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ON STRICT REGENERATION

by

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1. INTRODUCTION

Our object is to review the fundamental work of MAISONNEUVE [M1] in the light of its applications. In applications in the theories of Markov processes, semimarkov processes, etc., one is working on a large probability space which is not the canonical one on which [M] has worked. Moreover, the transformations from the given space to the canonical space of sawteeth functions of [M1] sometimes do not commute with the shift operators on the two spaces. For instance, in [M1;p. 167], the relation $\psi \circ \eta_t = \theta_t \circ \psi$ needed to prove the theorem on that page is not true; consequently, the interesting and important (and true) result that the local time constructed in [M1] for $\{t: X_t = x_0\}$ for a regular point x_0 of a standard Markov process X is the same as the usual local time at x_0 is left in doubt.

To remedy the situation we will work on one sample space furnished with one probability and one family of shifts. Our objective is to show that the results of [M1] hold in this setting. As such, the only differences from [M1] reside in technical aspects of the matter. Finally, we are incorporating into this numerous remarks and suggestions (and a new proof) on an earlier version which Maisonneuve has kindly communicated to us.

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2. DEFINITION OF REGENERATIVE SYSTEMS

Let $(\Omega,\underline{\underline{H}},P)$ be a complete probability space, and let $(\underline{\underline{H}}_t)$ be an increasing right continuous family of sub-σ-algebras of $\underline{\underline{H}}$ each $\underline{\underline{H}}_t$ containing the collection of all the negligible sets in $\underline{\underline{H}}$. Let (θ_t) be a family of shift operators on Ω such that

$$\theta_0\omega = \omega, \quad \theta_t\circ\theta_s(\omega) = \theta_{t+s}(\omega), \quad \theta_\infty(\omega) = \omega_\Delta$$

for every $\omega \in \Omega$, where ω_Δ is a distinguished point in Ω. Suppose that each θ_t is in $\underline{\underline{H}}_{s+t}/\underline{\underline{H}}_s$ for every s, i.e. $\theta_t^{-1}\underline{\underline{H}}_s \subset \underline{\underline{H}}_{s+t}$ for every s; consequently, $\theta_t \in \underline{\underline{H}}_\infty/\underline{\underline{H}}_\infty$. Finally, let $\underline{\underline{G}}$ be a sub-σ-algebra of $\underline{\underline{H}}$, and let $\mathbb{M}(\omega)$ be a Borel subset of $\mathbb{R}_+ = [0,\infty)$ for every $\omega \in \Omega$.

(2.1) DEFINITION. The collection $(\Omega,\underline{\underline{H}},\underline{\underline{H}}_t,\theta_t,\underline{\underline{G}},\mathbb{M},P)$ is said to be a strictly regenerative system provided that the following hold.

(2.2) Regularity. For every $\omega \in \Omega$, $\mathbb{M}(\omega)$ is right closed;

$$\mathbb{M}(\omega_\Delta) = \emptyset; \quad P\{\mathbb{M} = \emptyset\} < 1.$$

(2.3) Measurability. The mapping $(t,\omega) \to \theta_t\omega$ is in $\underline{\underline{R}}_+ \otimes \underline{\underline{H}}/\underline{\underline{H}}$. Considered as a subset of $\mathbb{R}_+\times\Omega$, the set $\mathbb{M}$ (whose section at ω is $\mathbb{M}(\omega)$) is in $\underline{\underline{R}}_+ \otimes \underline{\underline{G}}$, and is progressively measurable relative to $(\underline{\underline{H}}_t)$.

(2.4) Homogeneity. For every $\omega \in \Omega$ and $t \in \mathbb{R}_+$,

$$\mathbb{M}(\theta_t\omega) = (\mathbb{M}(\omega) - t) \cap \mathbb{R}_+.$$

(2.5) Regeneration. For every $Z \in b\underline{\underline{G}}$ and every stopping time T of $(\underline{\underline{H}}_t)$ such that $T(\omega) \in \mathbb{M}(\omega)$ for a.e. $\omega \in \{T < \infty\}$, the conditional expectation of $Z\circ\theta_T$ given $\underline{\underline{H}}_T$ is equal to a constant almost surely on $\{T < \infty\}$; moreover, this constant does not depend on T. $\square$

For any stopping time T of $(\underline{\underline{H}}_t)$, the mapping $\omega \to \theta_T\omega$ is the composition of the mappings $\omega \to (T(\omega),\omega)$ and $(t,\omega) \to \theta_t\omega$, which are in $\underline{\underline{H}}/\underline{\underline{R}}_+ \otimes \underline{\underline{H}}$ and $\underline{\underline{R}}_+ \otimes \underline{\underline{H}}/\underline{\underline{H}}$, the latter measurability being assumed in (2.3). Hence, $\theta_T \in \underline{\underline{H}}/\underline{\underline{H}}$, and the conditional expectation of $Z \circ \theta_T$ given $\underline{\underline{H}}_T$ is well defined. Regeneration assumption is that

$$E[Z\circ\theta_T|\underline{\underline{H}}_T] = E^o(Z) \qquad \text{a.s. on } \{T < \infty\} \tag{2.6}$$

for some constant $E^o[Z]$ whenever the stopping time T is such that $T \in \mathbb{M}$ a.s. on $\{T < \infty\}$.

It is clear that $Z \to E^o[Z]$ is an expectation operator. Let P^o be the corresponding probability measure; this is defined on $(\Omega,\underline{\underline{G}})$. The following proposition relates P^o to P by a direct formula. We define

$$S(\omega) = \inf \mathbb{M}(\omega), \qquad \omega \in \Omega, \tag{2.7}$$

that is, S is the starting point of $\mathbb{M}$.

(2.8) PROPOSITION. The stopping time S is such that $S(\omega) \in \mathbb{M}(\omega)$ for every $\omega \in \{S < \infty\}$. For any $\Lambda \in \underline{\underline{G}}$,

$$P^o(\Lambda) = P\theta_S^{-1}(\Lambda)/P\{S < \infty\}. \tag{2.9}$$

In particular,

$$P^o\{0 \in \mathbb{M}\} = 1. \tag{2.10}$$

PROOF. It is clear that S is a stopping time. If $S(\omega) < \infty$, either $S(\omega)$ is isolated and therefore must belong to $\mathbb{M}(\omega)$ in order to be $\inf \mathbb{M}(\omega)$, or else, if it is not isolated, $S(\omega) \in \mathbb{M}(\omega)$ since $\mathbb{M}(\omega)$ is right closed by the regularity assumption (2.2).

Let $\Omega_0 = \{0 \in \mathbb{M}\} = \{S = 0\}$. Note that

(2.11) $$\{S < \infty\} = \{S \in \mathbb{M}\} = \theta_S^{-1}\{0 \in \mathbb{M}\} \cap \{S < \infty\}.$$

By the regeneration property at S, we get that

$$P\{S < \infty\} = P\{S < \infty\}P^o\{0 \in \mathbb{M}\},$$

from which (2.10) follows, since $P\{S < \infty\} = 1 - P\{\mathbb{M} = \emptyset\} > 0$ by (2.2). In view of (2.10), we need to show (2.9) only for $\Lambda \in \underline{\underline{G}}$ such that $\omega_\Delta \notin \Lambda$. But then

$$P(\theta_S^{-1}\Lambda) = E[I_\Lambda \circ \theta_S;\ S < \infty] = P\{S < \infty\}P^o(\Lambda)$$

as desired. □

(2.12) EXAMPLE. Let $(\Omega, \underline{\underline{H}}, \underline{\underline{H}}_t, \theta_t, X_t, P^x)$ be a standard Markov process with state space $(E, \underline{\underline{E}})$, and let $x_0 \in E$ be a regular point for $\{x_0\}$. Let $\underline{\underline{G}} = \sigma(X_t,\ t \geq 0)$ and $\mathbb{M} = \{t\colon X_t = x_0\}$. Then, for any initial measure μ, the system $(\Omega, \underline{\underline{H}}, \underline{\underline{H}}_t, \theta_t, \underline{\underline{G}}, \mathbb{M}, P^\mu)$ is strictly regenerative. In this case, the probability P^o is the restriction of P^{x_0} to $\underline{\underline{G}}$. □

(2.13) EXAMPLE. Let $(\Omega, \underline{\underline{H}}, P)$ be a complete probability space and let (X_t) be a Chung process with (countable) state space E. Introduce the shifts θ_t such that $X_t \circ \theta_s = X_{t+s}$ for all t and s, and let $(\underline{\underline{H}}_t)$ be the natural history generated by (X_t), and set $\underline{\underline{G}} = \underline{\underline{H}}_\infty$. Let $i \in E$ be fixed and set $\mathbb{M} = \{t\colon X_t = i\}$. Then, $(\Omega, \underline{\underline{H}}, \underline{\underline{H}}_t, \theta_t, \underline{\underline{G}}, \mathbb{M}, P)$ is a strictly regenerative system. □

Let M be the indicator process of $\mathbb{M}$, that is, for every ω and t,

(2.14) $$M_t(\omega) = 1_{\mathbb{M}(\omega)}(t).$$

Define $\underline{\underline{F}}_t^o = \sigma(M_s;\ s \leq t)$, $\underline{\underline{F}}^o = \sigma(M_s;\ s \geq 0)$. Let $\underline{\underline{F}}$ be the completion of

$\underline{\underline{F}}^o$ with respect to P and P^o; (every negligible set $A \in \underline{\underline{F}}$ has the form $A = B \cup C$, where $B = A \cap \{S = 0\}$ is P^o-negligible, and $C = A \cap \{S > 0\}$ is P-negligible). Let $\underline{\underline{F}}_t$ be the σ-algebra generated by $\underline{\underline{F}}^o_{t+} = \bigcap_{s>t} \underline{\underline{F}}^o_s$ and all the negligible sets of $\underline{\underline{F}}$.

A stopping time T of $(\underline{\underline{H}}_t)$ is said to be a <u>regeneration time</u> if $T(\omega) \in \mathbb{M}(\omega)$ for P — a.e. $\omega \in \{T < \infty\}$. A stopping time T of $(\underline{\underline{F}}_t)$ is said to be a <u>recurrence time</u> if $T(\omega) \in \mathbb{M}(\omega)$ for P^o — a.e. $\omega \in \{T < \infty\}$.

It is possible that a stopping time T of $(\underline{\underline{F}}_t)$ be both a recurrence time and a regeneration time; for example, if $T(\omega) \in \mathbb{M}(\omega)$ for all $\omega \in \{T < \infty\}$, then this holds. However, in general, T can be a regeneration time and not a recurrence time, or vice versa. For example, suppose P is such that $\mathbb{M} = \{3,5,7,9,\dots\}$ almost surely; then, $P^o\{\mathbb{M} = \{0,2,4,6,\dots\}\} = 1$. Now, $T = 5$ is a regeneration time but not a recurrence time; $T = 2$ is a recurrence time but not a regeneration time.

3. REGENERATION UNDER P^o.

Throughout this section $(\Omega, \underline{\underline{H}}, \underline{\underline{H}}_t, \theta_t, \mathbb{M}, P)$ is a fixed regenerative set, and P^o, $\underline{\underline{F}}_t$, $\underline{\underline{F}}$, etc. are as defined in Section 2.

(3.1) PROPOSITION. If T is a regeneration time and U is a recurrence time, then $T + U \circ \theta_T$ is a regeneration time.

PROOF. Let T and U be as described. We need to show that

$$a = P\{M_{T+U\circ\theta_T} = 0, \quad T + U\circ\theta_T < \infty\}$$

is equal to zero. Since $M_{T+U\circ\theta_T} = M_U \circ \theta_T$,

$$a = P\{M_U \circ \theta_T = 0, \quad U \circ \theta_T < \infty; \ T < \infty\}.$$

Since U is a stopping time of $(\underline{\underline{F}}_t)$, the random variables M_U and U are in $\underline{\underline{F}}$. Hence, we may apply the regeneration property at T to get

$$a = P\{T < \infty\}P^o\{M_U = 0, \ U < \infty\}.$$

Since U is a recurrence time the last term is zero. □

The following shows that the regeneration property holds also under P^o.

(3.2) PROPOSITION. Let T be a recurrence time. Then, for any $Z \in b\underline{\underline{F}}$,

$$E^o[Z \circ \theta_T | \underline{\underline{F}}_T] = E^o[Z] \qquad P^o - \text{ a.s. on } \{T < \infty\}.$$

PROOF. By Proposition (2.8), $0 \in \mathbb{M}$ a.s. (P^o); hence, it is enough to show that

$$E^o[Y \cdot Z \circ \theta_T] = E^o[Y]E^o[Z] \tag{3.3}$$

for every $Y \in b\underline{\underline{F}}_T$ such that $Y = 0$ on $\{T < \infty\} \cup \{\omega_\Delta\}$. By formula (2.9),

$$(3.4) \qquad \ell hs(3.3) = E[Y \circ \theta_S \cdot Z \circ \theta_U]/P\{S < \infty\}$$

where $U = S + T \circ \theta_S$. Since S is a regeneration time, by the preceding proposition, U is a regeneration time. Note that $Y \circ \theta_S \in b\underline{\underline{F}}_U$ and that $Y \circ \theta_S = 0$ on $\{U = \infty\} = \{S = \infty\} \cup (\{T \circ \theta_S = \infty\} \cap \{S < \infty\})$. Hence, by the regeneration property (2.5) applied at U,

$$(3.5) \qquad rhs(3.4) = E^o[Z]E[Y \circ \theta_S;\ U < \infty]/P\{S < \infty\}.$$

Finally, noting that $Y \circ \theta_S \cdot I_{\{U < \infty\}} = Y \circ \theta_S \cdot I_{\{S < \infty\}}$ since $Y = 0$ $\{T = \infty\}$, we have

$$(3.6) \qquad E[Y \circ \theta_S;\ U < \infty] = P\{S < \infty\}E^o[Y]$$

by the regeneration property applied at S. Now (3.3) follows from (3.4), (3.5) and (3.6). □

The preceding proposition together with (2.10) imply the following zero-one law.

(3.7) PROPOSITION. For any $A \in \underline{\underline{F}}_0$, $P^o(A) = 0$ or 1.

PROOF. Let T be 0 on A and $+\infty$ on $\Omega \setminus A$. Then T is a stopping time of $(\underline{\underline{F}}_t)$. Since $0 \in \mathbb{M}$ a.s.(P^o), T is a recurrence time. Since $I_A = I_A \cdot I_A \circ \theta_T$, by the preceding proposition, $P^o(A) = P^o(A)P^o(A)$. □

The following consequence of this zero-one law separates the theory into two very different cases. Below, by discrete we mean that there are no finite accummulation points; and by perfect we mean that every point is an accummulation point.

(3.8) THEOREM. Either $\mathbb{M}$ is almost surely discrete or else $\mathbb{M}$ is almost surely perfect.

PROOF. Let $R = \inf \mathbb{M} \cap \mathbb{R}_0 = \inf\{t > 0: t \in \mathbb{M}\}$. The event $\{R > 0\}$ is in $\underline{\underline{F}}^o_{0+}$, and hence, $P^o\{R > 0\}$ is 0 or 1 by the zero-one law (3.7).

Suppose $P^o\{R > 0\} = 1$. Define

$$(3.9) \qquad R_0 = S, \quad R_{n+1} = R_n + R\circ\theta_{R_n}, \qquad n = 0,1,\ldots.$$

Since $\mathbb{M}$ is right closed R is both a regeneration time and a recurrent time. It follows from Proposition (3.1) that each R_n is a regeneration time. By the regeneration property applied at R_n, we obtain

$$(3.10) \quad P\{R \circ \theta_{R_n} \in B | \underline{\underline{H}}_{R_n}\} = P^o\{R \in B\} \qquad \text{on } \{R_n < \infty\},$$

for every Borel $B \subset \overline{\mathbb{R}}_+$. Hence, (R_n) is a renewal process under P, which implies by the well known results that $\left(\bigcup_n [R_n]\right) \cap [0,t]$ is finite for every t. It is clear that $\mathbb{M}$ is simply $\bigcup_n [R_n]$ in this case and we are finished in showing that $\mathbb{M}$ is discrete if $P^o\{R > 0\} = 1$.

Next suppose that $P^o\{R = 0\} = 1$. Let $\mathbb{D}$ be the set of all right end points of the intervals contiguous to $\mathbb{M}$. Since every point of $\mathbb{M}\setminus\mathbb{D}$ is a point of left-accummulation of $\mathbb{M}$ by the definition of $\mathbb{D}$, it is sufficient to show that every point of $\mathbb{D}$ is a point of right accummulation of $\mathbb{M}$ in order to show that $\mathbb{M}$ is perfect.

Let $R_{\varepsilon n}$ be the right extremity of the n^{th} interval amongst those contiguous intervals whose lengths exceed ε. Then, each $R_{\varepsilon n}$ is a stopping time, and

$$\mathbb{D} = \bigcup_{n,\varepsilon} [R_{\varepsilon n}]$$

where the union is over all integers n and rationals ε. In view of this, it is enough to show that each $R_{\varepsilon n}$ is a point of right accummulation of $\mathbb{M}$. But this follows easily: each $R_{\varepsilon n}$ is a regeneration time and

$$(3.11) \qquad P\{R \circ \theta_{R_{\varepsilon n}} = 0 | \underline{\underline{H}}_{R_{\varepsilon n}}\} = P^{o}\{R = 0\} = 1$$

a.s. on $\{R_{\varepsilon n} < \infty\}$. □

(3.12) REMARK. The preceding theorem remains the same with respect to the measure P^{o} also. A direct proof follows, mutadis mutandis, the proof above except that in (3.10) and (3.11) we need to replace P by P^{o} and $\underline{\underline{H}}$ by $\underline{\underline{F}}$.

The structure of a discrete regeneration set is simply that of a renewal process. We will, for this reason, concentrate on the perfect case in the remainder of this note.

4. STRUCTURE OF PERFECT REGENERATION SETS

Throughout this section $(\Omega, \underline{\underline{H}}, \underline{\underline{H}}_t, \theta_t, \mathbb{M}, P)$ is a regenerative set, P^o, $\underline{\underline{F}}$, $\underline{\underline{F}}_t$, etc. are as defined earlier, and $\mathbb{M}$ is <u>perfect</u>. We may, and do, suppose that $\mathbb{M}(\omega)$ is perfect for every ω. We define R as before and set $N_t = t + R \circ \theta_t$, that is,

$$(4.1) \qquad R = \inf\{t > 0: t \in \mathbb{M}\}, \qquad N_t = \inf\{s > t: s \in \mathbb{M}\}.$$

Then, N_t is the time of next regeneration after t. Note that $R = N_0$, and that R and N_t are both recurrence times and regeneration times, indeed, $R(\omega) \in \mathbb{M}(\omega)$ on $\{R < \infty\}$ and $N_t(\omega) \in \mathbb{M}(\omega)$ on $\{N_t < \infty\}$.

(4.2) PROPOSITION. The process (N_t) is right continuous and quasi-left-continuous.

PROOF. Right continuity is immediate. To show the quasi-left-continuity, let (T_n) be a sequence of stopping times of $(\underline{\underline{H}}_t)$ such that $T_n \uparrow T$. We need to show that $N_{T_n} \uparrow N_T$ a.s. This can fail only if $T_n(\omega) < T(\omega)$ for all n and $T(\omega)$ is the left end point of a contiguous interval of $\mathbb{M}(\omega)$ for a ω set of positive measure. Thus, all we need to show is that

$$a = P\{R \circ \theta_T > \varepsilon;\ T_n < T \quad \text{for all } n,\ T_n \uparrow T\}$$

is zero for every $\varepsilon > 0$. On the set in question, $N_{T_n} < T$ for every n; and hence for any $\delta > 0$,

$$a \le \lim_n P\{R \circ \theta_T > \varepsilon;\ N_{T_n} < T,\ N_{T_n} > T - \delta\}$$

$$\le \lim_n P\{Q_\varepsilon \circ \theta_{N_{T_n}} \le \varepsilon + \delta\}$$

where Q_ε is equal to ε plus the left end point of the first contiguous interval whose length exceeds ε. By the regeneration property applied at N_{T_n}, we see that

$$a \leq P\{N_{T_n} < \infty\} P^o\{Q_\varepsilon \leq \varepsilon + \delta\}, \tag{4.3}$$

and $P^o\{Q_\varepsilon \leq \varepsilon + \delta\} \to 0$ as $\delta \to 0$ because of the perfectness of $\mathbb{M}$ coupled with the fact that $0 \in \mathbb{M}$ a.s. (P^o). □

(4.4) REMARK. Proposition (4.2) is true under P^o also. Therefore, whenever $T_n \nearrow T$, $N_{T_n} \nearrow N_T$ almost surely (P) and almost surely (P^o).

(4.5) PROPOSITION. There is an increasing continuous process (B_t) adapted to $(\underline{\underline{F}}_t)$ such that the support of the measure dB_t is indistinguishable from $\bar{\mathbb{M}}$ under P^o.

PROOF. Let $A_t = 1 - \exp(-N_t)$; and let (B_t) be the predictable projection of (A_t) relative to $(\underline{\underline{F}}_t, P^o)$. That is, (B_t) is an increasing predictable process such that

$$E^o\left[\int_0^\infty Z_s dB_s\right] = E^o\left[\int_0^\infty Z_s dA_s\right] \tag{4.6}$$

for every $(\underline{\underline{F}}_t, P^o)$-predictable process (Z_t).

By the preceding proposition, strengthened by Remark (4.4), $A_{T_n} \nearrow A_T$ a.s. (P^o) whenever a sequence (T_n) of stopping times of $(\underline{\underline{F}}_t)$ increases to T. By (4.6) this implies that (B_t) does not charge any predictable times, and hence, (B_t) must be continuous.

Let $\bar{\mathbb{K}}$ be the support of dB_t; that is,

$$\bar{\mathbb{K}} = \{t \geq 0 : B_{t+\varepsilon} - B_{t-\varepsilon} > 0 \quad \text{for all } \varepsilon > 0\} \tag{4.7}$$

where, for the purposes of this definition $B_{-u} = 0$ for all $u > 0$. Now, define

$$Q_t = \inf\{s > t : B_s > B_t\}. \tag{4.8}$$

Then, each Q_t is a stopping time, and we have

$$\bar{K} = \overline{\bigcup_{\substack{t \\ \text{rational}}} [Q_t]}; \qquad \bar{\mathbb{M}} = \overline{\bigcup_{\substack{t \\ \text{rational}}} [N_t]}. \tag{4.9}$$

Hence, to complete the proof, it is sufficient to show that

$$Q_t = N_t \quad \text{a.s. } (P^o), \quad t \geq 0. \tag{4.10}$$

Taking $Z = I_{(t,N_t]}$ in (4.6), we obtain

$$E^o[B_{N_t} - B_t] = E^o[A_{N_t} - A_t] = 0,$$

since $A_{N_t} = A_t$ by the fact that $N_{N_t} = N_t$ because of the perfectness of $\mathbb{M}$. Hence, $B_t = B_{N_t}$ a.s. (P^o), which implies that $N_t \leq Q_t$ a.s. (P^o). Next, taking $Z = I_{(N_t,Q_t]}$ in (4.6), we obtain that

$$A_{N_t} = A_{Q_t} \quad \text{a.s. } (P^o) \tag{4.11}$$

since $B_t = B_{N_t} = B_{Q_t}$ by the definition (4.8) of Q_t and the continuity of B. Since $A_{N_t} = A_t$, (4.11) implies that $N_t = Q_t$ a.s. (P^o) thus proving (4.10). $\square$

The next proposition shows that there is a local time for $\mathbb{M}$ which is further additive.

(4.12) PROPOSITION. There exists an increasing continuous process $\mathring{L}$ adapted to $(\underline{\underline{F}}_t)$ such that the support of $d\mathring{L}_t$ is P^o-indistinguishable from $\bar{\mathbb{M}}$, and that for every recurrence time T and all $t \geq 0$

$$\mathring{L}_{T+t} = \mathring{L}_T + \mathring{L}_t \circ \theta_T \quad \text{a.s. } (P^o). \tag{4.13}$$

We shall need the following lemma in the proof.

(4.14) LEMMA. Let t be fixed, and suppose that the random variables $Z, Z' \in b\underline{\underline{F}}$ satisfy

$$E^o[Z|\underline{\underline{F}}_t] = E^o[Z'|\underline{\underline{F}}_t].$$

Then, for any recurrence time T,

$$E^o[Z \circ \theta_T|\underline{\underline{F}}_{T+t}] = E^o[Z' \circ \theta_T|\underline{\underline{F}}_{T+t}] \quad \text{a.s. } (P^o) \text{ on } \{T < \infty\}.$$

PROOF. Let $X \in b\underline{\underline{F}}_T$ and $Y \in b\underline{\underline{F}}^o_t$, and suppose that $X = 0$ on $\{T = \infty\}$. Then, by Proposition (3.2),

$$E^o[X \cdot Y \circ \theta_T \cdot Z \circ \theta_T] = E^o[X]E^o[YZ],$$

$$E^o[X \cdot Y \circ \theta_T \cdot Z' \circ \theta_T] = E^o[X]E^o[YZ'].$$

By the hypothesis concerning Z and Z', $E^o[YZ] = E^o[YZ']$. Hence, the left-hand sides of the two expressions above are equal. This completes the proof since the random variables of form $X \cdot Y \circ \theta_T$ with $X \in b\underline{\underline{F}}_T$ and $Y \in b\underline{\underline{F}}^o_t$ generate $\underline{\underline{F}}_{T+t}$ up to negligible sets. □

(4.15) PROOF of (4.12). Let B be as in Proposition (4.5) and define

$$\mathring{L}_t = \int_0^t e^s \, dB_s. \tag{4.16}$$

All the assertions are immediate from the properties of B except for (4.13).

Let T be a recurrence time. Since $Z = A_\infty - A_t = \exp(-N_t)$ and $Z' = B_\infty - B_t = \int_t^\infty e^{-s}\, d\mathring{L}_s$ satisfy the hypothesis of the preceding lemma, we have

$$(4.17) \qquad E^o[\exp(-N_t \circ \theta_T)\,|\,\underline{\underline{F}}_{T+t}] = E^o\Big[\int_t^\infty e^{-s}\, d(\mathring{L}_s \circ \theta_T)\,|\,\underline{\underline{F}}_{T+t}\Big]$$

a.s. (P^o) on $\{T < \infty\}$. Now consider the supermartingale $(X_{T+t})_{t \geq 0}$ defined on $(\Omega, \underline{\underline{F}}, P^o)$ (adapted to $\underline{\underline{F}}_{T+t})_{t \geq 0})$ by

$$X_{T+t} = E^o[\exp(-N_{T+t})\,|\,\underline{\underline{F}}_{T+t}].$$

Since $\mathbb{M}$ is perfect and T is a recurrence time, $N_{T+t} = T + N_t \circ \theta_T$ a.s.(P^o). Thus, by (4.17), P^o – a.s.,

$$(4.18) \qquad X_{T+t} = e^{-T} E^o\Big[\int_t^\infty e^{-s} d(\mathring{L}_s \circ \theta_T)\,|\,\underline{\underline{F}}_{T+t}\Big].$$

On the other hand, by the definitions of X_{T+t} and B,

$$(4.19) \qquad X_{T+t} = E^o\Big[\int_{T+t}^\infty e^{-s} d\mathring{L}_s\,|\,\underline{\underline{F}}_{T+t}\Big]$$

$$= e^{-T}\, E^o\Big[\int_t^\infty e^{-s} d(\mathring{L}_{T+t} - \mathring{L}_T)\,|\,\underline{\underline{F}}_{T+t}\Big]$$

a.s. (P^o) again. It follows from the uniqueness of the increasing process generating a supermartingale that the processes

$$(\mathring{L}_t \circ \theta_T)_{t \geq 0}, \qquad (\mathring{L}_{T+t} - L_T)_{t \geq 0}$$

are P^o-indistinguishable on $\{T < \infty\}$. This was the desired result. □

(4.20) REMARK. The negligible set in (4.13) depends on T but not on t.

Therefore t can be replaced by random variables.

Note that $\mathring{L}$ is left undefined on $\Omega \setminus \{S = 0\} = \Omega \setminus \{0 \in \mathbb{M}\}$ since $0 \in \mathbb{M}$ a.s. (P^o). We now define a local time which extends $\mathring{L}$ to all of Ω. Define, for every $\omega \in \Omega$ and $t \geq 0$,

$$(4.21) \qquad L_t(\omega) = \begin{cases} 0 & \text{if } t < S(\omega), \\ \mathring{L}_{t-S(\omega)}(\theta_{S(\omega)}\omega) & \text{if } t \geq S(\omega). \end{cases}$$

Since $\theta_S\omega$ is in $\{0 \in \mathbb{M}\}$ on which $\mathring{L}$ is well defined, (4.21) makes good sense. Note that $L = \mathring{L}$ on $\{0 \in \mathbb{M}\}$.

(4.22) PROPOSITION. The process (L_t) is continuous, increasing, and adapted to $(\underline{\underline{F}}_t)$. The support of the measure (dL_t) is indistinguishable (with both P and P^o) from $\overline{\mathbb{M}}$.

PROOF. The first statement is immediate from (4.21) and the similar properties of $\mathring{L}$. To show the second, first note that the support of (dL_t) is the set $S + \overline{\mathbb{K}}\circ\theta_S$ where $\overline{\mathbb{K}}(\omega)$ is the support of $d\mathring{L}_t(\omega)$ for $\omega \in \{0 \in \mathbb{M}\}$. On the other hand, $\overline{\mathbb{M}} = S + \overline{\mathbb{M}}\circ\theta_S$. Under P^o, $L = \mathring{L}$ and $P^o\{\overline{\mathbb{K}} \neq \mathbb{M}\} = 0$ by (4.12). Now, by the regeneration property applied at S,

$$P\{S + \overline{\mathbb{K}} \circ \theta_S \neq S + \overline{\mathbb{M}} \circ \theta_S\} = P^o\{\overline{\mathbb{K}} \neq \overline{\mathbb{M}}\}P\{S < \infty\} = 0,$$

as required. $\square$

Define

$$(4.23) \qquad S_t(\omega) = \inf\{s \geq 0 \colon L_s(\omega) > t\}.$$

Then, each S_t is both a recurrence time and a regeneration time. Note that $S_0 = S$, and that

(4.24) $$S_t = S + S_t \circ \theta_S.$$

(4.25) PROPOSITION. For every t and u,

(4.26) $$S_{t+u}(\omega) = S_t(u) + S_u \circ \theta_{S_t}(\omega) \quad \text{a.e. } \omega$$

both for P and P^o. The process (S_t) is a strictly increasing (right continuous) Lévy process.

PROOF. Since S_t is a recurrence time, the definition (4.23) together with the additivity result (4.12) imply that (4.26) is true for P^o - almost every ω. This fact together with (4.24) imply that (4.26) holds for P - a.e. ω also.

It follows from (4.26) and the regeneration property applied at the regeneration (resp. recurrence) time S_t that the process (S_t) has stationary and independent increments under P (resp. under P^o). It follows from Lebesgue's theorem on time changes (see DELLACHERIE, p. 91) that (S_t) is strictly increasing and right continuous because (L_t) is continuous and increasing. □

(4.27) THEOREM. Let $\mathbb{L}$ be the set of all left extremities of the intervals contiguous to $\mathbb{M}$. Then, $\mathbb{M}\setminus\mathbb{L} = \overline{\mathbb{M}}\setminus\mathbb{L}$ is indistinguishable (both under P and P^o) from the image $\mathbb{K} = \{s\colon S_t = s \text{ for some } t\}$ of an increasing Lévy process (S_t).

PROOF. Let (S_t) be as before. Then $\mathbb{K}$ is equal to the support $\overline{\mathbb{K}}$ of (dL_t) less the left extremities of the intervals contiguous to $\overline{\mathbb{K}}$. Thus, $\mathbb{M}\setminus\mathbb{L}$ is precisely $\mathbb{K}$ except on the set $\{\overline{\mathbb{K}} \neq \overline{\mathbb{M}}\}$, which is of measure 0 as was shown in Proposition (4.22). □

We next regularize the paths of S such that the additivity property (4.26) holds for all ω and t and u. We use the procedure due to WALSH for this purpose. Write $\hat{\theta}_t$ for θ_{S_t}, and let

$$S'_t(\omega) = \operatorname{ess}\limsup_{u \searrow 0} S_{t-u}(\hat{\theta}_u\omega) + S(\omega), \qquad t > 0, \tag{4.28}$$

$$S'_0(\omega) = \operatorname{ess}\limsup_{t \searrow 0} S'_t(\omega). \tag{4.29}$$

To apply the procedure of WALSH, we note that

a) S'_t is equal to S_t almost surely,

b) and that the mapping $(u,\omega) \to S_{t-u}(\hat{\theta}_u\omega)$ is $\lambda \times P^o$ and $\lambda \times P$ measurable, where λ is the Lebesgue measure on $\mathbb{R}$; (we use the fact that S is a Lévy process for this second point). It follows that S' is indistinguishable (P and P^o) from S and that for almost every ω,

$$S'_{t+u}(\omega) = S'_t(\omega) + S_u(\hat{\theta}_t\omega) \tag{4.30}$$

for all t and u. Moreover, we may remove the exceptional set altogether to have (4.30) for all t, u, and ω.

We now define

$$L'_t(\omega) = \inf\{u: S'_u(\omega) > t\}. \tag{4.31}$$

(4.32) THEOREM. The process (L'_t) is indistinguishable from (L_t). Moreover, for almost every ω (P or P^o),

$$L'_{t+s}(\omega) = L'_t(\omega) + L'_s(\theta_t\omega) \tag{4.33}$$

for all t and s.

PROOF. The process (L_t) is related to (S_t) through the formula (4.31) with primes deleted. Hence, L' is indistinguishable from L since S' is indistinguishable from S. To show (4.33), first take ω in the set $\{S = S'\}$ and let t be a point of right increase for $L'(\omega)$. Then,

$$(4.34) \qquad S'_{L'_t(\omega)}(\omega) = t, \qquad \hat{\theta}_{L'_t(\omega)}(\omega) = \theta_{S_{L'_t(\omega)}(\omega)}(\omega) = \theta_t\omega.$$

Now, using (4.31) and the additivity (4.30), we obtain

$$L'_{t+s}(\omega) = L'_t(\omega) + L'_s(\hat{\theta}_{L'_t(\omega)}\omega),$$

which is the same as (4.33) in view of (4.34). Still for $\omega \in \{S = S'\}$, if t is not a point of right increase, letting $Q_t(\omega) = u$ be the first time $L'(\omega)$ increases after t, we get $L'_t(\omega) = L'_u(\omega)$. If $u \geq t+s$, then $L'_{t+s}(\omega) = L'_u(\omega)$ and $L'_s(\theta_t\omega) = 0$ so we are finished. If $u < t+s$, then

$$\begin{aligned} L'_{t+s}(\omega) &= L'_{u+(t+s-u)}(\omega) \\ &= L'_u(\omega) + L'_{t+s-u}(\theta_u\omega) = L'_t(\omega) + L'_{t+s-u}(\theta_u\omega); \end{aligned}$$

and since $u-t$ is a point of right increase for $L'(\theta_t\omega)$,

$$\begin{aligned} L'_s(\theta_t\omega) &= L'_{(u-t)+(t+s-u)}(\theta_t\omega) \\ &= L'_{u-t}(\theta_t\omega) + L'_{t+s-u}(\theta_t\theta_{u-t}\omega) \\ &= 0 + L'_{t+s-u}(\theta_u\omega). \end{aligned}$$

So, $L'_{t+s}(\omega) = L'_t(\omega) + L'_s(\theta_t\omega)$ again. □

We summarize this section in the following

(4.35) MAIN THEOREM. There is an increasing continuous adapted process (L_t) such that for a.e. ω

$$L_{t+s}(\omega) = L_t(\omega) + L_s(\theta_t\omega)$$

for all t and s, and

$$\mathbb{K}(\omega) = \{t: L_{t+\varepsilon}(\omega) - L_t(\omega) > 0 \quad \text{for all } \varepsilon > 0\}$$

is exactly equal to $\mathbb{M}(\omega)\setminus\mathbb{L}(\omega)$. Moreover, $\mathbb{K}(\omega)$ is the range of the path $S(\omega)$ of a process (S_t) which is a (right continuous) strictly increasing Lévy process. This (S_t) and (L_t) are the inverses of each other.

Going back to Example (2.12) we now can give the following application.

(4.36) PROPOSITION. Let $(\Omega,\underline{\underline{H}},\underline{\underline{H}}_t,\theta_t,X_t,P^x)$ be a standard Markov process with state space $(E,\underline{\underline{E}})$, and let x_0 be a regular point in E. Let (L_t) be the process constructed above for the regenerative system $(\Omega,\underline{\underline{H}},\underline{\underline{H}}_t,\theta_t,\mathbb{M},P^\mu)$ where $\mathbb{M} = \{t: X_t = x_0\}$ and μ is a fixed initial measure. Then, (L_t) is a continuous additive functional of the process X whose support is $\{x_0\}$; more precisely, (L_t) is the local time of X at x_0 satisfying

$$E^x \int_0^\infty e^{-t}\, dL_t = E^x(e^{-R}).$$