere preferences $\overline{P}_3$. He can do better by casting ballot $P_3' = (c\ b\ a) \in \mathcal{P}$: $v_M(P_1', P_2', P_3') = c$, his second most preferred alternative. Therefore member three's optimal ballot is dependent on the other members ballots and, consequently, v_M is not straightforward. Therefore by itself restriction of $\mathcal{P}$ is insufficient to guarantee either strategy-proofness or strategy-proofness's generalization, straightforwardness.

5. Conclusions

If, because of the substantive nature of the alternatives, the profile of true preferences is in fact single-peaked, then the problem which Arrow [1] raised with his impossibility theorem vanishes: majority rule suffices to order social elements of S into a transitive social ordering which satisfies Arrow's conditions for being an acceptable social welfare function. Unfortunately, as our examples show, the existence of a profile of single-peaked true preferences does not make the problem of manipulation vanish in like manner. This makes it difficult for a group to realize the potential for strategy-proof decisions which, as Theorem 2 shows, single-peaked preference profiles create. To achieve strategy-proofness the group must both realize that preferences are single-peaked and then, before it votes among the elements of S, agree to restrict individuals from casting ballots which are not single-peaked.

If a group considers a sequence of issues such that members' preferences over the alternatives contained within each issue are certain to be single-peaked, then the requirement for such agreement may be easily met. The group can make a single once and for all decision that requires the casting of single-peaked ballots. If, however, the sequence of issues which the group considers has more variety, then this becomes a very difficult requirement because a once and for all decision is inappropriate. The decision to restrict the set of admissible ballots must be made anew for each issue. But making the restriction of the admissible ballot set itself an issue is self-defeating. By voting strategically on the subsidiary question of whether to restrict or not to restrict the admissible ballot set individuals may successfully manipulate the group's final decision among the elements of S. The existence of this possibility is a prima-facie violation of the concept of strategy-proofness.

Footnotes

1. The strong assumptions (a) that the group contains an odd number of members and (b) that members' preferences are always strict orders are justifiable because we are showing a negative result in this note. If single-peaked preferences necessarily induce revelation of true preferences when the group has an odd number of members and each member's preferences are a strict order, then in general, when the number of members may be odd or even and preferences may be weak or strict, single-peakedness will not necessarily induce preference revelation.

2. For the history of this and other related voting procedures, see Fishburn [4].

3. Black [2] intoduced the concept of single-peakedness. For further discussion of it, see Arrow [1, ch. 7], Sen [10, chs. 10 and 10*], and Kramer [6].