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# A NOTE ON THE CORE OF A GAME WITHOUT SIDE-PAYMENTS*

by

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ABSTRACT

The present note considers a game without side-payments which allows for infinitely many agents. A sufficient condition is given for the game to have the non-empty core.

The non-emptiness of the core of a balanced game without side-payments has been demonstrated by two different approaches. One approach, originated by H. SCARF [1967], uses a considerably elaborate algorithm. This result was applied by Y. KANNAI [1969] to the case in which there is a countably infinite set of agents, but the game has an extension which is also an extension of a game generated by the finite subsets of agents. The other approach is due to H. SCARF in his unpublished paper and to J.-P. AUBIN reported in I. EKELAND [1974]; assuming convexity as well as balancedness, it reduces the problem to Kakutani's fixed-point theorem. The purpose of the present note is to apply the latter technique to the case where an infinite set of agents is given.

Let $(A, \mathcal{A}, \nu)$ be a $\sigma$-finite, non-negative measure space fixed throughout the present note; $A$ is a set, $\mathcal{A}$ is a $\sigma$-algebra of subsets of $A$, and $\nu$ is a $\sigma$-additive non-negative measure defined on $\mathcal{A}$ such that $A$ is a union of some sequence $\{E_n\}_n$ in $\mathcal{A}$ with $\nu(E_n) < \infty$. The intended interpretation of this space is that $A$ is the set of all agents, $\mathcal{A}$ is the family of all coalitions, and $\nu(S)$ is the fraction of the totality of agents belonging to the coalition $S$ for each $S \in \mathcal{A}$ if $\nu(A) = 1$, or $\nu(S)$ is the cardinality of $S$ for each $S \in \mathcal{A}$ (i.e., $\nu$ is the counting measure) if $A$ is a countable set. Denote by $L_1$ ($L_\infty$, resp.) the family of all equivalence classes of $\nu$-integrable scalar functions (the family of all equivalence classes of $\nu$-essentially bounded $\nu$-measurable scalar functions, resp.) on $A$, where two functions are equivalent if their difference is a $\nu$-null function. It is well-known that $L_\infty$ is the adjoint of $L_1$ endowed with the $\|\ \|_1$ - topology (see, e.g., N. DUNFORD and J.T. SCHWARTZ [1958, IV. 8.5, p.289]). Denote by $\langle \cdot , \cdot \rangle$ the natural pairing of $L_1$ and $L_\infty$; i.e., for every $p \in L_1$ and every $u \in L_\infty$, $\langle p,u \rangle \equiv \int_A u(a)\ p(a)\ d\nu(a)$. For every $S \in \mathcal{A}$, define $L_1^S \equiv \{p \in L_1 \mid p(a) = 0,\ \nu\text{-a.e. in } A\backslash S\}$, and define $L_\infty^S$ similary. An element $u \in L_\infty$ is interpreted as a utility allocation; an agent $a \in A$ enjoys his utility level $u(a)$ under the allocation $u$. A game is a correspondence,

$$V: \mathcal{A} \to L_\infty,$$

such that $V(S) \subset L_\infty^S$ for every $S \in \mathcal{A}$. The set $V(S)$ is interpreted as the set of all utility allocations the coalition $S$ can make for its members by its own effort, regardless of actions of agents outside $S$. The core of a game $V$ is the set of all utility allocations $u \in L_\infty$ such that (1) $u \in V(A)$ and (2) for any $S \in \mathcal{A}$ for which $\nu(S) > 0$ there exists no $u' \in V(S)$ such that $u'|_S > u|_S$, where $[u'|_S > u|_S$ $(u'|_S \geq u|_S$, resp.)] means $[u'(a) > u(a)$ $(u'(a) \geq u(a)$, resp.) for $\nu$-a.e. in $S]$. The condition (1) says that an allocation $u$ in the core is feasible, and (2) says that it is stable in the sense that no coalition with positive measure can improve upon it.

For every $S \in \mathcal{A}$, denote by $\chi_S$ its characteristic function: $A \to \mathbb{R}$; i.e., $\chi_S(a) = 1$ if $a \in S$, and $\chi_S(a) = 0$ if $a \notin S$. Denote by $\underset{\sim}{1}$ the constant map : $A \to \mathbb{R}$, $a \mapsto 1$. For a pairing $\langle E,F \rangle$, denote by $w(E,F)$ ($m(E,F)$, resp.) the weak $(E,F)$ topology (the Mackey topology for $E$, resp.); i.e., the family of all polars of finite subsets of $F$ is a local base for $w(E,F)$, and the family of all polars of $w(F,E)$-compact circled convex subsets of F is a local base for $m(E,F)$ (see J. L. KELLEY and I. NAMIOKA, et al. [1963, p.141, p.173]). The following assumptions are made in I. EKELAND [1974]:

(H) For every $S \in \mathcal{A}$, there exists a convex, $m(L_\infty, L_1)$-compact set $k(S)$ in $\{u \in L_\infty \mid u \geq 0\}$ such that $V(S) = k(S) - \{u \in L_\infty^S \mid u \geq 0\}$.

(B) Balancedness. For every finite sequence $\{\alpha_i, S_i\}$ in $\mathbb{R}_+ \times \mathcal{A}$ such that $\sum_i \alpha_i \chi_{S_i} \leq \underset{\sim}{1}$, it is true that $\sum_i \alpha_i V(S_i) \subset V(A)$.

Example. Let $\mathcal{S}$ be the space of all $C^\infty$, monotone, convex preferences on a consumption set in $\mathbb{R}^\ell$, endowed with the $C^\infty$ compact-open topology; it is a separable metric space (T. ICHIISHI [1974, Theorem 4.1]). An element of $\mathcal{S}$ is a $C^\infty$ map from the consumption set to the non-negative part of the unit sphere in $\mathbb{R}^\ell$, and is customarily called an indirect demand function. An economy is a probability measure defined on the Borel $\sigma$-algebra of subsets of $\mathcal{S} \times \mathbb{R}^\ell$. By A. MAS-COLELL [1972, Footnote 11] and T. ICHIISHI [1974, Corollary 5.2], the set of all elements in $\mathcal{S}$ that can be represented by concave utility functions is residual in $\mathcal{S}$. By T. ICHIISHI [1976, Theorem (1)], the set of all economies in which a preference can be represented by a concave function almost everywhere is residual in the space of all economies endowed with the weak topology. One is, therefore, justified to consider an economy consisting of economic agents who have concave utility functions. Then, the corresponding market game (without side-payments) satisfies convex-set-valuedness of V and balancedness.

Define $\Lambda \equiv \{p \in L_1 \mid p \geq 0\}$, and $P \equiv \{p \in L_1 \mid p > 0\}$. One can prove a straightforward:

Lemma 1. Let $\nu$ be a $\sigma$-finite, non-negative measure. Then, $P \neq \phi$

Proof of Lemma 1. Put $A = \bigcup_{n=1}^{\infty} E_n$, with $0 \leq \nu(E_n) < \infty$. Without loss of generality, one may assume $E_m \cap E_n = \phi$ if $m \neq n$. For every $n$ such that $\nu(E_n) = 0$, define $p(a) \equiv 1$ for every $a \in E_n$. If $\nu(E_n) > 0$, define $p(a) \equiv 2^{-n}/\nu(E_n)$ for every $a \in E_n$. Then, $p(a) > 0$ for every $a \in A$, and

$$\int_A |p| d\nu = \int_A p d\nu = \sum_{n:\nu(E_n) > 0} 2^{-n}.$$

The right-hand side is finite; therefore, $p \in P$.

Q.E.D.

For each $f, g \in P$, define $\Lambda(f,g) \equiv \{p \in \Lambda \mid f \leq p \leq g\}$.

The final assumption in the present note restricts a class of measure spaces $(A, \mathcal{O}, \nu)$. For $\tilde{u} \in L_\infty$, denote by $K(\tilde{u})$ the set $\{u \in L_\infty \mid 0 \leq u \leq \tilde{u}\}$.

(C) For every $f, g \in P$ and every $\tilde{u} \in L_\infty$, at least one of the following two conditions holds: (C.1) $\Lambda(f,g)$ is $\| \ \|_1$ - compact; or (C.2) $K(\tilde{u})$ is $m(L_\infty, L_1)$ -compact.

Remark. If $\nu(\{a\}) > 0$ for every $a \in A$, then both (C.1) and (C.2) are satisfied: One applies N. DUNFORD and J. T. SCHWARTZ [1958, IV.8.13, p. 295] to establish (C.1) from Lemma 2. To show (C.2), take any sequence $\{f^k\}_k$ in $L_1$ which converges to 0, with respect to $w(L_1,L_\infty)$. Again by N. DUNFORD and J. T. SCHWARTZ [1958, IV.8.13, p.295], $\{|f^k(\cdot)|\}_k$ converges to 0 with respect to $w(L_1,L_\infty)$; so, $0 \leq \lim_k \operatorname{Sup}_{u \in K(\tilde{u})} |\langle f^k, u\rangle| \leq \lim_k < |f^k(\cdot)|, \tilde{u} > = 0$. By A. GROTHENDIECK [1953, p.134], $K(\tilde{u})$ is $m(L_\infty,L_1)$ -conditionally compact. But by the same argument as in the proof of Lemma 2, $K(\tilde{u})$ is $w(L_\infty,L_1)$-closed, hence is $m(L_\infty,L_1)$ -closed.

Theorem. Let $(A,\mathcal{Q},\nu)$ be a $\sigma$-finite, non-negative measure space which satisfies (C). Suppose a game $V$ satisfies the conditions (H), and (B). Then, it has non-empty core.

Lemma 2. Let $\nu$ be a non-negative measure. Then $\Lambda(f,g)$ is $w(L_1,L_\infty)$ - compact for every $f, g \in P$.

Proof of Lemma 2. Let $\{p^k\}_k$ be a net in $\Lambda(f,g)$ which converges to $p^o \in L_1$, with respect to $w(L_1,L_\infty)$. Suppose there exists $T \in \mathcal{Q}$ with $\nu(T) > 0$ such that $p^o(a) < f(a)$ for every $a \in T$. Since $\chi_T \in L_\infty$, $\langle p^k, \chi_T\rangle \to \langle p^o, \chi_T\rangle$, so $\langle p^o, \chi_T\rangle \geq \langle f, \chi_T\rangle$; a contradiction. Therefore, $p^o \geq f$, and similarly $p^o \leq g$. Thus, $\Lambda(f,g)$ is $w(L_1,L_\infty)$ - closed.

Since $0 \leq p \leq g$ for every $p \in \Lambda(f,g)$, where $g \in L_1$, it is easy to check that the countable additivity of the integral $\int_E p\,d\nu$ is uniform with respect to $p \in \Lambda(f,g)$. By N. DUNFORD and J. T. SCHWARTZ [1958, IV. 8.9, p. 292], $\Lambda(f,g)$ is $w(L_1,L_\infty)$- sequentially compact.

But $w(L_1,L_\infty)$ - closedness and $w(L_1,L_\infty)$ - sequential compactness together imply $w(L_1,L_\infty)$ - compactness, by J. L. KELLEY and I. NAMIOKA, et. al. [1963, Theorem 17.12, p. 159].

Q.E.D.

Lemma 3. Let $\nu$ be a $\sigma$-finite, non-negative measure. Then, $\cup\{\Lambda(f,g) \mid f,g \in P\}$ is $\|\ \|_1$ - dense in $\Lambda$.

Proof of Lemma 3. Take any $p \in \Lambda$, and define $A_n \equiv \{a \in A \mid p(a) \geq 1/n\}$. Fix any $f \in P$ (which is possible by Lemma 1), and define $f_n \in P$ by $f_n(a) \equiv f(a)/n$ if $a \in A \backslash A_n$, $f_n(a) \equiv p(a)$ if $a \in A_n$. It is easy to show $\|f_n - p\|_1 \to 0$.

Q.E.D.

For every $p \in \Lambda$, define a function $\varphi(\cdot,p)\colon \mathcal{Q} \to \mathbb{R}$ by:

$$\varphi(S,p) \equiv \text{Max}\,\{\langle p,u\rangle \in \mathbb{R} \mid u \in k\ (S)\}.$$

Fix any $f, g \in P$ with $f \leq g$, any $p \in \Lambda(f,g)$, and any finite subset $\mathcal{F}$ of $\mathcal{Q}$ such that $A \in \mathcal{F}$. Let $d$ be the cardinality of $\mathcal{F}$. Consider a primary program:

$$M = \operatorname*{Inf}_{u \in L_\infty} \left\{ \langle p,u \rangle \;\middle|\; \begin{array}{l} \forall\, S \in \mathcal{F};\ \langle p^S,u \rangle \geq \varphi(S,p) \\ u \geq 0 \end{array} \right\}$$

where $p^S$ is the projection of $p$ into $L_1^S$. Note that for the given $\mathcal{F}$, the map $\{\langle p^S,\cdot\rangle \mid S \in \mathcal{F}\} : u \mapsto \{\langle p^S,u\rangle \mid S \in \mathcal{F}\}$ may be regarded as a linear transformation from $L_\infty$ to $\mathbb{R}^d$, $\{\varphi(S,p) \mid S \in \mathcal{F}\}$ is a member in $\mathbb{R}^d$. $\mathbb{R}^d$ is the adjoint of $\mathbb{R}^d$. For the adjoint transformation $B : \mathbb{R}^d \to L_1$ defined by:

$$\langle Bx,u \rangle = \sum_{S \in \mathcal{F}} \langle p^S,u \rangle\, x_S,$$

one can consider the dual program:

$$M' = \operatorname*{Inf}_{x \in \mathbb{R}^d} \left\{ -\sum_{S \in \mathcal{F}} \varphi(S,p)\, x_S \;\middle|\; \begin{array}{l} -Bx \geq -p \\ x \geq 0 \end{array} \right\}.$$

Note that all the conditions of J. L. KELLEY and I. NAMIOKA, et al. [1963, Theorem 21.1, pp. 199-200] are satisfied, so that the dual program is well defined. Moreover, it is easy to see:

$$Bx = \sum_{S \in \mathcal{F}} x_S\, p^S, \quad \text{for every } x \in \mathbb{R}^d.$$

One can define the sub-value $m'$ (with respect to $\|\ \|_1$) of the dual program by:

$$m' = \inf \underline{\lim}_{n} \left\{ -\sum_{S \in \mathcal{F}} \varphi(S,p)\, x_S^n \;\middle|\; \begin{array}{l} \{x^n,q^n\}_n \text{ is a sequence in } \mathbb{R}^d \times L_1 \text{ such that} \\ x^n \geq 0,\ q^n \geq 0 \text{ for every } n, \\ -Bx^n - q^n \text{ converges to } -p \text{ with respect to } \|\ \|_1. \end{array} \right\}$$

Since the dual program is consistent, $m' \leq M'$. A program is called convergent, if there is a feasible point which gives rise to the optimal value.

Lemma 4. Fix any $f, g \in P$ such that $f \leq g$. Assume (H) and (B). For every $p \in \Lambda(f,g)$ and every finite subset $\mathcal{F}$ of $\mathcal{A}$ such that $A \in \mathcal{F}$, the above dual program is convergent and $M' = m' = -\varphi(A,p)$.

Proof of Lemma 4. By the formula for $Bx$, $[p(a) > 0$, $\nu$-a.e. in $A]$ implies that for each feasible point $x \in \mathbb{R}^d$, $\{x_S, S\}_{S\in\mathcal{F}}$ is a finite sequence in $\mathbb{R}_+ \times \mathcal{A}$ such that $\sum_{S\in\mathcal{F};S\ni a} x_S \leq 1$, $\nu$ - a.e. in $A$, or $\sum_{S\in\mathcal{F}} x_S \chi_S \leq \underset{\sim}{1}$. By (H) and (B), $-\sum_{S\in\mathcal{F}} \varphi(S,p)\, x_S \geq -\varphi(A,p)$. On the other hand, the point $\bar{x} \in \mathbb{R}^d$, defined by $\bar{x}_A = 1$, $\bar{x}_S = 0$ for every $S \in \mathcal{F}\backslash\{A\}$, is feasible and it gives rise to the value $-\varphi(A,p)$. Therefore, $M' = -\varphi(A,p)$, and one only needs to show that $m' \geq -\varphi(A,p)$. Take a sequence $\{x^n, q^n\}$ in $\mathbb{R}^d \times L_1$ such that $x^n \geq 0$, $q^n \geq 0$, for every $n$, $\| \sum_{S\in\mathcal{F}} x^n_S\, p^S + q^n - p \|_1 \to 0$, and that $m' = -\lim_n \sum_{S\in\mathcal{F}} \varphi(S,p)\, x^n_S$. If $\nu(S) = 0$, then $\varphi(S,p) = 0$, so one may assume without loss of generality $x^n_S = 0$ for every $n$. If $\nu(S) > 0$, then for every $\varepsilon > 0$, $x^n_S \leq 1 + \varepsilon$ for all $n$ sufficiently large. The set $\bigcup_n \{x^n\}$ is, therefore, bounded in $\mathbb{R}^d$, and has a subsequence, still denoted by the superscript $n$, which converges to $x^o \in \mathbb{R}^d_+$. Since $q^n \geq 0$ for every $n$, $\sum_{S\in\mathcal{F}} x^o_S \chi_S \leq \underset{\sim}{1}$. Again by (H) and (B), $-\sum_{S\in\mathcal{F}} \varphi(S,p)\, x^o_S \geq -\varphi(A,p)$. But the left-hand side is precisely $m'$.

Q.E.D.

Proof of the Theorem. The 1st Step. Fix any $f, g \in P$ with $f \leq g$. Fix any $p \in \Lambda(f,g)$ and any finite subset $\mathcal{F}$ of $\mathcal{A}$ such that $A \in \mathcal{F}$, and consider the primary problem and its dual (mentioned in the paragraph prior to Lemma 4). Then, by Lemma 4 and R. J. DUFFIN (1956, Theorem 1, p.161], $M = \varphi(A,p)$. Later (in the 4th step), the primary problem will be shown to be convergent.

The 2nd Step. Fix any $S \in \mathcal{A}$. Then, the map $\varphi(S,\cdot): \Lambda \to \mathbb{R}$, $p \mapsto \varphi(S,p)$ is continuous in $(\Lambda, w(L_1,L_\infty))$. Indeed, let $X \equiv (\Lambda, w(L_1,L_\infty))$; $Y \equiv (k(S), m(L_\infty L_1))$; $\Psi(\cdot,\cdot): X \times Y \to \mathbb{R}$, $(p,u) \mapsto \langle p,u \rangle$; $\Gamma: X \to Y$, $p \mapsto Y$. Then $\Psi$ is continuous in $X \times Y$, because of the Mackey topology, $\Gamma(p)$ is non-empty and compact for every $p \in X$ by (H). So, by C. BERGE [1963, Theorem 1, p. 115; Theorem 2, p. 116], $\varphi(S,\cdot) \equiv \text{Max}\,\{\Psi(\cdot,u) \mid u \in \Gamma(\cdot)\}$ is continuous in X.

The 3rd Step. Fix any $S \in \mathcal{A}$ such that $\nu(S) > 0$. Note that $V(S)$ is $w(L_\infty,L_1)$-closed in $L_\infty^S$. Then, using N. DUNFORD and J. T. SCHWARTZ [1958, V.2. 10, p. 417], one can repeat the same argument as in I. EKELAND [1974, pp. 67-68], to establish:

$$V(S) = \{u \in L_\infty^S \mid \forall\, p \in \Lambda;\ \varphi(S,p) - \langle p,u \rangle \geq 0\}.$$

The 4th Step. By (H), there exists $\tilde{u} \in L_\infty$ with $\tilde{u} \geq 0$ such that $u \in V(A)$ implies $u \leq \tilde{u}$. By (B), for every $S \in \mathcal{A}$, $u \in V(S)$ implies $u \leq \tilde{u}$. Now, fix any $f, g \in P$ with $f \leq g$, and any finite subset $\mathcal{F}$ of $\mathcal{A}$ such that $A \in \mathcal{F}$. Then, there exists $p_{f,g} \in \Lambda(f,g)$ and $u_{f,g} \in K(\tilde{u})$ such that

$$\forall\, p \in \Lambda(f,g);\ \varphi(A,p) - \langle p,u_{f,g} \rangle \geq 0 \tag{i}$$

$$\forall\, S \in \mathcal{F};\ \langle p_{f,g}, u_{f,g}^S \rangle - \varphi(S,p_{f,g}) \geq 0 \tag{ii}$$

Indeed, the proof is precisely in the spirit of I. EKELAND [1974, pp. 70-72]: (A sketch of the proof is given here for the case where (C.2) is satisfied. A symmetric argument holds for the other case, since $K(\tilde{u})$ is $w(L_\infty, L_1)$-compact by N. DUNFORD and J. T. SCHWARTZ [1958, V.4.3, p. 424]). Define a correspondence $\Gamma': \Lambda(f,g) \to L_\infty$ by $\Gamma'(p) \equiv \{u \in L_\infty |\ u$ minimizes $\langle p,\cdot\rangle$ subject to: $\forall S \in \mathcal{F}$; $\langle p, u^S\rangle - \varphi(S,p) \geq 0,\ u \geq 0\}$, and define a correspondence $G: L_\infty \to \Lambda(f,g)$ by $G(u) \equiv \{p \in \Lambda(f,g) |\ p$ minimizes $\varphi(A,p) - \langle p,u\rangle$ on $\Lambda(f,g)\}$. Observe that $\Gamma'(p) \cap K(\tilde{u}) \neq \emptyset$ for every $p \in \Lambda(f,g)$. To see this, fix $p \in \Lambda(f,g)$, let $H$ be a generic member of the hyperplanes in $L_\infty$ determined by $p$, i.e., $\{u \in L_\infty |\ \langle p,u\rangle = c\}$, and define $C \equiv \{u \in L_\infty |\ \forall S \in \mathcal{F};\ \langle p^S,u\rangle - \varphi(S,p) \geq 0,\ u \geq 0\}$, and $C_S \equiv \{u \in C \cap K(\tilde{u}) |\ u|_{A\setminus S} = \tilde{u}|_{A\setminus S},\ \langle p^S,u\rangle \geq \varphi(S,p)\}$ for every $S \in \mathcal{F}$. The set of all points in $L_\infty$ that satisfy the constraint of $\Gamma'(p)$ is $C$. Note that $\tilde{u} \in C$; so, start with $H$ that passes through $\tilde{u}$. There are two possibilities: (iii) The hyperplane $H$ supports $C$; or (iv) otherwise. If (iii) is the case, $\tilde{u} \in \Gamma'(p)$. If (iv) is the case, shift $H$ downwards (i.e., decrease the value of $c$ in the definition of $H$). Then, $H$ will come to the place at which $H$ contains the relative boundary $\{u \in C_S |\ \langle p^S,u\rangle = \varphi(S,p)\}$ of $C_S$ for some $S \in \mathcal{F}$. Denote by $\mathcal{F}_1$ the set of all such $S$. Note that for every $u \in H \cap C_S$ with $S \notin \mathcal{F}_1$, $\langle p^S,u\rangle > \varphi(S,p)$. Again, there are two possibilities, (iii) or (iv). If (iii) is the case, any point $u$ in $C_S$ with $S \in \mathcal{F}_1$ such that $\langle p^S,u\rangle = \varphi(S,p)$ is in $\Gamma'(p)$. The possibility (iii) is the case only if there is $H$ further down such that any $u$ in $H \cap K(\tilde{u})$ for which $\langle p^S,u\rangle = \varphi(S,p)$ and $(u|_S, \tilde{u}|_{A\setminus S}) \in C_S$ is in $C$ for every $S \in \mathcal{F}_1$. Since $\#\mathcal{F} < \infty$, after a finite number of steps this process stops (in fact, it stops when the condition for $A$ becomes binding, in view of the 1st step), at which H supports C and $\Gamma'(p) \cap K(\tilde{u}) \neq \phi$. So, define $\Gamma(p) \equiv \Gamma'(p) \cap K(\tilde{u})$. Both $G(u)$ and $\Gamma(p)$ are both non-empty and convex for each $(p,u) \in \Lambda(f,g) \times L_\infty$. One can easily show that the graph of $G$ is closed in $(L_\infty, m(L_\infty,L_1)) \times (\Lambda(f,g), w(L_1,L_\infty))$. Also, using continuity of $\langle , \rangle$, the 1st step and the 2nd step, one can show closedness of the graph of $\Gamma$ in $(\Lambda(f,g), w(L_1 L_\infty)) \times (K(\tilde{u}), m(L_\infty,L_1))$. In view of Lemma 2 and (C.2), one can apply the generalized Kakutani's fixed-point theorem (see K. FAN [1952, Theorem 1, p. 122] or I. L. GLICKSBERG [1952, p. 177]) to the correspondence: $\Lambda(f,g) \times K(\tilde{u}) \to \Lambda(f,g) \times K(\tilde{u})$, $(p,u) \mapsto G(u) \times \Gamma(p)$, to obtain a fixed point $(p_{f,g}, u_{f,g})$ in $\Lambda(f,g) \times K(\tilde{u})$. By the 1st step, it satisfies (i) and (ii).

<u>The 5th Step</u>. For each $(f,g) \in P \times P$ with $f \leq g$, obtain a point $(p_{f,g}, u_{f,g}) \in \Lambda(f,g) \times K(\tilde{u})$ that satisfies (i) and (ii). Direct $P \times P$ by: $(f,g) \leq (f',g')$ if and only if $[f' \leq f$ and $g' \geq g]$. Since $K(\tilde{u})$ is $w(L_\infty,L_1)$ -compact, the net $\{u_{f,g}\}$ has a subnet which converges to some point $u_{\mathcal{F}} \in K(\tilde{u})$. From (i), for every $p \in \Lambda(f,g)$ and every $(f',g')$ that comes after $(f,g)$, $\varphi(A,p) - \langle p, u_{f',g'} \rangle \geq 0$, or $\varphi(A,p) - \langle p, u_{\mathcal{F}} \rangle \geq 0$. So, for every $p \in \bigcup_{f,g} \Lambda(f,g)$, $\varphi(A,p) - \langle p, u_{\mathcal{F}} \rangle \geq 0$. By Lemma 3 this inequality holds actually for every $p \in \Lambda$. Then, by the 3rd step, $u_{\mathcal{F}} \in V(A)$. Without loss of generality, one may assume $u_{\mathcal{F}} \in k(A)$. The condition (ii) says that for any $S \in \mathcal{F}$ there exists no $u' \in V(S)$ such that $u'|_S > u_{f,g}|_S$. Consequently, for any $S \in \mathcal{F}$, there exists no $u' \in V(S)$ such that $u'|_S > u_{\mathcal{F}}|_S$.

<u>The 6th Step</u>. Direct the family of all finite subsets of $\mathcal{Q}$ by inclusion (i.e., $\mathcal{F}$ comes before $\mathcal{F}'$ if $\mathcal{F} \subset \mathcal{F}'$). Then $\{u_{\mathcal{F}}\}$ is a net in $k(A)$. By (H) the net has a subnet which converges to $\bar{u} \in k(A)$ with respect to $m(L_\infty,L_1)$. It is straightforward to check that $\bar{u}$ is in the core of $V$.

Q. E. D.

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