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COASE'S THEOREM AND THE CHOICE OF LIABILITY RULES */

by

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I. INTRODUCTION

Coase's proposition, namely that

> "It is necessary to know whether the damaging business is liable or not for damage caused since without the establishment of this initial delineation of rights there can be no market transactions to transfer and recombine them. But the ultimate result [i.e., the allocation of resources] (which maximizes the value of production) is independent of the legal position if the pricing system is assumed to work without cost" [4, p. 8],

has many echoes in recent literature. Gifford and Stone [6], for example, showed, using calculus that Coase's proposition is valid, while other writers, like Tybout [13] and Randall [12], showed its invalidity in the long run. Moreover, Marchand and Russell[10] have claimed to prove that even in the short run, Coase's conclusion is wrong when the cost functions are not separable. These contradictory results seem to remain, as yet, unresolved [5 , 7 , 8 , 11].

In Section II of this paper, we shall present a proof, using game-theoretic solutions, to Coase's proposition. We shall also point out, where the mistake in Marchand and Russell's paper lies. The analysis here generalizes, simplifies, clarifies and gives a more rigorous proof to Coase's arguments. Moreover, unlike in other papers, it can be adapted both to partial equilibrium (i.e., two firms, each a member of a different competitive industry) as well as to general equilibrium analysis (i.e., for the case of two industries). Introducing many different agents, each causing externalities can also be dealt with, in our representation. These agents do not necessarily have to be firms (whose goal is maximization of profits), but can also represent individuals (each having a utility function which, unlike profits,are not transferable).

In Section III we show that if the intersection of the "cores of the two extreme liability rules" is not empty, a profit distribution will be agreed upon, by the negotiating parties, without reference to the status quo point, i.e., in the absence of liability rules. The question of "fair" distribution is discussed in Section IV, making use of the spirit of Shapley's value.

Following Coase:

> "A good example of the problem under discussion is afforded by the case of straying cattle which destroys crops growing on neighboring properties. Let us further suppose, that without fencing between the properties, an increase in the size of the cattle raiser's herd increases the total damage to the farmer's crops" [4, p. 2].

Throughout the paper, we shall use this example with the following notations:

A - the cattle raiser

B - the farmer

x - the amount of cattle raised

$C(x)$ - the cost function of x, when no crops are produced

y - the amount of crops produced

$m(y)$ - the cost function of y, when no cattle are raised

$Z(x,y)$ - the damage function, in units of crops.

Note that everywhere, $0 \leq Z(x,y) \leq y$. Hence, we do not (and cannot) assume "separable costs" (see [3, 10]). A plausible assumption (though we do not use it) is that for all $y: Z(0,y) = 0$. Moreover, from the above inequality, for all $x: Z(x,0) = 0$. The reciprocal nature of externalities is thus apparent.

We shall assume, unless otherwise specified, that the prices of crops P_x, and cattle P_y, are given and constant.

II. COASE'S THEOREM AND THE CORE .

II.1 Proof of Coase's Proposition.

> "With costless market transactions, the decision of the courts concerning liability for damage would be without effect on the allocation of resources" ([4], p. 10).

In order to prove this statement, we use a game theoretic approach.

Suppose that the liability on A decided by the courts is $P_y \cdot A(x,y)$, and that on B is $P_y \cdot B(x,y)$, where, by definition of a liability law:

$$(1) \qquad A(x,y) + B(x,y) \equiv Z(x,y) .$$

(Coase, as all the other writers, considered only the extreme cases where $A(x,y) \equiv Z(x,y)$ or $A(x,y) \equiv 0$). If there is no co-operation between A and B, the maximum A can get is:

$$(2) \qquad v(A) = \max_{x} \; [P_x x - C(x) - P_y A(x,y)] .$$

Similarly:

$$(3) \qquad v(B) = \max_{y} \; [P_y y - m(y) - P_y B(x,y)] .$$

If, however, A and B do co-operate, their total benefit is given by:

$$v(A,B) = \max_{x,y} \; [P_x x - C(x) - P_y A(x,y) + P_y y - m(y) - P_y B(x,y)] .$$

Using (1):

$$(4) \qquad v(A,B) = \max_{x,y} \; [P_x x - C(x) + P_y y - m(y) - P_y Z(x,y)] .$$

It is thus obvious that $v(A,B)$ does not depend on either $A(x,y)$ or $B(x,y)$.

Assuming unique solutions, (so that there is only one efficient resource allocation), denote the solution to (2) by:

$$(5) \qquad x = \bar{x}(y).$$

the solution to (3) by:

$$(6) \qquad y = \bar{y}(x).$$

And the solution to (4) by:

$$(7) \qquad x = x^*,\ y = y^*.$$

By (5) and (6), if the liability rules are given by $A(x,y)$ and $B(x,y)$, the resource allocation is $x = \bar{x}$ and $y = \bar{y}$, where $\bar{x} = \bar{x}(\bar{y})$ and $\bar{y} = \bar{x}(\bar{y})$.

Coase's proposition is that whatever the liability rules are, if negotiations with binding contracts are allowed and costless, the resource allocation will be (x^*,y^*),i.e., Max $v(A,B)$ will be achieved.

To prove the above conjecture, consider the two cases:

(a) The liability laws $A(x,y)$ and $B(x,y)$ are such, that $\bar{x} = x^*$ and $\bar{y} = y^*$. That is, no negotiations are required to achieve Pareto-optimum. Obviously, Coase's proposition holds in this case.

(b) $\bar{x} \neq x^*$ and/or $\bar{y} \neq y^*$. Since the solution to (4) is unique, we have:

$$(8) \qquad v(A,B) > v(A) + v(B).$$

[Using Buchanan and Stubblebine's terminology [3], (when $A(x,y) \equiv 0$) (8) is equivalent to a "Pareto-relevant externality"].

In this case, negotiations can increase both parties' profits, and hence, argues Coase, the resource allocation will not be $(\bar{x},\bar{y})$, giving rise to $v(A)$ and $v(B)$, but rather (x^*,y^*), which brings to the maximum joint profits, $v(A,B)$.

The concept most frequently used for solutions in economics, which is borrowed from game theory, is the core (Definition 6 in Appendix). Inequality (8) guarantees the existence of a non-empty core for the game: $[\{A,B\},v]$. That is, co-operation will be mutually advantageous. (There is a net gain of $v(A,B) - v(A) - v(B)$). Which point in the core will be the actual solution depends upon the bargaining abilities of A and B, and, obviously, upon the status quo point. This point is defined by the liability rules. */

Hence, if we accept the core as the solution to the co-operative game (i.e., where A and B can sign a binding contract and transfer bribes or compensation), Coase's proposition quoted in the introduction has been proved; any allocation in the core is, by definition, Pareto-optimum **/ and thus so is the unique resource allocation. As $v(A,B)$ is independent of the liability rule, the efficient resource allocation will prevail for any $A(x,y)$ and $B(x,y)$. However, $v(A)$ and $v(B)$ do depend on the liability rule, and hence, the income distribution changes according to the liability rules.

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*/ The status quo point is the solution of the competitive game under the liability law $\{A(x,y),B(x,y)\}$, i.e., it is given by the point $(v(A),v(B))$.

**/ Note that in our example, Pareto-optimum is equivalent to the maximum joint profits, since we consider only the "welfare" of the cattle raiser and the farmer, given the prices are constant.

Figure 1 illustrates the situation. The line KL represents $v(A,B)$, i.e., it is the locus of the payments (α,β) (for A and B, respectively) such that $\alpha + \beta = v(A,B)$. This line is drawn irrespective of the liability rules!

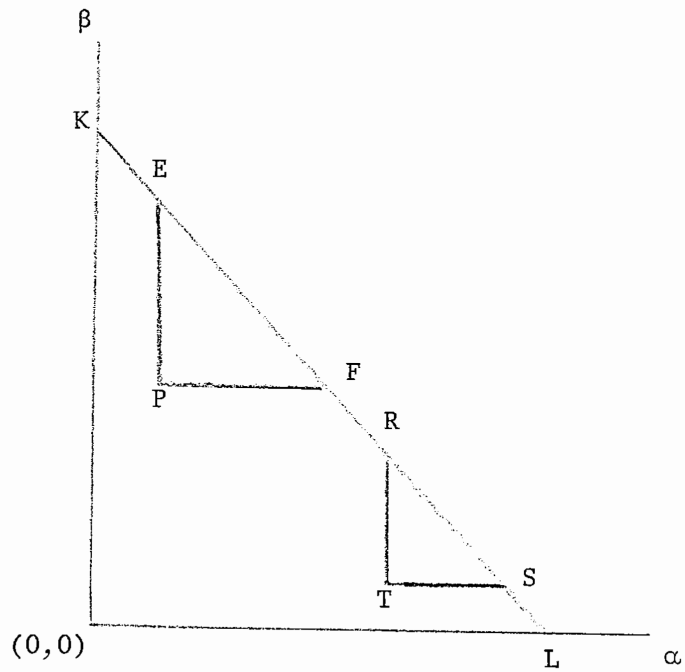

FIGURE 1

By definition, in order to get $v(A,B)$ dollars, the production of crops and cattle must be efficient. The liability rules specify the status quo points by influencing $v(A)$ and $v(B)$.

Let T represent the status-quo point in the competitive case, where the cattle raiser is not liable for the damage caused (i.e., $A(x,y) \equiv 0$), and let P represent the status quo point, when the farmer is fully compensated (i.e., $B(x,y) \equiv 0$). If the situation is as represented by Figure 1, the income distributions of the two extreme liability laws, (after negotiations took place, and an agreement had been reached), must differ since the core in the first case is given by EF, and in the second by RS, and we have $EF \cap RS = \emptyset$. However, in both cases $v(A,B)$ is achieved via the production of (x^*,y^*).

II.2 Remarks .

Remark 1: Marchand and Russell state:

> "Thus, only in the case in which there is joint decision making is there an optimal allocation of resources. If legal forces make firm A totally liable, or if firm A either receives or does not receive the bribe, there is a misallocation of resources: in these cases the allocation of resources is nonoptimal." [10,p. 615]

The reason they derive this conclusion is due to a wrong notion of how the compensation is made. They assume (p. 613-614) that the farmer bribes the cattle raiser to reduce his production from $\hat{x}$ (the competitive allocation) to x, by offering him the difference in the (farmer's) costs, that will result. In that case, the farmer will always incur the "pure competitive costs" $(m(\hat{y}) + P_y z(\hat{x},\hat{y}))$, where his best policy is to produce $\hat{y}$ (assuming unique solutions). The farmer will, therefore, offer no bribe payments!

The following is the appropriate bribe scheme in the spirit of Marchand and Russell's idea, in which the farmer gains nothing due to the reallocation.

For each x, the farmer solves:

$$(9) \qquad \operatorname*{Max}_{x} \, [P_y y - m(y) - P_y z(x,y)] \equiv L(x).$$

and offers the cattle raiser the amount of:

$$\gamma(x) = L(x) - L(\hat{x}).$$

The problem that the cattle raiser, thus faces, is:

$$\max_x P_x x - C(x) + \gamma(x). \tag{10}$$

Denote the solution to (10) by $x = \tilde{x}$. The farmer, after knowing the cattle-raiser's choice of $\tilde{x}$, maximizes:

$$\max_y P_y y - m(y) - P_y z(\tilde{x},y) - \gamma(\tilde{x}). \tag{11}$$

Denote the solution to (11) by $y = \tilde{y}$. It is easily seen that the profits under $\tilde{x},\tilde{y}$ and $\gamma(\tilde{x})$ are represented by the point S on KL in Figure 1, which is, obviously, a Pareto-optimum solution. The source of the mistake of Marchand and Russell lies in the fact that they choose inappropriate $\gamma(x)$ and moreover substitute it in (11), i.e., consider $\gamma(x)$ as a variable for the farmer. However, in the negotiation process, the farmer commits himself to $\gamma(x)$, and thereafter regards this "bribe payment" as a lump-sum transfer, on which he has no influence.

Remark 2: No loss of generality is caused by our assumption that only the cattle-raiser inflicts damage upon the farmer. In fact, we could consider the general case in which the cost functions depend on both outputs, i.e., the cost function for x, when y units of crops are produced is $\tilde{c}(x,y)$ and similarly, $\tilde{m}(x,y)$. Define:

$$c(x) \equiv \tilde{c}(x,0) \quad \text{and} \quad m(y) \equiv \tilde{m}(0,y).$$

The value of the total damage is given by:

$$\tilde{z}(x,y) \equiv \tilde{c}(x,y) + \tilde{m}(x,y) - c(x) - m(y),$$

and the above analysis carries over to this case. The only difference is that in the case of pure competition we shall no longer have $A(x,y) \equiv 0$, but rather $A(\hat{x},\hat{y}) \equiv \tilde{c}(\hat{x},\hat{y}) - c(\hat{x})$.

Remark 3: The reason that we analyzed only a two-person game is due to tradition. The generalization to an n-person game is immediate if we accept the core as the solution of the game and assume that it is non-empty. Moreover, we have a necessary and sufficient condition for the non-emptiness of the core, namely, the game has to be balanced. (Definition 7 in Appendix). [Note, that in particular it is not sufficient to have the analog of (8), namely:

$$v(A) + v(B) + \ldots + v(K) < v(A,B,\ldots,K) \quad]$$

Remark 4: Buchanan and Stubblebine [3]

> "find no cause for discussing production and consumption externalities separately. Essentially the same analysis applied in either case, (and hence), firms may be substituted for individuals and production functions for utility function".

In our model, however, since utilities are not trasnferable and/or comparable, we cannot use side-payments games in the case of consumption externalities. Fortunately, the core is defined also for games without side payments, which is the appropriate, even though more complicated, tool to analyse this case.

Remark 5: Although the analysis is a partial equilibrium one, taking p_x and p_y as constants, in view of Remark 4 we can easily transfer to the general equilibrium analysis, where prices change according to the amounts produced. Under standard assumptions, the prices vary continuously with the quantities produced, i.e., the correspondence θ, $\theta(x,y) = (p_x, p_y)$ is continuous. For each price system Remark 4 above applies. Hence, the farmer and the cattle raiser will choose an allocation $(\tilde{x},\tilde{y})$ such that there is no other production plan (x,y) which is mutually preferred. Though the calculation is more difficult, the model, in essence, remains unchanged.

Remark 6: Both Baumol [2] and Inada and Kuga [9] noted that in the presence of externalities the concavity of the production function and the concavity of the transformation curve are not preserved. In our model this is immaterial, since the maximum joint profit is well defined, no matter what shape the transformation curve has as long as it remains compact. In contrast to the use of calculus techniques, where the second order conditions are violated, our analysis holds also for corner solutions.

III. ACCEPTABLE SOLUTIONS PRIOR TO SPECIFYING THE LIABILITY RULES.

As proved above, if co-operation is allowed, a Pareto-optimum allocation would be achieved, whatever the liability rules are. Nevertheless, in order to negotiate the parties must know what these rules are. (I.e., their status quo point). The immediate question that arises, is how politically difficult would it be for the government to issue the liability rules? Put in another way, under what conditions can we expect the negotiating parties to reach a general consensus as to the profit distribution before the liability rules are given? (According to Coase, this can never be done.)

Our game-theoretic approach offers some insight to this problem. Each player has some influence upon the government. Since a vote on the liability rules has not been actually taken, each player has expectations as to the status quo point, that would have been reached. Obviously, if both parties have the same expectations, it is immaterial whether or not these liability rules were actually issued. The status quo point is well defined, and the solution will be in the core, relative to this agreed-upon status quo point. It seems, however, that generally each party over-estimates its powers and the pressue it can impose on the government. Thus, each considers a more favourable starting term, from its point of view, then would have been realized, were a vote to be taken. Even if we assume that each party knows its "objective" powers, for the sake of bargaining, it would argue as if more favourable liability rules could be achieved.

Let us consider, therefore, the extreme case in which each firm assumes that it will bear no cost due to the externality. That is, using Figure 1, the cattle raiser thinks of the status quo being T, where he achieves α_0 dollars, and the farmer assumes he can enforce P to be the status quo whereby he gets β_0 dollars. There is, in this case, no reason to suppose that a solution would be reached, unless the government does issue some liability rules. This is due to the fact that there is no feasible vector that dominates (α_0, β_0).

Suppose, however, that the extreme status quo points are $\tilde{T}$, $\tilde{p}$ such as represented by Figure 2.

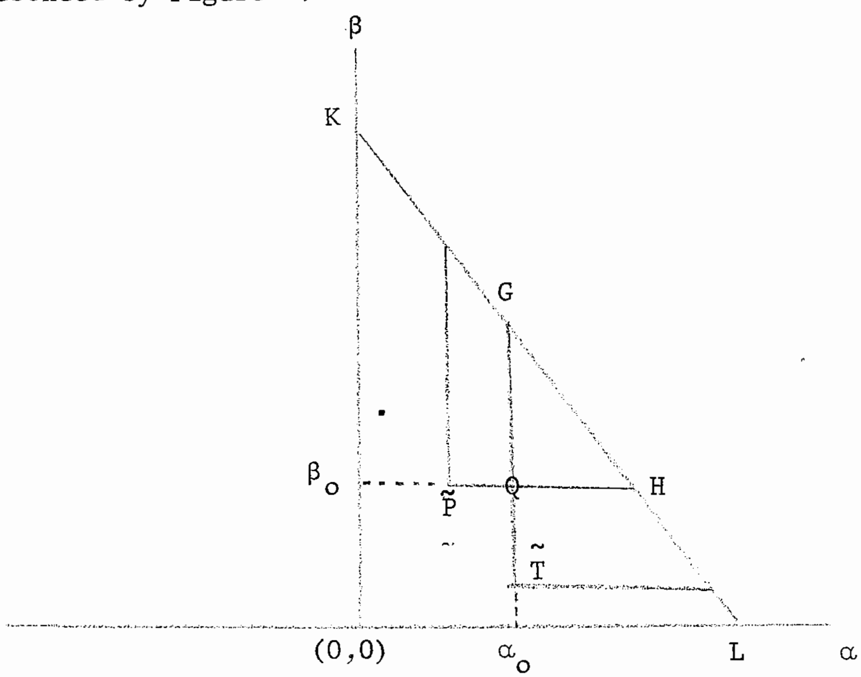

FIGURE 2

The status quo point, according to the "wishful thinking" of the participants, (which is an impossible one), is given by Q. The core of this game is non-empty, and hence, any income distribution represented by a point on GH, would be a reasonable outcome. (Again, if we accept the core as the appropriate solution when negotiations take place.) On GH, any point is prefered by both parties to the competitive allocation <u>regardless</u>

<u>of the liability rules</u>!

Formally, a consensus can be reached in the absence of the liability rules if the core, relative to one extreme liability rule L_0, (say when the cattle raiser is not liable for the damage), intersects the core of the game when the other extreme liability rule L_1 is assumed, (namely, when the farmer is fully compensated). From Figures 1 and 2, it is clear that two main factors influence the intersection of the cores:

(i) The difference of the gains to the parties in the two extreme liability rules, L_0 and L_1.

(ii) The joint gain from negotiation.

More specifically, it is easier to reach an acceptable solution, in the absence of specified liability rules, the smaller (i) is and the greater is (ii). To give a rigorous proof to the above statement, note that the cores intersect if and only if (α_0, β_o) is in the set OLK (Figure 2), i.e., iff:

$$\alpha_0 + \beta_0 < v(A,B). \tag{12}$$

By definition, if a Pareto relevant externality exists (see e.g. (8) above), we have:

$$v(A,B) > \alpha_0 + \beta_1 \quad \text{and} \quad v(A,B) > \alpha_1 + \beta_0. \tag{13}$$

Now, if $\beta_0 - \beta_1$ or $\alpha_0 - \alpha_1$ is relatively small, (12) follows from (13). That is, in order to reach a consensus prior to the issue of liability rules, the net <u>joint</u> gain from co-operation should exceed <u>one of the</u>

firm's net gain from the two extreme liability rules. Formally, since $\beta_0 > \beta_1$, $[v(A,B) - \alpha_o - \beta_0 > \beta_0 - \beta_1]$ implies that $v(A,B) - \alpha_0 - \beta_1 > \beta_0 - \beta_1$, or: $v(A,B) > \alpha_0 + \beta_0$. It should be noted that this is only a sufficient (but not a necessary) condition for the cores to intersect.

IV. "FAIR" PROFIT DISTRIBUTIONS .

Suppose an arbitrator had to decide on the income distribution of the farmer and of the cattle raiser when they co-operate (and hence a Pareto optimal resource allocation is achieved). Which is a "fair" distribution? The solution for fair allocations in game-theory is the Shapley value, a concept introduced now more frequently into economic theory. (See, for example, Aumann amd Kurz [1]). There are two main advantages of this value over the core (for games with side payments): (i) it always exists, (ii) it is unique. While (i) is not relevant to our analysis, since, by definition the core is non-empty in the presence of externalities, (ii) may be considered as an important improvement. Rather than give the axiomatic approach, we shall present the statistical one, for the definition of the value. If we assume that when a firm j joins a given structure of environment,*/ T, $j \notin T$, it gets the difference $v(T \cup \{j\}) - v(T)$ (i.e., its marginal contribution), the Shapley value, $\phi_j(v)$, is the expected value of the payments in the formation of all such environments (coalitions), which are assumed to be equiprobable. This approach particularly appeals to our case, since, as Coase argues:

> "We are dealing with a problem of a reciprocal nature. To avoid the harm to B would inflict harm on A. The real question that has to be decided is: should A be allowed to harm B or should B be allowed to harm A?" [4, p. 2].

It should, therefore, be immaterial as to whether the farmer was the first to join the environment or it was the cattle raiser.

For a two-person game, such as we are considering, with the characteristic function v one immediately obtains the Shapley value:

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*/ Environment here means a given set of (already existing) firms.

$$
(14) \qquad
\begin{aligned}
\phi_A(v) &= \frac{v(A,B) + v(A) - v(B)}{2} = v(A) + \tfrac{1}{2}[v(A,B) - v(A) - v(B)] \\
\phi_B(v) &= \frac{v(A,B) + v(B) - v(A)}{2} = v(B) + \tfrac{1}{2}[v(A,B) - v(A) - v(B)],
\end{aligned}
$$

i.e., the profits $[v(A,B) - v(A) - v(B)]$ are equally shared. In our case, however, as shown above, $v(A)$ and $v(B)$ are not defined unless the liability rule is given, and hence we cannot directly apply Shapley's value. In particular, which $v(A)$ and $v(B)$ should be used in (14)?

Surprisingly, unlike in the case of the core, the value does not depend on the existing liability rule! Following the spirit of the arguments for justifying the value as a solution, if A is the first to enter the locality, he contributes α_0 (i,e., the profits of A when he is not liable for the damage), and if he joins as the second, he contributes $v(A,B) - \beta_0$ (since B was there first, and hence his profits were β_0). Hence, since the two possibilities are equiprobable:

$$
\begin{aligned}
\tilde{\phi}_A &= \tfrac{1}{2}\alpha_o + \tfrac{1}{2}[v(A,B) - \beta_o] = \alpha_o + \tfrac{1}{2}[v(A,B) - \alpha_o - \beta_o] \\
\tilde{\phi}_B &= \tfrac{1}{2}\beta_o + \tfrac{1}{2}[v(A,B) - \alpha_o] = \beta_o + \tfrac{1}{2}[v(A,B) - \alpha_o - \beta_o].
\end{aligned}
$$

Note that it may well be the case the $v(A,B) < \alpha_0 + \beta_0$, thus A or B will receive less profits than he would under his most favorable liability rule.

V. APPENDIX - DEFINITIONS.

Let N be a finite non-empty set, and let S be a non-empty subset of N. An S vector x^S is a real function defined on S whose value at $i \in S$ is x_i^S. The superscript N will be omitted. E_+^S denotes the non-negative orthant of the Euclidean space E^S of all the S vectors. Let x^S and y^S be two S vectors. $x^S \geqq y^S$ if $x_i \geq y_i$ for all $i \in S$ and $x^S > y^S$ if $x_i > y_i$ for all $i \in S$.

Definition 1: An n person cooperative game without side payments is a pair $(N;v)$ where N is a set with n members, and v is a function which assigns to each non empty subset S of N a non-empty subset $v(S)$ of E_+^S so that:

(i) $v(S)$ is compact

(ii) If $x^S \in v(S)$ and $0 \leqq y^S \leqq x^S$ then $y^S \in v(S)$

(iii) $v(i) = 0$ for all $i \in N$.

N is the set of players of the game and v is its characteristic function.

$v(S)$ is the set of all possible solutions for the players in coalition S, if and when S formed. In particular, for $x^S \in v(S)$, $(x^S)_i$ is the "utility" individual $i \in S$ receives.

Definition 2: A cooperative game with side payments is the pair $(N;\mu)$ which is a cooperative game without side payments $(N;v)$ where v is given by: $v(S) = [x^S \in E_+^S \mid \sum_{i \in S} x_i^S \leq \mu(S)]$. $\mu(S)$ is called the value of the coalition S.

In a game with side payments, a transfer of α units of individual i to individual j, increases the "utility" of player j by exactly α units. If x^S represents the profit distirubiton (rather than the utilities) among the members of coalition S, a game with side payments is the appropriate notion to be used.

<u>Definition 3:</u> x^S is called a payoff for the coalition S if $x^S \in \bar{v}(S)$ where

$$\bar{v}(S) = [y^S \in v(S) \mid \text{ there exists no } z^S \in v(S) \text{ such that } y^S < z^S]$$

That is, a payoff $x^S \in v(S)$ is a feasible utility (profit) distribution which is also Pareto optimal relative to coalition S.

<u>Definition 4:</u> x is called a <u>payoff</u> if $x \in \bar{v}(N)$.

<u>Definition 5:</u> Let x be a payoff. Coalition S_o is called a <u>blocking coalition</u> if there exists $y \in v(S_0)$ such that $y_i > x_i$ for all $i \in S_o$.

In words, S_o is a blocking coalition for the proposed payoff x, if by acting on its own, it can improve(relative to x) the utility of each of its members.

<u>Definition 6:</u> The <u>core</u> of the game (N:v) is the set of all payoffs for which there is no blocking coalition. That is, x is in the core if and only if, $x \in \bar{v}(N)$ and for all $S \subseteq N$, $y^S > x^S$ implies $y^S \notin \bar{v}(S)$.

The reason why the core is considered to contain all possible outcomes of the game is apparent. If a payoff is not in the core, there exists at least one group of people who have the incentive (and the power) to depart, form its own coalition, and benefiting thereby. In other words, for a solution to be reached via negotiations, it has to obtain the general consensus, hence be in the core.

<u>Definition 7:</u> A set of coalitions $\{S_k\}_{k=1}^{t}$ is called <u>balanced</u>, if there exist "weights" $\{\gamma_k\}_{k=1}^{t}$ such that for all k, $\gamma_k \geq 0$ and for every individual $i \in N$, $\sum_{k | i \in S_k} \gamma_k = 1$.

<u>Theorem:</u> A game with side payments $(N;\mu)$ has a non-empty core, if and only if for all balanced collection $(\{S_k\}_{k=1}^{t}, \{\gamma_k\}_{k=1}^{t})$: $\sum_{k=1}^{t} \gamma_k \mu(S_k) \leqq \mu(N)$.