

DISCUSSION PAPER NO. 177
CHOICE AND SELECTION IN INCOME
CONTINGENT LOAN PLANS

by

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October 1975

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ABSTRACT

A major concern of an institution offering income contingent loan plans is the level of participation by individuals, particularly those with high income expectations. When an institution considers offering a package of plans, and allows participants to choose any plan from that package, then the institution's problem of determining the package is affected by an additional factor -that of individuals' choices. In this paper the problem of designing such a package of plans is formulated as an integer programming problem. This integer programming formulation, however, has a large number of variables and constraints. Taking advantage of the special structure of the integer programming problem an enumerative algorithm is presented to obtain the optimal package.

Introduction

Income contingent loan plans are a relatively new means of credit for financing students' investment in higher education. The basic concept underlying income contingent loans is that of linking the repayments and future income of individual borrowers. Johnstone [5] provides an excellent description and discussion of a few income contingent loan proposals. Currently, such plans are being offered by a few universities for the benefit of their students. The universities have not tried to generate profit through these plans but have sought to break even for the total cohort of students participating in their plans. Because of the relationship between repayments and income, and the institutions desire to break even, individuals with higher incomes pay back more than those with lower incomes. Consequently, one of the major concerns of institutions has been the level of participation in such plans, particularly by students with high income expectations. If one sets the plan parameters without anticipating this selection process, there could be a concentration of students with low income expectations participating in the plan, thereby leading to the financial failure of such plans. This process is often referred to as adverse selection and has been examined for individual plans by Jain [3].

In this paper we consider the selection process when a multitude of plans are available to students instead of restricting our attention to individual plans. With individual plans, the only choice available to students is whether to participate or not participate in the plan. In a situation when a number of plans are available the individual borrower has the option of

choosing the plan that is most desirable amongst the ones offered. From the institution's view point, it can offer a number of plans with varying plan parameters, such as the repayment horizon, the exit out option, and the repayment formula connecting income and repayments. It thus has the option of offering a large number of plans with different combinations and values of the above parameters. For administrative convenience the institution would like to limit the number of plans it offers. The problem that the institution then faces is the selection of a certain number of plans from the many considered such that, if each student chose the most desirable plan amongst the ones offered, the institution would at least break even on each of the plans. In order for the institution to evaluate different combinations of the plans it could offer, a criterion is required. In this paper we assume that the appropriate criterion for evaluating plans is known. Jain and Wagner [4] have discussed three different criteria that the institution might use to select plans.

Model Formulation

We consider the problem of selection of plans by the institution in a deterministic environment with perfect knowledge about future incomes of the participants, interest rates, and the utility functions of the individuals. We let N denote the number of different plans that the institution considers, K the maximum number of plans it wishes to select, and M the number of individuals to whom the plans are to be offered. Under the assumptions made, the institution can determine u_{ij} , v_{ij} , and z_{ij} , the utility

to the individual, the present value of repayments minus the principal amount, and the value of the objective criteria, if individual j chooses plan i [Note that the value of objective criterion, z_{ij} , could be v_{ij} if the institution wishes to maximize the present value of repayments minus the amount loaned, or could be u_{ij} , which under certain assumptions would be a social welfare function.]

A model for the above selection process is formulated next. Define

$$Y_i = \begin{cases} 0 & \text{if plan } i \text{ is not offered by the institution} \\ 1 & \text{if plan } i \text{ is offered by the institution for } i = 1, 2, \dots, N, \end{cases}$$
$$X_{ij} = \begin{cases} 0 & \text{if plan } i \text{ is not chosen by individual } j \\ 1 & \text{if plan } i \text{ is chosen by individual } j \text{ for } i = 1, 2, \dots, N, \\ & \text{and } j = 1, 2, \dots, M. \end{cases}$$

We have to ensure that each individual chooses that plan from the ones offered, that maximizes the individual's utility. This condition can be decomposed into two sets of constraints, namely, no individual chooses a plan that is not offered, and that individuals choose plans that maximize their utility. Thus we have

$$X_{ij} - Y_i \leq 0 \text{ for } j = 1, 2, \dots, M, i = 1, 2, \dots, N,$$

and

$$\sum_{i=1}^N u_{ij} X_{ij} - u_{ij} Y_i \geq 0 \text{ for } j = 1, 2, \dots, M, i = 1, 2, \dots, N.$$

Further, we also want to ensure that each individual chooses not more than one plan, and that the institution does not offer more than a maximum number (K) of plans. These conditions can be represented by

$$\sum_{i=1}^N X_{ij} \leq 1 \quad \text{for } j = 1, 2, \dots, M,$$

and

$$\sum_{i=1}^N Y_i \leq K$$

Finally, we add the constraint that the institution does not make a loss on any plan that is offered. Since plans that are not offered, cannot be chosen, this condition can be represented by

$$\sum_{j=1}^M v_{ij} X_{ij} \geq 0 \quad \text{for } i = 1, 2, \dots, M.$$

Thus, the choice and selection process can be represented by the following integer programming problem.

$$(1) \quad \text{Maximize} \quad \sum_{i=1}^N \sum_{j=1}^M z_{ij} X_{ij}$$

subject to

$$(2) \quad X_{ij} - Y_i \leq 0 \quad \text{for } j = 1, 2, \dots, M, i = 1, 2, \dots, N,$$

$$(3) \quad \sum_{i=1}^N u_{ij} X_{ij} - u_{ij} Y_i \geq 0 \quad \text{for } j = 1, 2, \dots, M, i = 1, 2, \dots, N,$$

$$(4) \quad \sum_{i=1}^N X_{ij} \leq 1 \quad \text{for } j = 1, 2, \dots, M$$

$$(5) \quad \sum_{i=1}^N Y_i \leq K$$

$$(6) \quad \sum_{j=1}^M v_{ij} X_{ij} \geq 0 \quad \text{for } i = 1, 2, \dots, M.$$

$$(7) \quad 0 \leq X_{ij} \leq 1, 0 \leq Y_i \leq 1, X_{ij}, Y_i \text{ integer variables.}$$

For a reasonable problem with about 30-50 plans being considered and 1000 individuals the formulation would give between 30,000-50,000 variable and an even larger number of constraints. These problems are clearly too large to solve by conventional integer programming techniques. We exploit the special structure of the problem and devise an enumeration scheme for obtaining the solution. We first observe the following:

Proposition I. If each individual has strictly ordered preferences amongst the plans, that is, no u_{ij} 's are equal for a fixed j , then restricting Y_i to be integer would imply that in any feasible solution to (1) - (7), X_{ij} would also be integer.

Proof: Follows immediately from constraint set (3).

Since most individuals would just choose one plan to participate in, it is not unreasonable to assume that their preferences over the plans are a strict order. Henceforth, we assume this. With this assumption and the above proposition, the number of integer variables is considerably reduced. In the example considered earlier when 30-50 plans were being considered for 1000 individuals, the number of integer variables that we have to consider decreases from 30,000 to 50,000 to 30-50. In general, we are left with N integer variables, where N is the number of plans being

considered by the institution. With this reduction there are a finite number $\sum_{k=0}^K \binom{N}{k}$ of solutions in the Y variables that satisfy (5), and

exhaustive enumeration provides a finite procedure for determining the optimal combination of plans to offer. However, $\sum_{k=0}^K \binom{N}{k}$ can turn out to

be a large number. When $N = 30$, and $K = 5$, we have about 175,000 possible combinations which would take a long time to evaluate. We therefore provide an implicit enumeration technique with an additive algorithm to obtain an optimal solution.

The Enumerative Algorithm

The underlying enumeration procedure upon which the algorithm for determining an optimal solution is super-imposed is that of elementary tree search as described in [1, 2]. Some path along the tree of solutions is traced until either a new solution is obtained or a node is reached which yields information that all solutions in which that particular node is included may be ruled out of consideration. Thereupon the process backtracks to the unique node that immediately precedes the one ruled out, and embarks on a different path, unless none are left and it becomes necessary to backtrack further. Once the process is pushed back to the starting node, and information is obtained that forbids tracing out any more branches of the tree, the procedure terminates.

We add one extra Y variable, Y_0 , to the formulation. Y_0 is a variable associated with not offering a plan and has the corresponding z_{ij} , u_{ij} , and v_{ij} equal to zero. With Y_0 added as a variable we can always be

sure of always having a feasible solution, namely, the plan Y_0 by itself. Also constraint (5) now becomes

$$(8) \quad \sum_{i=0}^N Y_i \leq K + 1$$

To illustrate the branching process more precisely we will use the following standard notation and conventions [1, 2]. The term i will be used to denote $Y_i = 1$, and the complementary term \bar{i} will be used to denote $Y_i = 0$. We define a solution sequence to be a sequence of the integer variables in which (i) no term appears more than once, (ii) the corresponding 0-1 assignment to some or all of the Y variables is well defined (i.e., not both i and \bar{i} can appear in the sequence and for each i , $0 \leq i \leq N$), and (iii) the number of terms i is less than or equal to $K + 1$. The index i or alternately the variable Y_i will be said to be free if neither i nor \bar{i} appears in the solution sequence. At any point in the tree search, the best feasible solution obtained until that point is stored, together with its objective function value and called the incumbent solution. A terminal solution sequence is defined to be a solution sequence for which there exists no 0-1 assignment of the free variables that will produce a feasible solution better than the incumbent. Before proceeding to give the algorithm we define the following sets for a solution sequence corresponding to node n in the enumeration tree:

S^n is the solution sequence, i.e., a set of indices i or \bar{i} , $0 \leq i \leq N$, some of which may be underlined.

F^n is the set of free variables, i.e., set of indices such that neither i nor \bar{i} is in the solution sequence.

R^n is the set of variables fixed at level 1, i.e., set of indices i such that i is in the solution sequence.

C^n is the set of variables fixed at level 0, i.e., or alternately set of indices such that \bar{i} is in the solution sequence.

For a given solution sequence S^n corresponding to node n , we partition the set of individuals into the following mutually exclusive and exhaustive subsets.

$$(9) \quad A_i^n = \{ j \mid u_{ij} \geq u_{kj} \text{ for all } k \in F^n \cup R^n \} \text{ for each } i \in R^n,$$

and

$$(10) \quad B^n = \{ j \mid j \notin A_i^n \text{ for any } i \in R^n \}.$$

Thus the set A_i^n consists of individuals who would choose the i^{th} plan (which is at level 1 in the solution sequence), regardless of whether any additional plans are added to the solution sequence. We also let

$$(11) \quad u_{i,N+1}^n = \sum_{i \in A_i^n} u_{ij} ,$$

$$(12) \quad v_{i,N+1}^n = \sum_{i \in A_i^n} v_{ij} ,$$

and

$$(13) \quad z_{i,N+1}^n = \sum_{i \in A_i^n} z_{ij} .$$

for each $i \in R^n$, and set $u_{i,N+1}^n, v_{i,N+1}^n$, and $z_{i,N+1}^n = 0$ for each $i \notin R^n$.

Further, we denote the current lower bound at node n by L^n .

An algorithm for determining an optimal solution to the problem (1) - (7) is provided next. In this algorithm the special structure of the problem is exploited to eliminate certain branches from consideration. Initially

all the plans except the one with the 0^{th} index are free. The 0^{th} plan, which corresponds to not offering any plan is fixed at level 1, and is the initial incumbent solution.

Step 0. Let $n = 0$, $S^n = \underline{0}$, $R^n = \{0\}$, $C^n = 0$, $F^n = \{1, 2, \dots, M\}$, and $L^n = 0$. Determine A_0^0 and B^0 , and let $u_{0, N+1} = v_{0, N+1} = z_{0, N+1} = 0$.

Step 1. Form a tableau with rows corresponding to the indices in $R^n \cup F^n$ and the columns corresponding to individuals in B^n . If the number of indices in R^n equals $K + 1$, form the tableau with rows corresponding to indices in R^n only. In each cell (k, j) of the tableau, corresponding to plan k and individual j , store z_{kj} , v_{kj} , and u_{kj} . Add to this tableau another column (column $N + 1$) and store $u_{i, N+1}$, $v_{i, N+1}$, and $z_{i, N+1}$ where these values are given by (11) - (13).

Step 2. For each column $j \in B^n$ determine the unique index i , such that $u_{ij} > u_{kj}$ for $k \neq i$. Set the corresponding $X_{ij} = 1$. For each row i in the tableau let

$$z_i = \sum_{\{j \in B^n \text{ and } X_{ij} = 1\}} z_{ij} + z_{i, N+1}$$

$$v_i = \sum_{\{j \in B^n \text{ and } X_{ij} = 1\}} v_{ij} + v_{i, N+1}$$

Step 3. Upper Bound Check

The upperbound of the objective function for all nodes along this branch equals $\sum_{i=1}^M z_i$. If this value is less than L^n go to

Step 8 else go to Step 4.

Step 4. Feasibility Check for Current Solution

If $v_i \geq 0$ for each i , and if the number of plans in the set $\{i | X_{ij} = 1 \text{ and } i \in R^n\}$ is less than or equal to $K + 1$, then the current solution is feasible. If the current solution is feasible go to Step 7, else go to Step 5.

Step 5. Infeasibility Check for Current Solution Sequence

If $v_i \geq 0$ for each $i \in R^n$ go to Step 6. Otherwise for each $i \in R^n$, such that $v_i < 0$, check whether

$$\sum_{j \in B^n} \text{maximum} \{0, v_{ij}\} + v_i \geq 0.$$

If the above holds go to Step 6, otherwise go to Step 8.

Step 6. Branching

Determine $k \in F^n$, such that $z_k = \text{maximum}_{i \in F^n} z_i$. Let $n = n + 1$, augment the old solution sequence by adding index k to its right. Then $R^{n+1} = R^n \cup \{k\}$, $C^{n+1} = C^n$, and $F^{n+1} = F^n - \{k\}$. Determine A_i^{n+1} for each $i \in R^{n+1}$, and B^{n+1} . Let $L^{n+1} = L^n$, and go to Step 2.

Step 7. Incumbent Solution

If $\sum_{i=1}^M z_i > L^n$, let $L^{n+1} = \sum_{i=1}^M z_i$, and store the solution

sequence and values of the X_{ij} 's, otherwise let $L^{n+1} = L^n$. Go to Step 8.

Step 8. Backtracking

Determine the rightmost term in the solution sequence that is not underlined. If none exists, terminate, else place the complement of that term in the solution sequence, underline it, and remove all

terms to the right of it. Replace n by $n+1$ and update S^{n+1} , F^{n+1} , C^{n+1} , R^{n+1} , A^{n+1} for $i \in F^{n+1}$, B^{n+1} , and let $L^{n+1} = L^n$. Go to Step 2.

The above algorithm terminates in a finite number of steps, since no node is ever repeated. The proof of the finiteness of the algorithm is similar to the proof given in [2].

Some Remarks

1. Observe that the tableau contracts in size as we go down a branch. This is because the variables corresponding to plans fixed at level 0 can be excluded from the tableau. Further, the columns corresponding to individuals who choose from the fixed plans need not be considered when branching down from a node.

2. When there are W_i number of individuals in group i , having the same utility and income expectations, then the problem becomes

$$\begin{aligned} \text{Maximize} \quad & W_1 z_{11} X_{11} + W_2 z_{12} X_{12} + \dots + W_M z_{1M} X_{1M} \\ & + W_1 z_{21} X_{21} + W_2 z_{22} X_{22} + \dots + W_M z_{2M} X_{2M} \\ & + \dots \\ & + W_1 z_{N1} X_{N1} + W_2 z_{N2} X_{N2} + \dots + W_M z_{NM} X_{NM} \end{aligned}$$

subject to

$$(10) \quad X_{ij} - Y_i \leq 0 \text{ for } j = 1, 2, \dots, M$$

$$\text{for } i = 1, 2, \dots, N$$

$$(11) \quad \sum_{j=1}^M v_{ij} W_j X_{ij} \geq 0 \text{ for } i = 1, 2, \dots, N$$

$$(12) \quad \sum_{i=1}^N u_{ij} X_{ij} - u_{ij} Y_i \geq 0 \text{ for } i = 1, 2, \dots, N$$

$$\text{and } j = 1, 2, \dots, M$$

$$(13) \quad \sum_{i=1}^N X_{ij} \leq 1$$

$$(14) \quad \sum_{i=1}^N Y_i \leq K$$

$$0 \leq X_{ij} \leq 1$$

$$0 \leq Y_i \leq 1$$

X_{ij} and Y_i integer.

Multiplying (12) by W_j we have

$$\sum_{i=1}^N u_{ij} W_j X_{ij} - u_{ij} W_j Y_i \geq 0$$

Now if we replace $v_{ij} W_j$ by V_{ij} and $u_{ij} W_j$ by U_{ij} we have a problem which is of the same form as the one considered.

3. A variety of combinatorial constraints such as not offering a plan in conjunction with another, or offering at least one from certain classes of plans, can be very easily taken care of in the enumeration tree. For example, if the institution does not want to offer both the plans i_0 and i_1 , then as soon as i_0 or i_1 enters the solution sequence, \bar{i}_1 or \bar{i}_0 can be respectively added to the solution sequence.

4. If the institution wishes to offer exactly K plans then the test in Step 4 can be modified to check whether the number of indices in J is exactly $K + 1$ or not before allowing it to take the place of the incumbent.

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