Discussion Paper No. 182

MODELS OF NEW PRODUCT DIFFUSION
THROUGH ADVERTISING AND WORD-OF-MOUTH

by

Joe A. Dodson, Jr. and Eitan Muller*

Revised
April 1976

*We wish to acknowledge our appreciation for helpful comments by Nancy Schwartz on earlier drafts and programming assistance by Jerry Wharton.

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A simple growth model for the timing of initial purchase of new products is developed. A special case of the model has been tested against data for consumer durables by Bass (1969). The model is extended by incorporating the effects of repeat purchasing and thus is applicable to frequently purchased products.

The model postulated here is powerful in the sense that it is based on a minimum of premises. Yet, these premises lead to a sales response curve which is characteristic of that observed for a broad range of products.

The theoretical framework for the model is derived from the stochastic models which have found widespread application in epidemiology. The behavioral assumptions are made explicit and the implications of these assumptions for the shape of the product growth curve are derived.

I. Growth Model Without Repeat Sales

Assume that we are dealing with a new product that has just entered the market. We are concerned with predicting buyer purchase timing. That is, we are interested in the growth curve of initial purchases of the product. At some initial state, z_0 people have already bought the product. Defining $z(t)$ as the cumulative number of purchasers at time t, $z(0) = z_0$. This number is of course an integer, and it will change over time by integral amounts. Nonetheless, if $z(t)$ is large, we can reasonably consider it to be a continuous variable in the sense that the change in $z(t)$ over a small time interval will be small compared to the magnitude of $z(t)$.

We begin with a plausible assumption that the rate at which consumers buy the product is proportional to the number who have already made a purchase.

$$(1) \qquad \frac{dz(t)}{dt} = \gamma z(t)$$

where γ is the constant conversion rate.

What does this simple premise imply about the time history of the growth in cumulative sales? The solution to equation (1) is

$$(2) \qquad z(t) = z_0 e^{\gamma t}$$

which produces exponential growth in the number of purchasers. (See Figure 1.)

The early history of the growth curve for many populations corresponds very closely to the exponential growth predicted. However, data obtained over longer time periods is likely to exhibit a definite leveling-off effect. This effect can be obtained by assuming a fixed population of consumers, N. Defining $x(t)$ as the number of potential purchasers of the product,

$$(3) \qquad x(t) + z(t) = N.$$

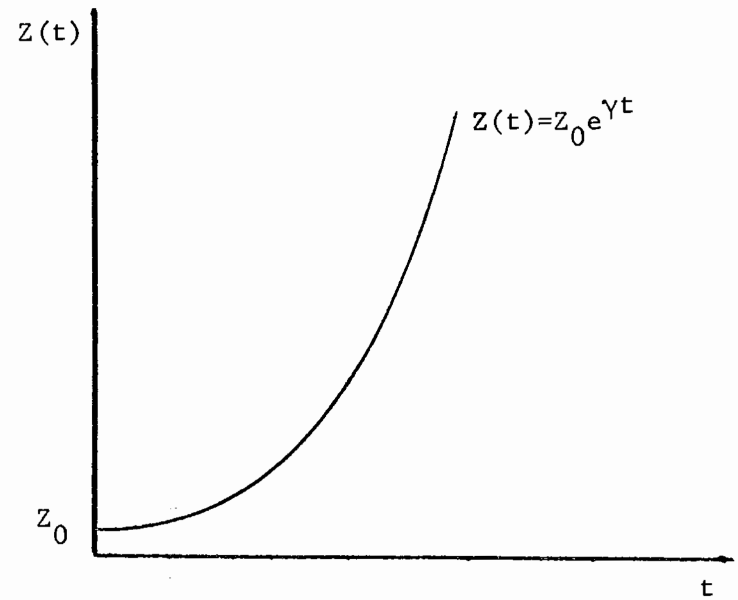

Figure 1
Constant Proportionate Growth Rate of Purchasers

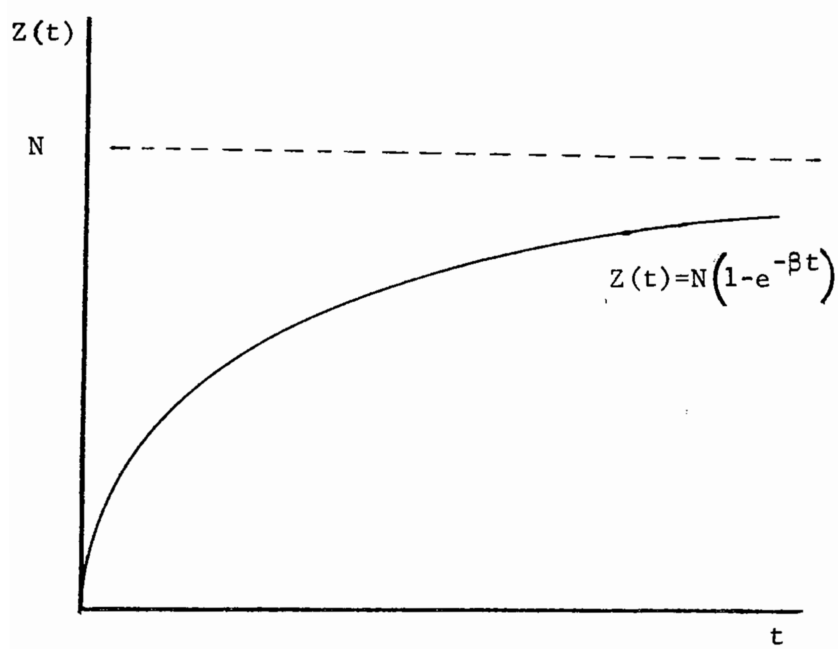

Figure 2
Constant Proportionate Decay of Nonpurchasers

Let us suppose in place of (1) that

$$(4) \qquad \frac{dx(t)}{dt} = - \quad \beta x(t)$$

that is, that a constant proportion of the potential consumers purchase the product each period. Equation (4) implies

$$(5) \qquad z(t) = N(1-e^{-\beta t})$$

given that X(0)=N. (See Figure 2.)

This type of exponential growth to some asymptote has been suggested in the sales models by Haines (1964) and Fourt and Woodlock (1960). Yet neither of these models can account for the type of growth pattern found in Figure 3. This type of pattern is characteristic of the growth curves found throughout the literature on adoption and diffusion of new ideas and new products. The concepts underlying such a pattern remain imprecisely defined and the models are generally verbal. The predominant theories are discussed by Rogers (1962).

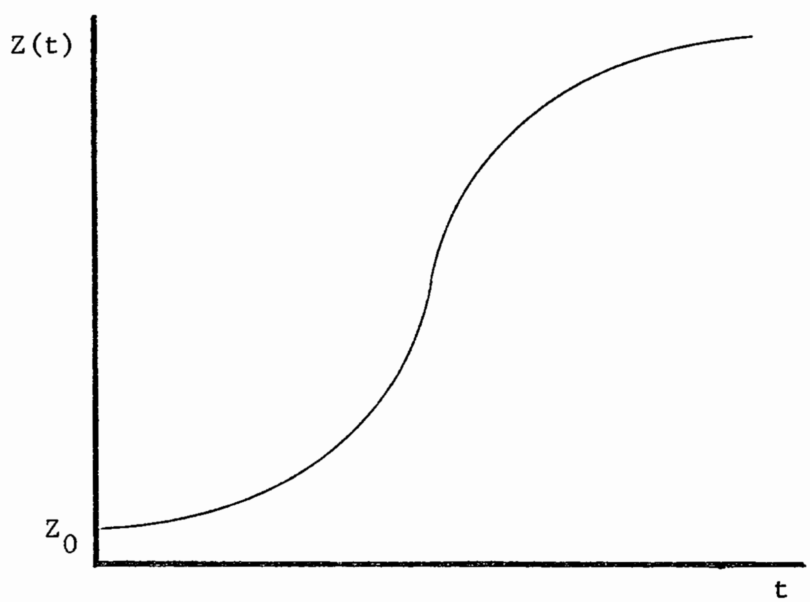

Figure 3
"Typical" Adoption Pattern

A model proposed by Lavidge and Steiner (1961) suggests that consumers move through stages, described by Palda (1966) as a hierarchical sequence of effects. This sequence of effects can be represented in the following form. Let

$N(t)$ be the number of people in the market at time t,

$x(t)$ be the number of potential buyers who are unaware of the existence of the product, at time t,

$y(t)$ be the number of potential buyers who become aware of the product but do not purchase the product, and

$z(t)$ be the number of customers who have purchased the product.

By definition

$$x(t) + y(t) + z(t) = N(t) \tag{6}$$

Consider the simple case where there are a fixed number, N, of potential consumers of a product. After buying the product, a purchaser is removed from the group of potential consumers. This model may be descriptive of such products as color television sets, refrigerators, washing machines, and other types of consumer durables with longer replacement cycles. To be considered as a realistic model of such a market, the time frame over which sales would be predicted would have to be shorter than the replacement cycle and the size of the market N would have to be stable. Nonetheless, these assumptions do not seem excessively restrictive for the type of products mentioned above.

One variable influencing the rate of adoption of a new product is word-of-mouth (Whyte, 1954). Early adopters of a new product or new idea interact with other less innovative members of the group. The potential adopters are influenced in their purchase timing by the early adopters of the product. Also, a certain proportion of potential customers may purchase the product independently of the influence of word-of-mouth. This inducement to purchase may represent the effects of distribution, promotion, advertising, and other forms of marketing effort.

The general model can be formulated as

$$\text{(7a)} \qquad \frac{dx(t)}{dt} = -\beta x(t)\big(y(t)+z(t)\big) - \mu x(t)$$

$$\text{(7b)} \qquad \frac{dy(t)}{dt} = \beta x(t)\big(y(t)+z(t)\big) + \mu x(t) - \gamma y(t)$$

$$\text{(7c)} \qquad \frac{dz(t)}{dt} = \gamma y(t)$$

where β reflects the impact of word-of-mouth and μ and γ reflect the effects of the marketing efforts of the firm. More precisely, the explanation of this set of equations is:[1]

(a) The people who know $y(t)+z(t)$ contact and inform a total of $b\big(y(t)+z(t)\big)$, out of which only a fraction $x(t)/N$ are newly informed. Thus $\beta=b/N$. In addition, out of the total number of people informed through advertising μN, only a fraction $x(t)/N$ are newly informed.

(b) The number of people who know but did not yet buy is increased by those newly informed $\beta x(t)\big(y(t)+z(t)\big) + \mu x(t)$ and decreased by those who buy $\gamma y(t)$.

(c) The number of people who buy the product is $\gamma y(t)$. Note that in the case of a durable, if we assume that each consumer buys exactly one unit, equation (c) describes the sales, i.e., denoting the sales by $s(t)$, we have

$$s(t)=\frac{dz(t)}{dt}\ .$$

To solve this system of equations, we begin by solving (7a). Since $x(t) + y(t) + z(t) = N$, (7a) can be rewritten as

$$\frac{dx(t)}{dt} = -\ \beta x(t)\left(N-x(t)\right) - \mu x(t) \tag{8}$$

Thus,

$$\frac{1}{(\beta x(t)-\rho)\,x(t)} \cdot \frac{dx(t)}{dt} = 1 \tag{9}$$

where $\rho = \beta N + \mu$ and

$$x(t) = \frac{\rho e^{-\rho t}}{\left(\mu/N\right)+ \beta e^{-\rho t}} \tag{10}$$

To solve (7c), note that

$$\frac{dz(t)}{dt} = \gamma\left(N-x(t)-z(t)\right) \tag{11}$$

or

$$\frac{dz(t)}{dt} + \gamma z(t) = \gamma\left(N-x(t)\right) \tag{12}$$

The solution to (11) or (12) is

$$s(t) = \left(e^{-\gamma t}\right) \gamma N \left(1+\gamma\rho \int_0^t \frac{e^{\gamma\tau}}{b+\mu e^{(b+\mu)\tau}}\, d\tau\right) - \frac{\gamma N\rho}{b+\mu e^{(b+\mu)t}} \tag{13}$$

where $b = \beta N$. The exact shape of the sales curve depends on the relative values of the model parameters. However, if a solution exists for $s'(t)=0$, then two possibilities exist and are illustrated in Figures 4 and 5. For $b<\mu+\gamma$, the growth in sales can be represented by the curve in Figure 4. If $b>\mu+\gamma$, there are two inflection points, see Figure 5 .

The differences in the shape of the sales curve depend upon the relationship between b and $\mu+\gamma$. b is the contact coefficient and reflects the relative impact of the diffusion process on the growth in sales. μ and γ reflect the impact of a firm's marketing efforts, which independently influence the growth in sales. Thus, the nature of the growth pattern depends upon the relative importance of word-of-mouth versus marketing effort in obtaining new product trial.

At this point, additional insight into the model can be obtained by considering some special cases of the model.

A) Case I. Consider first the case in which $y(t)=0$ for all t. This might be the case when the new introduction provides such a significant improvement over existing alternatives that everyone who becomes aware of its existence adopts it. Equation (7a) becomes

$$\frac{dx(t)}{dt} = -\beta x(t)\, z(t) - \mu x(t) \tag{14}$$

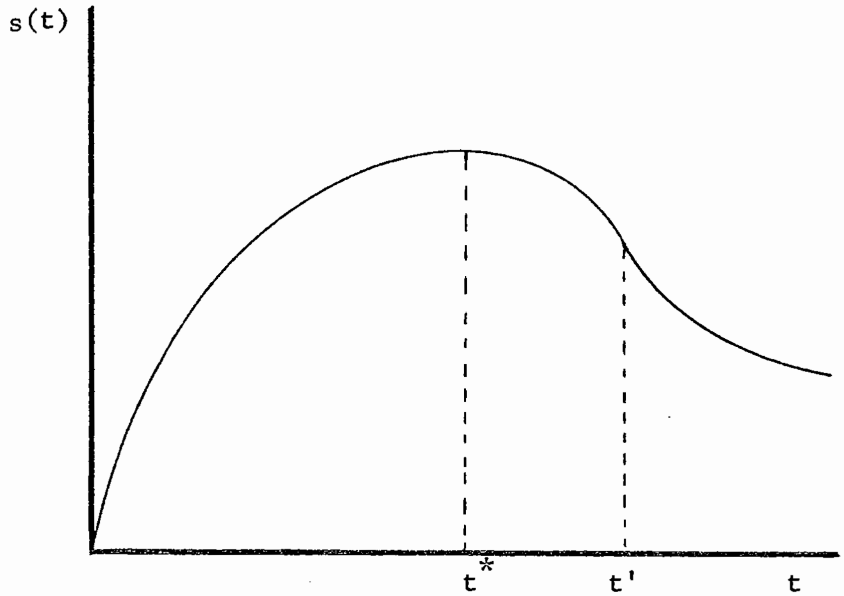

Figure 4
The Sales Curve when $b<\mu+\gamma$

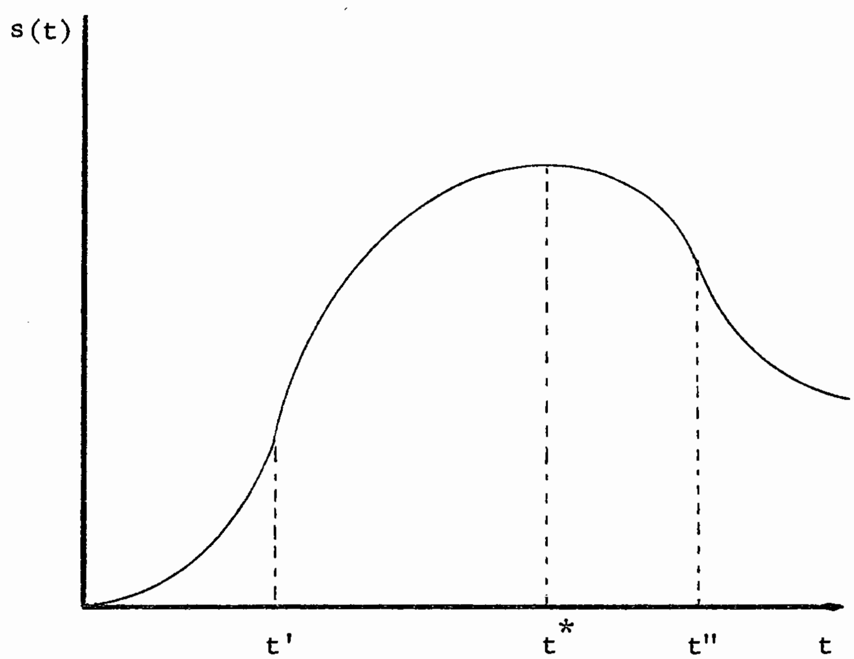

Figure 5
The Sales Curve when $b>\mu+\gamma$

Substituting x(t)+z(t)=N into (14) yields

$$\text{(15)} \qquad \frac{dz(t)}{dt} = \beta\Big(N-z(t)\Big)\ z(t) + \mu\Big(N-z(t)\Big)$$

Since $\frac{dz(t)}{dt}$ is the rate of sales at time t, s(t), the last equation, can be written as

$$\text{(16)} \qquad s(t) = a' + b'z(t) + c'z^2(t)$$

where $a=\mu N$, $b'=\beta N-\mu$, and $c'=-\beta$. This is equivalent to the equation estimated by Bass (1969). Bass' coefficients of innovation and imitation are equivalent to μ and β. Bass showed that a discrete analogue of equation (20) gave good predictions of initial purchase timing for eleven consumer durables.

The general solution of (15) is

$$\text{(17)} \qquad z(t) = \frac{N-Ne^{-\rho t}}{1+\frac{\beta N}{\mu}e^{-\rho t}}$$

where $\rho=\beta N+\mu$. This solution provides insight into the relationship between the penetration curve and the relative values of b, the contact coefficient, and μ, which represents the influence of a firm's promotional activities. When $b>\mu$ the growth in penetration follows the "typical" penetration curve shown in Figure 3. However, when $b<\mu$, i.e., when promotional activities dominate the market conversion mechanism, then penetration follows the curve shown in Figure 2.

To find the time t^*, at which the sales rate reaches its peak, we differentiate s(t) and set it equal to zero, yielding

$$t^* = \frac{1}{\rho}\ln\left(\frac{\beta N}{\mu}\right) \text{ and } s(t^*) = \frac{\rho^2}{4\beta}$$

Since $\rho=\beta N+\mu$, the following observation can readily be made. The higher the advertising effort μ, the sooner the peak will arrive and the larger the peak will be.

B) <u>Case II</u>. The second case considered is one for which $\mu=0$ and $z(t)\ll y(t)$. This occurs when the number of people who actually buy the new product is much smaller than those who know about the product but did not yet buy. This may be representative of a new introduction which faces considerable resistance to trial. Equation (7a) then becomes

$$\text{(18)} \qquad \frac{dx(t)}{dt} = -\beta x(t)\, y(t).$$

Dividing (18) by (7c) yields

$$\text{(19)} \qquad \frac{dx(t)}{dz(t)} = -(\beta/\gamma)x(t)$$

Thus,

$$\text{(20)} \qquad x(t) = x_0 e^{-(\beta/\gamma)z(t)}$$

Equation (7c) can be rewritten as

$$\text{(21)} \qquad \frac{dz(t)}{dt} = \gamma\left(N-x(t)-z(t)\right)$$

Substituting (20) into (21) yields

$$\text{(22)} \qquad s(t) = \gamma N-\gamma x_0 e^{-(\beta/\gamma)z(t)} - \gamma z(t)$$

This equation, from an empirical point of view, is a generalization of the Bass model since approximation of the exponential function by a second degree polynomial will produce an equation which is equivalent to (16). One should note that equations (16) and (22) were derived under completely different assumptions and thus are applicable to different situations. Note also that these models are concerned only with the timing of initial purchase. In the next section these models are modified to accommodate repeat purchasing.

II. Growth Model With Repeat Sales

To this point the discussion has concentrated on the development of a model which represents the growth in first purchases of a product. It is worthwhile to consider how one might incorporate repeat sales into the model of new product penetration. Repeat sales become an important consideration when one is dealing with low-priced, frequently-purchased, branded products, or durable products for a long enough time frame for repeat sales to become a significant proportion of total sales.

Let $x(t)$ and $y(t)$ be defined as before. Let $z_i(t)$ be the number of persons who buy the product for the i^{th} time. Assuming that a fixed proportion of consumers move from state to state, the repeat sales model becomes

$$\text{(23a)} \qquad \frac{dx(t)}{dt} = -\mu x(t)$$

$$\text{(23b)} \qquad \frac{dy(t)}{dt} = \mu x(t) - \gamma_1 y(t)$$

(23c) $$\frac{dz_1(t)}{dt} = \gamma_1 \, y(t) - \gamma_2 \, z_1(t)$$

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(23i) $$\frac{dz_i(t)}{dt} = \gamma_i \, z_{i-1}(t) - \gamma_{i+1} \, z_i(t)$$

.
.
.

The cumulative penetration is $\sum_{i=1}^{\infty} z_i$ and cumulative sales is

(24) $$S(t) = \sum_{i=1}^{\infty} i \, z_i(t)$$

The sales rate at time t, $\frac{dS(t)}{dt}$, denoted by s(t) is

(25) $$s(t) = \sum_{i=1}^{\infty} i \, \frac{dz_i(t)}{dt} = \gamma_1 y + \sum_{i=1}^{\infty} \gamma_{i+1} \, z_i(t)$$

The solution of equation (23a) is given by

(26) $$x(t) = Ne^{-\mu t}$$

assuming x(0)=N. Substitution into (23b) yields

(27) $$y(t) = \frac{\mu N}{\gamma_1 - \mu} \left(e^{-\mu t} - e^{-\gamma_1 t} \right)$$

Substituting into (23c) and solving yields

(28) $$z_1(t) = \gamma_1 \, \mu N \left[\frac{e^{-\mu t}}{(\gamma_1 - \mu)(\gamma_2 - \mu)} - \frac{e^{-\gamma_1 t}}{(\gamma_2 - \gamma_1)(\gamma_1 - \mu)} + \frac{e^{-\gamma_2 t}}{(\gamma_2 - \gamma_1)(\gamma_2 - \mu)} \right]$$

In general,

$$(29) \qquad z_k(t) = \left(\prod_{i=1}^{k} \gamma_i\right) N \left[\frac{e^{-\mu t}}{\prod_{i=1}^{k+1} (\gamma_i - \mu)} + \right.$$

$$\left. + \sum_{j=1}^{k+1} (-1)^j e^{-\gamma_j t} \Big/ \left((\gamma_j - \mu) \prod_{\substack{i=1 \\ i<j}}^{k+1} (\gamma_j - \gamma_i) \prod_{\substack{i=1 \\ i>j}}^{k+1} (\gamma_i - \gamma_j) \right) \right]$$

assuming $z_i(0)=0$, for all i.

A) <u>Constant Trial and Repeat Rate</u>. In the special case where $\gamma_i=\bar{\gamma}$, for i=1,2, ... then

$$(30) \qquad s(t) = \bar{\gamma} y(t) + \bar{\gamma} \sum_{i=1}^{\infty} z_i(t)$$

Assuming $x(t) + y(t) + \sum_{i=1}^{\infty} z_i(t)=N$, we get

$$(31) \qquad s(t) = \bar{\gamma} \left(N-x(t) \right)$$

and the model reduces to

$$(32a) \qquad \frac{dx(t)}{dt} = - \mu x(t)$$

$$(32b) \qquad s(t) = \bar{\gamma} \left(N-x(t) \right)$$

where s(t) is defined, as before, as the sales rate. Solving (32) yields

$$(33) \qquad s(t) = \bar{\gamma} N (1-e^{-\mu t})$$

revealing that the assumption of a constant trial rate ($\bar{\gamma}$) equal to the repeat rates (γ_i) at each repeat level, i=1,2, ..., yields a monotonically increasing growth curve, see Figure 6. Sales for the product (or brand) would approach an asymptotic level of $\bar{\gamma}N$ and the brand could expect a long-run purchase rate of $0 \leq \bar{\gamma} \leq 1$ under stable market conditions.

B) Constant Repeat Rate. There are many reasons to believe that the trial rate γ_1 is determined by factors, including promotion, distribution, etc., which make it unique and different from the repeat rates. Thus, for $\gamma_1 \neq \bar{\gamma}$, $\gamma_i = \bar{\gamma}$, i=2,3, ..., we have

(34a) $$\frac{dx(t)}{dt} = -\mu x(t)$$

(34b) $$\frac{dy(t)}{dt} = \mu x(t) - \gamma_1 y(t)$$

(34c) $$s(t) = \gamma_1 y(t) + \bar{\gamma}\left(N - x(t) - y(t)\right)$$

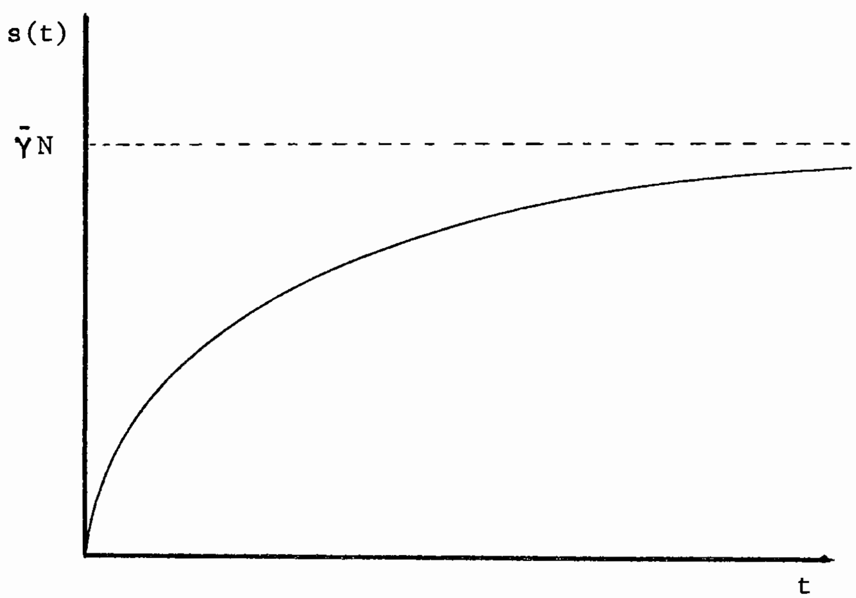

Figure 6
Sales For a Moderate Repeat Purchase Rate
$\gamma_1 > \bar{\gamma} > \mu$ or $\gamma_1 < \bar{\gamma} < \mu$

A unique solution to (38) depends on the relative value of the three parameters μ, γ_1, $\bar{\gamma}$. In general,

$$(35) \qquad s(t) = \bar{\gamma}N + \frac{\gamma_1 N(\mu - \bar{\gamma})}{(\gamma_1 - \mu)} e^{-\mu t} - \frac{\mu N(\gamma_1 - \bar{\gamma})}{(\gamma_1 - \mu)} e^{-\gamma_1 t}$$

$$(36) \qquad = \bar{\gamma}N + \frac{N}{(\gamma_1 - \mu)} \left(\gamma_1(\mu - \bar{\gamma}) e^{-\mu t} - \mu(\gamma_1 - \bar{\gamma}) e^{-\gamma_1 t} \right)$$

The relative values of the three parameters μ, γ_1, and $\bar{\gamma}$ reflect the impact of three different dimensions of marketing effort. Let us call μ the awareness rate, that is the rate at which potential buyers are becoming aware of the product or brand. Awareness can be achieved through advertising, distribution, etc. The trial rate, γ_1, is the rate at which potential consumers are converted to purchasers of the product. Conversion is influenced by promotion, pricing, etc. The repeat rate, $\bar{\gamma}$, is the critical determinant of a new product's or brand's success in the marketplace and depends to a large degree on the perceived value of the product and its need-satisfying characteristics.

The cumulative penetration $Z(t) = \sum_{i=1}^{\infty} z_i(t)$ can easily be found by solving equations (34a) and (34b) for $x(t)$ and $y(t)$, respectively, and plugging into

$$(37) \qquad Z(t) = N - x(t) - y(t)$$

Thus cumulative penetration is given by

$$(38) \qquad Z(t) = N - N\left(\frac{1}{\gamma_1 - \mu}\right)\left(\gamma_1 e^{-\mu t} - \mu e^{-\gamma_1 t}\right)$$

The shape of the cumulative penetration curve depends only on N, γ_1, and μ.[2] Its general form is given by the S-curve shown in Figure 3. Note that the cumulative penetration is independent of $\bar{\gamma}$ and is strictly a function of the influence of μ and γ_1 on the diffusion process.

However, the shape of the sales curve depends on all three parameter values. Since $s(0)=0$ and $s(\infty)=\bar{\gamma}N$, the form of the growth curve can be determined by examining the signs of $s'(t)$ and $s''(t)$ for alternative relative values of μ, γ_1, and $\bar{\gamma}$. If unique values of the first and second derivatives of $s(t)$ exist at $t>0$, then they are given by

$$t^{*} = \frac{1}{\gamma_1 - \mu} \ln\left(\frac{\gamma_1 - \bar{\gamma}}{\mu - \bar{\gamma}}\right) \tag{39}$$

when $s'(t)=0$ and

$$t^{**} = \frac{1}{\gamma_1 - \mu} \ln\left(\frac{\gamma_1}{\mu}\right)\left(\frac{\gamma_1 - \bar{\gamma}}{\mu - \bar{\gamma}}\right) \tag{40}$$

when $s''(t)=0$. Note that $t^{**} > t^{*}$ since

$$t^{**} = t^{*} + \frac{1}{\gamma_1 - \mu} \ln\left(\gamma_1 / \mu\right) \tag{41}$$

The three general shapes of $s(t)$ are therefore easily traced out to be the following:

1. $s(t)$ is increasing at a decreasing rate if the repeat purchase rate is moderate, i.e., $\gamma_1 > \bar{\gamma} > \mu$, or $\gamma_1 < \bar{\gamma} < \mu$ (see Figure 6).

2. $s(t)$ is increasing at a decreasing rate and then decreasing until it stabilizes at a lower level below the peak, if $\gamma_1 > \bar{\gamma}$ and $\mu > \bar{\gamma}$, i.e., the repeated purchase rate is low (see Figure 7).

3. When the repeat purchase rate is relatively large, i.e., $\bar{\gamma} > \mu$ and $\bar{\gamma} > \gamma_1$, two possibilities exist:

a) $\bar{\gamma} < \gamma_1 + \mu$, i.e., γ_1 is moderately large and we get the same shape as in Figure 6.

b) $\bar{\gamma} > \gamma_1 + \mu$, then s(t) is increasing, first at an increasing rate then at a decreasing rate, and we get the curve shown in Figure 8.

To summarize, we observe that the repeat rate $\bar{\gamma}$ is the critical determinant of the success of a new brand and the sole determinant of the long-run sales rate a new brand can expect to obtain. And, it can be seen from equation (38) that a primary effect of advertising is to decrease the time between introduction and peak sales.

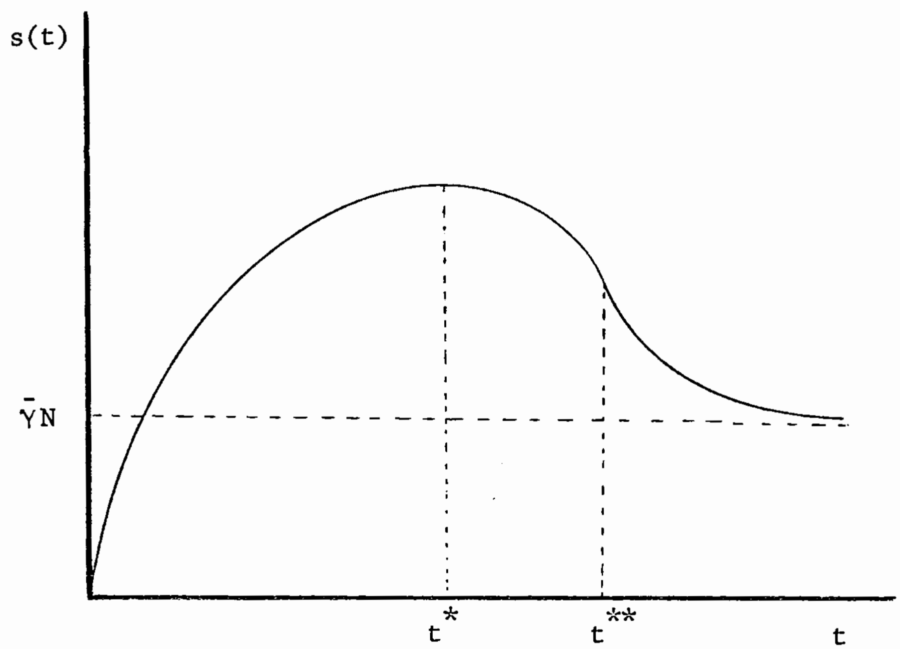

Figure 7
Sales For a Small Repeat Purchase Rate
$\gamma_1 > \bar{\gamma}$ and $\mu > \bar{\gamma}$

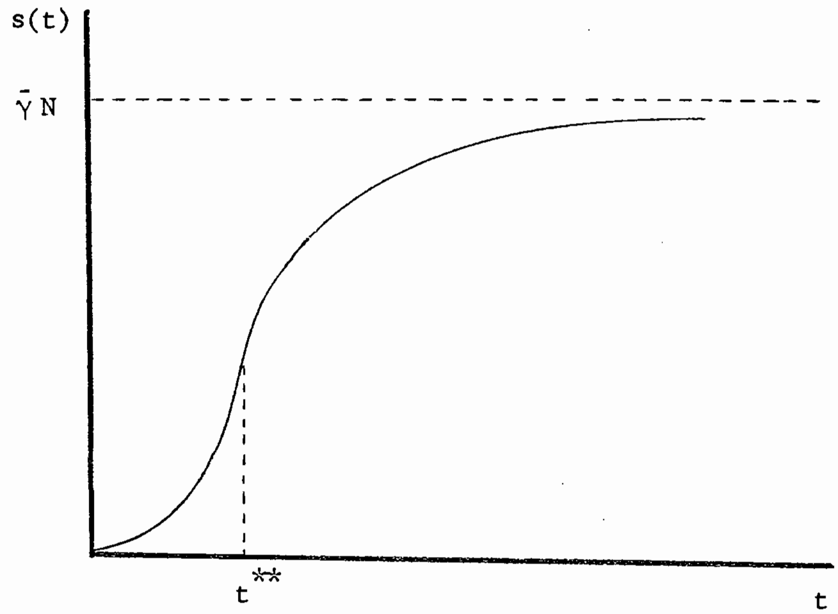

Figure 8
Sales For a Large Repeat Purchase Rate
$\bar{\gamma} > \gamma_1 + \mu$

Application and Empirical Issues

In this section the models which have been proposed are used in two distinct and unique applications.

A) Advertising In The Professions. Consider the case where the product offered is a service performed by a professional, e.g., a doctor or a lawyer.

Let

$x(t)$ = the number of people who do not know of a specific service

$y(t)$ = the number of people who know of the service but did not use it yet.

$z(t)$ = the number of cumulative uses of that service.

In this section, we show that the introduction of advertising will yield the greatest benefits for the younger professionals, e.g., doctors, lawyers.

The introduction of advertising can be dealt with in the following manner. Initially assume that the only mechanism at work is the influence of word-of-mouth. In accordance with our previous discussion, this mechanism can be described by the following set of differential equations.

(42a) $$\frac{dx(t)}{dt} = -\beta x(t)\Big(N-x(t)\Big)$$

(42b) $$\frac{dy(t)}{dt} = \beta x(t)\Big(N-x(t)\Big)-\gamma_1 y(t)$$

(42c) $$\frac{dz(t)}{dt} = \gamma_1 y(t)+\gamma_2\Big(N-x(t)-y(t)\Big)$$

when we do introduce advertising, we will have both mechanisms at work, i.e., the model, at that point, will change to

(43a) $$\frac{dx(t)}{dt} = -\beta x(t)\Big(N-x(t)\Big)-\mu x(t)$$

(43b) $$\frac{dy(t)}{dt} = \beta x(t)\Big(N-x(t)\Big)-\gamma_1 y(t)+\mu x(t)$$

(43c) $$\frac{dz(t)}{dt} = \gamma_1 y(t)+\gamma_2\Big(N-x(t)-y(t)\Big)$$

Denote by Z_W the cumulative number of visits, according to the model, with word-of-mouth only, and by Z_A the cumulative number of visits to the same professional service, according to the model, which includes advertising. In addition, denote the time of the introduction of advertising by T.

Proposition 1: For $t>T$, $Z_A-Z_W>0$.

The proof is given in Appendix B as part of the proof of Proposition 2.

Define the lag L as the time at which the professional began establishing his practice. In addition, choose the origin (t=0) such that for any professional in the market, his lag L would be positive (see Figure 9). Note that the larger L is, the "younger" the professional is. As before, define $b=\beta N$ as the contact coefficient. Regarding L as a parameter, we propose the following:

Proposition 2: For $t>T$ and $\mu>b$,

Z_A-Z_W is an increasing function of L.

The proof is given in Appendix B. Z_A-Z_W is the difference between the cumulative number of visits to a professional, when he is allowed to (and indeed does) advertise, and the cumulative number of visits he would have had if he did not advertise. The proposition states that the younger the professional is (in terms of years of practice), the larger this difference becomes.

This sheds some light on an old argument, stated by R.A. Kessel (1971) that "...the American Medical Association has particularly powerful sanctions over those (doctors) who are most likely to be price cutters. Those are young doctors trying to establish a practice." It simply states that the ban on advertising imposed by the AMA is most harmful to younger doctors and thus is one of the "sanctions" it employs. In addition, it shows the usefulness of the models introduced earlier to areas other than new product introduction, which has been the most obvious application.

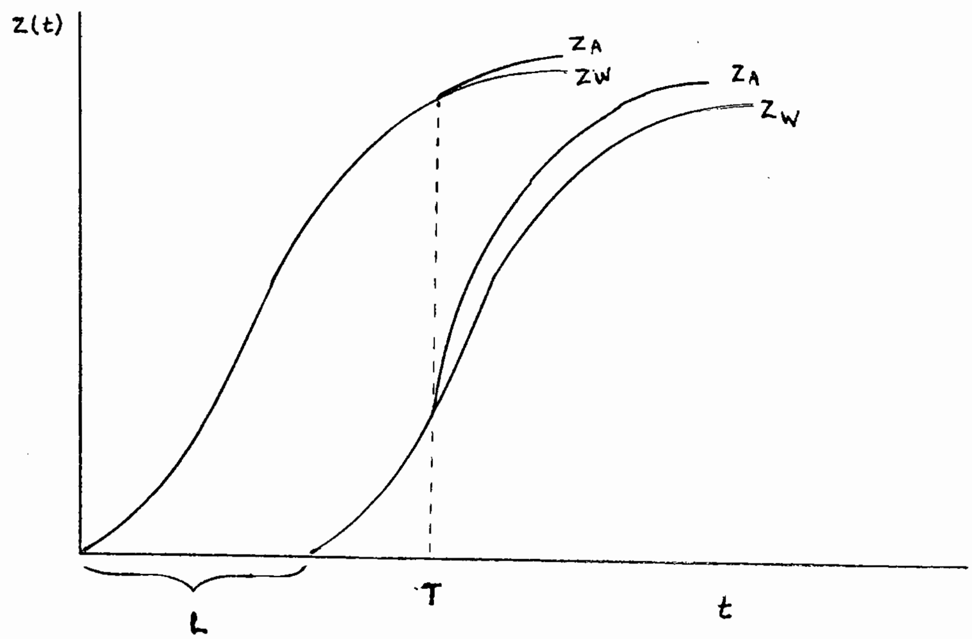

Figure 9
The Effect of Advertising on The Cumulative Visits

B) <u>New Brand Introduction</u>. With the increased importance of bringing new products to the marketplace, it is critical that producers have an accurate indication of progress. The growth model with repeat sales provides us with a tool for monitoring a new product introduction in the market for frequently-purchased branded goods. From equation (38c) we get

(44) $$s(t) = a+bx(t)+cy(t)$$

where $a=\bar{\gamma}N$, $b=-\bar{\gamma}$, and $c=\gamma_1-\bar{\gamma}$. Thus the values of γ_1, $\bar{\gamma}$, and N can be estimated from time series data using the discrete analog of equation (44). This requires data on s(t)=sales at t, x(t)=number of people unaware of the product at time t, and y(t)=number of consumers aware but not having purchased the product at time t. This data can be obtained through the commercial data collection services which operate consumer panels and provide the facility for collecting such information on a continuing basis.

An alternative parameter estimation procedure has been adopted here in order to illustrate implementation of the model. This example uses data from the Chicago Tribune Consumer Panel and represents the purchase history of 459 panel families over an eight year period.[3] The Wondra brand of flour was introduced into the Chicago Metropolitan Area market in August, 1963. Without data on x(t) and y(t) estimation of the model parameters proceeds by grid search. Given s(t), N, and t, the values for μ, γ_1, and $\bar{\gamma}$, from equation (40) which minimize the residual sum of squares, can be determined. The best fit was obtained for values of N=149, $\hat{\mu}=.400$, $\hat{\gamma}_1=.400$, and $\hat{\bar{\gamma}}=.066$, (R.S.SQ.=706). This estimate describes a growth curve with peak sales occurring at $\hat{t}^*=2.99$, three months after introduction, and long-run purchase rate of $\hat{\bar{\gamma}}N=9.849$. The actual and model estimated sales are plotted for the product's first twenty-four months in Figure 10. This represents the type of growth exhibited in Figure 7 where $\hat{\gamma}_1>\hat{\bar{\gamma}}$ and $\hat{\mu}>\hat{\bar{\gamma}}$.

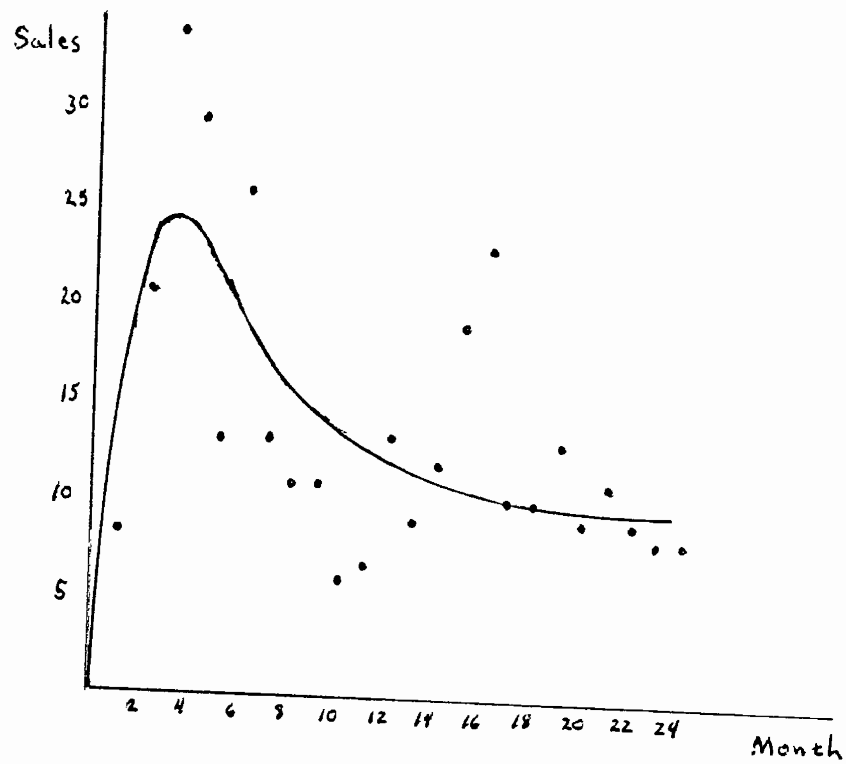

Figure 10
The Sales of Wondra Brand
in its First 24 Months

<u>Footnotes</u>

1. See Gould (1970) for a related model and an expanded discussion.

2. Equation (38) represents an alternative to the exponential function used by Fourt and Woodlock (1960) and Parfitt and Collins (1968) where $Z(t)=N\left(1-e^{-\alpha t}\right)$ and α represents the conversion parameter.

3. Dodson (1975) provides a more detailed description of the data.

Appendix A

If we denote the solution of the first equation in model (40a), $\frac{dx(t)}{dt} = \beta x(t)\big(N-x(t)\big)$, by x_1, and demand that at $t=L$, $x(L) = N-1$, i.e., nobody knows of the doctor at time L, except one person who begins the process of diffusion, then

$$\text{(A-1)} \qquad x_1 = \frac{N}{1+\frac{1}{N-1}\, e^{b(t-L)}}$$

where $b = \beta N$. And if we denote the solution of the first equation in model (41a), $\frac{dx(t)}{dt} = -\beta x(t)\big(N-x(t)\big) - \mu x(t)$ by x_2, requiring that at $t=T$, $x_1(T) = x_2(T)$, i.e., T is the time of the introduction of advertising, then

$$\text{(A-2)} \qquad x_2 = \frac{\rho N}{b+e^{\rho(t-T)}\left[\mu+\frac{\rho}{N-1}\, e^{b(T-L)}\right]}$$

where $\beta N+\mu=\rho$. It is easy to check that if $t>T$, then $x_1-x_2>0$, which is the result one would expect since $x(t)$ is the number of people who do not know of the specific doctor.

It is somewhat more tedious, but still straightforward, to check that if $t>T$ and $\mu>b$, x_1-x_2 is an increasing function of L, i.e., $\partial(x_1-x_2)/\partial L > 0$.

If we continue denoting the model with word-of-mouth only by 1, and the model which includes advertising by 2, then we have

$$\text{(A-3)} \qquad \frac{dz_2(t)}{dt} - \frac{dz_1(t)}{dt} = (\gamma_1-\gamma_2)\ (y_2-y_1) + \gamma_2(x_1-x_2)$$

since for both models, $y_i = \frac{1}{\gamma_1} \; -\left(\frac{dx_i(t)}{dt} - \frac{dy_i(t)}{dt}\right)$, $i=1,2$ then

$$\text{(A-4)} \qquad y_2-y_1 = \frac{1}{\gamma_1}\left[\frac{dx_1(t)}{dt} + \frac{dy_1(t)}{dt} - \frac{dx_2(t)}{dt} - \frac{dy_2(t)}{dt}\right]$$

upon substituting (A-4) into (A-3), integrating from T to t, and noting that at $t=T$, $z_1=z_2$, $x_1=x_2$, $y_1=y_2$, we get:

$$\text{(A-5)} \qquad z_2-z_1 = \frac{\gamma_1-\gamma_2}{\gamma_1}\left[x_1+y_1-(x_2+y_2)\right] + \gamma_2\int_T^t (x_1-x_2)d\tau$$

since the solution of $\frac{dy(t)}{dt} + \gamma_1 y(t) = -\frac{dx(t)}{dt}$ is

$$\text{(A-6)} \qquad y(t) = ce^{-\gamma_1 t} - e^{-\gamma_1 t}\int_T^t e^{\gamma_1\tau}\frac{dx(\tau)}{d\tau}d\tau$$

where c is a constant, then

$$\text{(A-7)} \qquad x_1+y_1-(x_2+y_2) = \gamma_1 e^{-\gamma_1 t}\int_T^t e^{\gamma_1\tau}(x_1-x_2)d\tau$$

and upon substitution into (A-5), and multiplying by $e^{\gamma_1 t}$, we get

$$\text{(A-8)} \qquad e^{\gamma_1 t}(z_2-z_1) = (\gamma_1-\gamma_2)\int_T^t e^{\gamma_1\tau}(x_1-x_2)d\tau + \gamma_2 e^{\gamma_1 t}\int_T^t (x_1-x_2)d\tau$$

Denote $A = e^{\gamma_1 t}(z_2-z_1)$. Instead of proving $z_2-z_1>0$, it is enough to prove that $A>0$, which is easily done by noting that at $t=T$, $A=0$ and for $t>T$, $dA/dt>0$. This proves Proposition 1.

In the same manner, denote $B = e^{\gamma_1 t}\left(\partial(z_2-z_1)/\partial L\right)$. In order to prove that $B>0$ (for $t>T$), we fix L at L_0, note that at $t=T$, $B=0$ and for $t>T$, $dB/dt>0$, since $\partial(x_1-x_2)/\partial L>0$. Since it is true for any L_0, this completes the proof of Proposition 2.