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INDIVIDUAL DECISIONS AND GROUP DECISIONS: THE FUNDAMENTAL DIFFERENCES

by

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1. Introduction

A commonplace among participants is that group decisions are neither rational nor straightforward. What is rare is understanding that elimination of the self-contradictory and roundabout nature of group decisions is logically impossible. Groups in general cannot make decisions in the same rational, straightforward manner that an individual can. Our purpose is to establish the formal truth of this last statement. Complementing this purpose is a secondary, expository purpose: we seek to make the recent literature on strategy-proof voting procedures, Arrow's impossibility theorem, and their relationship to each other more accessible than it currently is.

The development of our discussion is in five steps. After establishing basic notation and definitions in Sections 2, 3, and 4, we introduce in Section 5 the classic problem of rational group choice which Kenneth J. Arrow formulated in Social Choice and Individual Values.[1] We show that a correspondence exists between the properties which the rational choices of individuals exhibit and the properties which group choices would exhibit if they did satisfy the rationality and independence of irrelevant alternative conditions which Arrow postulated. In other words, if a voting procedure did satisfy Arrow's conditions, then the group's decisions would have the same properties which the usual theory of rational individual choice under certainty predicts for individual's decisions.

In Section 6 we consider the incentives which individuals within a group face when the voting procedure that the group uses satisfies Arrow's conditions of rationality, independence of irrelevant alternatives, and non-negative response. We show that such a voting procedure is necessarily strategy-proof. This means that the individuals within the group never have a positive incentive to misrepresent their preferences. Within groups that use voting procedures that do not satisfy Arrow's conditions such a positive incentive may occur. For example, an individual voting in an election where a Democrat, a Republican, and a minor party candidate are on the ballot may decide to vote for his second choice, the Democrat, instead of his first choice, the minor party candidate, because he thinks that a vote for the minor party candidate would be a wasted vote on a hopeless candidacy.

In Section 7 we present and prove Arrow's famous impossibility theorem[2]. It states that no voting procedure exists which satisfies Arrow's rationality and independence of irrelevant alternative conditions plus certain minimal standards of democratic responsiveness. This result shows definitively that group decision making is fundamentally different from individual decision making. No matter what voting procedure is used, the properties of group decisions cannot match the properties of a rational individual's decisions.

In Section 8 we ask what can be salvaged from the negativism of Arrow's result. In particular we ask if any voting procedures exist which are strategy-proof? In other words, does a democratic voting procedure exist which, though it does not duplicate the rationality of individual decision making, at least induces within the group a straightforward decision process where no individual has an incentive to misrepresent his preferences? The answer is negative: we reprove the impossibility theorem for strategy-proof voting procedures that Gibbard and Satterthwaite independently formulated and proved.[3]

The question which we consider in Section 9 is the relationship which exists between these two impossibility results. We show a strong relationship between the two: for a broad and attractive class of voting procedures the Arrow conditions of rationality, independence of irrelevant alternatives, non-negative response, and citizens' sovereignty are equivalent to the conditons of strategy-proofness and citizens' sovereignty. This result, while original to this paper, is closely related to an earlier result of Satterthwaite.[4] An implication of this result is that any attempt to circumvent the Arrow impossibility theorem by weakening the conditions which he postulated for voting procedures is likely to create a voting procedure which has incentive problems.

The model which we use in deriving these results is the standard social choice model that dates from Arrow's original work.[5] This is a simple model which does not have much structure. In Section 10, the concluding section, we consider how robust these results are when additional structure is added. It is at this point in the paper where we report some of the work which has been done beyond that of Arrow, Gibbard, and Satterthwaite.[6] The tentative conclusions arrived at in this section are that both the Arrow and the Gibbard-Satterthwaite impossibility results are quite robust. The robustness, however, of the equivalence between Arrow's conditions and strategy-proofness is largely unknown. Further work needs to be done.

Since an explicit purpose of this paper is to make known results more accessible we have sought to make the content of and the motivation behind the various definitions, assumptions, and theorems as clear as possible. The proofs have been written with a minimal number of skipped steps. Unfortunately this does not mean that reading each proof is easy. Despite our best efforts some steps appear to be intractably subtle.

2. Basic Formulation

A group is a set N of $|N| \geq 1$ individuals whose task is to select a single alternative from that set X of alternatives which are feasible. This feasible set X, which the group accepts as given, is a subset of the universal set S which contains all conceivable alternatives, whether feasible or not. Both the number of members $|N|$ in the group and the number of alternatives $|S|$ in the universal set are assumed to be finite. Each individual $i \in N$ is rational and has preferences P_i that are a strict order on S, i.e. P_i is complete, asymmetric, and transitive. If $x, y \in S$ and $i \in N$, then $x\ P_i\ y$ means that individual i prefers that the group select alternative x instead of alternative y. The collection $P = (P_1, \ldots, P_n)$ of all individuals' preferences is called the preference profile.

The group makes its choice among elements of the feasible set by voting. Each individual reveals a preference ordering P_i and the resulting profile of revealed preferences $P = (P_1, \ldots, P_m)$ is inserted into the voting procedure which calculates the group's choice. A voting procedure (VP) is a function $v(P\,|X)$ whose arguments are the profile of revealed preferences P and the feasible set X. It is a singlevalued mapping which selects one element of the feasible set X to be the group's choice.[7] Thus $v(P|X) = x$ where $x \in X \subset S$.

When an individual reveals a preference ordering P_i for insertion into the VP $v(P|\,X)$ he may or may not accurately reveal his true preferences P_i. Any attempt through direct regulation to require him to reveal his true preferences is certain to fail because his true preferences are purely internal to him and thus are unobservable. If an individual's true preferences are P'_i and he does reveal them truly, then $P'_i = P_i$ and we call P_i his sincere strategy If he misrepresents his true preferences, then $P'_i \neq P_i$ and we call P_i a sophisticated strategy.[8] If every individual in the group plays his sincere strategy, then P, the profile of revealed preferences, is called the sincere strategy profile. Presumably an individual bases his decisions between playing his sincere strategy and playing a sophisticated strategy on his estimate as to which strategy is most likely to produce a group choice which he prefers.

We adopt four assumptions in this basic model which are important and deserve explicit discussion. The effects of weakening these assumptions are discussed in Section 10. Our first assumption is that individuals are never indifferent between two or more alternatives. Their preferences are always strict orderings on S. We make this assumption, with its lack of realism, mainly because it simplifies the paper's proofs and exposition. Moreover, as we report in Section 10, relaxation of this assumption does not significantly change the results which we derive.

The second assumption is that any strict order P_i on S is admissible as either an individual's true preferences or his revealed preferences. This prevents the imposition of any sort of structure on individual preferences. For example, economic theory often suggests that preferences of all individuals should share a common convexity structure such as single-peakedness.

These first two assumptions are easily formalized jointly. Let p be the collection of all possible strict orderings on the universal set S and let p^n be the n-fold cartesian product of p. Given this notation, the assumptions that indifference is inadmissible and preferences are unrestricted over the set of strict orderings implies that any strict ordering $P_i \in p$ is admissible as either an individual's true preferences or revealed preferences. Similarly any profile $P \in p^n$ is admissible as a group's preference profile.

The third assumption is that the outcome of a VP must be a single alternative. This excludes from our consideration any VP which selects as its outcome either a set of alternatives or a lottery among several alternatives. Our justification is that generally groups must select for action a single alternative.[9] For example, for any particular activity during a given year one and only one budget can be implemented.

The third assumption is that any set $X \subset S$ is admissible as a feasible set. This assumption prevents the feasible set from having any specific structure. Such structure is often called for by economic theory. For example, in consumer theory the feasible set is normally limited to those sets $X \subset S$ which are "budget triangles".

3. The Functions Ψ, θ, and Δ.

In this section we define three functions which we use constantly throughout the paper. The function Ψ_X, defined for any feasible set $X \subset S$, is a mapping from p onto the elements of S. It has the property that $x = \Psi_X(P_i)$ if and only if $x \in X$ and $x \; P_i y$ for all $y \in X - \{x\}$. In other words, Ψ_X picks out the maximal element of X. For example, if $P_i = (w \; x \; y \; z)$ and $X = \{x, y\}$, then $\Psi_X(P_i) = x$, given that $P_i = (w \; x \; y \; z)$ means that individual prefers w over x, x over y, etc. Turning to the function θ_W, let $W \subset S$ be any feasible set. Define θ_W to be the projection mapping from the set of strict orderings defined on S to the set of strict orderings defined on W. Thus θ_W has the property that if $x, y \in W$ and if $P_i' = \theta_W(P_i)$, then $x \; P_i' \; y$ if and only if $x \; P_i \; y$. The function constructs a new strict order P_i' by simply deleting from P_i those elements of S that are not contained in W, e.g. if $P_i = (w \; x \; y \; z)$ and $W = \{x, z\}$, then $\theta_W(P_i) = (x \; z)$. Since no confusion can result, let $\theta_W(P) = [\theta_W(P_1), \ldots, \theta_W(P_n)]$.

The third function is Δ_W. Let $W \subset S$ be any feasible set. Let Δ_W have the properties that if $P_i' = \Delta_W(P_i)$, then $x \; P_i' y$

a. if $x, y \in W$ and $x \; P_i \; y$;

b. if $x, y \in S - W$ and $x \; P_i \; y$; or

c. if $x \in W$ and $y \notin W$.

Thus $P_i' = \Delta_W(P_i)$ reshuffles the ordering P_i by moving the elements of the feasible set W to the top of the ordering P_i'. The relative rankings of the elements within W remain unchanged as do the rankings of the elements within the complement of W. For example, if $P_i = (w \; x \; y \; z)$ and $W = \{y, z\}$, then $\Delta_W(P_i) = (y \; z \; w \; x)$. Let $\Delta_W(P) = [\Delta_W(P_i), \ldots, \Delta_W(P_n)]$.

4. Two Illustrative Voting Procedures

In the sections that follow we formally define several properties which a VP may or may not have. Understanding of these properties is facilitated if we relate the formal definitions to specific VPs and determine which of the properties they satisfy. Therefore in this section we define two positionalist voting procedures: the first we label type α and the second we label type β[10]. They are denoted respectively by $v_\alpha(P|X)$ and $v_\beta(P|X)$.

Consider $v_\alpha(P|X)$ first. Given a profile $P = (P_1, \ldots, P_i, \ldots, P_n)$, a universal set S, and a feasible set X, each alternative $x \in S$ receives $(|S| - k-1)$ points for each preference ordering P_i in which it is ranked k positions from the top. The points each alternative receives are summed and the winner is that alternative, from among the alternatives contained in the feasible set X, which received the most points. If two alternatives receive the same number of points, then individual one's preference ordering P_1 is used to break the tie. Suppose, for example, that $|W| = 3$, $S = \{w, x, y, z\}$, and

$$\begin{aligned} P_1 &= (w\ x\ y\ z), \\ P_2 = P_3 &= (x\ z\ w\ y). \end{aligned} \tag{4.1}$$

The rule for v_α assigns x eight points, w five points, z four points, and y one point. Therefore $v_\alpha(P|\{w, z\}) = w$.[11]

The VP $v_\beta(P|X)$ modifies the above rule by (a) assigning points only to alternatives contained in the feasible set X and (b) assigning points on the basis of an alternative's position relative to the other elements of X. Thus, given the preference ordering $P_1 = (w\ x\ y\ z)$, alternative x is assigned two points if $X = \{w, x, y, z\}$ or $\{x, y, z\}$, one point if $X = \{w, x, y\}$, $\{w, x, z\}$, $\{x, y\}$, or $\{x, z\}$, and zero points if $X = \{w, x\}$. Consequently, if P is defined by (4.1), then $v_\beta(P|\{w, z\}) = z \neq v_\alpha(P|\{w, z\})$.

5. Arrow's Aggregation Problem

In Social Choice and Individual Values Arrow essentially asked if the usual economic theory of rational choice on the part of individuals can be generalized to a theory of rational group choice. In this section we give this question formal meaning by first deriving some important properties which rational individual choice possesses. We then show that the generalizations of these properties which are appropriate for group choice are precisely those properties which, according to Arrow, any acceptable voting procedure must possess. As mentioned in Section 2, a rational individual has a complete, transitive, and asymmetric preference ordering $P_i \in p$ over the universal set of alternatives S. When he must pick one element from a feasible set $X \subset S$ he picks that alternative within X which he ranks highest on his preference ordering P_i, i.e. he picks the alternative $\Psi_X(P_i)$.

Rational individual choice has three properties which interest us here: independence of irrelevant alternatives, independence of non-optimal alternatives, and preference revelation. Independence of irrelevant alternatives means that an individual while making a choice within the feasible set X can disregard his preferences over the alternatives within the infeasible set $\bar{X} = S - X$.[12] Therefore independence of irrelevant alternatives implies that if his preferences over the elements within X remain constant, then his choice will remain fixed on one particular alternative regardless of how much his preferences over the elements in $\bar{X}$, the complement of X, might vary. Clearly rational individual choice satisfies independence of irrelevant alternatives because, if $\theta_X(P_i) = \theta_X(P_i')$ for two preference orderings P_i, $P_i' \in p$, then $\Psi_X(P_i) = \Psi_X(P_i')$. Satisfaction of this property gives the individual an important degree of efficiency in the making of his choices because it eliminates his need to ask himself hypothetical questions concerning his preferences over the "irrelevant" alternatives contained in the infeasible set $\overline{X}$.

The second property in which we are interested is independence of non-optimal alternatives.[13] Suppose that an individual chooses alternative x when the feasible set is X. Now suppose that the feasible set is shrunk without deleting alternative x to some proper subset Y of X. Thus $x \in Y \subset\subset X$. If this deletion of non-optimal alternatives does not cause the individual's choice to change from x to another alternative contained in the reduced feasible set Y, then his choice satisfies independence of non-optimal alternatives. Clearly any individual whose choices are rational satisfies this property because if $x \in Y \subset X$ and $\Psi_X(P_i) = x$, then the definitions of Ψ_X and Ψ_Y imply $\Psi_Y(P_i) = x$.

The third property in which we are interested is preference revelation. An individual's preference ordering $P_i \in p$ is not directly observable. Only his actual choices from within varying feasible sets are observable. Nevertheless these observed choices can be used in the following manner to construct the rational individual's preference ordering. Let $\Phi(X)$ be an individual's observed choice given that the feasible set is $X \subset S$. Assume that his choice always consists of a single alternative. Given a particular feasible set X, define the binary relation P_i' such that $x P_i' y$ if $x = \Phi(X)$ and $y \in X - \{x\}$. If the individual is rational, if $P_i \in p$ is his preferences, and if X is varied over a sufficient number of subsets of S, then the resulting binary relation P_i' will be complete, asymmetric, transitive, and identical to his preference ordering P_i.

The properties of rationality, independence of irrelevant alternatives, independence of non-optimal alternatives, and preference revelation may each be generalized stratightforwardly from the case of individuals to the case of groups.[14] We begin this process by using Arrow's concept of a social welfare function to formalize the idea of rational group choice. A <u>social welfare function</u> (SWF) is any function u: $p^n \to p$ that associates each preference profile $P = (P_1,\ldots,P_n) \in p^n$ with a unique strict group preference ordering $P_N = u(P) \in p$. If, for some profile $P \in p^n$, $P_N = u(P)$, then $x\ P_N\ y$ and $x\ u(P)\ y$ are equivalent notations signifying that the group prefers x over y. A SWF u(P) <u>underlies</u> a VP $v(P|X)$ if and only if, for all $P \in p^n$ and all $X \subset S$, $v(P|X) = \Psi_X[u(P)]$.

> Rationality (R). A VP $v(P|X)$ satisfies condition R if and only if a SWF u(P) exists which underlies it.

In other words, for a rational VP $v(P|X)$ and a given profile $P \in p^n$, the strict ordering $P_N = u(P)$ generated by the VP's underlying SWF rationalizes the group's choices as X varies in exactly the same manner that an individual's preference ordering P_i rationalize his choices $\Phi(X)$.

The VP v_α is an example of a VP that satisfies R while the VP $v_\beta(P|X)$ is an example of a VP that does not. This latter assertion is easily checked by analyzing the choices which v_β makes when $|N| = 3$, $P_1 = (x\ y\ z)$, $P_2 = (z\ x\ y)$, $P_3 = (y\ z\ x)$, and **X** varies. Observe that $v_\beta(P|\{x, y\}) = x$, $v_\beta(P|\{y, z\}) = y$, and $v_\beta(P|\{x, z\}) = z$. For an ordering $P_N = u(P)$ to rationalize these choices, it would have to have the properties that $x\ P_N\ y$, $y\ P_N\ z$, and $z\ P_N\ x$. This, however, violates transitivity. Therefore no SWF u(P) exists which rationalizes v_β and hence v_β does not satisfy R.

The next concept we extend to group choice is independence of irrelevant alternatives.

> Independence of Irrelevant Alternatives (IIA). A VP $v(P|X)$ satisfies IIA if and only if $v(P|X) = v(P'|X)$ for all feasible sets $X \subset S$ and all pairs of preference profiles $P, P' \in p^n$ which have the property that $\theta_X(P) = \theta_X(P')$.

The condition requires that the group's choices depend only on how the individuals rank the alternatives within the feasible set, not how they rank the infeasible alternatives. It is a necessary condition for efficient group decision-making because if the condition were violated, then each time the group needed to make a decision among a small number of feasible alternatives every member of the group would have to construct his preferences over the perhaps enormous, universal set S of alternatives.

Notice that rationality is neither a sufficient nor a necessary condition for a VP $v(P|X)$ to satisfy IIA. For example, the VP $v_\alpha(P|X)$ satisfies R but not IIA while the VP $v_\beta(P|X)$ satisfies IIA but not R. That v_α does not satisfy IIA may be seen by considering the two profiles, P and P':

$$P_1 = (x\ y\ z\ w),\ P_2 = (y\ x\ z\ w),\ P_3 = (w\ z\ y\ x), \tag{5.1}$$

$$P_1'\ (x\ z\ w\ y),\ P_2' = (y\ x\ z\ w),\ P_3' = (w\ z\ y\ x). \tag{5.2}$$

Examination shows that $\theta_{\{x,y\}}(P) = \theta_{\{x,y\}}(P')$, $v_\alpha(P|\{x,y\}) = y$, and, in violation of IIA, $v_\alpha(P'|\{x,y\}) = x$. This independence of R and IIA is a break in the parallelism between group and individual choice; for individual choice satisfaction of R guarantees satisfaction of IIA.

We extend independence of non-optimal alternatives as follows:

Independence of Non-Optimal Alternatives (INOA). A VP $v(P|X)$ satisfies INOA if and only if, for all $P \in p^n$ and all $W \subset S$, $v(P|W) = v(P|Y)$ whenever $Y \subset W$ and $v(P|W) \in Y$.

Notice that if a VP $v(P|X)$ satisfies R, then it must satisfy INOA. Not as evident is the fact that if a VP $v(P|X)$ satisfies INOA, then it also satisfies R. Sen and others have proven this result previously.[15] In Section 7 we reprove this result as an integral part of the proof of Theorem 4. Moreover if a VP $v(P|X)$ satisfies INOA and is therefore rational, then its underlying SWF $u(P)$ is revealed by its observed choices $v(P|X)$ in exactly the same manner that a rational individual's preferences P_i are revealed by his observed choices $\Psi_X(P_i)$.

The question Arrow asked was: can a VP $v(P|X)$ satisfy simultaneously the independent conditions R and IIA?[16] If such a VP does exist, then groups can in effect make choices in the same manner as individuals. Certain trivial examples of such VPs exist. For instance, if an individual i is the dictator of a group, then $v(P|X) = \Psi_X(P_i)$. Clearly $v(P|X)$ satisfies the conditions R and IIA. We want to rule out such trivial solutions. Therefore, as Arrow did, we specify three additional properties which any non-trivial VP should possess.

Non-Dictatorship (ND). A VP $v(P|X)$ satisfies ND if and only if no feasible set $W \subset S$ exists such that, for some $i \in N$ and all $P \in p^n$, $v(P|W) = \Psi_W(P_i)$.

Citizens' Sovereignty (CS). A VP $v(P|X)$ satisfies CS if and only if, for every $W \subset S$, and every $x \in W$, a profile $P \in p^n$ exists such that $v(P|W) = x$.

Non-Negative Response (NNR). For any $x \in S$, let $Y = S - \{x\}$ and let $P, P' \in p^n$ be any two profiles which have the properties that, for all $y \in Y$ and all $i \in N$,

a. $\theta_Y(P_i) = \theta_Y(P_i')$ and

b. $x P_i' y$ if $x P_i y$.

A VP $v(P|X)$ satisfies NNR if and only if $v(P'|W) = x$ for all feasible sets $W \subset S$ such that $v(P|W) = x$.

Condition ND rules out the possibility of one individual dominating the decision process, Condition CS insures that the group can actually choose any alternative within the feasible set X, and Condition NNR rules out perverse VPs which in effect weight some individuals' preferences negatively. More specifically NNR requires that if the only change in switching from profile P to profile P' is that on some individual preference orderings within P' alternative x has moved up relative to other alternatives, then, if the group's choice in the original situation is x, it must remain x in the final situation. In other words, NNR requires that moving x up in the individual's preference orderings can not hurt the chances of x to be the group's choice.

The four conditions of R, IIA, NNR, and CS are equivalent to three alternative conditions: R, monotonic binarity (MB), and Pareto optimality (PO). The usefulness of this equivalence is that conditions MB and PO used together considerably simplify some of the proofs which follow. Moreover, MB, as its name implies, makes explicit the binarity which is inherent in the Arrow requirements that a satisfactory VP satisfy R, IIA, NNR, CS, and N.D. Therefore in the remainder of this section we define condition MB, prove that R and MB are equivalent to R, IIA and NNR, define condition PO, and show that R, MB, and CS are equivalent to R, MB, and PO.

The basis for condition MB is the concept of decisive set. Consider a specific rational VP $v(P|X)$ and its underlying SWF $u(P)$, a specific pair of alternative $x, y \in S$, and a specific subset of individuals $B \subset N$. Let $\mathcal{P}_{xy}(B)$ be the collection of all profiles $P = (P_1, \ldots, P_n) \in p^n$ such that, for all $i \in B$, $x \, P_i \, y$. Thus if $P \in \mathcal{P}_{xy}(B)$, then every individual contained in B prefers x to y. The subset of individuals $B \subset N$ is <u>decisive for</u> x <u>over</u> y if and only if $P \in \mathcal{P}_{xy}(B)$ implies that $x \, u(P) \, y$. In other words, if B is decisive for x over y, then in a qualitative sense the individuals in B form a majority for x against y since their agreement on (x,y) assures that the group ranking u(P) reflects their opinion. An important property of this definition is that if two subsets of individuals, B and C, satisfy $B \subset C$ and if B is decisive for x over y, then C is also decisive for x over y because $\mathcal{P}_{xy}(C) \subset \mathcal{P}_{xy}(B)$ when $B \subset C$. Thus montonicity with respect to subsets of individuals is built into the definition of decisiveness.

> Monotonic Binarity (MB). A rational VP $v(P|X)$ and its underlying SWF $u(P)$ satisfies MB if and only if, for any profile $P \in p^n$ and for any pair x, $y \in S$, a group ordering of $x \, u(P) \, y$ implies that the subset of individuals $B = \{i \mid i \in N \text{ and } x \, P_i y\}$ is decisive for x over y.

In other words, MB states that if initially a subset of individuals $B \subset N$ is able to make the group follow its opinion that x is preferred to y, then as long as those individuals continue to agree that x is preferred to y, the group will continue to rank x over y. Nothing, whether it be conversion of former opponents to the view that x is preferred to y or changes in how individuals rank alternatives other than x and y, can change the group preference from x over y to y over x. Consequently MB is both a binarity and monotonicity condition.

Theorem 1 is an equivalence between conditions R, IIA, and NNR and conditions R and MB. Nevertheless conditions IIA and NNR are not equivalent to condition MB. For example, the VP v_β satisfies IIA and NNR but does not satisfy either MB or R because it does not calculate the winning alternative through a series of binary comparisons.

> Theorem 1. Conditions R, IIA, and NNR are equivalent to conditions R and MB.

Proof. We show first that R, IIA, and NNR imply MB. Suppose to the contrary, that some rational VP $v(P|X)$ and its underlying SWF $u(P)$ satisfies IIA and NNR but does not satisfy MB. Since MB is not satisfied, a profile $P \in p^n$ and pair of alternatives $x, y \in S$ exist such that (a) $x\ u(P)\ y$ and (b) $B = \{i | i \in N \text{ and } x\ P_i\ y\}$ is not decisive for x over y. Therefore a profile $P' \in p^n$ exists such that $P' \in \mathcal{P}_{xy}(B)$ and $y\ u(P')\ x$. Consider the sequence

$$u(P_1, P_2, \ldots, P_n) = P_N(0),$$

$$u(P_1', P_2, \ldots, P_n) = P_N(1),$$

$$\cdots$$

$$u(P_1', P_2', \ldots, P'_{i-1}, P_i, P_{i+1}, \ldots, P_n) = P_N(i-1), \tag{4.1}$$

$$u(P_1', P_2', \ldots, P'_{i-1}, P_i', P_{i+1}, \ldots, P_n) = P_N(i),$$

$$\cdots$$

$$u(P_1', P_2', \ldots, P_n') = P_N(n).$$

Since x $P_N(0)$ y and y $P_N(n)$ x, a $j \in N$ must exist such that x $P_N(j-1)$ y and y $P_N(j)$ x. Recall that $B = \{i | i \in N \text{ and } x\ P_i\ y\}$ and $P' \in \mathscr{P}_{xy}(B)$; therefore, for all $i \in N$, x P_iy implies that x P_i'y. Consequently, for the critical individual j, three possibilities exist:

a. x P_j y and x P_j' y,

b. y P_j x and y P_j' x,

c. y P_j x and x P_j' y.

Each of these possibilities contradict the assumption that IIA and NNR are fulfilled. If either (a) or (b) is true, then the switch from x $P_N(j-1)$ y to y $P_N(j)$ x violates IIA. These conclusions can be seen by letting the feasible set be $X = \{x,y\}$. Therefore $v(P_1',\ldots,P_{j-1}',P_j,P_{j+1},\ldots P_n | X) = v(P^*/P_j | X) = \Psi_X[P_N(j-1)] = x$ and $v(P_1',\ldots,P_{j-1}',P_j',P_{j+1},\ldots,P_n | X) = v(P^*/P_j' | X) = \Psi_X[P_N(j)] = y$.[17] Note that for cases (a) and (b) $\theta_X(P^*/P_j) = \theta_X(P^*/P_j')$. Therefore IIA implies that if $v(P^*/P_j | X) = x$, then, contrary to assumption, $v(P^*/P_j' | X) = x$. If case (c) is true, then the switch from $v(P^*/P_j | X) = x$ to $v(P^*/P_j' | X) = y$ violates NNR because the relevant difference between P_j and P_j' is that x is moved up relative to y. NNR states that such a change should result in the group's choice remaining x. Therefore, no matter which case is true, either IIA or NNR is violated if MB is not satisfied.

We now show that MB implies IIA. We omit the proof that MB implies NNR because that proof parallels the proof that MB implies IIA. Suppose that some VP $v(P|X)$ and its SWF u (P) satisfy MB but not IIA. Therefore a profile $P \in \mathbf{p}^n$, a second profile $P' \in p^n$, and a subset $X \subset S$ exist such that (a) $v(P|X) = \Psi_X[u(P)] = x$,

(b) $v(P'|X) = \Psi_X[u(P')] = y$, and $\theta_X(P) = \theta_X(P')$. Recall that v satisfies MB. Therefore, for each $z \in X$ such that $z \neq x$, the set of individuals $B(z) = \{j | j \in N \text{ and } x P_j z\}$ is decisive for x over z. In particular the set B(y) is decisive for x over y. Because $\theta_X(P') = \theta_X(P)$ and $(x,y) \subset X$, the set of individuals $B'(y) = \{j | j \in N \text{ and } x P'_j y\}$ is identical to the set B(y). Consequently B'(y) is decisive for x over y. This means that $x\, u(P')\, y$, which in turn means that $v(P'|X) \neq y$ since both x and y are elements of X. Therefore our original assumption that $v(P'|X) = y$ is contradicted. ||

Given a feasible set $X \subset S$ and a preference profile P, Pareto optimality requires that if an alternative $y \in X$ is unanimously ranked below a second alternative $x \in X$, then y is not the group's choice.

> Pareto Optimality (PO). A VP $v(P|X)$ satisfies PO if and only if, for any feasible set $W \subset S$ and any pair $x, y \in W$, $v(P|W) \neq y$ whenever, for all $i \in N$, $x P_i y$.

Two results may easily be derived concerning PO. First and most obvious, if a VP $v(P|X)$ satisfies PO, then it satisfies CS. Second, if a VP $v(P|X)$ satisfies R, IIA, NNR, and CS, then it satisfies PO. Suppose this latter result were not true. Let $v(P|X)$ and its underlying SWF $u(P)$ satisfy R, IIA, NNR, and CS, but not PO. Theorem 1 allows MB to be substituted for IIA and NNR. Since $v(P|X)$ does not satisfy PO, a feasible set $W \subset S$, a pair $x, y \in W$, and a profile $P \in p^n$ exist such that $v(P|W) = \Psi_W[u(P)] = y$ and, for all $i \in N$, xP_iy. Because $v(P|X)$ satisfies CS a profile $P' \in p^n$ exists such that $v(P'|W) = \Psi_W[u(P')] = x$. Condition MB therefore implies that the set $B = \{j | j \in N \text{ and } xP'_jy\}$ is decisive for x over y. Note that $P \in \mathcal{P}_{xy}(B)$. Therefore MB implies that $xu(P)y$. This implies that $v(P|W) = \Psi_W[u(P)] \neq y$, a contradiction of our assumption that v violates PO at profile P. Consequently if v satisfies R, IIA, NNR, and CS, then it must also satisfy PO. This result, the fact that PO implies CS, and Theorem 1 together imply that the four basic conditions of R, IIA, NNR, and CS are equivalent to the three conditions of R, MB and PO.

6. Strategy-proofness of Voting Procedures

In the preceding section we have followed Arrow in arguing that an acceptable VP should satisfy the five properties of R, IIA, NNR, CS, and ND. In setting up these criteria we have tacitly assumed that the VP is strategy-proof.[18] Strategy-proofness means that every individual has an incentive to report his true preferences for insertion into the VP.[19] If a VP is not strategy-proof, then an individual may on occasion have an incentive to misrepresent his true preferences with the goal of manipulating the group's choice to his personal advantage. The possibility of such manipulation creates a severe problem in the design of acceptable VPs because a VP which gives "fair" choices when individuals honestly report their preferences may, with reference to individuals' true preferences, give quite arbitrary choices when individuals falsely report their preferences.

Consequently when we set up requirements, as we did in the previous section, that a VP $v(P|X)$ have properties such as R, IIA, and NNR, then we are assuming that individuals will in fact honestly report their true preferences. If we do not make this assumption, then a theory must be constructed as to how individuals will misrepresent their preferences. Suppose, given a particular VP $v(P'|X)$, such a theory takes the form that $P' = \omega(P|X)$ where $P' \in p^n$ is the preference profile the individuals actually report for insertion into the voting procedure, $P \in p^n$ is the individual's true preference profile, X is the feasible set, and ω is the function which describes how individuals misrepresent their true preferences. Given the functions v and ω, the function we really want to evaluate for optimality is the composition of ω and v: $f(P|X) = v[\omega(P|X)|X]$.[20] For example, it is not important that the function $v(P'|X)$ satisfy rationality; what is important is that the function $f(P|X)$ satisfy rationality because only $f(P|X)$ reveals if the group's choices are rational with respect to the individuals' true preferences. Determination of the function $f(P|X)$, however, is difficult because neither the true preference profile nor the function ω are observable. A common example of this problem is the free rider problem of public finance. Mechanisms which give Pareto optimal allocations of public goods with respect to reported preferences do not necessarily give Pareto optimal allocations with respect to true preferences.

Therefore, before exploring the actual construction of voting procedures $v(P|X)$ which satisfy R, IIA, NNR, CS, and ND, we should check to see if such a voting procedure is necessarily strategy-proof. If the answer is positive, then the arguments in favor of requiring Arrow's five conditions are greatly strengthened. On the other hand, if the answer is negative, then the usefulness of constructing such a VP must be reconsidered.

In this section we prove that the answer is positive: the three conditions of R, IIA, and NNR are by themselves sufficient to insure the strategy-proofness of a VP.[21] As a result any voting procedure which satisfies all five conditions of R, IIA, NNR, CS, and ND is strategy-proof. The formal definition of strategy-proofness depends on the concept of manipulability. A VP $v(P|X)$ is <u>manipulable</u> <u>at</u> <u>profile</u> $P \in p^n$ if and only if, for some feasible set $W \subset S$ and some individual $i \in N$, a preference ordering exists such that

$$v(P/P_i'|W) \; P_i \, v(P/P_i|W) \qquad (6.1)$$

where $P/P_i' \equiv (P_1,\ldots,P_{i-1},P_i',P_{i+1},\ldots,P_n)$ and $P/P_i \equiv P \equiv (P_1,\ldots,P_{i-1},P_i,P_{i+1},\ldots,P_n)$.

Strategy-Proofness (SP). A VP $v(P|X)$ satisfies SP if and only if no profile $P \in p^n$ exists at which v is manipulable.

The VPs v_α and v_β are example of VPs which do not satisfy SP. Consider the profile

$$P_1 = (w\ x\ y\ z),\ P_2 = (x\ w\ y\ z),\ P_3 = (x\ w\ y\ z). \qquad (6.2)$$

If $X = S = \{w, x, y, z\}$, then this profile P is manipulable by individual one: $v_\alpha(P/P_1) = x$, $v_\alpha(P/P_i') = w$ where $P_i' = (w\ y\ z\ x)$, and $v_\alpha(P/P_1')\ P_1\ v_\alpha(P/P_1)$. Therefore v_α does not satisfy SP. Moreover, because $v_\beta(P|S) = v_\alpha(P|S)$ for all $P \in p^n$, the example proves also that v_β does not satisfy SP.

Interpretation of the definition for SP is straightforward. If $v(P|X)$ is not strategy-proof, then a $P/P_i \in p^n$, an $i \in N$, and a $P_i' \in p$ exist such that v is manipulable at P. Let the ordering P_i be his sincere strategy and let the ordering P_i' be a sophisticated strategy. As relation (6.1) shows, individual i can improve the outcome according to his true preferences P_i through substitution of the sophisticated strategy P_i' for his sincere strategy P_i. Alternatively, in the language of game theory, if a VP $v(P|X)$ is strategy-proof, then every sincere preference profile $P = (P_1,\ldots,P_n) \in p^n$ is a Nash equilibrium.

Theorem 2. If a VP $v(P|X)$ satisfies R, IIA, and NNR, then it also satisfies SP.

Proof. Consider a VP $v(P|X)$ and its underlying SWF $u(P)$ which satisfy R, IIA, and NNR. According to Theorem 1 we can substitute R and MB for R, IIA and NNR. Suppose v is not strategy-proof. Therefore an individual $i \in N$, a profile $P/P_i \in p^n$, an ordering P_i', and a feasible set $X \subset S$ exist such that

$$v(P/P_i'|X) \;\; P_i \;\; v(P/P_i|X). \tag{6.3}$$

Relation (5.2) may be rewritten as

$$\Psi_X[u(P/P_i')] \;\; P_i \;\; \Psi_X[u(P/P_i)] \tag{6.4}$$

because v is rational. Let $v(P/P_i'|X) = \Psi_X[u(P/P_i')] = \Psi_X(P_N') = y$ and $v(P/P_i|X) = \Psi_X[u(P/P_i)] = \Psi_X(P_N) = x$. Relation (6.3) implies that $y \; P_i \; x$. Relation (6.4) implies that $x \; P_N \; y$ and $y \; P_N' \; x$.

Because v satisfies MB and $x \; P_N \; y$, the definition of condition MB implies that the set of individuals $B = \{j | j \in N \text{ and } x \; P_j \; y\}$ is decisive for x over y. Note that individual i is not a member of B because $y \; P_i \; x$. Recall that $\mathcal{P}_{xy}(B)$ is the collection all profiles $P \in p^n$ such that, for all $j \in B$, $x \; P_j \; y$. Consequently $P/P_i' \in \mathcal{P}_{xy}(B)$. The definition of decisiveness therefore implies that $x \; P_N' \; y$. This conclusion, however, contradicts our earlier conclusion, which followed from the assumption that if v is not strategy-proof, then it can not satisfy both R and MB. ||

7. Arrow Impossibility Theorem

We have argued in the previous two sections that a voting procedure which satisfies the five conditions of R, IIA, NNR, CS, and ND is very desirable. Such a voting procedure would be strategy-proof and would allow groups to make democratic decisions in a manner which is conceptually analogous to the decision making of rational individuals. But can this desirable combination or properties be in fact achieved? Unfortunately Arrow's impossibility theorem shows that construction of such a voting procedure is impossible when the number of alternatives is at least three.[22] Thus group decisions generally cannot achieve the same level of rationality that individual decisions can.

In this section we reprove Arrow's theorem. His formulation of the theorem stated that when $|S| \geq 3$ no voting procedure exists that satisfies R, IIA, NNR, CS, and ND. We showed in Section 5 that the four conditions of R, IIA, NNR, and CS are equivalent to the three conditions of R, MB, and PO. We therefore may restate Arrow's theorem in the equivalent form which is convenient for our proof: when $|S| \geq 3$ no voting procedure exists which satisfies R, MB, PO, and ND.

The proof which we use relies heavily on the concept of dictatorial sets of individuals. For a voting procedure which satisfies R, we define a subset of individuals $B \subset N$ to be dictatorial if and only if, for every pair of alternatives $x,y \in S$, the subset B is decisive for x over y. In other words, if the subset $B \subset N$ is dictatorial and, for any arbitrary pair of alternatives $x,y \in S$, xP_iy for all $i \in B$, then $x\ u(P)y$ irrespective of the preferences of those individuals who do not belong to the subset B. Notice that when $|B| = 1$ the dictatorial set B is simply a dictatorial individual as defined in condition ND.

The plan underlying our proof is as follows.[23] Let the VP $v(P|X)$ and its underlying SWF $u(P)$ satisfy R, MB, and PO. Clearly $v(P|X)$ recognizes certain subsets of individuals as being decisive over particular pairs of alternatives. For example, PO implies that the set N is decisive for any x over any y. In Lemma 1 we show that any subset of individuals $B \subset N$ which is decisive over some pair of alternatives must be decisive over all possible pairs of alternatives. In other words, any subset B which is decisive for a particular x over a particular y must be a dictatorial subset. This is an intuitively attractive result because it states that all alternatives must be treated symmetrically by the VP $v(P|X)$; there can be no special alternatives towards which $v(P|X)$ is biased either for or against.[24]

Suppose some subset $B \subset N$ is a dictatorial subset and suppose with respect to a pair $x,y \in S$ the subset B is not unanimous. In particular, suppose that B may be partitioned into two non-empty sets B_1 and B_2 where, for all $i \in B$, $x \; P_i \; y$ and, for all $i \in B_2$, $y \; P_i \; x$. How is this conflict between factions of a dictatorial subset to be settled? Will the ordering $x \; u(P) \; y$ be selected by the VP or will $y \; u(P) \; x$ be selected? One or the other must be selected. Will the choice depend on the preferences of those individuals who are not members of either B_1 or B_2? Lemma 2 shows that either B_1 or B_2 must be a dictatorial subset. Thus, if the conflict is resolved in favor of B_1, then B_1 not only is favored in the choice of x over y, it is favored absolutely in all choices over all pairs alternatives. Subsets B_1 and B_2 can not share power. Individuals who are neither members of B_1 nor members of B_2 can never have any power to influence the selection among a set of feasible alternatives.

Recall that PO implies that the set of all individuals N is necessarily a dictatorial set. Consequently we can repeatedly partition N into smaller and smaller subsets, retaining at each repetition the dictatorial subset while discarding the non-dictatorial subset. Since N is finite this process leads eventually to identification of a dictatorial subset containing but a single individual.[25] This individual is a dictator and therefore the theorem is proved: no VP exists which satisfies R, MB, PO, and ND.

> Lemma 1. Consider a VP $v(P|X)$ which satisfies R, MB, and PO. Let $u(P)$ be its underlying SWF and let $x,y \in S$ be any pair of alternatives. If $|S| \geq 3$, then any set of individuals $B \subset N$ which is decisive for x over y is a dictatorial set.

Proof. Let $|S| \geq 3$ and let $v(P|X)$ and its underlying SWF $u(P)$ satisfy R, MB, and PO. Suppose $B \subset N$ is decisive for the pair $x,y \in S$. Pick an arbitrary $z \in S$ where $z \neq x$ and $z \neq y$. Our first step is to show that B must be decisive for x over z. Define the profile $P \in p^n$ to have the following properties:

a. $x \, P_i \, y \, P_i \, z$ for all $i \in B$ and

b. $y \, P_j \, z \, P_j \, x$ for all $j \notin B$.

B is decisive for x over y; therefore $x \, u(P) y$. For every $i \in N$, $y \, P_i \, z$; therefore PO implies $y \, u(P) z$. Transitivity of $u(P)$ then implies that $x \, u(P) z$. Notice that P is constructed such that $\{i | i \in N \text{ and } x \, P_i z\} = B$. Consequently condition MB, which by assumption is satisfied, implies that B is decisive for x over z.

Our second step is to show that B must be decisive for z over y. Define $P' \in p^n$ such that

a. $zP'_i \, x \, P'_i \, y$ for all $i \in B$ and

b. $yP'_j \, z \, P'_j \, x$ for all $j \notin B$.

Decisiveness of B for x over y implies that x $u(P')$ y and PO implies that z $u(P')$ x. Therefore, by transitivity, z $u(P')$ y. P' is constructed such that $\{i \mid i \in N \text{ and } z\, P_i' y\} = B$. Therefore, because v satisfies MB, B is decisive for z over y.

These two results can now be used repetitively to show that decisiveness on one pair $\{x,y\}$ implies decisiveness on all pairs $\{w,z\}$:

a. If B is decisive for x over y, then B is decisive for x over z and for z over y.

b. If B is decisive for x over z then B is decisive for w over z.

c. If B is decisive for z over y, then B is decisive for z over w.

Consequence (b) and (c) together imply that if B is decisive for x over y, then it is decisive for w over z and for z over w whenever the pairs $\{x,y\}$ and $\{w,z\}$ are not identical. The only remaining question concerning the dictatoriality of the set B is if decisiveness for x over y implies decisiveness for y over x. The answer is positive because decisiveness of B for x over y implies decisiveness of B for w over z which, in turn, implies decisiveness of B for y over x. ||

> Lemma 2. Consider a VP $v(P|X)$ which satisfies R, MB, and PO. Let $u(P)$ be its underlying SWF. Suppose $|S| \geq 3$ and suppose a set of individuals $B \subset N$, $|B| \geq 2$, exists which is dictatorial. If B is partitioned into two non-empty subsets B_1 and B_2, then either B_1 or B_2 is itself a dictatorial set.

Proof. Suppose that $|S| \geq 3$ and that the VP $v(P|X)$ satisfies R, MB, and PO. Let $B \subset N$, $|B| \geq 2$, be a dictatorial subset. Without loss of generality consider any triple of alternatives $\{x,y,z\} \subset S$. Arbitrarily partition B into two non-empty subsets B_1 and B_2. Define a profile $P \in P^n$ such that:

a. $xP_i y\ P_i z$ for all $i \in B_1$,

b. $zP_j x\ P_j y$ for $j \in B_2$,

c. $yP_k z\ P_k x$ for all $k \notin B$.

Clearly x u(P)y because B is dictatorial. Consequently two possibilities exist for u(P):

(i) If x u(P) y u(P) z, then B_1 is decisive for x over z because v satisfies MB and $B_1 = \{i | i \in N \text{ and } x P_i z\}$. Therefore, by Lemma 1, B_1 is dictatorial.

(ii) If z u(P) x u(P) y, then B_2 is decisive for z over y because v satisfies MB and $B_2 = \{i | i \in N \text{ and } z P_i y\}$. Therefore, by Lemma 1, B_2 is dictatorial.

The only other ordering for u(P) which would be consistent with the PO induced requirement that x u(P) y is x u(P) z u(P) y. Nevertheless this ordering leads to a contradiction and therefore is not a real possibility. By the reasoning of (i) above, x u(P) z implies that B_1 is a dictatorial set. By the reasoning of (ii) above z u(P) y implies that B_2 is a dictatorial set. This is a contradiction because, by construction, B_1 and B_2 are disjoint; consequently only one or the other can be dictatorial. Hence situation (i) where B_1 is dictatorial and situation (ii) where B_2 is dictatorial are the only two consistent possibilities. ||

This last lemma essentially proves Arrow's theorem. Let $|S| \geq 3$. Assume that $v(P|X)$ satisfies R, MB, and PO. This, as we commented earlier, is equivalent to assuming that $v(P|X)$ satisfies R, IIA, NNR, and CS. Condition PO implies that the set N of all individuals is a dictatorial set. The set N can be partitioned into two subsets B_1 and B_2 and, according to Lemma 2, necessarily one of them will be a dictatorial subset. Suppose it is B_1 that is dictatorial. B_1 can be partitioned into two subsets B_{11} and B_{12}. Lemma 2 implies that either

B_{11} or B_{12} is dictatorial. If the number of individuals contained in N is finite, then this process when repeated leads eventually to identification of a dictatorial subset which contains a single dictatorial individual.

> Theorem 3 (Arrow). Consider a VP $v(P|X)$ which satisfies R, IIA, NNR, and CS. If $|S| \geq 3$, then an individual $i \in N$ exists such that $v(P|X) = \Psi_X(P_i)$ for all $P \in p^n$ and all $X \subset S$.

> Corollary 1 (Arrow). If $|S| \geq 3$, then no VP $v(P|X)$ exists which satisfies R, IIA, NNR, CS, and ND.

8. The Impossibility Theorems for Strategy-Proof Voting Procedures

Arrow's impossibility theorem shows that group decisions are intrinsically different from individual decisions. Given this result the question is what properties should a voting procedure have. The properties of R, IIA, NNR, CS, ND, and SP, cannot be jointly achieved. Which of these properties should be abandoned in order to make the search for an acceptable, if not perfect, VP viable?

Traditional social choice theory has been primarily concerned with obtaining possibility results through the weakening of one or more of the five conditions of R, IIA, NNR, CS, and ND. For a comprehensive example of this approach, see Sen's excellent book.[26] Our approach here is different We think that regardless of how any of the first five conditions are relaxed, the sixth condition of SP should be retained if possible.[27] As shown in Section 5, if a VP does not satisfy SP, then its actual properties, as opposed to its nominal properties, become very hard to ascertain. Therefore in this section our goal is to investigate the possibility of constructing VPs subject only to the conditions of SP, CS, and ND.

The conclusion of this investigation is a theorem equally as negative as Arrow's impossibility theorem. Gibbard and Satterthwaite have shown independently that when $|S| \geq 3$ no VP $v(P|X)$ exists which satisfies SP, CS, and ND.[28] We prove this theorem in three steps using the general approach that Gibbard [7] originally used.[29] Within the first two steps we consider only VPs which belong to the class of "normal" voting procedures. Normal voting procedures are VPs whose functional forms are restricted in a natural manner which is to be formalized below. In the proof's first step we show that if a normal VP satisfies SP and CS, then it satisfies independence of non-optimal alternatives (INOA).

As we commented in Section 5, INOA is a necessary and sufficient condition for a VP to be rational.

Therefore, in the proof's second step, we use INOA to show that a normal, strategy-proof VP does reveal an asymmetric group preference ordering as the feasible set varies. Consequently any normal voting procedure which satisfies SP and CS also satisfies R. Moreover, in the same step, we show that such a voting-procedure must also satisfy IIA and NNR. Therefore any normal VP which satisfies SP and CS also satisfies R, IIA, and NNR. But Arrow's impossibility theorem states that when $|S| \geq 3$ any VP which satisfies R, IIA, NNR, and CS is dictatorial. Consequently, no normal voting procedure that satisfies SP, CS, and ND exists when $|S| \geq 3$. In the proof's third step we show that when $|S| \geq 3$ no VP of any type exists which satisfies SP, CS, and ND. Its proof follows from the result that no normal VP satisfying SP, CS, and ND exists when $|S| \geq 3$. The results of these three steps are respectively summarized as Lemma 3, Theorem 4, and Theorem 5.

A VP $v(P|X)$ is <u>normal</u> if and only if a function $\mu(P)$ exists such that $v(P|X) = \mu[\Delta_X(P)]$. Recall from Section 3 that Δ_X moves the elements of the feasible set to the top of each individual preference ordering. An intuitive justification for this definition is as follows. Consider a function $\nu(P)$ whose domain is p^n and whose range is S. In effect ν is a VP defined only for the situation where the feasible and universal sets are identical. How can ν be generalized into a VP which is defined for all feasible sets $X \subset S$? For a given feasible set $X \subset S$ a natural means to do this is to reshuffle each individual preference ordering P_i by moving, without disturbing their relative positions, all feasible alternatives to the top of the individual's ordering and moving all infeasible alternatives to the bottom. Provided ν satisfies PO, this reshuffling effectively eliminates all infeasible alternatives from contention for selection as the group's choice. This preference profile of rearranged preference orderings, which is just $P' = \Delta_X(P)$, may then be inserted into ν to calculate the outcomes. In other words, define $v(P|X)$ such that $v(P|X) = \nu[\Delta_X(P)]$. So defined, $v(P|X)$ is a normal voting procedure.

> Lemma 3. If a normal VP satisfies SP, then it satisfies INOA.

Proof. Suppose $v(P|X)$ is a normal VP that satisfies SP, but does not satisfy INOA. Therefore a profile $P \in p^n$, two feasible sets $Z \subset W \subset S$, and two alternatives $x,y \in Z$ exist such that $v(P|W) = \mu[\Delta_W(P)] = \mu(P') = x$ and $v(P|Z) = \mu[\Delta_Z(P)] = \mu(P'') = y$ where $P' = \Delta_W(P)$ and $P'' = \Delta_Z(P)$. Notice that $\mu(P') = v(P'|S) = x$ and $\mu(P'') = v(P''|S) = y$ because $\Delta_S(P') = P'$ and $\Delta_S(P'') = P''$.

We now show that v is manipulable when the feasible set is S. Consider the sequence

$$v(P_1',P_2',\ldots,P_n'|S) = x$$
$$v(P_1'',P_2',\ldots,P_n'|S) = x$$
$$\cdots$$
$$v(P_1'',\ldots,P_{i-1}'',P_i',P_{i+1}',\ldots,P_n'|S) = x = v(P^*/P_i'|S) \qquad (8.1)$$
$$v(P_1'',\ldots,P_{i-1}'',P_i'',P_{i+1}',\ldots,P_n'|S) = y' = v(P^*/P_i''|S)$$
$$\cdots$$
$$v(P_1'',\ldots,P_{n-1}'',P_n'') = y.$$

An $i \in N$ must exist such that $v(P^*/P_i'|S) = x$ and $v(P^*/P_i''|S) = y'$, $x \neq y'$, because $v(P'|S) \neq v(P''|S)$. Two possibilities exist: either $y' \in Z$ or $y' \notin Z$.

Suppose first that $y' \in Z$. Recall that $x \in Z$, $P_i' = \Delta_W(P_i)$, and $P_i'' = \Delta_Z(P_i)$. The definition of the function Δ implies that (a) if $x\ P_i\ y'$, then $x\ P_i'\ y'$ and $x\ P_i''\ y$ and (b) if $y'\ P_i\ x$, then $y'\ P_i'\ x$ and $y'\ P_i''\ x$. If (a) is the case, then $v(P^*/P_i'|S)\ P_i''\ v(P^*/P_i''|S)$. If (b) is the case, then $v(P^*/P_i''|S)\ P_i'\ v(P^*/P_i'|S)$. Thus in either case v is manipulable.

Suppose now that $y' \notin Z$. Because $x \in Z$ and $P_i'' = \Delta_Z(P_i)$, $x\ P_i''\ y'$. Therefore $v(P^*/P_i'|S)\ P_i''\ v(P^*/P_i''|S)$ and v is manipulable. Consequently, if v is normal and does not satisfy INOA, then it cannot satisfy SP. $||$

Theorem 4 states that every normal, strategy-proof VP satisfies R, IIA, and NNR. Theorem 3, Arrow's impossibility theorem, states that when $|S| \geq 3$ every VP which satisfies R, IIA, NNR and CS has a dictator. Therefore when $|S| \geq 3$ every normal strategy-proof VP which satisfies CS has a dictator. This result is stated in two alternative forms as Corollaries 2 and 3.

> Theorem 4. If a normal voting procedure satisfies SP, then it also satisfies conditions R, IIA, and NNR.
>
> Corollary 2. Consider a normal VP $v(P|X)$ which satisfies SP and CS. If $|S| \geq 3$, then an individual $i \in N$ exists such that $v(P|X) = \Psi_X(P_i)$ for all $P \in \rho^n$ and all $X \subset S$.
>
> Corollary 3. If $|S| \geq 3$, then no normal VP exists which satisfies SP, CS, and ND.

Proof of Theorem 4. First we show that a normal VP $v(P|X)$ which satisfies SP is rational. We do this by ascertaining that, for each profile $P \in p^n$, v reveals a complete, asymmetric and transitive ordering. Recall that if v is rational, then its underlying SWF $u(P)$ may be constructed as follows: for any $W \subset S$, if $v(P|W) = x$, then $x\ u(P)\ y$ for all $y \in W - \{x\}$. Moreover Lemma 3 is applicable: $v(P|X)$ satisfies INOA because it is normal and strategy-proof. Completeness of $u(P)$ is guaranteed by allowing the feasible set W to vary over all possible two element subsets of S. Asymmetry of $u(P)$ follows from INOA. Suppose asymmetry is not satisfied. Therefore a profile $P \in p^n$, two feasible sets $W, Z \subset S$, and a distinct pair $x,y \in W \cap Z$ exist such that

(a) $v(P|W) = x$ and (b) $v(P|Z) = y$. Application of INOA to (a) implies that $v(P|W \cap Z) = x$ and to (b) implies that $v(P|W \cap Z) = y$. But this is a contradiction because $v(P|W \cap Z)$ is singlevalued and must equal either x or y, not both. Thus $v(P|X)$ satisfies asymmetry.

Suppose $u(P)$ as revealed by $v(P|X)$ does not satisfy transitivity: a profile $P \in p^n$ and triple $x,y,z \in S$ exist such that $x\ u(P)\ y\ u(P)\ z\ u(P)\ x$. In other words, feasible sets $X,Y,Z \subset S$ must exist such that

$$v(P|X) = x \quad \text{and} \quad x,y \in X, \tag{8.2}$$

$$v(P|Y) = y \quad \text{and} \quad y,z \in Y, \text{ and} \tag{8.3}$$

$$v(P|Z) = z \quad \text{and} \quad z,x \in Z. \tag{8.4}$$

Let $W = \{x,y,z\}$. By definition, $v(P|W) \in W$. Suppose, without loss of generality, that $v(P|W) = z$.

Let $Y' = \{y,z\} \subset Y$. Since v satisfies INOA, $v(P|Y) = y$ implies that $v(P|Y') = y$. Similarly since $Y' \subset W$, $v(P|W) = z$ implies that $v(P|Y') = z$. But this is a contradiction: $v(P|Y')$ cannot equal both y and z. Therefore if the ordering revealed by v is not transitive, then v cannot satisfy INOA. This completes the proof that every normal, strategy-proof $v(P|X)$ reveals a SWF that is complete, asymmetric and transitive, i.e. v necessarily satisfies R.

The remaining question is: if v is normal and satisfies SP, then does it necessarily satisfy IIA and NNR as well as R. The equivalence result of Theorem 1 allows substitution of MB for IIA and NNR. Therefore suppose that $v(P|X)$ and its underlying SWF $u(P)$ are normal, satisfy SP and R, but do not satisfy MB. Therefore profiles $P', P'' \in p^n$ and distinct alternatives $x,y \in S$ exist such that $x\ u(P')\ y$, $y\ u(P'')\ x$, and $P'' \in \mathcal{P}_{xy}(B)$ where $B = \{i | i \in N \text{ and } xP_i' y\}$. Let $W = \{x,y\}$. Consider the sequence:

$$
\begin{aligned}
v(P'|W) &= \Psi_W[u(P_1',P_2',\ldots,P_n')] = x \\
&\quad \Psi_W[u(P_1'',P_2',\ldots,P_n')] = x \\
&\quad \ldots \\
&\quad \Psi_W[u(P_1'',\ldots,P_{i-1}'',P_i',P_{i+1}',\ldots,P_n')] = v(P^*/P_i'|W) = x \qquad (8.5) \\
&\quad \Psi_W[u(P_i'',\ldots,P_{i-1}'',P_i'',P_{i+1}',\ldots,P_n')] = v(P^*/P_i''|W) = y \\
&\quad \ldots \\
v(P''|W) &= \Psi_W[u(P_1'',\ldots,P_{n-1}'',P_n'')] = y
\end{aligned}
$$

At least one such $i \in N$ must exist because $v(P|W) \in W$ for all $P \in p^n$. Two possibilities exist: either (a) $xP_i'y$ or (b) $yP_i'x$. If $xP_i'y$ is the case, then $x\ P_i''\ y$ because $P'' \in \mathcal{P}_{xy}(B)$. Therefore $v(P^*/P_i'|W)\ P_i''\ v(P^*/P_i''|W)$. If $y\ P_i'\ x$ is the case, then $v(P^*/P_i''|W)\ P_i'\ v(P^*/P_i'|W)$. Thus in either case v is manipulable if $v(P|X)$ does not satisfy MB. Therefore, if v is normal and satisfies SP, v must necessarily satisfy R and MB.||

If we remove the assumption that $v(P|X)$ is normal, then we must weaken Theorem 3 correspondingly. Theorem 3 stated that if $|S| \geq 3$, $v(P|X)$ is normal, and $v(P|X)$ satisfies SP and CS, then an $i \in N$ exists such that for all $P \in p^n$ and all $X \subset S$, $v(P|X) = \Psi_X(P_i)$. If the VP satisfies SP and CS, but is not necessarily normal, then we can prove that an $i \in N$ exists such that, for all $P \in p^n$, $v(P|S) = \Psi_S(P_i)$. In other words, $v(P|X)$ may be dictatorial when the feasible set is S, but not dictatorial when the feasible set is a proper subset of S. For example, consider the following VP $v(P|X)$ which is not normal. Suppose $S = \{w,x,y,z\}$ and $N = \{1,2,3\}$. Let individual one make decisions dictatorially when X contains four elements, let individual two make decisions dictatorially when X contains three elements, and let the majority of N make decisions when X contains two elements. This VP satisfies SP, but does not fit the form $v(P|X) = \Psi_X(P_i)$.[30]

> Theorem 5 (Gibbard and Satterthwaite). Consider a VP $v(P|X)$ which satisfies SP and CS. If $|S| \geq 3$, then an individual $i \in N$ exists such that $v(P|S) = \Psi_S(P_i)$ for all $P \in p^n$.

Corollary 4. If $|S| \geq 3$, then no VP $v(P|X)$ exists which satisfies SP, CS, and ND.

Proof of Theorem 5. Let $|S| \geq 3$. Suppose $v(P|X)$ satisfies SP and CS. Therefore, for feasible set S, no $P \in p^n$ exists at which $v(P|S)$ is manipulable. Define the function $\nu(P|X)$ such that, for all $P \in p^n$ and all $W \subset S$, $\nu(P|W) = v[\Delta_W(P)|S]$. Assume for the moment that ν is a legitimate, normal VP. In particular, assume that, for all $P \in \rho^n$ and all $W \subset S$, $\nu(P|W) \in W$. The strategy-proofness of $v(P|X)$ suggests that ν is also SP. The truth of this can be ascertained by supposing that ν does not satisfy SP. Therefore, an $i \in N$, a $P \in p^n$, a $P_i' \in p$, and a $W \subset S$ exist such that

$$\nu(P/P_i'|W) \; P_i \; \nu(P/P_i|W). \tag{8.6}$$

Substitution gives

$$v[\Delta_W(P/P_i')|S] \quad P_i \quad v[\Delta_W(P/P_i)|S]. \tag{8.7}$$

Let $v[\Delta_W(P/P_i')|S] = v[P^o/P_i^{o\prime}|S] = x$ and $v[\Delta_W(P/P_i)|S] = v[P^o/P_i^o|S] = y$ where $P_i^{o\prime} = \Delta_W(P_i')$ and $P^o = \Delta_W(P)$. Relation (8.7) implies that $x \; P_i \; y$. Consequently $x \; P_i^o \; y$ because $x,y \in W$, $P_i^o = \Delta_W(P_i)$, and the function Δ_W preserves the ordering of the elements of W. Therefore substitution into (8.7) gives $v[P^o/P_i^{o\prime}|S] \quad P_i^o \quad v[P^o/P_i^o|S]$. Thus, if ν did not satisfy SP, then v would also not satisfy SP. Therefore ν satisfies SP.

Because $|S| \geq 3$ and ν is normal and satisfies SP, Theorem 3 implies that some $i \in N$ exists such that $\nu(P|X) = v[\Delta_X(P)|S] = \Psi_X(P_i)$ for all $P \in p^n$. Let $P' = \Delta_X(P)$ and $P_i' = \Delta_X(P_i)$. Therefore $v(P'|S) = \Psi_X(P_i)$. Note that $\Psi_X(P_i) = \Psi_S[\Delta_X(P_i)] = \Psi_S(P_i')$. Hence $v(P'|S) = \Psi_S(P_i')$. In other words, $v(P|X)$ has a dictator when the feasible set is S.

This conclusion depends on the validity of our assumption that $\nu(P|X)$ is a normal VP. We can show that this assumption is true when v satisfies SP and CS by supposing that ν is not a normal VP. This can only be true if a $P' \in p^n$ and a $W \subset S$ exist such that $\nu(P'|W) \notin W$. The first possibility we must consider is the extreme one that, for some $W \subset S$ and all $P \in P^n$, $\nu(P|W) \notin W$. Since v satisfies CS, a profile $P' \in p^n$ exists such that $v(P'|S) = x \in W$. Nevertheless, by assumption, $\nu(P'|W) = v[\Delta_W(P')|S] \notin W$. Let $P'' = \Delta_W(P')$. Consider the sequence

$$v(P_1'', P_2'', \ldots, P_n''|S) \notin W$$

$$v(P_1', P_2'', \ldots, P_n''|S) \notin W$$

$$\ldots$$

$$v(P_1', \ldots, P_{i-1}', P_i'', P_{i+1}'', \ldots, P_n''|S) = v(P^*/P_i''|S) = y \notin W \qquad (8.8)$$

$$v(P_1', \ldots, P_{i-1}', P_i', P_{i+1}'', \ldots, P_n''|S) = v(P^*/P_i'|S) = x' \in W$$

$$\ldots$$

$$v(P_1', \ldots, P_{n-1}', P_n'|S) = x \in W.$$

Notice that $x' \; P_i'' \; y$ because $P_i'' = \Delta_W(P_i')$, $y \notin W$, and $x' \in W$. Therefore $v(P^*/P_i'|S) \; P_i'' \; v(P^*/P_i''|S)$ which contradicts the assumption that v satisfies SP. Consequently, for all $W \subset S$, there must exist a $P' \in p^n$ such that $\nu(P'|W) \in W$.

Now consider the less extreme possibility that, for some $P' \in p^n$ and some $W \subset S$, $\nu(P'|W) = v(\Delta_W(P')|S) \notin W$. The result derived immediately above states that a $P'' \in p^n$ exists such that $\nu(P''|W) = v[\Delta_W(P'')|S] = x \in W$. Let $P^{o\prime} = \Delta_W(P')$ and let $P^{o\prime\prime} = \Delta_W(P'')$. Consideration of a sequence like (8.8) shows that an $i \in N$ must exist such that

$$v(P^*/P_i^{o\prime}|S) = y \notin W \tag{8.9}$$

$$v(P^*/P_i^{o\prime\prime}|S) = x' \in W \tag{8.10}$$

Notice that $x' \; P_i^{o\prime} \; y$ because $P_i^{o\prime} = \Delta_W(P_i')$, $y \notin W$, and $x' \in W$. Therefore $v(P^*/P_i^{o\prime\prime}|S) \; P_i^{o\prime} \; v(P^*/P_i^{o\prime}|S)$, a contradiction of the assumption that v satisfies SP. Therefore if v satisfies SP and CS, then, for all $W \subset S$ and all $P \in p^n$, $\nu(P|W) = v[\Delta_W(P)|S] \in W$, i.e. ν is a legitimate, normal VP. ||

9. An Equivalence Theorem

Theorem 2 states that any VP which satisfies R, IIA, and NNR also satisfies SP. Theorem 4 states that a normal VP which satisfies SP also satisfies R, IIA, and NNR. These two results suggest that in some sense condition SP is equivalent to conditions R, IIA, and NNR. This suggestion is correct: a VP $v(P|X)$ satisfies R, IIA, NNR, and CS if and only if it is normal and satisfies SP and CS. Therefore, if our interest is limited only to those "well behaved" VPs that are normal and satisfy CS, then SP is in fact equivalent to R, IIA, and NNR. In other words, the inability of groups to make decisions that satisfy R, IIA, and NNR and the incentives which necessarily exist within groups for individuals to misrepresent their preferences are two views of the same phenomenon.

This equivalence allows the impossibility theorems of Arrow and of Gibbard and Satterthwaite to be mutually derived from each other. In this paper we proved Arrow's theorem directly and from it, using the method of Gibbard, derived Gibbard and Satterthwaite's theorem. Symmetrically Satterthwaite proved Gibbard and Satterthwaite's theorem constructively and from it derived the Arrow's theorem.[31]

> Theorem 6. A VP $v(P|X)$ satisfies R, IIA, NNR, and CS if and only if it is a normal VP which satisfies SP and CS.

This theorem is most closely related to a correspondence theorem of Satterthwaite.[32] It is also related to a formal result of Pattanaik and to the less formal, but prescient, discussion of Vickrey.[33]

Proof of Theorem 6. Theorem 2 states that any VP which satisfies R, IIA, and NNR also satisfies SP. Theorem 4 states that a normal VP which satisfies

SP also satisfies R, IIA, and NNR. Consequently what we need to show in order to prove the theorem is that any VP $v(P|X)$ which satisfies R, IIA, and NNR is normal. We can do this if we assume that $v(P|X)$, in addition to satisfying R, IIA, and NNR, satisfies CS.

The demonstration that R, IIA, NNR, and CS imply that $v(P|X)$ is normal works as follows. Let $v(P|X)$ satisfy R, IIA, NNR, and CS and let $u(P)$ be the SWF which underlies $v(P|X)$. Note that the definitions of θ and Δ together imply that, for all $P \in p^n$ and $W \subset S$, $\theta_W[\Delta_W(P)] = \theta_W(P)$. Therefore, for all $P \in p^n$ and all $W \subset S$,

$$v(P|W) = \Psi_W[u(P)] = \Psi_W\{u[\Delta_W(P)]\} = v[\Delta_W(P)|W)] \tag{9.1}$$

because $\theta_W(P) = \theta_W[\Delta_W(P)]$ and v satisfies IIA.

In Section 5 we showed that if $v(P|X)$ satisfies R, IIA, NNR, and CS, then it also satisfies PO. Suppose $v(P|X)$ satisfies PO, but $v[\Delta_W(P)|S] = \Psi_S\{u[\Delta_W(P)]\} \notin W$ for some $P \in p^n$ and some $W \subset S$. Let $\Delta_W(P) = P'$. The definition of Δ implies that, for all $x \in W$, for all $y \notin W$, and for all $i \in N$, $x\ P_i'y$. PO requires that if a $x \in W$ exists such that, for all $i \in N$, $x\ P_i\ y$, then $v(P|W) \neq y$. Consequently $v[\Delta_W(P)|S] = v(P'|S) \notin W$ violates PO. Therefore, since $v(P|X)$ satisfies PO, $\Psi_S\{u[\Delta_W(P)]\} \in W$.

Because $\Psi_S\{u[\Delta_W(P)]\} \in W$, $\Psi_W[u[\Delta_W(P)]] = \Psi_S\{u[\Delta_W(P)]\}$ for all $P \in p^n$ and all $W \subset S$. Substitution into (8.1) therefore gives

$$v(P|W) = \Psi_W\{u[\Delta_W(P)]\} = \Psi_S\{u[\Delta_W(P)]\} = v[\Delta_W(P)|S]. \tag{8.2}$$

In other words, $v(P|X)$ is a normal VP because it can be written in the form of a normal VP.

Therefore we have a generalization of Theorem 2: if $v(P|X)$ satisfies R, IIA, NNR, and CS, then it is normal, satisfies SP, and satisfies CS. A trivial generalization of Theorem 4 is: if $v(P|X)$ is normal and satisfies SP and CS, then it satisfies R, IIA, NNR, and CS. Together these two results imply Theorem 6.||

10. Conclusions

The model analyzed in this paper can easily be criticized on at least five grounds.

1. The set of admissible feasible sets includes all subsets X contained in the universal set S. Within many models economic theory suggests that this set should be restricted, for example, to "budget triangles."

2. Our definition of a voting procedure requires that in every circumstance one and only one alternative be selected as the group choice. This rules out the possibility that whenever two alternatives are "tied" in the sense of receiving exactly symmetrical rankings within the profile of revealed preferences, then one of the two alternatives is chosen by means of a fair lottery.

3. The Nash equilibrium concept we use in defining strategy-proofness excludes the possibility that an individual who, in the Nash sense, can manipulate a voting procedure at some preference profile will decide not to play the sophisticated strategy because other individuals can counter his manipulation with their own sophisticated strategies. Moreover the Nash concept only considers manipulation by individuals; it does not consider manipulation by coalitions.

4. The set of admissible individual preference orderings does not allow individuals to be indifferent between alternatives.

5. The set of admissible individual preference orderings includes all possible strict orderings of the alternatives. Within many models economic theory suggests that all preference orderings which do not satisfy a convexity requirement should be excluded from the set of admissible preference orderings.

In addition to these five criticisms a voluminous literature exists which criticizes and defends the model that Arrow originally postulated in Social Choice and Individual Values. Consideration of that literature is beyond this paper's scope. For a review of it, see Fishburn's review paper and Sen's book.[34]

In this section we consider the five points listed above. Our concern is the robustness of the results which we have presented in this paper. The conclusions which we arrive at are easily summarized. The impossibility theorems of Arrow and of Gibbard and Satterthwaite are quite robust when the model's assumptions are weakened. The robustness of our equivalence theorem (Theorem 6), however, is uncertain. Further work is necessary on it.

The first point listed above concerns our assumption that every set $X \subset S$ is admissible as a feasible set. This is quite likely a significant assumption in regard to the equivalence result of Theorem 6.[35] The reason for our belief has two parts. First, revealed preference theory plays an important role in the proof of Lemma 3 and thus, indirectly, in the proof of Theorem 6. Second, the assumption of unrestricted feasible sets is critical in revealed preference theory as comparison of Sen, who made our assumption, and Richter, who made a more restrictive assumption, shows.[36]

The second point concerns our requirement that VPs select a single alternative. The two impossibility theorems appear very robust when this assumption is relaxed. Arrow's original statement of his theorem allowed set-valued, as opposed to singlevalued VPs. Barbera, Gibbard, and Kelly have used different approaches to extend Gibbard and Satterthwaite's theorem to the case of VPs which admit lotteries among the several alternatives as their outcomes.[37] The robustness of Theorem 6, our equivalence theorem, is not clear. Speaking somewhat approximately, Barbera showed that if set-valued VPs are admissable, then condition MB is a necessary, but not sufficient, condition for a VP to satisfy SP.[38]

The third point concerns the nature of our Nash equilibrium definition of strategy-proof voting procedures. Pattanaik considered several plausible equilibrium concepts where an individual does consider the possibility that other individuals will counter his sophisticated strategy with sophisticated strategies of their own. For each of these equilibrium concepts he proved a generalization of the impossibility theorem of Gibbard and Satterthwaite. In addition, Pattanaik has investigated the effects of altering the definition of strategy-proofness so that a strategy-proof voting procedure not only must prevent manipulation by individuals, but must also prevent it by coalitions. Not surprisingly he has shown that such strengthening of the definition reinforces the impossibility result.[39]

The fourth point concerns our limitation of admissible individual preferences to strict orderings. We make this assumption, which excludes the possibility of an individual being indifferent between alternatives, primarily because it simplifies our exposition. In their initial papers Arrow, Gibbard, and Satterthwaite each proved their impossibility results for the case where indifference is admissible. Moreover, Satterthwaite's correspondence theorem provides a guide for generalizing Theorem 6 (our equivalence theorem) to the case where individual indifference is admissible.[40]

The fifth and last point concerns our assumption that the set of admissible preferences includes every possible strong ordering of the elements in the universal set. This is a strong assumption because it prevents the imposition of a convexity requirement or any other reasonable structure onto individual preferences. A great deal of research has been done concerning how much this assumption must be weakened in order to convert Arrow's theorem from an impossibility result to a possibility result. For example, single-peaked preferences, value-restricted preferences, extremal restricted preferences, and limited agreement restricted preferences are each sufficient to guarantee that majority rule satisfies R, IIA, NNR, CS, and ND. For a review of this literature and some of its implication see Kramer.[41]

Pattanaik has shown, in effect, that if a VP $v(P|X)$ satisfies conditon R, IIA, NNR, then it is strategy-proof no matter how much the set of admissible preference profiles is restricted.[42] Thus, if both admissible true preferences and admissible revealed preferences are restricted sufficiently to allow construction of a VP $v(P|X)$ which satisfies R, IIA, NNR, CS, and ND, then that VP is also strategy-proof. For example, majority rule is strategy-proof when preferences are single-peaked. If, however, a less rigorous restriction than single-peakedness is placed on admissible preferences and admissible revealed preferences, then the impossibility result may continue to hold. For example, Hurwicz has shown an impossibility result for a case where preferences are restricted to be convex and selfish.[43]

It is not clear whether Theorem 6 (the equivalence theorem) remains valid when preferences are restricted. The result of Pattanaik reported immediately above establishes the implication in one direction.[44] Nevertheless, to our knowledge, no proof has been constructed which shows that, when the admissible preference set is restricted, a normal VP which satisfies SP and CS also satisfies R, IIA, and NNR. The problem which prevents our proof of that result for the unrestricted preference case from being successfully generalized arises in the proof of Lemma 3. Suppose a profile of strict orderings $P \in p^n$ is an admissible preference profile. The proof of Lemma 3 requires that, for any $W \subset S$, the profile $P' = \Delta_W(P)$ also be an admissible preference profile. When the set of admissible preference orderings is restricted there is no guarantee that P' falls within the admissible set. Consequently, when admissible preferences are restricted, it is an open question whether a normal VP which satisfies SP and CS also satisfies R, IIA, and NNR.

FOOTNOTES

[1] (2nd ed.; New Haven and London: Yale University Press, 1963).

[2] Arrow, Social Choice and Individual Values, p. 54.

[3] Allan Gibbard, "Manipulation of Voting Schemes: A General Result," Econometrica, XLI (July, 1973), 587-602; Mark A. Satterthwaite, "The Existence of a Strategy Proof Voting Procedure: A Topic in Social Choice Theory," (unpublished Ph.D. dissertation, University of Wisconsin-Madison, 1973); Mark A. Satterthwaite, "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions." Journal of Economic Theory, X (April, 1975), 187-217.

[4] "Strategy-Proofness and Arrow's Conditions," Theorem 2.

[5] Arrow, Social Choice and Individual Values.

[6] Arrow, Social Choice and Individual Values; Gibbard, "Manipulation of Voting Schemes;" Satterthwaite, "Strategy-proofness and Arrow's Conditions."

[7] This definition of VP is different from the previous definitions of Gibbard ("Manipulation of Voting Schemes") and Satterthwaite ("Strategy-Proofness and Arrow's Conditions"). Their definitions endow each VP with only a single argument: the profile of revealed preferences P_i. Here we follow the useful lead of Edi Karni and David Schmeidler ("Independence of Non-Feasible Alternatives and Independence of Non-Optimal Alternatives," forthcoming in Journal of Economic Theory) who defined VPs to include the feasible set X as a second argument.

[8] Robin Farquharson in Theory of Voting (New Haven: Yale University Press, 1969) introduced the terms sincere strategy and sophisticated strategy.

[9] Gibbard ("Manipulation of Voting Schemes," Section 2) makes an excellent defense of this assumption.

[10] A substantial literature exists concerning the Borda Count and its variants, e.g. Peter Gardenfors, "Positionalist Voting Functions," Theory and Decision, IV (1973), 1-24.

[11] The tie-breaking feature of $v_\alpha(P|X)$ works as follows. Suppose the profile of revealed preferences is the voting paradox: $P_1 = (x\ y\ z)$, $P_2 = (z\ x\ y)$, $P_3 = (y\ z\ x)$. Each alternative receives a total of three points. Therefore individual one's revealed preferences are controlling: $v_\alpha(P|\{x,y,z\}) = x$, $v_\alpha(P|\{y,z\}) = y$, etc.

[12] We define independence of irrelevance alternatives in the same manner as Arrow (Social Choice and Individual Values). Other authors in other contexts have defined the term differently. For a history of the term, see Paramesh Ray, "Independence of Irrelevant Alternatives," Econometrica, XLI (Sept. 1973), 987-91.

[13] We adopt the label "independence of non-optimal alternatives" from Karni and Schmeidler ("Non-Feasible Alternatives and Non-Optimal Alternatives").

[14] The conditions R, IIA, NNR, CS, and ND contained in this paper are equivalent to the conditions which Arrow (Social Choice and Individual Values) defined. The only difference between his conditions and ours is that he defined his conditions exclusively in terms of SWFs while we define the conditions primarily in terms of VPs. In this respect our presentation is similar to that of Bengt Hansson, "Voting and Group Decision Functions," Synthese, XX (1969), 526-37.

[15] Theorem 5 in Amartya K. Sen, "Choice Functions and Revealed Preference," Review of Economic Studies, XXXVIII (1971), 307-317. Our Condition INOA is identical to Sen's Property α. The sufficiency of INOA for R depends critically on the VP's singlevaluedness.

[16] Arrow, Social Choice and Individual Values.

[17] We implicitly introduce the notation that $(P^*/P_j) \equiv (P'_1,\ldots,P'_{j-1}, P_j, P_{j+1},\ldots,P_n)$ and $(P^*/P'_j) \equiv (P'_1,\ldots,P'_{j-1}, P'_j, P_{j+1},\ldots,P_n)$.

[18] Arrow (Social Choice and Individual Values, p. 7) explicitly recognized that he was making this assumption.

[19] No term for strategy-proofness is standard throughout the literature. Gibbard ("Manipulation of Voting Schemes") calls it non-manipulability, Hurwicz calls it incentive compatibility, and Schmeidler and Sonnenschein call it cheat proofness. See Leonid Hurwicz, "On Informationally Decentralized Systems" in Decision and Organization, ed. by C. B. McGuire and R. Radner (Amsterdam: North-Holland, 1972), 297-336; David Schmeidler and Hugo Sonnenschein, "The Possibility of a Cheat Proof Social Choice Function: A Theorem of A. Gibbard and M. Satterthwaite," (Discussion Paper No. 89, Center for Mathematical Studies in Economics and Management Science, Northwestern University, May 1974).

[20] For discussion of this problem from two different points of view, see Jean Blin and Mark Satterthwaite, "An Impossibility Theorem for Deterministic Organizations," (Discussion Paper No. 129, Center for Mathematical Studies in Economics and Management Science, Northwestern University, January 1975); Theodore Groves and John Ledyard, "A Incentive Mechanism for Efficient Resource

Allocation in General Equilibrium with Public Goods," (Discussion Paper No. 119, Center for Mathematical Studies in Economics and Management Science, Northwestern University, December 1974).

21 This result is essentially a restatement of Lemma 7 in Satterthwaite, "Strategy-proofness and Arrow's Conditions."

22 Arrow, Social Choice and Individual Values, p. 59.

23 Many proofs of Arrow's theorem have been constructed. See, for example, Julian H. Blau, "The Existence of Social Welfare Functions," Econometrica, XXV (1957), 302-313; Peter C. Fishburn, "Arrow's Impossibility Theorem: Concise Proof and Infinite Voters," Journal of Economics Theory, II (1970), 103-106; Amartya K. Sen, Collective Choice and Social Welfare, (San Francisco: Holden-Day: 1970), ch. 3*.

24 This property is known as "neutrality." It was introduced in the context of majority rule by Kenneth O. May, "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision," Econometrica, XX (1952), 680-684.

25 If N contained an infinite number of individuals, then this process would not necessarily identify a dictatorial individual. Fishburn (Arrow's Impossibility Theorem") and Alan P. Kirman and Dieter C. Sondermann ("Arrow's Theorem, Many Agents, and Invisible Dictators," Journal of Economic Theory, V [Oct. 1972], 267-277) have used this fact to show that Arrow's theorem is not strictly valid when the number of individuals is infinite.

26 Collective Choice and Social Welfare.

27 The strength of our argument in favor of condition SP depends critically on our implicit assumption that individuals are in some manner being asked to reveal their preferences. If we drop this assumption, then the question of

weakening the Arrow conditions becomes paramount. For example, an official who wishes to make consistent decisions based on his perceptions of his constitutents' preferences would be interested in how the Arrow conditions may be weakened to obtain possibility results. Amartya K. Sen ("Social Choice Theory: A Re-examination," [paper presented at third world congress of the Econometric Society, Toronto, Canada, August 21, 1975]) calls the former problem the "committee decision" problem and the latter problem the "social welfare judgement" problem.

[28] Gibbard, "Manipulation of Voting Schemes"; Satterthwaite, "Existence of a Strategy Proof Voting Procedure"; Satterthwaite, "Strategy-proofness and Arrow's Conditions."

[29] Gibbard, "Manipulation of Voting Schemes."

[30] If $v(P|W)$ satisfies SP and CS, then, for each $W \subset S$ such that $|W| \geq 3$, an $i \in N$ exists such that $v(P|W) = \Psi_W(P_i)$ for all $P \in p^n$. For a proof of this, in a slightly different context, see Satterthwaite, "Strategy-proofness and Arrow's Conditions," p. 202.

[31] Gibbard, "Manipulation of Voting Schemes"; Satterthwaite, "Strategy-proofness and Arrow's Conditons."

[32] Satterthwaite, "Strategy-proofness and Arrow's Conditions," Theorem 2.

[33] Prasanta K. Pattanaik, "On the Stability of Sincere Voting Situations," Journal of Economic Theory, VI (1973), 558-574, Theorem 2: William Vickrey, "Utility, Strategy, and Social Decision Rules," Quarterly Journal of Economics, 74 (1960), 507-535.

[34] Peter C. Fishburn, "Social Choice Functions," SIAM Review, XVI (1974), 63-90; Sen, Collective Choice and Social Welfare.

[35] This assumption is clearly not important in regard to the impossibility theorem of Gibbard and Satterthwaite because Satterthwaite ("Strategy-proofness and Arrow's Conditions") proved it directly for the case of an arbitrary feasible set.

[36] Sen, "Choice Functions and Revealed Preference"; M. K. Richter, "Revealed Preference Theory," Econometrica, XXXIV (1966), 635-645.

[37] Salvador Barbera, "Strategy and Social Choice," (unpublished Ph.D. dissertation, Northwestern University, 1975), p. 21; Allan Gibbard, "Manipulation of Schemes that Mix Voting with Chance," Discussion Paper No. 175, Center for Mathematical Studies in Economics and Management Science, Northwestern University, Oct. 1975; Jerry S. Kelly, "Strategy-proofness and Social Choice Functions with Large Image Sets," Syracuse University, mimeographed, 1975.

[38] Barbera, "Strategy and Social Choice," pp. 69-83.

[39] Pattanaik, "On the Stability of Sincere Voting Situations"; Prasanta K. Pallanaik, "Threats, Counter-threats, and Strategic Voting," forthcoming in Econometrica; Prasanta K. Pattanaik, "Counter-threats and Strategic Manipulation under Voting Schemes," Working Paper No. 130, Delhi School of Economics, University of Delhi, April 1974; Prasanta K. Pattanaik, "Collective Rationality and Strategy-proofness of Group Decision Rules," Working Paper No. 138, Delhi School of Economics, University of Delhi, August 1974.

[40] Arrow, Social Choice and Individual Values; Gibbard, "Manipulation of Voting Schemes;" Satterthwaite, "Strategy-proofness and Arrow's Conditions," Theorems 1′ and 2′.

[41] Gerald H. Kramer, "On a Class of Equilibrium Conditions for Majority Rule," Econometrica, XLI (March 1973), 285-97.