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SINGULARITY THEORY OF UTILITY MAPPINGS I: DEGENERATE MAXIMA AND PARETO OPTIMA

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SINGULARITY THEORY OF UTILITY MAPPINGS – I

Degenerate Maxima and Pareto Optima

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Many papers have described effective necessary and sufficient conditions for Pareto optimality using calculus techniques. The sufficient conditions are usually some sort of non-degeneracy conditions on the second derivatives of the utility mappings. In this paper, we shall investigate the optima not covered by these tests – optima which we call 'degenerate'. In addition to discussing methods for determining the optimality of such points, we shall look at two basic questions concerning degenerate optima. The first question is to determine for 'most' utility mappings the structure of the subset of the degenerate optima within the set of all optima points. In particular, do they form a subset of measure zero? The second question concerns the relative abundance of utility mappings which give rise to these degenerate optima. For example, are there open sets of utility mappings which admit non-empty submanifolds of degenerate optima?

As indicated by our choice of words like 'most' and 'generically', we will be studying open-dense or residual subsets of utility mappings. Since there is a bit of arbitrariness in passing from an observed preference ordering to an analytical utility function, it certainly makes sense to study residual subsets of utility mappings since, intuitively speaking, the probability of choosing a utility mapping that is not in a given residual set is zero.

The first question mentioned above asks for the size of the degeneracy set in the set of all optima. We shall show that it is usually not only a measure zero but it even forms a lower-dimensional submanifold. Simon–Titus (1975) give a very brief sketch of a proof of this fact. We are giving another proof here for two reasons. First of all, the ideas and concepts in our proof are much more elementary. For example, unlike Simon–Titus, we do not use the differential geometry of differential forms and wedge products. Secondly, while others,

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such as Smale, Simon, Titus, Wan, have used general singularity theory to obtain results on optima, we begin with this paper a systematic study of the singularity theory of utility mappings in the space of utility mappings. We shall begin to develop, essentially from first principles, and to apply the singularity theory of the space of utility mappings. This project ties in with the challenge of Thom (1972) to develop singularity theories for special categories of functions.

The second question raised in our introductory paragraph asks a complementary question to the first. Once it is known that, when degenerate optima exist, they usually lie on submanifolds of lower dimension, it is natural to question whether a utility mapping with degenerate optima can always be perturbed to obtain a new utility mapping possessing only non-degenerate optima. If this would be the case, then an argument based on the arbitrariness of the choice of utility functions could be forcefully presented to consider only those utility mappings for which the optima are discernable by second-order calculus techniques! However, in the last sections of this paper we show that, for economies with three or more agents, degenerate optima persist, i.e., they exist for open subsets of utility mappings. Indeed, the more agents there are in the economy, the more derivatives one may have to check to find all the Pareto optima.

This latter fact is in sharp contrast to the situation of mappings into $\mathbf{R}^1$ – where an open-dense set of functions have only non-degenerate critical points – and even to the situation of mappings into $\mathbf{R}^2$ – where an open-dense set of mappings have only non-degenerate Pareto optima.

Most of the research for this paper was done while the second author was visiting the Mathematics Department at Northwestern University. He would like to express his appreciation to both the Mathematics Department and the Center for Math Research in Economics at Northwestern for their hospitality. In addition, some of the ideas of this paper were generated at discussions at the 1974 Rencontre in Mathematical Economics at the University of Warwick, and at a 1975 conference at Oberwolfach.

1. Mathematics and economics background

In this section, we will recall some of the basic definitions and concepts involved. For further details, see Debreu (1959), Malinvaud (1972), Simon–Titus (1975), or the papers of Smale.

Let $\mathbf{R}_+$ denote the set of positive real numbers and $\overline{\mathbf{R}}_+$ the set of non-negative real numbers. We will work in economies with a agents and c commodities, where $2 \leqq a, c < \infty$. A holding of the kth agent is a vector $x^k = (x_1^k, \ldots, x_c^k)$ in $\overline{\mathbf{R}}_+^c$ where x_j^k designates the amount of the jth commodity held by the kth person. We shall assume that the total amount of each commodity is fixed. Thus, our state space is $\Omega \equiv \Omega_{c,a} \equiv \{(x^1, x^2, \ldots, x^a) \in (\overline{\mathbf{R}}_+^c)^a \,|\, x^1 + \ldots + x^a = b\}$, where b is a fixed vector in $\mathbf{R}_+^c$. Notice that Ω is the subset defined by the intersection of a given $c(a-1)$-dimensional affine subspace with the closure of the positive orthant of $\mathbf{R}^{ca}$. By the choice of this subspace, it follows that Ω is compact.

We'll assume further that each agent's preferences are summarized to be a smooth utility function $u^k:\Omega \to \mathbf{R}$, so that if x and y are in Ω, the kth agent prefers commodity bundle x to commodity bundle y if and only if $u^k(x) \geqq u^k(y)$. For simplicity, we will take 'smooth' to mean C^∞ throughout this paper, although each result holds with weaker smoothness assumptions. We will make two rather important classical assumptions on our utility functions: no externalities and no satiation. The first assumption implies that the kth agent's preferences depend only on his own holdings, i.e., there is $\bar{u}^k:\overline{\mathbf{R}}^c_+ \to \mathbf{R}$ such that $u^k(x^1, \ldots, x^a) = \bar{u}^k(x^k)$, where $\bar{u}^k$ is C^∞ on a neighborhood of $\overline{\mathbf{R}}^c_+$. The second assumption means that no $\bar{u}_k$ has a critical point in $\overline{\mathbf{R}}^c_+$. In addition, we will make no assumptions on the convexity or monotonicity of $\bar{u}_k$.

Let $C^\infty_u(\Omega_{c,a}\mathbf{R}^a)$ denote the space of all the utility mappings $u = (u^1, \ldots, u^a)$ where each u^k satisfies the above hypothesis. A topology for $C^\infty_u(\Omega_{c,a}\mathbf{R}^a)$ will be discussed in section 3.

For simplicity, write $y \succ x$ for $x, y \in \Omega$ if $u^k(y) \geqq u^k(x)$ for all choices of k and $u^j(y) > u^j(x)$ for some j. A commodity bundle $x \in \Omega$ is a local Pareto optimum (LPO) if there is some neighborhood W of x in Ω such that no $y \in W$ satisfies $y \succ x$. If W can be chosen to be Ω, then x is a Pareto optimum (PO).

For a given $u \in C^\infty(\Omega, \mathbf{R}^a)$ and $x \in \Omega$, one would like necessary and sufficient conditions for x to be an LPO for u. Let $Du(x)$ represent the derivative of u at x, either as a linear map from the tangent space $T_x\Omega$ to $\mathbf{R}^a$, or as a Jacobian matrix. If one uses $x^1, \ldots, x^{a-1}$ as coordinates for $\Omega_{c,a}$, then the matrix $Du(x)$ for this coordinate system is

$$\begin{bmatrix} D\bar{u}^1(x^1) & 0 & \ldots & 0 \\ 0 & D\bar{u}^2(x^2) & \ldots & \vdots \\ \vdots & \vdots & \ldots & 0 \\ \vdots & \vdots & \ldots & D\bar{u}^{a-1}(x^{a-1}) \\ -D\bar{u}^a(x^a) & -D\bar{u}^a(x^a) & \ldots & -D\bar{u}^a(x^a) \end{bmatrix},$$

where $x^a = b - x^1 - \ldots - x^{a-1}$. Since no $D\bar{u}^k(x^k)$ is zero (the non-satiation assumption), this matrix has rank a or rank $(a-1)$. In the latter case, there are non-zero $\lambda_1, \ldots, \lambda_a$ in $\mathbf{R}$ with $\lambda_i D\bar{u}^i(x^i) = \lambda_a D\bar{u}^a(x^a)$ for all i or alternatively with $\sum_i \lambda_i Du^i(x) = 0$ on $T_x\Omega$. (See Lemma 7 in section 4.) Let

$$\begin{aligned} S(u) &= \{x \mid Du(x) \text{ has rank } < a\} \\ &= \{x \mid Du(x) \text{ has rank } (a-1)\} \\ &= \{x \mid \text{there are non-zero } \lambda_1, \ldots, \lambda_a \text{ with } \textstyle\sum_i \lambda_i Du^i(x) = 0\}. \end{aligned}$$

This is the 'singularity set' of u. Let $\theta(u)$ be the following subset of $S(u): \theta(u) = \{x \in \Omega \mid \exists \lambda_1, \ldots, \lambda_a$ all positive with $\sum \lambda_i Du^i(x) = 0\}$. Proposition 1 below states that a necessary condition for $x \in \mathring{\Omega}$ (the interior of Ω) to be a PO or LPO is that $x \in \theta(u)$.

Before sufficient conditions can be given, we'll need to introduce a few more concepts. Let $K_x \equiv$ kernel $Du(x) = \bigcap_i$ kernel $Du^i(x)$. It is easily seen that K_x is the tangent space to $\bigcap_i u^{i^{-1}}(u^i(x))$. Let $x \in \theta(u)$, and let $\lambda_1, \ldots, \lambda_a \in \mathbf{R}_+$ be such that $\sum \lambda_i Du_i(x) = 0$. Let F_x^2 be $\sum_i \lambda_i D^2 u_i(x)$ restricted to $K_x \times K_x$. Finally, let $\Gamma(u) = \{x \in \theta(u) \mid F_x^2$ is a degenerate bilinear map, i.e., it has determinant zero$\}$. In the following proposition, we collect some of the basic results of the above references regarding necessary and sufficient conditions for optimality.

Proposition 1. Let $u \in C_u^3(\Omega_{c,a}, \mathbf{R}^a)$ and let $x \in \mathring{\Omega}$.

(a) *If x is an* LPO, *$x \in \theta(u)$.*
(b) *F_x^2 is intrinsic in that it does not depend on coordinatization of Ω.*
(c) *If $x \in \theta(u)$ and $x \notin \Gamma(u)$, then x is an* LPO *if and only if F_x^2 is negative definite on K_x.*
(d) *Let $U_x^{(k)} = \bigcap_{i \neq k} u^{i^{-1}}(u^i(x))$. $U_x^{(k)}$ is a submanifold of Ω of codimension $a-1$. If $x \in \theta(u)$, then $T_x U_x^{(k)} = K_x$ and the index of $D^2(u^k \mid U_x^{(k)})$ is the index of F_x^2.*
(e) *If x is a (local) maximum for* every *$u^k \mid U_x^{(k)}$, then x is a* (LPO) PO.
(f) *If $x \in \theta$ and x is a local maximum for some $u^k \mid U_x^{(k)}$, then x is an* LPO.

An essential mathematical tool in all these investigations is the implicit function theorem. For completeness, we will state this theorem in a very useful form. See Golubitsky–Guillemin (1973, sec. 1.2) or Edwards (1973, sec. 3.3).

Proposition 2. (Implicit Function Theorem). Let $F: \mathbf{R}^m \to \mathbf{R}^n$ be a C^r map with $m \geqq n$. Let $b \in \mathbf{R}^n$ and W be an open set in $\mathbf{R}^m$ such that, for all $x \in W \cap F^{-1}(b)$, $DF(x)$ has rank n. Then, $W \cap F^{-1}(b)$ is a codimension n C^r-submanifold of $\mathbf{R}^m$; and if $y \in W \cap F^{-1}(b)$, $T_y F^{-1}(b)$ is the kernel of $DF(y)$.

For example, most of Proposition 1(d) follows immediately from Proposition 2 and the fact that rank $Du(x) \geqq a-1$.

2. Degenerate minima

Parts (e) and (f) of Proposition 1 indicate that one can often reduce the problem of optimizing k functions on Ω to that of maximizing a single function on a submanifold of Ω. For the latter problem, one can use many classical techniques, such as those of Lagrange or of Kuhn–Tucker. Most of these techniques, however, find only non-degenerate maxima. In a later paper, we plan to treat extensively calculus methods for finding degenerate maxima. In this section, we describe a simple method in this direction.

Since we are studying local maxima, we will assume that we have a smooth function $f\colon \mathbf{R}^m \to \mathbf{R}$ with $f(0) = 0$ and $Df(0) = 0$. Point 0 is a non-degenerate maximum if and only if the Hessian $D^2f(0)$ is negative-definite. Notice that this is a condition on the values of the derivatives of order 2 when evaluated at $(0, 0)$. In the following statement we generalize this condition to include degenerate critical points.

Theorem 3. *Suppose that* $f\colon (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ *is a real analytic function with* $Df(0, 0) = (0, 0)$. *Let* p *and* q *be the smallest integers such that*

$$\frac{\partial^p f}{\partial y^p}(0) \neq 0 \quad \textit{and} \quad \frac{\partial^q f}{\partial x^q}(0) \neq 0.$$

Suppose that

(1) *both* p *and* q *are non-zero, finite, and even,*

(2) *both* $\dfrac{\partial^p f}{\partial y^p}(0)$ *and* $\dfrac{\partial^q f}{\partial x^q}(0)$ *are negative,*

(3) $\dfrac{\partial^{m+n} f}{\partial x^m \partial y^n}(0) = 0$ *for all* (m, n) *such that* $mp+nq < pq$, *and*

(4) *neither of the equations*

$$\sum_{mp+nq=pq} \frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0)u^n = 0, \qquad \sum_{mp+nq=pq} (-1)^m \frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0)u^n = 0$$

admits a real-valued solution u.

Then, $(0, 0)$ is a strict local maximum of f.

If $p = q = 2$, then Hypotheses (2) and (4) are equivalent to the usual condition that $D^2f(0)$ be negative-definite. Notice that if Hypothesis (3) is strengthened to include all (m, n) such that $mp+nq \leqq pq$, $m \neq q$, and $n \neq p$, then Hypothesis (4) is trivially satisfied by Hypothesis (2) and the parity of p and q.

Proof. The proof uses some simple ideas of algebraic functions. See Hille (1962, pp. 105–109) for all the concepts employed here. In $\mathbf{R}^2$, consider

$$N = \left\{(m, n) \,\middle|\, \frac{\partial^{m+n} f}{\partial x^m \partial y^n}(0) \neq 0\right\}.$$

This is called the Newton Diagram of f. Let C be the convex hull of N, and let L denote the line segments on ∂C, the boundary of C, joining the y axis to the x axis and separating the origin from the interior of C in $\mathbf{R}^2_+$. According to Hypothesis (3), L consists of the line segment from $(0, p)$ to $(q, 0)$. Write $(p, q) = r(a, b)$, where a and b are relatively prime integers and integer $r \geqq 1$. According to Puiseux's Theorem [Walker (1950, pp. 98–105) or Hille (1963)] there is a complex analytic function $h(t) = \sum_{k=0}^{\infty} d_k t^{k+b}$ such that the germ of the zero set of f in C^2 is given by

$$x = t^a, \qquad y = h(t),$$

$$x = t^a, \qquad y = h(wt),$$

$$\vdots$$

$$x = t^a, \qquad y = h(w^{b-1}t),$$

where $w = e^{2\pi i/b}$. Furthermore, if f has a real zero set at $(0, 0)$, there is a real analytic choice of h. One finds the coefficients of h by solving

$$f\left(t^a, \sum_0^{\infty} \alpha_k t^{k+b}\right) = 0$$

for $\alpha_0, \alpha_1, \ldots$. However, by Hypothesis 4, this equation does not admit a real value for α_0. Therefore, the germ of the real zero set of f at $(0, 0)$ is $(0, 0)$ itself; that is, there is a neighborhood U of $(0, 0)$ in $\mathbf{R}^2$ such that $f^{-1}(0) \cap U = \{(0, 0)\}$. Consequently, f cannot change sign in U; which means that $(0, 0)$ is a strict local extrema of f. That it is a local maximum follows from Hypothesis (2).

Remarks

1. An examination of the proof of Puiseux's Theorem (for example, Hille) shows that it is an easy task to generalize the above theorem, but the resulting statement becomes somewhat cumbersome. The idea of using Puiseux's Theorem comes from Simon–Titus (forthcoming) where the existence of real Puiseux series is also discussed.

2. While Hypothesis (1) and (2) of Theorem 3 are clearly necessary for $(0, 0)$ to be a maximum, they are not sufficient, even if the mixed partials on the line $mp+nq = pq$ vanish. To see this, notice that $g(x, y) = -y^2+x^2y-x^6$ is negative semi-definite at $(0, 0)$ on every straight line through $(0, 0)$; but g changes sign at $(0, 0)$ on the curve $y = x^3$. On the other hand, $f(x, y) = -y^2+x^4y-x^6$ does have a local maximum at $(0, 0)$ by Theorem 3.

3. Transversality theory in $C_u^\infty(\Omega_{c,a}, \mathbf{R}^a)$

The theory of 'singularities of mappings' as developed by Thom, Malgrange, Mather, and others is a powerful tool for studying the local behavior of mappings in $C^k(\Omega, \mathbf{R}^a)$; the vector space of all C^k maps from Ω to $\mathbf{R}^a$. As we mentioned earlier, several authors have used this theory to study the problem of optimizing a mapping in an economics situation. However, in general, these papers (see section 1) used singularity theory in $C^k(\Omega, \mathbf{R}^a)$ to study problems in $C_u^k(\Omega, \mathbf{R}^a) = \{$restriction of $C^k(\Omega, \mathbf{R}^a)$ to utility maps$\}$. That is, they used utility maps with externalities to discuss utility maps without externalities.

As we mentioned in the introduction, we will correct this situation here by developing, essentially from first principles, a singularity theory for $C_u^k(\Omega, \mathbf{R}^a)$. In general, this is somewhat more difficult than the setting where externalities are admitted, since in this case only special perturbations of a given utility mapping are permitted. Thus the structure of these maps must be emphasized. On the other hand, many of the results we shall need for a singularity theory of utility maps follow immediately from existing statements in the literature. While we will not reprove these statements, we will outline some of the proofs for reasons of completeness. Concurrently with the development of the theory, we shall employ it to obtain either new proofs of existing statements, or new results concerning utility mappings.

In this paper we shall not be concerned with the value of the total resources of the economy as represented by vector b. Consequently by a simple translation we can, and will, assume that Ω is a compact subset of the *vector space*

$$\{(x^1, \dots, x^a) \in (\mathbf{R}^c)^a \mid \textstyle\sum x^i = 0\}.$$

Let f and g be in $C_u^k(\Omega, \mathbf{R}^a)$, let $x \in \Omega$, and let r be an integer less than k. Write $f \sim_{r,x} g$ if $f(x) = g(x)$ and if all partial derivatives of order $\leq r$ of f and of g agree at x, i.e., if the rth order Taylor expansions of f and of g about x agree. An equivalence class under $\sim_{r,x}$ is called an r-jet at x. The set of all such equivalence classes at all $x \in \Omega$ is called the space of r-jets of utility maps from Ω to $\mathbf{R}^a$, $J_u^r(\Omega, \mathbf{R}^a)$. An element in $J_u^r(\Omega, \mathbf{R}^a)$ will be written as $j^r f(x)$. In fact, $J_u^r(\Omega, \mathbf{R}^a)$ is an open subset of a finite-dimensional vector space and is isomorphic to $\Omega \times \mathbf{R}^a \times L_u(\Omega, \mathbf{R}^a) \times \dots \times L_u^r(\Omega, \mathbf{R}^a)$, where $L_u^j(\Omega, \mathbf{R}^a)$ denotes the space of symmetric j-linear maps from Ω to $\mathbf{R}^a$ that arise as jth derivatives of a utility mapping. For example, by the computations of section 1, $J_u^1(\Omega, \mathbf{R}^a)$ is the space of all matrices of the form

$$\begin{bmatrix} A_1 & \cdots & \cdots & 0 \\ 0 & A_2 & \cdots & \vdots \\ 0 & \vdots & \cdots & A_{a-1} \\ -A_a & -A_a & \cdots & -A_a \end{bmatrix},$$

where the A_i are non-zero row vectors in $\mathbf{R}^c$ corresponding to the $D\bar{u}_i(x^i)$. A necessary part of this study will be an examination of the structure of $L^k_u(\Omega, \mathbf{R}^a)$ in greater detail. So, we will postpone a more precise definition of these spaces until that time when they arise in a natural fashion. In particular, we will describe $L^1_u(\Omega, \mathbf{R}^a)$ more thoroughly in section 4.

Coordinatize Ω by $(x^1, \ldots, x^{a-1})$, where $x^k = (x^k_1, \ldots, x^k_c) \in \mathbf{R}^c$. This induces a coordinate system on $J^r_u(\Omega, \mathbf{R}^a) \cong \Omega \times \ldots \times L^r_u(\Omega, \mathbf{R}^a)$ which we will write as

$$(x^1_1, \ldots, x^1_c, \ldots, x^{a-1}_c;\ \ y^1_1, \ldots, y^a;\ \ v^1_1, \ldots, v^1_c, \ldots,$$

$$v^a_1, \ldots, v^a_c, v^1_{11}, v^1_{12}, \ldots, v^a_{11}, \ldots, v^a_{cc}, v^1_{111}, \ldots, v^a_{\underbrace{c1, \ldots, c}_{r}}).$$

So if $\xi = j^r u(x)$ in $J^r_u(\Omega, \mathbf{R}^a)$, then

$$x^i_j(\xi) = x^i_j, \qquad y^i(\xi) = u^i(x) = \bar{u}^i(x^i),$$

$$v^i_j(\xi) = \frac{\partial u^i}{\partial x^i_j}(x) = \bar{u}^i_j(x), \ldots, \qquad v^i_{j,k}(\xi) = \frac{\partial^2 \bar{u}^i}{\partial x^i_j \partial x^i_k}(x^i)$$

$$\equiv \bar{u}^i_{j,k}(x), \quad \text{etc.}$$

One defines the Whitney C^r-topology on $C^k_u(\Omega, \mathbf{R}^a)$ by using r-jets, as follows. Give $J^r_u(\Omega, \mathbf{R}^a)$ the topology it inherits as an open subset of a finite-dimensional vector space. Then, give $C^k_u(\Omega, \mathbf{R}^a)$ the topology for which the following is a base of open subsets: $\{f \in C^k_u(\Omega, \mathbf{R}^a) \mid j^r f(\Omega) \subset V\}$ for all open sets V in $J^r_u(\Omega, \mathbf{R}^a)$. When one takes the union of all these bases as r goes to infinity, one obtains a base of open subsets for the Whitney C^∞-topology.

It turns out that two utility maps f and g are 'close' in the Whitney topology of $C^k_u(\Omega, \mathbf{R}^a)$ if and only if there is a 'small' positive function $\eta : \Omega \to \mathbf{R}$ with

$$\sup_{0 \leqq i \leqq k} \| D^i f(x) - D^i g(x) \| \leqq \eta(x),$$

where $D^i f(x)$ denotes the ith derivative of f at x.

We now turn to an examination of j^k and state, without proof, some of its important properties. It follows from its definition that if f is a given utility map, then $j^k f$ is a mapping from Ω to $J^k_u(\Omega, \mathbf{R}^a)$. It is important to note that it is a smooth mapping; that is, it is in $C^\infty(\Omega, J^k_u(\Omega, \mathbf{R}^a))$.

While most computations in the study of optima are done with the Taylor series, $j^k f$, the ultimate goal is to relate these statements to the actual choice of mappings f. This is particularly the case in this paper where the goal is to obtain generic statements. Thus, it will be necessary to know whether this process of assigning a utility mapping to an element in $J_v^k(\Omega, \mathbf{R}^a)$ can be done in a continuous fashion. That it can be is the content of the following statement.

Lemma 4. The mapping $j^k : C_u^\infty(\Omega, \mathbf{R}^a) \to C^\infty(\Omega, J_u^k(\Omega, \mathbf{R}^a))$ *defined by* $f \to j^k f$ *is a continuous map in the Whitney* C^∞ *topology.*

The proof of this statement is a modification of the one found in Golubitsky–Guillemin (1973) – but enough of a modification that we will sketch its proof. The idea is to take a basic open set $V \subset C^\infty(\Omega, J_u^k(\Omega, \mathbf{R}^a))$ and show that $(j^k)^{-1}(V)$ is an open set in $C_u^\infty(\Omega, \mathbf{R}^a)$. Our basic open set V has the form

$$V = \{g \in C^\infty(\Omega, J_u^k(\Omega, \mathbf{R}^a)) \,|\, j^m g(\Omega) \subset W\},$$

for some integer m and some open set W in $J^m(\Omega, J_u^k(\Omega, \mathbf{R}^a))$.

By the definition of the Whitney C^∞-topology, we need only show that $(j^k)^{-1}(V)$ is an open subset of $J_u^n(\Omega, \mathbf{R}^a)$ for some n, where the natural choice is $n = k+m$. So, identity $g \in V$ with $j^k u$ for some map $u : \Omega \to \mathbf{R}^a$. Note that u must satisfy $j^m(j^k u)(\Omega) \subset W$. To finish the proof by showing that $(j^k)^{-1}(V)$ defines an open set in $J_u^{k+m}(\Omega, \mathbf{R}^a)$, one relates $j^{m+k}(u)$ with $j^m(j^k u)$ by noting is smooth and that $j^m(j^k u)$ depends only on partial derivatives of order $\leqq k+m$.

All of this is standard, as in Golubitsky–Guillemin (1973, sec. II.3). The necessary modification in the proof is to show that one can choose a u in $C_u^\infty(\Omega, \mathbf{R}^a)$ to represent g. That is, one must show that $u(x)$ is a utility mapping. But, since $g(x) \in J_u^k(\Omega, \mathbf{R}^a)$ for each x, the projection of g onto the factor $L_u^1(\Omega, \mathbf{R}^a)$ is a matrix of the form

$$\begin{bmatrix} \mathbf{A}_1 & 0 & & 0 \\ 0 & \mathbf{A}_2 & 0 & 0 \\ & & & \mathbf{A}_{a-1} \\ -\mathbf{A}_a & -\mathbf{A}_a & & -\mathbf{A}_a \end{bmatrix},$$

where the $\mathbf{A}_i$ are all non-zero elements of $\mathbf{R}^c$. So

$$\frac{\partial u_i}{\partial x_j^h}(x) = 0 \quad \text{for} \quad i \neq h \quad \text{and} \quad \forall x \in \Omega.$$

This fact and the equation

$$x^a = -\sum_{i=1}^{a-1} x^i$$

imply that u must be a utility mapping.

Next, we recall some basic facts about transversality theory. Let $f \in C^\infty(\Omega, \mathbf{R}^a)$ and let N be a smooth submanifold of $\mathbf{R}^a$. Then, f is *transverse* to N ($f \pitchfork N$) if, whenever $f(x) \in N$, then $Df(x)(T_x\Omega) + T_{f(x)}N = T_{f(x)}\mathbf{R}^a$. Since the tangent space of a point in a vector space can be identified with the vector space, we can write this equation as

$$Df(x)(\Omega) + T_{f(x)}N = \mathbf{R}^a,$$

i.e., the image of Ω under f fills up a complementary space to N in $\mathbf{R}^a$ at x. We now list some simple consequences of transversality.

Proposition 5. (a) *If f is transverse to N, then $f^{-1}(N)$ is a submanifold of Ω with the codimension of $f^{-1}(N)$ in Ω equal to the codimension of N in $\mathbf{R}^a$.*

(b) *Thom Transversality Theorem – Let P be a finite-dimensional manifold, and let N be a submanifold of $\mathbf{R}^a$. Let $F: \Omega \times P \to \mathbf{R}^a$ be a C^k map with F transverse to N and $k > \dim\Omega - \dim N$. Then, there is a residual subset of $p \in P$, such that the map $F_p: \Omega \to \mathbf{R}^a$ defined by $F_p(x) = F(x, p)$, is transverse to N.*

A subset of P is 'residual' if it is the countable intersection of open-dense subsets of P, e.g., the irrational numbers in $\mathbf{R}$. 'Residual' is stronger than 'dense' but weaker than 'open-dense'. In particular, we say a property in a space A is 'generic' if it holds for a residual subset of A.

See Golubitsky–Guillemin (1973) or Simon–Titus (1975) for the proof of Proposition 5 and for a more complete discussion of transversality. Part (a) of Proposition 5 is a direct consequence of Proposition 2. One writes N as $g^{-1}(0)$ locally, where $g: \mathbf{R}^a \to \mathbf{R}^n$, $n = \operatorname{codim} N$, and 0 is a regular value of g. Then, $f^{-1}(N)$ is $(gf)^{-1}(0)$ and the transversality condition implies that 0 is a regular value of gf. Part (b) is an application of Sard's Theorem.

The next step in the theory is Thom's Jet Transversality Theorem, a beautiful synthesis of the notions of jet and transversality. Here, we mimic Thom's proof of this theorem to demonstrate the corresponding theorem for utility mappings. We exploit the compactness of Ω to obtain a slightly stronger statement, particularly when the theorem is applied to optima problems.

Theorem 6. Let Σ be any submanifold of $J^r_u(\Omega, \mathbf{R}^a)$. The set $\mathscr{B}$ of utility maps u in $C^\infty_u(\Omega, \mathbf{R}^a)$ such that $j^r u: \Omega \to J^r_u(\Omega, \mathbf{R}^a)$ is transverse to Σ is a residual subset of $C^\infty_u(\Omega, \mathbf{R}^a)$. If Σ is closed, then $\mathscr{B}$ is open-dense.

Proof. We will first prove the theorem for closed Σ. It is rather easy to see that $\mathscr{B}$ is open. For, transversality implies either non-intersection or that certain submatrices have non-zero determinant. Since Ω is compact, Σ is closed, and j^r is continuous, it follows that $\mathscr{B}$ is open.

Thus only the density needs to be established. Let P_r be the vector space of all polynomial mappings of degree $\leq r$ from $\mathbf{R}^c$ to $\mathbf{R}$. Let $\mathscr{P}_r = P_r \times \ldots \times P_r$

be the polynomial maps of degree $\leqq r$ in $C_u^\infty(\Omega, \mathbf{R}^a)$. That is, if $p \in \mathscr{P}_r$ and $(x^1, \ldots, x^a) \in \Omega$, then $p(x^1, \ldots, x^a) = (p^1(x^1), \ldots, p^a(x^a))$ where each $p^j \in P_r$ and $x^a = -\sum_1^{a-1} x^i$. Let f be an arbitrary mapping in $C_u^\infty(\Omega, \mathbf{R}^a)$. Consider the map $F: \Omega \times \mathscr{P}_r \to J_u^r(\Omega, \mathbf{R}^a)$ defined by $F(x,p) = j^r(f+p)(x)$. Therefore, $j^r(f+p)(x) \cong j^r f(x) + p(x)$ is the Taylor series of order r of $f+p$. But, $\Omega \times \mathscr{P}_r$ can clearly be viewed as the tangent space at $j^r f(x)$ in the vector space $J_u^r(\Omega, \mathbf{R}^a)$. Consequently, F is a submersion, i.e., $DF(x,p)$ is onto at each point. In other words, every point in $J_u^r(\Omega, \mathbf{R}^a)$ is a regular value of F, and $F: \Omega \times \mathscr{P}_r \to J_u^r$ is transverse to any submanifold of $J_u^r(\Omega, \mathbf{R}^a)$. By Proposition 5(b), there is a $p \in \mathscr{P}_r$ arbitrarily close to zero with $j^r(f+p)$ transverse to Σ. This shows the density of transversal jets.

Assume Σ is not closed. Cover Σ by a countable number of open set $\{V_i\}$ such that $V_i \subset \Sigma$ for all i. The above argument shows that if $\mathscr{B}_i$ is the set of utility maps transverse to V_i, then $\mathscr{B}_i$ contains an open dense set. ($\bar{V}_i$ is used to obtain the 'open' part. V_i, a submanifold of Σ, is used to obtain the denseness.) Since $\mathscr{B} = \cap \mathscr{B}_i$, $\mathscr{B}$ is residual.

Notice that if $\bar{\Sigma}$ can be covered by a finite union of closed submanifolds, or closed submanifolds with boundary, the dimension of each bounded by the dimension of Σ, then the argument shows that $\mathscr{B}$ contains an open-dense set.

4. Generic Properties of $C_u^\infty(\Omega_{c,a}, \mathbf{R}^a)$

We shall now apply Theorem 6. As our first application we'll show for an open-dense subset of $C_u^\infty(\Omega_{c,a}, \mathbf{R}^a)$, $S(u)$ and $\Theta(u)$ are $(a-1)$-dimensional submanifolds in Ω, if they are non-empty. This theorem is an improvement (open-dense replacing residual) of a result which was first proved in Smale (1974) and later in Simon–Titus (1975).

Our first step in this direction is to describe $L_u^1(\Omega, \mathbf{R}^a)$ more carefully. As noted in sections 1 and 3, $L_u^1(\Omega, \mathbf{R}^a)$ is the set of all linear maps from $(\mathbf{R}^c)^a$ to $\mathbf{R}^a$ which assume the form

$$A = \begin{bmatrix} \mathbf{A}_1 & 0 & \vdots & \cdots & 0 \\ 0 & \mathbf{A}_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \vdots & \vdots & \cdots & \mathbf{A}_{a-1} \\ -\mathbf{A}_a & -\mathbf{A}_a & \vdots & \cdots & -\mathbf{A}_a \end{bmatrix},$$

where vectors $\mathbf{A}_i \in \mathbf{R}^c$ are all non-zero. The non-vanishing of the $\mathbf{A}_i$'s corresponds to the non-satiation condition imposed upon the u_i's. The form of the matrix corresponds to our study of utility maps *without* externalities. Since the specification of 'a' vectors from $\mathbf{R}^c$ completely determines a given map in this space, the dimension of $L_u^1(\Omega, \mathbf{R}^a)$ is ca, and it can be identified in a natural smooth fashion with $\mathbf{R}_*^{ca} = (R^c)^a - \{$coordinate planes defined by $\mathbf{A}_i = \mathbf{0}$, for $i = 1, 2, \ldots, a\}$. That $L_u^1(\Omega, \mathbf{R}^a)$ is a manifold, actually a submanifold of $L^1(\Omega, \mathbf{R}^a)$, now follows from this identification. We can now state the key lemma.

Lemma 7. Let S denote the subset of $L_u^1(\Omega, \mathbf{R}^a)$ consisting of linear maps of corank 1, *and let θ denote the subset of S defined by $\mu_1\mathbf{A}_1 = \ldots = \mu_a\mathbf{A}_a$ where the scalars μ_i are all positive. Then, S and θ are codimension $(a-1)(c-1)$ closed submanifolds of $L_u^1(\Omega, \mathbf{R}^a)$; and $\Omega \times \mathbf{R}^a \times S$ and $\Omega \times \mathbf{R}^a \times \theta$ are codimension $(a-1)(c-1)$ submanifolds of $J_u^1(\Omega, \mathbf{R}^a)$.*

We shall provide two proofs of this lemma to illustrate two different approaches to the subject of singularities. The first proof emphasizes the structure of $L_u^1(\Omega, \mathbf{R}^a)$ while the second is an analytical proof emphasizing the behavior of a given utility map.

Proof. We first characterize the set S by claiming that if $[a \times c(a-1)]$ matrix A has rank $a-1$, then there exist (unique) non-zero scalars λ_i such that $\mathbf{A}_i = \lambda_i\mathbf{A}_a$ for $i = 1, \ldots, a$. For, the first $(a-1)$ rows of A, $(\mathbf{A}_1, \mathbf{0}, \ldots, \mathbf{0})$, $(\mathbf{0}, \mathbf{A}_2, \mathbf{0}, \ldots,)$, $(0, \ldots, 0, \mathbf{A}_{a-1})$ are linearly independent since the $\mathbf{A}_i$ are non-zero. Since A has rank $(a-1)$, the last row is a linear combination of the first $(a-1)$ rows in $\mathbf{R}^{c(a-1)}$. That is, there is $(\mu_1, \ldots, \mu_{a-1}) \neq \mathbf{0}$ with $(-\mathbf{A}_a, \ldots, -\mathbf{A}_a) = \mu_1(\mathbf{A}_1, \mathbf{0}, \ldots, \mathbf{0}) + \ldots + \mu_{a-1}(\mathbf{0}, \ldots, \mathbf{0}, \mathbf{A}_{a-1})$. So, $-\mathbf{A}_a = \mu_i\mathbf{A}_i$. Since $\mathbf{A}_a \neq \mathbf{0}$, each μ_i is non-zero. Let $\lambda_i = -1/\mu_i$.

The above shows that S can be identified with the submanifold of $\mathbf{R}_*^{ca}$ given by $(\lambda_1\mathbf{A}_a, \ldots, \lambda_{a-1}\mathbf{A}_a, \mathbf{A}_a)$ where non-zero $\mathbf{A}_a \in \mathbf{R}^c$ and scalars λ_i are non-zero. This submanifold is of dimension $c+(a-1)$; thus it follows that S is a submanifold of codimension $ac-c-(a-1) = (a-1)(c-1)$. That S is closed in $L_u^1(\Omega, \mathbf{R}^a)$ follows from the above identification. Subset θ is defined by the condition that all the λ_i's are positive. Thus the stated conclusion for θ follows immediately.

We now turn to the second proof of this lemma. According to the computations of section 1, $j^1(u) \in \Omega \times \mathbf{R}^a \times S$ if and only if

$$Du(x) =$$

$$\begin{bmatrix} \frac{\partial u^1}{\partial x_1^1}(x^1) & \ldots & \frac{\partial u^1}{\partial x_c^1}(x^1) & 0, 0, 0 & \ldots & & & & \\ & \ldots & 0 & 0 & \frac{\partial u^2}{\partial x_1^2}(x^2) & \ldots & \frac{\partial u^2}{\partial x_c^2}(x^2) & & \\ & & & & & \ldots & \frac{\partial u^{a-1}}{\partial x^{a-1}}(x^{a-1}) & \ldots & \frac{\partial u^{a-1}}{\partial x_c^{a-1}}(x^{a-1}) \\ -\frac{\partial u^a}{\partial x_1^a}(x^a) & \ldots & -\frac{\partial u^a}{\partial x_1^a}(x^a) & & & \ldots & -\frac{\partial u^a}{\partial x_1^a}(x^a) & \ldots & -\frac{\partial u^a}{\partial x_c^a}(x^a) \end{bmatrix}$$

has rank $(a-1)$ where $x^a = -\sum_{i=1}^{a-1} x^i$. Since each row is a scalar multiple of the last row, as seen in the first proof, all the following 2×2 submatrices of $Du(x)$ have zero determinant:

$$\frac{\partial u^1}{\partial x_1^1}\frac{\partial u^a}{\partial x_2^a}-\frac{\partial u^1}{\partial x_2^1}\frac{\partial u^a}{\partial x_1^a}=0,$$

$$\vdots$$

$$\frac{\partial u^1}{\partial x_1^1}\frac{\partial u^a}{\partial x_c^a}-\frac{\partial u^1}{\partial x_c^1}\frac{\partial u_a^a}{\partial x_1^a}=0,$$

$$\vdots$$

$$\frac{\partial u^{a-1}}{\partial x_1^{a-1}}\frac{\partial u^a}{\partial x_c^a}-\frac{\partial u^{a-1}}{\partial x^{a-1}}\frac{\partial u^a}{\partial x_1^a}=0.$$

We want to see that these equations define a submanifold of $J_u^1(\Omega, \mathbf{R}^a)$. Recall the coordinate system $x_1^1, \ldots, x_c^{a-1}$, $y_1^1, \ldots, y^a$, $v_1^1, \ldots, v_c^a$ defined above for $J_u^1(\Omega, \mathbf{R}^a)$. Let W_i $(1 \leqq i \leqq a)$ denote the open subset of $J_u^1(\Omega, \mathbf{R}^a)$ defined by $v_i^1 \neq 0$, $v_i^2 \neq 0, \ldots, v_i^a \neq 0$. If $Du(x)$ has corank 1, then $\exists\lambda_1, \ldots, \lambda_a$ all non-zero such that $\lambda_1 D\bar{u}_1(x^1) = \ldots = \lambda_a D\bar{u}_a(x^a) \neq \mathbf{0}$. This means that $W_1\cup\ldots\cup W_a$ covers S in $J_u^1(\Omega, \mathbf{R}^a)$. So, it suffices to show that $W_i \cap S$ is a submanifold for each i. For simplicity, we'll work with W_1.

Now, $W_1 \cap S$ is given by the $(a-1)(c-1)$ equations on J_u^1:

$$\begin{aligned}
A_2^1 &\equiv v_1^1 v_2^a - v_2^1 v_1^a = 0,\\
&\vdots\\
A_c^1 &\equiv v_1^1 v_c^a - v_c^1 v_1^a = 0,\\
A_2^2 &\equiv v_1^2 v_2^a - v_2^2 v_1^a = 0,\\
&\vdots\\
A_c^2 &\equiv v_1^2 v_c^a - v_c^2 v_1^a = 0,\\
A_2^{a-1} &\equiv v_1^{a-1} v_2^a - v_2^{a-1} v_1^a \equiv 0,\\
&\vdots\\
A_c^{a-1} &\equiv v_1^{a-1} v_c^a - v_c^{a-1} v_1^a = 0.
\end{aligned} \qquad (\neq)$$

To see that $W_1 \cap S$ is a codimension $(a-1)(c-1)$ submanifold of W_1, we need only show that zero is a regular value of the map

$$A \equiv (A_2^1, \ldots, A_c^1, A_2^2, \ldots, A_2^{a-1}, \ldots, A_c^{a-1}) : J_u^1(\Omega, \mathbf{R}^a) \to \mathbf{R}^{(a-1)(c-1)}$$

C

by Proposition 2. In other words, we must find an $(a-1)(c-1)$ non-singular square submatrix in $DA(\mathbf{x}, \mathbf{y}, \mathbf{v})$. However, if one holds fixed $\mathbf{x}, \mathbf{y}, v_1^2, v_1^3, \dots, v_1^a, v_2^a, \dots, v_c^a$ and differentiates with respect to the other variables, one finds that

$$\partial A/\partial(v_2^1, \dots, v_c^1, v_2^2, \dots, v_c^2, \dots, v_2^{a-1}, \dots, v_c^{a-1}) \text{ is diag } \{v_1^a, v_1^a, \dots, v_1^a\}.$$

Since $v_1^a \neq 0$ in W_1, zero is a regular value of A and $S \cap W_1$ is a codimension $(a-1)(c-1)$ submanifold of W_1.

Theorem 8. There exists an open-dense subset of u in $C_u^\infty(\Omega, \mathbf{R}^a)$ for which $\theta(u)$ and $S(u)$ are $(a-1)$-dimensional submanifolds of $\mathring{\Omega}$ (if they are non-empty).

Proof. This is a combination of Lemma 7 and Theorem 6. One first takes $\Omega \times \mathbf{R}^a \times S$ and then $\Omega \times \mathbf{R}^a \times \theta$ to be the Σ of Theorem 6. Since S and θ are closed submanifolds of $L_u^1(\Omega, \mathbf{R}^a)$, it follows from Theorem 6 that for each of the two sets described in the previous sentence, there exists an open-dense subset ($\mathscr{B}_\theta$ and $\mathscr{B}_s$) of utility mappings with the property that $j^1(u)$ is transverse to Σ. The open dense subset $\mathscr{B}_\theta \cap \mathscr{B}_s$ is the subset we want. According to Proposition 5(a), for such u, $S(u) = j^1(u)^{-1}(S)$ and $\theta(u) = j^1(u)^{-1}(\theta)$ are codimension $(a-1)(c-1)$ submanifolds of Ω. If either set is non-empty, then its dimension is $c(a-1)-(c-1)(a-1) = a-1$.

We now come to the main goal of this section: an estimate of the size of the degeneracy set $\Gamma(u)$ in $\Theta(u)$ for generic utility mapping u. We will need two lemmas.

Lemma 9. Let V be a finite-dimensional vector space. Let $p \equiv (p_1, p_2, \dots, p_k)$: $V \to \mathbf{R}^k$ be a polynomial on V with $Dp_1(x), Dp_2(x), \dots, Dp_k(x)$ all independent for some $x \in V$. Then, $p^{-1}(0)$ is an algebraic variety in V. In particular, it is a finite union of submanifolds of V, each of codimension greater than or equal to k.

Proof. This is a classical and simple result of analytic function theory. See Milnor (1968, pp. 14–15).

Lemma 10. Suppose that $u \in C_u^\infty(\Omega, \mathbf{R}^a)$ lies in the open-dense subset described in Theorem 8 [so, $\theta(u)$ is a submanifold of Ω]. Then, $x \in \Gamma(u)$ if and only if $x \in \Gamma(u)$ and there is a non-zero vector in $T_x\theta(u) \cap kernel\ Du(x)$.

Proof. Let us forget the special structures of u and Ω for a second. Let $x^* \in \theta(u)$. Since $Du(x^*)$ has rank $(a-1)$, one can find coordinates $(y_1, \dots, y_{a-1}, z_1, \dots, z_{(c-1)(a-1)})$ in a neighborhood W centered at x^* in Ω so that in these coordinates u takes the form

$$\begin{aligned} u_1(\mathbf{y}, \mathbf{z}) &= u_1(x_1^*)+y_1, \\ &\vdots \\ u_{a-1}(\mathbf{y}, \mathbf{z}) &= u_{a-1}(x_{a-1}^*)+y_{a-1}, \\ u_a(\mathbf{y}, \mathbf{z}) &= v(y_1, \dots, z_{(c-1)(a-1)}). \end{aligned}$$

This fact follows easily from the implicit function theorem and is sometimes called the 'rank theorem'. [For example, see Sternberg (1964, p. 40).]

In these coordinates,

$$Du(x^*) =$$

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & & & & \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \dfrac{\partial v}{\partial y_1} & \dfrac{\partial v}{\partial y_2} & & \dfrac{\partial v}{\partial y_{a-1}} & \dfrac{\partial v}{\partial z_1} & \dots & \dfrac{\partial v}{\partial z_{(a-1\ (c-1)}} \end{bmatrix}$$

Clearly, $(\alpha_1, \dots, \alpha_{a-1}, \beta_1, \dots)$ lies in the kernel of $Du(x^*)$ if and only if $\alpha_1 = \dots = \alpha_{a-1} = 0$. [Recall, $Du(x^*)$ has rank $a-1$.] Note that $x^* \in \theta(u)$ if and only if

$$\frac{\partial v}{\partial z_1}(0) = \dots = \frac{\partial v}{\partial z_{(a-1)(c-1)}}(0) = 0.$$

The second intrinsic derivative, $F_{x^*}^2$, defined on the kernel of $Du(x^*)$ is

$$\sum_{i=1}^{a} \lambda_i D^2 u_i(x^*)(0, \beta) = \lambda_a \left(\left(\frac{\partial^2 v}{\partial z_i \partial z_j} \right) \right)_{1 \leqq i, j \leqq (a-1)(c-1)} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{(a-1)(c-1)} \end{pmatrix}.$$

So, $x^* \in \Gamma(u)$ if and only if the matrix

$$\left(\left(\frac{\partial^2 v}{\partial z_i \partial z_j}(0) \right) \right)$$

is singular.

On the other hand, since $\theta(u)$ is defined by

$$\frac{\partial v}{\partial z_1} = \ldots = \frac{\partial v}{\partial z_{(a-1)(c-1)}} = 0,$$

the tangent space to $\theta(u)$ at x^* is the kernel of

$$D\left(\frac{\partial v}{\partial z_1}, \ldots, \frac{\partial v}{\partial z_{(a-1)(c-1)}}\right)(0),$$

i.e., (α, β) such that

$$\left(\left(\frac{\partial^2 v}{\partial y_i \partial z_k} \frac{\partial^2 v}{\partial z_j \partial z_k}\right)\right)\begin{pmatrix} \alpha_1 \\ \alpha_{a-1} \\ \beta_1 \\ \beta_{(a-1)(c-1)} \end{pmatrix} = 0.$$

So, $(\alpha, \beta) \in$ kernel $Du(x^*) \cap T_{x^*}\theta(u)$ if and only if $\alpha = 0$ and

$$\left(\left(\frac{\partial^2 v}{\partial z_j \partial z_k}\right)\right)\beta = 0.$$

In particular, β can be non-zero if and only if $x^* \in \Gamma(u)$.

Theorem 11. For an open-dense subset of u in $C_u^\infty(\Omega, \mathbf{R}^a)$, $\theta(u) \cap \mathring{\Omega}$ is empty or is an $(a-1)$-dimensional submanifold and $\Gamma(u) \cap \mathring{\Omega}$ is a finite union of lower-dimensional submanifolds. In particular, $\Gamma(u)$ is generically measure zero in $\theta(u)$.

Proof. Let us calculate the defining equations for $\Gamma(u)$. By Lemma 10, $x \in \Gamma(u)$ if and only if $x \in \theta(u)$ and $T_x\theta(u) \cap$kernel $Du(x) \neq \{0\}$. We want to describe this analytically. For simplicity of notation, we will work in detail for the case $a = 3$ and then write out the corresponding equations for general a at the end of this proof.

Let $U = (u, v, w) : \Omega \to \mathbf{R}^3$; then $DU(x, y, z)$ is

$$\begin{bmatrix} u_1(x) & \ldots & u_c(x) & 0 & \ldots & 0 \\ 0 & \ldots & 0 & v_1(y) & \ldots & v_c(y) \\ -w_1(z) & \ldots & -w_c(z) & -w_1(z) & \ldots & -w_c(z) \end{bmatrix}$$

where $z = -x-y$ and $v_j(y)$ stands for $(\partial v/\partial y_j)(y)$. It follows from Lemma 7 that if $x \in \theta(u)$ and if the kth component of ∇u is non-zero, then so is the kth component of ∇v and ∇w. Since some component must be non-zero, assume without loss of generality that u_1, v_1 and w_1 are all non-zero. So, we shall work in the W_1 of Lemma 7.

Vector $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha_1, \ldots, \alpha_c, \beta_1, \ldots, \beta_c)$ lies in kernel of $Du(x, y, z)$ if and only if $u_1\alpha_1 + \ldots + u_c\alpha_c = 0$ and $v_1\beta_1 + \ldots + v_c\beta_c = 0$, i.e.,

$$\begin{aligned} \alpha_1 &= -\frac{u_2\alpha_2 + \ldots + u_c\alpha_c}{u_1}, \\ \beta_1 &= -\frac{v_2\beta_2 + \ldots + v_c\beta_c}{v_1}. \end{aligned} \tag{*}$$

Let

$$U^j \equiv \frac{u_j(x)}{u_1(x)}, \quad V^j \equiv \frac{v_j(y)}{v_1(y)}, \quad W^j \equiv \frac{w_j(-x-y)}{w_1(-x-y)}.$$

It follows from the condition $\lambda_i D\bar{u}_i(x) = \lambda_k D\bar{u}_k(x)$ that $(x, y) \in \theta(U)$ if and only if

$$U^j(x) - W^j(z) = 0 \quad \text{and} \quad V^j(y) - W^j(z) = 0,$$

for $j = 2, \ldots, c$. So, $T_{(x,y)}\theta$ is given by the kernel of $D(U^2 - W^2, \ldots, V^c - W^c)(x, y)$, i.e., by $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ such that

$$\begin{bmatrix} U^2_{x_1} + W^2_{z_1} & \ldots & U^2_{x_c} + W^2_{z_c} & W^2_{z_1} & \ldots & W^2_{z_c} \\ & & \vdots & & & \\ U^c_{x_1} + W^c_{z_1} & \ldots & U^c_{x_c} + W^c_{z_c} & W^c_{z_1} & \ldots & W^c_{z_c} \\ W^2_{z_1} & \ldots & W^2_{z_c} & V^2_{y_1} + W^2_{z_1} & \ldots & V^2_{y_c} + W^2_{z_c} \\ \vdots & & \vdots & & & \\ W^c_{z_1} & \ldots & W^c_{z_c} & V^c_{y_1} + W^c_{z_1} & \ldots & V^c_{y_c} + W^c_{z_c} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} = \mathbf{0}.$$

Now, $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ lies in the kernel of $DU(x, y) \cap T_{(x,y)}\theta(U)$ if and only if (*) holds and

$$\begin{bmatrix} \mathscr{U} + \mathscr{W} & \mathscr{W} \\ \mathscr{W} & \mathscr{V} + \mathscr{W} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \vdots \\ \alpha_c \\ \beta_2 \\ \vdots \\ \beta_c \end{bmatrix} = \mathbf{0}, \tag{**}$$

where

$$\mathcal{U} \equiv \begin{bmatrix} U^2_{x_2} - U^2_{x_1} U^2 & \dots & U^2_{x_c} - U^2_{x_1} U^c \\ \vdots & & \\ U^c_{x_2} - U^c_{x_1} U^2 & \dots & U^c_{x_c} - U^c_{x_1} U^c \end{bmatrix} \quad \text{at } x,$$

$$\mathcal{V} \equiv \begin{bmatrix} V^2_{y_2} - V^2_{y_1} V^2 & \dots & V^2_{y_c} - V^2_{y_1} V^c \\ \vdots & & \\ V^c_{y_2} - V^c_{y_1} V^2 & \dots & V^c_{y_c} - V^c_{y_1} V^c \end{bmatrix} \quad \text{at } y,$$

$$\mathcal{W} \equiv \begin{bmatrix} W^2_{z_2} - W^2_{z_1} W^2 & \dots & W^2_{z_c} - W^2_{z_1} W^c \\ \vdots & & \\ W^c_{z_2} - W^c_{z_1} W^2 & \dots & W^c_{z_c} - W^c_{z_1} W^c \end{bmatrix} \quad \text{at} \quad z = -x-y.$$

So by Lemma 10 and the above remarks, $(x, y) \in \Gamma(u)$ if and only if

$$\det \begin{pmatrix} \mathcal{U}(x) + \mathcal{W}(z) & \mathcal{W}(z) \\ \mathcal{W}(z) & \mathcal{V}(y) + \mathcal{W}(z) \end{pmatrix} \equiv D(x, y) = 0.$$

It is clear that this yields a polynomial equation in $J^2_u(\Omega, \mathbf{R}^a)$ since

$$u^3_1 v^3_1 w^3_1 \cdot D(x, y, u_x, u_{xx}, v_y, v_{yy}, w_z, w_{zz})$$

is a polynomial expression in the first- and second-order partial derivatives of u, v, and w. For example,

$$U^2_{x_2} - U^2_{x_1} U^2 = \frac{(u_1)^2 u_{22} - (u_2)^2 u_{11}}{u^3_1}$$

$$= \frac{(v^1_1)^2 v^1_{2,2} - (v^1_2)^2 v^1_{1,1}}{(v^1_1)^3} \, (j^2 U_{(x,y)}).$$

So, the defining equations for Γ in $J^2_u(\Omega, \mathbf{R}^a)$ are the $(a-1)(c-1)$ equations of $(\neq)$ in the second proof of Lemma 7 and the above polynomial on $(x, y, v^i_j, v^i_{j,k})$ induced by

$$(u^3_1 v^3_1 v^3_1) D(x, y, u_1, \dots, u_{cc}, v_1, \dots, w_{cc}) = 0.$$

The latter equation depends heavily on the second-order partial derivatives of u, v, and w, while the equations of $(\neq)$ were independent of each other and involved no second derivatives. Consequently, these $(a-1)(c-1)+1$ equations are independent at many points. By Lemma 9, our Γ is a finite union of submanifolds in $J^2_u(\Omega, \mathbf{R}^a)$, each of codimension at least $(a-1)(c-1)+1$; each of which can be described by algebraic equations. Since $\Gamma(U) = j^2(U)^{-1}(\Gamma)$, the theorem follows now from Theorem 5 and the comment which follows the proof. The measure zero statement follows from the smoothness of these lower dimensional submanifolds.

For the general case where a is any integer $\geqq 2$, (**) is replaced by

$$\begin{bmatrix} \mathscr{U}^1+\mathscr{U}^a & \mathscr{U}^a & \dots & \mathscr{U}^a \\ \mathscr{U}^a & \mathscr{U}^2+\mathscr{U}^a & \dots & \mathscr{U}^a \\ \mathscr{U}^a & \mathscr{U}^a & \dots & \mathscr{U}^{a-1}+\mathscr{U}^a \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \alpha^1_2 \\ \alpha^1_c \end{bmatrix} \\ \alpha^2_2 \\ \alpha^{a-1}_2 \\ \alpha^{a-1}_c \end{bmatrix} = \mathbf{0},$$

where $U = (u^1, \dots, u^a) : \Omega \to \mathbf{R}^a$,

$$U^{i,j}(x) = \frac{u^i_{x_j}}{u^i_{x_1}}(x^i),$$

and

$$\mathscr{U}^i = ((U^{i,k}_{x^i_j} - U^{i,k}_{x^i_1} U^{i,j}))_{\substack{2 \leqq j \leqq c \\ 2 \leqq k \leqq c}}.$$

Otherwise, the same argument applies.

Remark. The last two sections have laid the foundations of the singularity theory of utility mappings. It would be a reasonable goal to see how much of the usual singularity theory as described in Golubitsky–Guillemin (1973) or Martinet (1974) carries over to the special theory of utility mappings; that is, by taking an approach for higher-order derivatives analogous to the first proof of Lemma 7. For example, Γ is probably a codimension $(a-1)(c-1)+1$ *submanifold* of $J^2_u(\Omega, \mathbf{R}^a)$. In fact, all the 'Thom–Boardman singularity sets' in $J^k(\Omega, \mathbf{R}^a)$ that arise in $J^k_u(\Omega, \mathbf{R}^a)$ are probably submanifolds. We hope to say more about this and its economic applications in later papers. We will look at the cases $a = 2$ and $a = 3$ more carefully in the next sections of this paper.

We conclude this section with a simple result to illustrate other types of statements which can result from Theorem 6. In the following, we determine the structure of optima satisfying some given functional relationship between the values of the utility functions and the Pareto point.

Corollary 12. Let $\mathbf{0}$ *be a regular value of smooth function* $f: \Omega \times \mathbf{R}^a \to \mathbf{R}^k$, $k \leqq c(a-1)+a$. *Let* $F(u) = \{\mathbf{x} \in \Omega \mid \mathbf{x}$ *is in* $\theta(u)$ *and* $f(\mathbf{x}, u(\mathbf{x})) = 0\}$. *There exists an open-dense subset of* $C^\infty_u(\Omega, \mathbf{R}^a)$ *such that* $F(u)$ *is an* $(a-1-k)$*-dimensional* *submanifold of* $\mathring{\Omega}$, *if it is non-empty. In particular, if* $k = a-1$, *then* $F(u)$ *is a union of isolated points, and if* $k > a-1$, *then* $F(u)$ *is empty.*

Proof. According to the inverse function theorem, $f^{-1}(\mathbf{0})$ defines a smooth codimension k closed submanifold of $\Omega \times \mathbf{R}^a$. Thus, from Lemma 7, $F = f^{-1}(\mathbf{0}) \times \theta$ forms a smooth codimension $(c-1)(a-1)+k$ closed submanifold of $J_u^1(\Omega, \mathbf{R}^a)$. According to Proposition 6, there is an open dense subset of utility mappings such that $j^1(u)$ is transverse to F. By Proposition 5, for such a choice of u, $F(u) = j^1(u)^{-1}(F)$ is a codimension $(a-1)(c-1)+k$ submanifold of Ω. If $F(u)$ is non-empty, then it is of dimension $c(a-1)-(a-1)(c-1)-k = a-1-k$. If $k = a-1$, it is of dimension zero. If $k > a-1$, it is empty.

The above imposed a special condition upon the $D\bar{u}(*)$. The statement can easily be generalized to some functional relationship between the $\mathbf{x}$, $u(x)$ and $Du(\mathbf{x})$.

Other examples follow from Theorem 6 by using higher-order jet spaces. For example, the condition that F_x^2 (see Proposition 1) is negative definite is an open condition in $L_u^2(\Omega, \mathbf{R}^a)$. Thus it follows from Theorem 6 that *for an open-dense set of utility mappings, the set of non-degenerate* LPO's *is either empty or it forms a smooth* $(a-1)$*-dimensional submanifold of* $\mathring{\Omega}$. While this statement is immediate, we defer a detailed proof until a future paper where we emphasize the structure of the higher-order jet spaces.

5. Degenerate optima in two-agent economies

Theorem 11 shows that for a residual set of utility mappings u, the degenerate singularity set $\Gamma(u)$ is very small in the singularity set $\Theta(u)$ – not only is it measure zero, but it is also lower-dimensional. To complete this current study of the degenerate singularity set, we need to look at one more question: are they removable in the sense that for generic u in $C_u^\infty(\Omega, \mathbf{R}^a)$ all the LPO's are in $\Theta(u)-\Gamma(u)$, where, according to Proposition 1, second-order tests give necessary and sufficient conditions for optimality? Namely, is there a 'near-by' u which has only non-degenerate optima?

In this section, we will consider these questions in economics with two agents. In this case, the generic $u: \Omega \to \mathbf{R}^2$ has $\Theta(u)$ a one-dimensional manifold and $\Gamma(u)$ a set of isolated points in $\Theta(u) \cap \mathring{\Omega}$. We will see that the generic u has no optima in $\Gamma(u)$ so that second-order tests give all the LPO's for most u.

Two basic facts make the study of $C_u^\infty(\Omega, \mathbf{R}^2)$ rather simple. First of all, the studies of $C_u^\infty(\Omega, \mathbf{R}^2)$ and $C^\infty(\Omega, \mathbf{R}^2)$ are equivalent. Secondly, H. Whitney (1955) has fully described the generic theory of singularities for maps into the plane. Wan (1975) has also applied Whitney's theory to study the generic $\Theta(u)$.

Lemma 13. For a residual set of maps f *in* $C^2(\Omega, \mathbf{R}^2)$, $Df(x)$ *never has rank zero. In searching for generic properties in* $C_u^\infty(C_{c,2}, \mathbf{R}^2)$, *it is equivalent to search for such properties in* $C^\infty(\mathbf{R}^c, \mathbf{R}^2)$.

Proof. Let $\Sigma^c = \{j^1f(x) \in J^1(\Omega, \mathbf{R}^2) \mid Df(x) = 0\} = \Omega \times \mathbf{R}^2 \times \{0\}$, a codimension $2(\dim \Omega)$ submanifold of $J^1(\Omega, \mathbf{R}^2)$. So, Σ^c is the space of all 1-jets $(x, f(x), Df(x))$ such that kernel $Df(x)$ is c-dimensional. By Thom's Jet Transversality Theorem, j^1f is transverse to Σ^c in $J^1(\Omega, \mathbf{R}^2)$ for a residual subset of f in $C^\infty(\Omega, \mathbf{R}^2)$. For such f, $(j^1f)^{-1}(\Sigma^c)$ is a codimension $2\cdot(\dim \Omega)$ submanifold in Ω [by Proposition 5(a)]. Since $2\cdot(\dim \Omega) > \dim \Omega$, this means that for such f, $(j^1f)^{-1}\Sigma^c = \emptyset$, i.e., $Df(x)$ never has rank zero.

The second sentence of the lemma follows by noticing that if $u = (u^1, u^2)$ lies in $C^k_u(\Omega, \mathbf{R}^2)$, then $u(x^1, x^2) = (\bar{u}^1(x^1), \bar{u}^2(x^2)) = (\bar{u}^1(x^1), \bar{u}^2(-x^1))$, where $x^1 \in \mathbf{R}^c$. So, given a map $f \in C^k(\mathbf{R}^c, \mathbf{R}^2)$ of rank $\geqq 1$ everywhere, one can think of f as being in $C^k_u(\Omega_{c,2}, \mathbf{R}^2)$ by writing it as $(f_1(x^1), f_2(x^2))$ where $x^1 \in \mathbf{R}^c \cong \Omega_{c,2}$ and $x^2 = -x^1$. ∎

For the remainder of this section, we will work without loss of generality in $C^\infty(\mathbf{R}^c, \mathbf{R}^2)$. To indicate why a generic f has no optima in its degeneracy set $\Gamma(f)$, we will outline briefly how one obtains 'normal forms' for mappings via charges of variables in the source. The prototype of all normal-form theorems is Morse's Lemma which states that if x^* is a non-degenerate critical point of $g \in C^3(\mathbf{R}^c, \mathbf{R})$, then on some neighborhood of x^* in $\mathbf{R}^c$, one can choose coordinates $(y_1, \ldots, y_c)$ in which g has the form

$$g(y_1, \ldots, y_c) = g(x^*) + \sum_{i=1}^{c} \varepsilon_i y_i^2,$$

where each $\varepsilon_i = \pm 1$.

For our study, we will need the following generalization of Morse's Lemma.

Proposition 14 (Decomposition Lemma). Let $g : \mathbf{R}^c \to \mathbf{R}$ be a C^∞ map with $g(0) = 0$ and $Dg(0) = 0$. Suppose that the Hessian of g,

$$\left(\left(\frac{\partial^2 g}{\partial x_i \partial x_j}(0)\right)\right)$$

has rank $p \leqq c$. Then, there is a coordinate system $(y_1, \ldots, y_c)$ on a neighborhood of 0 in $\mathbf{R}^c$ under which

$$g(y_1, \ldots, y_c) = Q(y_1, \ldots, y_p) + h(y_{p+1}, \ldots, y_c).$$

Here, Q is a non-degenerate quadratic form in $y_1, \ldots, y_p$; h depends only on $y_{p+1}, \ldots, y_c$; and all the first- and second-order partial derivatives of h vanish at 0.

Although the proof of this 'decomposition theorem' does not depend on the division theorem or on some other similarly powerful but abstract technique of singularity theory, it does rely heavily on the implicit function theorem and on some basic facts about differential equations. Consequently, we will omit the rather technical proof and refer the reader to Martinet (1974). We come now to the goal of this section.

Theorem 15. Let $\mathscr{B} \subset C_u^\infty(\Omega_{c,2}, \mathbf{R}^2)$ *be the open-dense subset described in Theorem 11. For* $u \in \mathscr{B}$, *there are no* LPO's *in* $\Gamma(u)$.

Proof. By Lemma 13, we can study $C^\infty(\mathbf{R}^c, \mathbf{R}^2)$ without loss of generality. The problem now is to find and study the generic singularities in $C^\infty(\mathbf{R}^c, \mathbf{R}^2)$. This task has been accomplished by Whitney (1955) and Morin (1965). See Martinet (1974) for an excellent exposition of this work – work that we will briefly outline now.

Suppose $x_0 \in S(u)$, i.e., $Du(x_0): \mathbf{R}^c \to \mathbf{R}^2$ has rank one. In singularity theory, one uses the notation $x_0 \in \Sigma^{c-1}(u)$ to emphasize that $Du(x_0)$ has $(c-1)$-dimensional kernel, i.e., corank $(c-1)$. By the 'rank theorem' again, there is a change of coordinates in a neighborhood W about x_0 under which x_0 corresponds to 0 and $u(y)$ has the form

$$u^1(y_1, \ldots, y_c) = y_1,$$

$$u^2(y_1, \ldots, y_c) = h(y_1, \ldots, y_c).$$

On W,

$$Du(\mathbf{y}) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ h_{y_1} & h_{y_2} & \ldots & h_{y_c} \end{pmatrix},$$

and $\Sigma^{c-1}(u) \cap W$ is $\{\mathbf{y} \mid h_{y_2} = \ldots = h_{y_c} = 0\}$. By our genericity assumption on u, $\Sigma^{c-1}(u)$ is a 1-manifold, i.e., 0 is a regular value of $(h_{y_2}, \ldots, h_{y_c})$.

Again using the notation of singularity theory, we will write $x_0 \in \Sigma^{c-1,k}(u)$ if $x_0 \in \Sigma^{c-1}(u)$ and if the kernel of $D(u|\Sigma^{c-1}(u))$ at x_0 has dimension k. [Recall $\Sigma^{c-1}(u)$ is a submanifold, so this makes sense.] As in Lemma 10, the latter condition is equivalent to the subspace $\ker Du(x_0) \cap T_{x_0}\Sigma^{c-1}(u)$ having dimension k. So, if $k = 0$, then $x_0 \notin \Gamma(u)$, i.e., x_0 is a non-degenerate singularity. In this case, x_0 is a non-degenerate critical point of $u^2 | u^{1^{-1}}(u(x_0))$; and one can easily use the Proposition 14 to find coordinates $(\tilde{y}_1, \ldots, \tilde{y}_c)$ about x_0 in which u has the form

$$u^1(\tilde{y}) = \tilde{y}_1,$$

$$u^2(\tilde{y}) = r(\tilde{v}_1) + Q(\tilde{v}_2, \ldots, \tilde{y}_c),$$

where Q is a non-degenerate quadratic and $r'(0) \neq 0$.

If x_0 is a degenerate point of $S(u)$, i.e., $x_0 \in \Gamma(u)$, then $k \geqq 1$. As shown in Lemma 10, the matrix $((h_{y_iy_j}(0)))_{i,j>1}$ is singular in this case. So, $\Gamma(u)$ is given by the c equations

$$h_{y_2} = 0, \quad h_{y_3} = 0, \ldots, h_{y_c} = 0, \quad \Delta \equiv \det((h_{y_iy_j}))_{i,j>1} = 0.$$

By our hypothesis on u, 0 is a regular value of these equations. Since $\Sigma^{c-1}(u)$ is a one-manifold, this means $\Gamma(u)$ is a zero-dimensional submanifold, i.e., a set of isolated points.

Notice that it is impossible for k to be $\geqq 2$ in the expression $\Sigma^{c-1,k}(u)$. This follows from the definition of $\Sigma^{c-1,k}(u)$ and from the fact that Σ^{c-1} is a one-manifold by the generic transversality assumptions we have made for u. So we need only investigate the singularities in $\Sigma^{c-1,1}(u)$, i.e., we can assume that $((h_{y_iy_j}(0)))_{i,j>1}$ has rank $(c-2)$. Thus, the map $(y_2, \ldots, y_c) \to h(0, y_2, \ldots, y_c)$ has a Hessian of corank one at $\mathbf{0}$. By Lemma 14, we can choose coordinates $\tilde{y}_1, \ldots, \tilde{y}_c$ so that $y_1 = \tilde{y}_1$ and $h(0, \tilde{y}_2, \ldots, \tilde{y}_c) = R(\tilde{y}_2) + Q(\tilde{y}_3, \ldots, \tilde{y}_c)$ where Q is a non-degenerate quadratic form. We have not yet used the fact that $\mathbf{0}$ is a regular value of $(h_{y_2}, \ldots, h_{y_c}, \Delta) : \mathbf{R}^c \to \mathbf{R}^c$. Since

$$\frac{\partial h_{y_i}}{\partial \tilde{y}_1}(0) = h_{\tilde{y}_i\tilde{y}_1}(0) = 0 \quad \text{for} \quad i > 1,$$

$\mathbf{0}$ a regular value implies that

$$\frac{\partial \Delta}{\partial \tilde{y}_1}(0) \neq 0.$$

A short calculation shows that this inequality implies that

$$\frac{\partial^3 h}{\partial \tilde{y}_2^3}(0) \neq 0,$$

i.e., that $R(\tilde{y}_2) = \tilde{y}_2^3 S(\tilde{y}_2)$ where $S(0) \neq 0$. Change variables by $\bar{y}_2 = \tilde{y}_2[S(\tilde{y}_2)]^{1/3}$ near 0. Now,

$$h(0, \bar{y}_2, \ldots, \tilde{y}_c) = \bar{y}_2^3 + Q(\tilde{y}_3, \ldots, \tilde{y}_c).$$

One can now use Proposition 1 to see that x_0 cannot be an LPO for u. For, $u^2 \,|\, (u^1)^{-1}(u^1(x_0))$ is $h(0, \bar{y}_2, \tilde{y}_3, \ldots, \tilde{y}_c)$ in our new coordinates. Since $\bar{y}_2^3 + Q(\tilde{y}_3, \ldots, \tilde{y}_c)$ cannot have a maximum at $\mathbf{0}$, x_0 cannot be an LPO for u.

If one is ambitious, one can continue the above analysis and, using the Malgrange–Mather Division Theorem, derive the following canonical form for u about a point in $\Sigma^{c-1,1}(u)$,

$$u^1(z_1, \ldots, z_c) = z_1$$

$$u^2(z_1, \ldots, z_c) = k_1(z_1) + k_2(z_1)z_2 + z_2^3 + Q(z_3, \ldots, z_c),$$

where Q is a non-degenerate quadratic and $k_2'(0) \neq 0$.

6. Stable degenerate optima in $C_*^\infty(\mathbf{R}^m, \mathbf{R}^a)$

In section 5, we saw that for the generic economy with two agents, second-derivative tests are sufficient in the search for all local Pareto optima. Our goal now is to begin a systematic study of this question for economies with a agents, $a \geqq 2$. We will first need to introduce some more ideas from general singularity theory. While doing this, we will treat degenerate optima in $C_*^\infty(\mathbf{R}^m, \mathbf{R}^a) \equiv \{f: \mathbf{R}^m \to \mathbf{R}^a \mid f \text{ is } C^\infty,\ m \geqq a, \text{ rank } Df(x) \geqq a-1 \text{ for all } x\}$. In the next section, we will specialize to utility mappings in $C_u^\infty(\Omega_{c,a}, \mathbf{R}^a)$.

Recall from the previous section that for $f \in C_*^\infty(\mathbf{R}^m, \mathbf{R}^a)$, $\Sigma^{m-a+1}(f) = \{x \in \mathbf{R}^m \mid \ker Df(x) \text{ has dimension } m-a+1\} = \{x \in \mathbf{R}^m \mid Df(x) \text{ has source corank } m-a+1 \text{ or target corank } 1\}$. This is the singularity set of f. By the Thom Jet Transversality Theorem this is generically a submanifold of $\mathbf{R}^m$. One continues this process as before. If $\Sigma^{m-a+1}(f)$ is a submanifold, look at $\Sigma^{m-a+1,k_1}(f) = \{x \in \Sigma^{m-a+1}(f) \mid \text{kernel } D(f \mid \Sigma^{n-a+1}(f)) \text{ has dim } k_1\}$. This turns out to be $\{x \in \Sigma^{m-a+1}(f) \mid \text{kernel } Df(x) \cap T_x\Sigma^{m-a+1}(f) \text{ has dimension } k_1\}$, as was sketched in Lemma 10, and again is a submanifold for generic f.

Continuing this process, one defines $\Sigma^{m-a+1,k_1,\ldots,k_r}(f)$, where $m-a+1 \geqq k_1 \geqq \ldots \geqq k_r$, as $\{x \in \Sigma^{m-a+1,\ldots,k_{r-1}}(f) \mid \Sigma^{m-a+1,\ldots,k_{r-1}}(f) \text{ is a submanifold and } \ker Df(x) \cap T_x\Sigma^{m-a+1,\ldots,k_{r-1}}(f) \text{ has dimension } k_r\}$. For a residual set of f, this is a submanifold whose dimension is easily computed. See Golubitsky–Guillemin (1973) or Boardman (1967).

As examples of stable degenerate optima, we will now focus on the sets $\Sigma^{m-a+1,1,1,\ldots,1,0}(f)$. These are the so-called Morin singularities [Morin (1965)] and are generalizations of the Whitney singularities examined in section 5. Consider for example, the following map $f = (f_1, \ldots, f_a)$ in $C_*^\infty(\mathbf{R}^m, \mathbf{R}^a)$:

$$\begin{aligned} f_1 &= x_1, \\ f_2 &= x_2, \\ &\vdots \\ f_{a-1} &= x_{a-1}, \\ f_a &= \delta x_a^r + x_1 x_a^{r-2} + x_2 x_a^{r-3} + \ldots + x_{r-2} + \sum_{i=a+1}^{m} \varepsilon_i x_i^2, \end{aligned} \qquad (\neq\neq)$$

where $\delta = \pm 1$, $\varepsilon_i = \pm 1$ for all i, and $r \leqq a+1$.

Proposition 16. [*Morin* (1965)]. *For the f of* $(\neq\neq)$, $\mathbf{0}$ *lies in*

$$\Sigma^{m-a+1,\underbrace{1,1,\dots,1}_{r-1},0}(f).$$

Furthermore, if $g \in C^{\infty}_{*}(\mathbf{R}^m, \mathbf{R}^a)$ *and if*

$$x^* \in \Sigma^{m-a+1,\underbrace{1,\dots,1}_{r-1},0}(g),$$

where all the

$$\Sigma^{m-a+1,\underbrace{1,\dots,1}_{k}}$$

are manifolds for $0 \leqq k \leqq r-1$, *then there are coordinates* $(x_1,\dots,x_m)$ *in a neighborhood U of* x^* *in* $\mathbf{R}^m$ *and coordinates in a neighborhood of* $g(x^*)$ *in* $\mathbf{R}^a$ *such that* $g|U$ *has the form* $(\neq\neq)$ *in these coordinates. There is also a neighborhood W of g in* $C^{\infty}_{*}(\mathbf{R}^m, \mathbf{R}^a)$ *such that for h in W,*

$$\Sigma^{m-a+1,\underbrace{1,\dots,1}_{r-1},0}(h)\cap W \neq \emptyset.$$

We will refer the reader to Morin (1965) for the proof. See also Golubitsky–Guillemin (1973) or Martinet (1974) for the case $a = m$. The first sentence is a simple calculation. The last sentence of the proposition follows immediately from the transversality conditions in the definition of $\Sigma^{m-a+1,1,1,\dots,1,0}(g)$ and from the openness of transversal intersection. The main steps in the proof of the middle sentence are the universal unfolding theorem and the decomposition lemma – both consequences of the Mather–Malgrange Division Theorem.

Note that, if one chooses r to be even, $\delta = -1$, and $\varepsilon_i = -1$ for all i in $(\neq\neq)$, then by Proposition 1, $\mathbf{0}$ is a degenerate Pareto optimum of f. For, $f_a|(f_1 = \dots = f_{a-1} = 0)$ is then $-x_a^r - \sum_{a+1}^m x_i^2$, which has $\mathbf{0}$ as a strict local maximum. By Morin's Theorem, $\mathbf{0}$ is a persistent optimum in that nearby functions will still have a degenerate optimum near zero. Furthermore, to test for optimality near $\mathbf{0}$ for f and mappings near f with calculus techniques, one needs to examine at least r derivatives, where $r \leqq a$ if a is even and $r \leqq a+1$ if a is odd. This leads naturally to the following conjecture:

Conjecture: For a residual subset G of $C^{\infty}_{*}(\mathbf{R}^m, \mathbf{R}^a)$, one needs to examine at most $(a+1)$st derivatives in using calculus techniques to find all the LPO's of an element in G.

Mather's theory of stable mappings indicates that this conjecture is true for $a \leqq 6$. A map $f \in C^{\infty}(\mathbf{R}^m, \mathbf{R}^a)$ is called 'stable' if there is a neighborhood W of f in $C^{\infty}(\mathbf{R}^m, \mathbf{R}^a)$ such that for any $f^1 \in W$ there are diffeomorphisms (i.e., smooth changes of variables) g of $\mathbf{R}^m$ and h of $\mathbf{R}^a$ such that $f^1 = hfg$. In Mather

(1970a), Mather shows that if f is a stable map and $x \in \mathbf{R}^m$ then there are coordinates in a neighborhood of x in which f is a polynomial of degree $\leqq a+1$. [In fact, this polynomial is $j^{a+1}f(x)$. See Martinet (1974) for another proof.] Furthermore, Mather (1970b) shows that stable maps form a residual subset of $C^\infty(\mathbf{R}^m, \mathbf{R}^a)$ for $a \leqq 6$ with the exception of $a = 6$, $m = 8$. The following proposition summarizes the discussion of this section.

Proposition 17. (a) *Let F be any dense subset of $C^\infty_*(\mathbf{R}^m, \mathbf{R}^a)$. Then, there exists $f \in F$ for which one must check* $(a+1)$st *order derivatives to find all the* LPO's *of f via calculus techniques.* (b) *If $a \leqq 6$ (except for $m = 8$, $a = 6$), there is a residual subset G of $C^\infty_*(\mathbf{R}^m, \mathbf{R}^a)$ with the property that one need only check the first $(a+1)$ derivatives of any f in G to find all the* LPO's *of such an f via calculus techniques.*

7. Persistent degenerate optima for utility mappings

In this section, we will prove the analogue of part (a) of Proposition 17 for the utility mappings in $C^\infty_u(\Omega_{c,a}, \mathbf{R}^a)$. So the more agents there are, the more derivatives one will have to check to find all the persistent LPO's. The special structure of utility mappings makes the construction of $\Sigma^{k,1,1,\dots,1,0}$-singularities a non-trivial matter. For example, one cannot construct persistent degenerate Pareto optima in $C^\infty_u(\Omega_{2,3}, \mathbf{R}^3)$ while holding $\bar{u}^1(x_1, x_2) = x_1$.

Theorem 18. There exists an open set $\mathscr{B}$ of $C^\infty_u(\Omega_{c,a}, \mathbf{R}^a)$ such that for $u \in \mathscr{B}$ one must check $(a+1)$st *order derivatives (or* ath *order if a is even) to find all the* LPO's *of u via calculus.*

We understand that Y.H. Wan has independently established a result similar to Theorem 18.

Proof. To simplify notation, we will prove this theorem for $c = 2$. Examples for $c > 2$ are easily constructed from the examples below. Furthermore, we will only carry out the details for $a = 3$, 4 and 5. We will indicate at the end of the proof how to carry out the construction for $a > 5$.

$c = 2, a = 3$: For simplicity, write u as (g, h, k), where g, h and k lie in $C^\infty_*(\mathbf{R}^2, \mathbf{R}^1)$. Recall that $\Omega = \{(x, y, z) \mid x+y+z = 0\}$; so (x, y) coordinatizes Ω. Without loss of generality, we can assume that $\mathbf{0} \in \Theta(u)$ and that $Dg(\mathbf{0}) = Dh(\mathbf{0}) = Dk(\mathbf{0}) = (1, 0)$. Since

$$Du(x, y) = \begin{bmatrix} g_{x_1} & g_{x_2} & 0 & 0 \\ 0 & 0 & h_{y_1} & h_{y_2} \\ -k_{z_1} & -k_{z_2} & -k_{z_1} & -k_{z_2} \end{bmatrix}, \tag{7.1}$$

$(\mathbf{x}, \mathbf{y}) \in \Theta(u)$ if and only if $G-K=0$ and $H-K=0$, where $G = g_{x_2}/g_{x_1}$, $H = h_{y_2}/h_{y_1}$, $K = k_{z_2}/k_{z_1}$. Furthermore, $\Theta(u) = \Sigma^2(u)$ is a manifold about $\mathbf{0}$ if and only if 0 is a regular value of $(G-K, H-K)$. In particular, we want the following matrix to have rank 2 at (0, 0):

$$\begin{pmatrix} G_{x_1}+K_{z_1} & G_{x_2}+K_{z_2} & K_{z_1} & K_{z_2} \\ K_{z_1} & K_{z_2} & H_{y_1}+K_{z_1} & H_{y_2}+K_{z_2} \end{pmatrix}. \tag{7.2}$$

If this happens, then (x, y) near (0, 0) lies in $\Sigma^{2,1}$ if and only if $\ker Du(x, y) \cap \ker D(G-K, H-K)(x, y)$ is one-dimensional. But $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \ker Du(x, y)$ if and only if $\alpha_1 = -G\alpha_2$ and $\beta_1 = -H\beta_2$ at (x, y); and $(-G\alpha_2, \alpha_2, -H\beta_2, \beta_2) \in \ker D(G-K, H-K)(x, y)$ if and only if

$$\begin{pmatrix} (G_{x_2}+K_{z_2})-G(G_{x_1}+K_{z_1}) & K_{z_2}-HK_{z_1} \\ K_{z_2}-GK_{z_1} & (H_{y_2}+K_{z_2})-H(H_{y_1}+K_{z_1}) \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{7.3}$$

Let $\mathscr{G} \equiv G_{x_2} - G_{x_1}G$, $\mathscr{H} \equiv H_{y_2} - H_{y_1}H$, $\mathscr{K} \equiv K_{z_2} - K_{z_1}K$. Then, if $(x, y) \in \Sigma^2(u)$, $(x, y) \in \Sigma^{2,1}(u)$ if and only if the matrix of (7.3) has rank one at (x, y): i.e., if and only if

$$\Delta(x, y) \equiv \det \begin{pmatrix} \mathscr{G}+\mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{K}+\mathscr{H} \end{pmatrix}$$

$$= \mathscr{G}\mathscr{H} + \mathscr{H}\mathscr{K} + \mathscr{G}\mathscr{K} = 0 \quad \text{at} \quad (x, y),$$

where some entry is non-zero. [Recall $G = H = K$ in $\Theta(u)$.] Furthermore, $\mathbf{0} \in \Sigma^{2,1}(u)$ and $\Sigma^{2,1}(u)$ is a manifold near 0 if $\mathbf{0}$ is a regular value of $(G-K, H-K, \Delta)$, i.e., if the following matrix has rank three at 0:

$$\begin{pmatrix} G_{x_1}+K_{z_1} & G_{x_2}+K_{z_2} & K_{z_1} & H_{y_2} \\ K_{z_1} & K_{z_2} & H_{y_1}+K_{z_1} & H_{y_2}+K_{z_2} \\ \Delta_{x_1} & \Delta_{x_2} & \Delta_{y_1} & \Delta_{y_2} \end{pmatrix} \tag{7.4}$$

Finally, if (x, y) near $\mathbf{0}$ is in the manifold $\Sigma^{2,1}(u)$, then $(x, y) \in \Sigma^{2,1,1}(u)$ if and only if $\ker Du(x, y) \cap \ker D(G-K, H-K, \Delta)(x, y)$ is one-dimensional. [Keep in mind that $T_{(x,y)}\Sigma^{2,1}(u)$ is $\ker D(G-K, H-K, \Delta)(x, y)$.] As above, this means that the matrix

$$\begin{pmatrix} \mathscr{G}+\mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{H}+\mathscr{K} \\ \Delta_{x_2}-K\Delta_{x_1} & \Delta_{y_2}-K\Delta_{y_1} \end{pmatrix}$$

has rank one at (x, y); for example that

$$\wedge \equiv \det \begin{pmatrix} \mathscr{K} & \mathscr{H}+\mathscr{K} \\ \Delta_{x_2}-K\Delta_{x_1} & \Delta_{y_2}-K\Delta_{y_1} \end{pmatrix}$$

is zero at (x, y). Finally, to see that $\mathbf{0} \in \Sigma^{2,1,1,1,0}(u)$ and that $\Sigma^{2,1,1}(u)$ is a zero-manifold near 0, one need only show that $\wedge(0) = 0$ and that $D(G-K, H-K, \Delta, \wedge)(\mathbf{0})$ has rank four.

To realize all this with a concrete example in $C_u^\infty(\Omega_{2,3}, \mathbf{R}^3)$, consider the utility mapping $u = (g, h, k)$, where

$$g(x_1, x_2) = x_1 + x_1 x_2,$$

$$h(y_1, y_2) = y_1 + y_1 y_2 - y_2^2,$$

$$k(z_1, z_2) = z_1 + z_1 z_2 - z_2^4.$$

Using the above notation, we find

$$G \equiv \frac{g_{x_2}}{g_{x_1}} = \frac{x_1}{1+x_2},$$

$$H \equiv \frac{h_{y_2}}{h_{y_1}} = \frac{y_1 - 2y_2}{1+y_2},$$

$$K \equiv \frac{k_{z_2}}{k_{z_1}} = \frac{z_1 - 4z_2^3}{1+z_2}.$$

Since $G(0) = H(0) = K(0) = 0$, 0 lies in $\Theta(u) \equiv \Sigma^2(u)$. Since $D(G-K, H-K)(0)$ is

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & -2 \end{pmatrix}$$

[as in (7.2)], $\Sigma^2(u)$ is a two-manifold about $\mathbf{0}$.

Similarly,

$$\mathscr{G} \equiv G_{x_2} - GG_{x_1} = \frac{-2x_1}{(1+x_2)^2},$$

$$\mathscr{H} \equiv H_{y_2} - HH_{y_1} = \frac{-2-2y_1+2y_2}{(1+y_2)^2},$$

$$\mathscr{K} \equiv K_{z_2} - KK_{z_1} = \frac{-2z_1 - 12z_2^2 - 4z_2^3}{(1+z_2)^2}.$$

Since

$$\Delta \equiv \mathscr{G}\mathscr{H}+\mathscr{G}\mathscr{K}+\mathscr{H}\mathscr{K} = 0 \quad \text{at} \quad \mathbf{0}, \mathbf{0} \in \Sigma^{2,1}(u).$$

Since

$$D(G-K, H-K, \Delta)(0) = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & -4 & 0 \end{pmatrix}$$

[as in (7.4)] has rank 3, $\Sigma^{2,1}(u)$ is a one-manifold around $\mathbf{0}$. Since

$$\begin{bmatrix} \mathscr{G}+\mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{H}+\mathscr{K} \\ \Delta_{x_2}+k\Delta_{x_1} & \Delta_{y_2}-K\Delta_{y_1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

has rank one, $\mathbf{0} \in \Sigma^{2,1,1}(u)$. Since

$$D(G-K, H-K, \Delta, \wedge)(0) = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & -4 & 0 \\ \wedge_{x_1} & -96 & \wedge_{y_1} & \wedge_{y_2} \end{bmatrix}$$

(where $\wedge$ is defined as above), $\Sigma^{2,1,1}(u)$ is a zero-manifold about $\mathbf{0}$ and $\mathbf{0} \in \Sigma^{2,1,1,0}(u)$.

We now apply Proposition 1 to see that $\mathbf{0}$ is an LPO for u. On $g = 0$, $x_1 = 0$; on $h = 0$, $y_1 = (y_2^2/1+y_2)$. So, $k|(g = h = 0)$ is

$$(-x_1-y_1)(1+z_2)-z_2^4 = -y_2^2\left[\frac{1-x_2-y_2}{1+y_2}\right]-(x_2+y_2)^4.$$

By inspection, it is clear that this expression is non-positive for (x_2, y_2) near $(0, 0)$. So $(0, 0)$ is a strict local maximum. One can also apply Theorem 3 here. By Proposition 1, $\mathbf{0}$ is an LPO for u.

Furthermore, since $\mathbf{0} \in \Sigma^{2,1,1,0}(u)$, $\mathbf{0}$ is a stable Morin singularity. By Proposition 16, $\Sigma^{2,1,1,0}(w) \neq \emptyset$ for all w close enough to u in $C_u^\infty(\Omega, \mathbf{R}^3)$. Applying the second sentence of Proposition 16, one finds that these $\Sigma^{2,1,1,0}(w)$'s will contain a (degenerate) LPO near $\mathbf{0}$. So, a whole open set of utility mappings in $C_u^\infty(\Omega, \mathbf{R}^3)$ have LPO's where fourth-order tests are needed to determine optimality. Such LPO's will be isolated points.

D

$c = 2$, $a = 4$: Apply the same analysis to $u = (f, g, h, k)$, where

$$f(w_1, w_2) = w_1 + w_1 w_2^2,$$

$$g(x_1, x_2) = x_1 + x_1 x_2 - x_2^2,$$

$$h(y_1, y_2) = y_1 + y_1 y_2 - y_2^2,$$

$$k(z_1, z_2) = z_1 + z_1 z_2 - z_2^4.$$

One easily checks that $\mathbf{0} \in \Sigma^{3,1,1,0}(u)$ and that $\Sigma^3(u)$, $\Sigma^{3,1}(u)$, $\Sigma^{3,1,1}(u)$, and $\Sigma^{3,1,1,0}(u)$ are all submanifolds near $\mathbf{0}$. Furthermore,

$$k|(f = g = h = 0) = \left(-\frac{x_2^2}{1+x_2} - \frac{y_2^2}{1+y_2}\right)(1+z_2) - (x_2+y_2+w_2)^4,$$

which has a strict local maximum at $\mathbf{0}$. By Proposition 1, $\mathbf{0}$ is a (degenerate) LPO for u. By Proposition 16, all utility mappings close enough to u will have an LPO in $\Sigma^{3,1,1,0}$. This time the degenerate optima lie on one-dimensional manifolds!

$c = 2$, $a = 5$: For this case, we will construct a stable example where one must examine the 6-jet to determine optimality. The utility mapping is a bit more complicated here. Consider the utility mapping $u = (f, g, h, j, k)$, where

$$f(v_1, v_2) = v_1(1 + \tfrac{1}{2}v_2^2 + \tfrac{1}{3}v_2^3 + \tfrac{1}{8}v_2^4) - 2v_2^2 - v_2^4 - \tfrac{2}{3}v_2^5 - \tfrac{1}{4}v_2^6,$$

$$g(w_1, w_2) = w_1(1 + w_2 + \tfrac{1}{6}w_2^3 + \tfrac{1}{4}w_2^5) - 2w_2^2 - 2w_2^3 - \tfrac{1}{3}w_2^5 - \tfrac{1}{2}w_2^7,$$

$$h(x_1, x_2) = x_1(1 + 2x_2 + \tfrac{1}{2}x_2^2 + x_2^4) - 2x_2^2 - 4x_2^3 - x_2^4 - 2x_2^6,$$

$$j(y_1, y_2) = y_1(1 + \tfrac{1}{2}y_2^2 + \tfrac{1}{3}y_2^5) - 2y_2^2 - y_2^4 - \tfrac{2}{3}y_2^5,$$

$$k(z_1, z_2) = z_1 - \tfrac{1}{2}z_2^2 - z_2^6.$$

At the end of this proof, we will discuss the rationale behind this choice of utility mapping.

As before, let $F = f_{v_2}/f_{v_1}$, $G = g_{w_2}/g_{w_1}, \ldots, K = k_{z_2}/k_{z_1}$. The equations for $\Theta(u)$ are $(F-K, G-K, H-K, J-K) = (0, 0, 0, 0)$. Since $F(\mathbf{0}) = \ldots = K(\mathbf{0}) = 0$, $\mathbf{0} \in \Theta(u)$. [Alternatively, note that $Df(\mathbf{0}) = \ldots = Dk(\mathbf{0}) = (1, 0)$.] Furthermore, $\Theta(u) = \Sigma^4(u)$ is a 4-manifold near $\mathbf{0}$ since $\mathbf{0}$ is a regular point of $(F-K, G-K, H-K, J-K)$. For,

$$D(F-K,\, G-K,\, H-K,\, J-K)(0)$$

$$= \begin{bmatrix} F_{v_1}+K_{z_1}(0) & F_{v_2}+K_{z_2} & K_{z_1} & K_{z_2} & K_{z_1} & K_{z_2} & K_{z_1} & K_{z_2} \\ K_{z_1}(0) & K_{z_2} & G_{w_1}+K_{z_1} & G_{w_2}+K_{z_2} & K_{z_1} & K_{z_2} & K_{z_1} & \mathbf{K}_{z_2} \\ K_{z_1}(0) & K_{z_2} & K_{z_1} & K_{z_2} & H_{x_1}+K_{z_1} & H_{x_2}+K_{z_2} & K_{z_1} & K_{z_2} \\ K_{z_1}(0) & K_{z_2} & K_{z_1} & K_{z_2} & K_{z_1} & K_{z_2} & J_{y_1}+K_{r_1} & J_{y_2}+K_{z_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -3 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & -3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -3 \end{bmatrix}$$

has full rank. Since the kernel of $Du(v, w, x, y)$ is $\{(-F\alpha, \alpha, -G\beta, \beta, -H\gamma, \gamma, -J\delta, \delta) \mid (\alpha, \beta, \gamma, \delta) \in \mathbf{R}^4\}$, $\mathbf{0} \in \Sigma^4(u)$ lies in $\Sigma^{4,1}(u)$ if and only if

$$\ker Du(\mathbf{0}) \cap T_0\Sigma^4(u) = \ker Du(\mathbf{0}) \cap \ker D(F-K,\, G-K,\, H-K,$$
$$H-K,\, J-K)(\mathbf{0})$$

is one-dimensional, i.e., if and only if

$$\begin{bmatrix} \mathscr{F}+\mathscr{K} & \mathscr{K} & \mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{G}+\mathscr{K} & \mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{K} & \mathscr{H}+\mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{K} & \mathscr{K} & \mathscr{J}+\mathscr{K} \end{bmatrix}$$

has rank three at $\mathbf{0}$, where as before, $\mathscr{F} \equiv F_{v_2} - F_{v_1}F, \ldots, \mathscr{K} \equiv K_{z_2} - K_{z_1}K$. Since this matrix is

$$\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$\mathbf{0} \in \Sigma^{4,1}(\mathbf{u})$. Again, let $\Delta = \mathscr{F}\mathscr{G}\mathscr{H}\mathscr{J} + \mathscr{F}\mathscr{G}\mathscr{H}\mathscr{K} + \mathscr{F}\mathscr{H}\mathscr{J}\mathscr{K} + \mathscr{F}\mathscr{G}\mathscr{J}\mathscr{K} + \mathscr{G}\mathscr{H}\mathscr{J}\mathscr{K}$, the determinant of the above matrix. Note that $\Sigma^{4,1}(u)$ is a 3-manifold about $\mathbf{0}$ since

$$D(F-K,\ G-K,\ H-K,\ J-K,\ \Delta)(0)$$

$$=\begin{bmatrix} 0 & -3 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -3 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & -3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -3 \\ -16 & 0 & 32 & 0 & 48 & 0 & -16 & 0 \end{bmatrix}$$

has full rank.

Similarly, $\mathbf{0} \in \Sigma^{4,1,1}(u)$ if and only if

$$\ker Du(0) \cap T_0\Sigma^{4,1}(u) = \ker Du(\mathbf{0}) \cap \ker D(F-K,\ G-K,\ H-K,\ J-K,\ \Delta)(\mathbf{0})$$

is one-dimensional, i.e., if and only if

$$\begin{bmatrix} \mathscr{F}+\mathscr{K} & \mathscr{K} & \mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{G}+\mathscr{K} & \mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{K} & \mathscr{H}+\mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{K} & \mathscr{K} & \mathscr{J}+\mathscr{K} \\ \Delta_{v_2}-F\Delta_{v_1} & \Delta_{w_2}-G\Delta_{w_1} & \Delta_{x_2}-H\Delta_{x_1} & \Delta_{y_2}-J\Delta_{y_1} \end{bmatrix}$$

has rank three at $\mathbf{0}$. Since $\Delta_{v_2}(\mathbf{0}) = \Delta_{w_2}(\mathbf{0}) = \Delta_{x_2}(\mathbf{0}) = \Delta_{y_2}(\mathbf{0}) = 0$, it does have rank three, and $\mathbf{0} \in \Sigma^{4,1,1}(u)$. Now, let L be the determinant of the lower 4×4 submatrix of the preceding matrix. One checks that

$$D(F-K,\ G-K,\ H-K,\ J-K,\ \Delta,\ L)(0)$$

$$=\begin{bmatrix} 0 & -3 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -3 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 & -3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & -3 \\ -16 & 0 & 32 & 0 & 48 & 0 & -16 & 0 \\ 2^9 & 0 & 7.4^4 & 0 & 9.4^5 & 0 & 0 & 0 \end{bmatrix}$$

which has full rank. So, $\Sigma^{4,1,1}(u)$ is a two-manifold.

Now, $\mathbf{0} \in \Sigma^{4,1,1}(u)$ if and only if

$$\ker Du(0) \cap \ker D(F-K,\ G-K,\ H-K,\ J-K,\ \Delta,\ L)(\mathbf{0})$$

is one-dimensional, i.e., if and only if

$$\begin{bmatrix} \mathscr{F}+\mathscr{K} & \mathscr{K} & \mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{G}+\mathscr{K} & \mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{K} & \mathscr{H}+\mathscr{K} & \mathscr{K} \\ \mathscr{K} & \mathscr{K} & \mathscr{K} & \mathscr{J}+\mathscr{K} \\ \Delta_{v_2}-F\Delta_{v_1} & \Delta_{w_2}-G\Delta_{w_1} & \Delta_{x_2}-H\Delta_{x_1} & \Delta_{y_2}-J\Delta_{y_1} \\ L_{v_2}-FL_{v_1} & L_{w_2}-GL_{w_1} & L_{x_2}-HL_{x_1} & L_{y_2}-JL_{y_1} \end{bmatrix}$$

has rank three at $\mathbf{0}$. Since $L_{v_2}(0) = L_{w_2}(0) = L_{x_2}(0) = L_{y_2}(0) = 0$, it does have rank three and $\mathbf{0} \in \Sigma^{4,1,1,1,1}(u)$. Let M be the determinant of the 4×4 submatrix composed of rows, 2, 3, 4, and 6 of the previous matrix. One checks as before that $D(F-K, G-K, H-K, J-K, \Delta, L, M)(0)$ has full rank at $\mathbf{0}$ and therefore $\Sigma^{4,1,1,1,1}(u)$ is a one-manifold near $\mathbf{0}$.

Similarly, one constructs a new matrix by adjoining the row $(M_{v_2}-FM_{v_1}, M_{w_2}-GM_{w_1}, M_{x_2}-HM_{x_1}, M_{y_2}-JM_{y_1})$ to the previous matrix. Since $M_{v_2}(\mathbf{0}) = \ldots = M_{y_2}(\mathbf{0}) = 0$, this new matrix has rank 3, and $\mathbf{0} \in \Sigma^{4,1,1,1,1,1}(u)$. One final computation illustrates that $\Sigma^{4,1,1,1,1,1}(u)$ is a zero manifold about $\mathbf{0}$ and $\mathbf{0}$ is in $\Sigma^{4,1,1,1,1,0}(\mathbf{u})$.

We now use Proposition 1 to verify that $\mathbf{0}$ is an LPO for u. We want to examine k restricted to $f = g = h = j = 0$. Simple calculations show that

$$f(v_1, v_2) = 0 \quad \Leftrightarrow \quad -v_1 = -2v_2^2,$$

$$g(w_1, w_2) = 0 \quad \Leftrightarrow \quad -w_1 = -2w_2^2,$$

$$g(x_1, x_2) = 0 \quad \Leftrightarrow \quad -x_1 = -2x_2^2,$$

$$j(y_1, y_2) = 0 \quad \Leftrightarrow \quad -y_1 = -2y_2^2.$$

But $k(\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}) = (-v_1-w_1-x_1-y_1)-\frac{1}{2}(v_2+w_2+x_2+y_2)^2-(v_2+w_2+x_2+y_2)^6$. On $f = g = h = j = 0$, $k(v_2, w_2, x_2, y_2) = -2(v_2^2+w_2^2+x_2^2+y_2^2) - \frac{1}{2}(v_2+w_2+x_2+y_2)^2-(v_2+w_2+x_2+y_2)^6$. For simplicity, let us drop the subscript 2 now and make the change of variables

$$a = \tfrac{1}{4}(v+w-x-y),$$

$$b = \tfrac{1}{4}(-v+w-x+y),$$

$$c = \tfrac{1}{4}(-v+w+x-y),$$

$$d = \tfrac{1}{4}(v+2+x+y) = -\tfrac{1}{4}z.$$