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# OPTIMAL CONTROL OF SERVICE QUALITY IN A QUEUEING SYSTEM

by

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ABSTRACT

The amount of time spent on servicing a customer is used as a surrogate measure of the quality of service provided. The expected reward from servicing is assumed to be a nondecreasing function of the service quality, while a linear holding cost is assumed for customers waiting in the system. The long run average return per unit time is maximized by optimally controlling the quality of service to be provided as a function of the workload facing the server for different waiting room capacities. The form of the optimal control policy for a given capacity is shown to be monotone and the effect of varying the system capacity on the optimal control policy is also examined. Furthermore, the design problem of selecting an optimal capacity is analyzed, assuming that optimal service control policy will be followed for each system capacity, thereby integrating the design and control problems into a unified framework.

# OPTIMAL CONTROL OF SERVICE QUALITY IN A QUEUEING SYSTEM

In stochastic control of queueing systems, two areas of control that have been considered in the literature are the arrival process and the service mechanism. Stidham and Prabhu [11] and Sobel [10] present an excellent synthesis and survey of the literature in this general area. In controlling the service mechanism, an important decision variable considered by Crabill [ 2 ], Sabeti [ 8 ], Schassberger [ 9 ] and others, is the speed or the rate at which customers are served. In all of these papers an implicit assumption is that each customer is rendered service of constant quality, so that the reward upon each service completion is identical.

In many situations the quality of service provided to a customer can be varied. Typical examples include R and D management, inspection sampling in a warehouse, and patient care in an emergency ward. In the R and D process, research ideas are generated randomly (see Gaver and Srinivasan [ 4 ]) and an important decision then is the amount of time and effort to be spent on the development of each idea. The longer the time spent in developing a particular idea, the greater is the reward from that idea. However, because of resource limitations, spending more time on one idea delays the development of other ideas. Due to obsolescence and actions of competitors trying to develop similar ideas, this delay leads to an opportunity cost for the firm. Consequently, the firm has to balance the expected reward from developing an idea more fully, and the costs associated with delay. In the inspection sampling example, the power of a sampling plan to distinguish between good and bad lots depends upon the sample size. The inspection crew has to balance the benefit of taking a large sample from a given lot against the cost of delaying the inspection of other lots. Finally, in an emergency ward of a hospital, the doctor has to decide on the amount of care to provide a patient and is again faced with the problem of balancing the benefit of extra care provided to the patient against the possible deterioration of the condition of other patients waiting to see him.

In all of the above examples, once a decision is made to provide a certain quality of service, it is impractical to change the decision during the service. Further, the quality of service provided can be reasonably measured by the amount of time spent on service. Due to storage limitations we also have a finite system capacity, beyond which new arrivals are turned away.

In this paper we consider the problem of optimally controlling the quality of service to be provided as a function of the work load facing the server. Optimal control policies are considered for different system capacities. Furthermore, the capacity design problem is analyzed, assuming that optimal control policies are followed for each value of the system capacity in a region of interest. Thus the design and control problems are integrated into a unified framework, leading to a higher order optimum.

In the next section we formulate the control problem as a semi-Markov decision process and establish the existence of an optimal stationary policy which maximizes the average net reward per unit time. In the second section we analyze the design problem of varying the system capacity and establish an upper bound on the optimal system capacity. Finally, we show that, for any capacity less than this upper bound, the optimal service quality is monotone nonincreasing in the amount of work load facing the server.

## 1. THE MODEL

We consider a single server facility, where the quality of service provided, as measured by the service duration d, can be selected from the compact set of actions $A = [D_1, D_2]$, where $0 < D_1 \leq D_2 < \infty$. Here $D_1$ may be interpreted as the minimum amount of service duration that must be provided and $D_2$ as the maximum amount. Customers from a homogeneous population are assumed to arrive according to a Poisson process with rate $\lambda > 0$, provided the system capacity N is not filled. Providing a quality of service duration d yields the expected customer reward $R(d)$, where $R(\cdot)$ is assumed to be a concave increasing function, reflecting the fact that the customer satisfaction increases with the quality of service, though at a decreasing rate due to saturation effects. The cost of waiting per unit time per customer is assumed to be a constant $c > 0$.

Given a system capacity N, the state of the system at any instant of time is represented by the number of people, n, in the system, where $n \in S^N = \{0,1,2,\ldots,N\}$, $S^N$ being the state space. Given that the state of the system at the beginning of a service is $n \in S^N \sim \{0\}$, if the service of duration $d \in A$ is chosen, then two things happen:

(i) The expected reward $R(d)$ is earned and the expected waiting cost, denoted by $C^N(n,d)$, is incurred. The term $C^N(n,d)$ has two components, the cost of waiting of the n customers for the service duration d and the expected cost of waiting of future customers arriving during the service.

(ii) The state of the system at the service completion epoch upon departure is m with probability $q^N_{n,m}(d)$, where

$$q^N_{n,m}(d) = \begin{cases} p_{m-n+1}(d) & \text{if } N-1 > m \geq n-1 \\ Q_{N-n-1}(d) & \text{if } m = N-1 > n-1 \\ 1 & \text{if } n = N,\ m = N-1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where

$$p_k(d) = \frac{e^{-\lambda d}(\lambda d)^k}{k!} \qquad k = 0,1,.... \tag{2}$$

and

$$Q_k(d) = \sum_{j=k+1}^{\infty} p_j(d) \qquad k = 0,1,... \tag{3}$$

When $n = 0$, no decision is made regarding the service duration, we wait until the first customer arrives. During this period no reward is collected nor any cost incurred. Thus, the state of the system at the decision epochs changes according to the Markov chain with the above transition probabilities.

A <u>policy</u> $\Delta_n = \{\Delta_m : m = 1,2,\ldots,\}$ is a sequence of decision rules $\Delta_m$ for choosing a duration at the $m^{th}$ decision epoch, where $\Delta_m(\cdot|n_0,d_0,\ldots,n_{m-1},d_{m-1},n_m)$ is conditional probability measure on the Borel subsets of A for each sequence of histories of past states and actions. A <u>stationary policy</u> $\delta:S^N \to A$ is a function specifying the service duration $\delta(n)$ whenever the system is found to be in state n, n = 1,...,N. Following the notation of Ross [7], if Z(t) denotes the net return accumulated in [0,t] and $n_0$ is the starting state, then

$$V_\Delta^1(n_0) = \liminf_{t \to \infty} E_\Delta\{Z(t)|n_0\}/t \tag{4}$$

is the long run average expected return starting in state n and following the policy $\Delta$. Further

$$V^1(n_0) = \sup_\Delta V_\Delta(n_0) \tag{5}$$

is the optimal average return, starting in state $n_0$.

If $Z_m$ denotes the net expected reward in the $m^{th}$ decision interval (which is $R(d) - C^N(n,d)$ if the interval starts with n in the system and lasts for duration d), then let

$$V_\Delta^2(n_0) = \liminf_{m \to \infty} E_\Delta[\sum_{j=1}^m Z_j \mid n_0]/[\sum_{j=1}^m d_j] \tag{6}$$

and

$$V^2(n_0) = \sup_\Delta V_\Delta^2(n_0). \tag{7}$$

From (1) it is clear that, for any given system capacity $N > 1$, the state $N$ is never reached, provided the system starts in some state other than $N$. Hence, the only states of relevance are $0,1,2,\ldots,N-1$. With these states, from (1) we can see that every stationary policy gives rise to an irreducible (and hence positive recurrent) embedded Markov chain, because, for any $d \in A$, $\infty > d > 0$. Therefore, by Theorem 2.10 of Stidham and Prabhu [11], it follows that $V_\delta^1(n) = V_\delta^2(n)$ for any stationary policy $\delta$. Further, by their corollary 2.15, there exists an average return optimal stationary policy $\delta_*^N(\cdot)$, a bounded function $h_n^N$ of the state $n$ and a constant $g^N$ such that

$$h_n^N = \max_{D_1 \le d \le D_2} \left\{ R(d) - C^N(n,d) - g^N d + \sum_{k=0}^{N-n-1} h_{n+k-1}^N \, p_k(d) + \left[1 - \sum_{k=0}^{N-n-1} p_k(d)\right] h_{N-1}^N \right\}, \quad n=1,2,\ldots,N-1 \tag{8}$$

and

$$h_0^N = h_1^N - \frac{g^N}{\lambda} \tag{9}$$

where (9) follows from the fact that in state 0 we have no reward or cost and wait for an average duration of $1/\lambda$ time units until an arrival. Here $g^N = V^1(n) = V^2(n)$ is the optimal average gain per unit time and $h_n^N$ is the relative value of starting in state $n$.

Alternatively, with the complete state space $S^N = \{0,1,\ldots,N\}$, although the embedded Markov chain is not irreducible, it is clear that under any stationary policy the mean recurrence time to go from any state to state 0 is finite because state 0 is accesible from every other state. Hence, from the semi-Markov decision version of Theorem 6.19 and Theorem 7.6 of Ross [ 7 ] the existence of an optimal stationary policy and numbers $g^N$ and $h_n^N$, $n = 0,1,2,\ldots,N$, is guaranteed, with the set of equations (8) and (9) appended by the $N^{th}$ equation

$$h_N^N = \underset{D_1 \le d \le D_2}{\mathrm{Max}} \{R(d) - C^N(N,d) - g^N d + h_{N-1}^N\} \tag{10}$$

Equation (10) turns out to be inconsequential and hence will be ignored in the subsequent analysis, because in the Markov chain embedded at the decision epochs, state N is never reached. Furthermore, the optimal stationary policy $\delta_*^N(\cdot)$ specifies the duration $\delta_*^N(n)$ in state n, which maximizes the right hand side of the functional equation (8).

To rewrite (8) in a more convenient form note that

$$\sum_{k=0}^{N-n-1} h_{n+k-1}^N \, p_k(d) + [1 - \sum_{k=0}^{N-n-1} p_k(d)] \, h_{N-1}^N$$

$$= p_0(d) \, h_{n-1}^N + p_1(d) \, h_n^N + \ldots + p_{N-n-1}(d) \, h_{N-2}^N - h_{N-1}^N \sum_{k=0}^{N-n-1} p_k(d) + h_{N-}^N$$

$$= h_{N-1}^N - p_0(d) \, [h_n^N - h_{n-1}^N] - [p_0(d) + p_1(d)][h_{n+1}^N - h_n^N]$$

$$- [p_0(d) + p_1(d) + p_2(d)][h_{n+2}^N - h_{n+1}^N] \,\ldots\ldots\ldots$$

$$\ldots\ldots - [\sum_{k=0}^{N-n-1} p_k(d)][h_{N-1}^N - h_{N-2}^N]$$

$$= h^N_{N-1} - \sum_{k=0}^{N-n-1} \Big[\sum_{k=0}^{k} p_j(d)\Big][h^N_{n+k} - h^N_{n+k-1}]$$

$$= h^N_{N-1} - \sum_{k=0}^{N-n-1} P_k(d)[h^N_{n+k} - h^N_{n+k-1}] \ ,$$

where $$P_k(d) = \sum_{j=0}^{k} p_j(d) = 1 - Q_k(d)\ , \quad k = 0,1,2,\ldots$$

Hence,

$$h^N_n = \underset{D_1\leq d\leq D_2}{\text{Max}} \Big\{R(d) - C^N(n,d) - g^N d + h^N_{N-1} - \sum_{k=0}^{N-n-1} P_k(d)[h^N_{n+k} - h^N_{n+k-1}\Big\}, \quad n = 1,2,\ldots,N-1 \tag{11}$$

From now on, for simplicity, we denote $\delta^N_*(n)$ by $d^N_n$ and $\delta^N_*$ by $d^N = (d^N_1,\ldots,d^N_N)$. With the optimal duration $d^N_n$ maximizing the right hand side of (11) we have

$$h^N_n = R(d^N_n) - C^N(n,d^N_n) - g^N d^N_n + h^N_{N-1}$$
$$- P_0(d^N_n)\ [h^N_n - h^N_{n-1}] - \sum_{k=1}^{N-n-1} P_k(d^N_n)[h^N_{n+k} - h^N_{n+k-1}]\ .$$

Since $h^N_n = h^N_{N-1} - \sum_{k=1}^{N-n-1} [h^N_{n+k} - h^N_{n+k-1}]$, we have

$$P_0(d_n^N)[h_n^N - h_{n-1}^N] = R(d_n^N) - C^N(n,d_n^N) - g^N d_n^N$$

$$+ \sum_{k=1}^{N-n-1} Q_k(d_n^N)[h_{n+k}^N - h_{n+k-1}^N],$$

$$n = 1,2,\ldots,N-2 \qquad (12)$$

$$P_0(d_{N-1}^N)[h_{N-1}^N - h_{N-2}^N] = R(d_{N-1}^N) - C^N(N-1,d_{N-1}^N) - g^N d_{N-1}^N \,. \qquad (13)$$

Similarly, using any $d \in [D_1,D_2]$ yields, (since $d_n^N$ is an action maximizing the right hand side of (11))

$$P_0(d)[h_n^N - h_{n-1}^N] \geq R(d) - C^N(n,d) - g^N d$$

$$+ \sum_{k=1}^{N-n-1} Q_k(d)[h_{n+k}^N - h_{n+k-1}^N]$$

$$n = 1,2,\ldots,N-1 \qquad (14)$$

$$P_0(d)[h_{N-1}^N - h_{N-2}^N] \geq R(d) - C^N(N-1,d) - g^N d \quad . \qquad (15)$$

In order to establish the structure of the optimal stationary policy and an upper bound on the optimal system capacity in the subsequent sections, some required properties of the waiting cost function $C^N(n,d)$ are shown below.

Lemma 1: The waiting cost $C^N(n,d)$ satisfies the following

(i) $$C^N(n,d) = cNd - \frac{c}{\lambda} \sum_{k=0}^{N-n} (N-n-k)Q_k(d), \; n \leq N \qquad (16)$$

which is convex and strictly increasing in d.

(ii) $$C^N(n,d) - C^N(n-1,d) = \frac{c}{\lambda}\left[\sum_{k=0}^{N-n} Q_k(d)\right] \quad n \le N \tag{17}$$

so that $C^N(n,d)$ is strictly increasing in n.

(iii) $C^N(n,d_2) - C^N(n,d_1)$ is increasing in n, whenever $d_2 \ge d_1$

(iv) $$C^{N+1}(n,d) - C^N(n,d) = \frac{c}{\lambda} \sum_{k=N-n+1}^{\infty} Q_k(d) \tag{18}$$

so that $C^N(n,d)$ is strictly increasing in N.

Proof:

(i) Clearly, $C^N(N,d) = cNd$, so that (i) is true for $n = N$.

Suppose that

$$C^N(n+1,d) = cNd - \frac{c}{\lambda}\sum_{k=0}^{N-n-1} (N-n-1-k)Q_k(d).$$

Now, conditioning on the time of arrival of the first customer during d and then unconditioning, the following recursive equation is obtained

$$C^N(n,d) = \int_0^d \left[\frac{cn}{\lambda} + C^N(n+1,d-t)\right]\lambda e^{-\lambda t}dt \tag{19}$$

$$= \int_0^d \left[\frac{cn}{\lambda} + cN(d-t) - \frac{c}{\lambda}\sum_{k=0}^{N-n-1} (N-n-k-1)Q_k(d-t)\lambda e^{-\lambda t}dt\right.$$

$$= (1-e^{-\lambda d})\left(\frac{cn}{\lambda} + cN\right) - cN\int_0^d t\lambda e^{-\lambda t}dt$$

$$- \frac{c}{\lambda}\sum_{k=0}^{N-n-1} (N-n-1-k)\int_0^d Q_k(d-t)\lambda e^{-\lambda t}dt$$

$$= cNd - \frac{c}{\lambda}\sum_{k=0}^{N-n} (N-n-k)Q_k(d)$$

since $\int_0^d Q_k(d-t)\lambda e^{-\lambda t}dt = Q_{k+1}(d)$.

To prove convexity and monotonicity of $C^N(n,d)$ in d we proceed by induction. Now $C^N(N,d) = cNd$ is clearly convex and increasing in d. Suppose $C^N(n+1,d)$ is convex and increasing in d. Since

$$C^N(n,d) = \int_0^d [\frac{cn}{\lambda} + C^N(n+1,d-t)]\lambda e^{-\lambda t}dt,$$

$C^N(n,d)$ is increasing in d. Also

$$\frac{\partial C^N(n,d)}{\partial d} = cne^{-\lambda d} + \int_0^d \frac{\partial C^N(n+1,d-t)}{\partial d} \lambda e^{-\lambda t}dt, \text{ since}$$

$C^N(n+1,0) = 0$. Further

$$\frac{\partial^2 C^N(n,d)}{\partial d^2} = - \lambda\, cne^{-\lambda d} + \frac{\partial C^N(n+1,0)}{\partial d} \lambda e^{-\lambda d} + \int_0^d \frac{\partial^2 C^N(n+1,d-t)}{\partial d^2} \lambda e^{-\lambda t}dt.$$

Since $\dfrac{\partial C^n(n+1,0)}{\partial d} = \lim_{\Delta t\to 0} \dfrac{C^N(n+1,\Delta t)-C^N(n+1,0)}{\Delta t} = (n+1)c > nc,$

$$\frac{\partial^2 C^N(n+1,d)}{\partial d^2} \geq 0 \text{ implies } \frac{\partial^2 C^N(n,d)}{\partial d^2} \geq 0,$$

completing the induction arguement.

(ii) Expression (17) follows from (16), so that monotonicity of $C^N(n,d)$ in n is clear.

(iii) $C^N(N,d_2) - C^N(N,d_1) = cN(d_2-d_1)$

while

$C^N(N-1,d_2) - C^N(N-1,d_1) = cN(d_2-d_1) - \frac{c}{\lambda}(e^{-\lambda d_1} - e^{-\lambda d_2})$

so that

$C^N(N-1,d_2) - C^N(n-1,d_1) \leq C^N(N,d_2) - C^N(N,d_1)$, for $d_2 \geq d_1$.

Suppose for $d_2 \geq d_1$,

$$C^N(n,d_2) - C^N(n,d_1) \leq C^N(n+1,d_2) - C^N(n+1,d_1),$$

then

$$C^N(n,d_2) - C^N(n,d_1) = \frac{cn}{\lambda}(e^{-\lambda d_1} - e^{-\lambda d_2})$$

$$+ \int_0^{d_2} C^N(n+1,d_2-t)\,\lambda e^{-\lambda t}dt - \int_0^{d_1} C^N(n+1,d_1-t)\,\lambda e^{-\lambda t}dt$$

$$= \frac{cn}{\lambda}(e^{-\lambda d_1} - e^{-\lambda d_2}) + \int_{d_1}^{d_2} C^N(n+1,d_2-t)\,\lambda e^{-\lambda t}dt$$

$$+ \int_0^{d_1} [C^N(n+1,d_2-t) - C^N(n+1,d_1-t)]\,\lambda e^{-\lambda t}dt.$$

Similarly,

$$C^N(n-1,d_2) - C^N(n-1,d_1) = \frac{c(n-1)}{\lambda}(e^{-\lambda d_1} - e^{-\lambda d_2})$$

$$+ \int_0^{d_1} [C^N(n,d_2-t) - C^N(n,d_1-t)]\lambda e^{-\lambda t}dt$$

$$+ \int_{d_1}^{d_2} C^N(n,d_2-t)\,\lambda e^{-\lambda t}dt.$$

Using the induction hypothesis and the fact that

$$C^N(n+1,d_2-t) \geq C^N(n,d_2-t) \text{ for all } t \leq d_2,$$

we get

$$C^N(n-1,d_2) - C^N(n-1,d_1) \leq C^N(n,d_2) - C^N(n,d_1)$$

proving (iii) for all $n \leq N$ by induction.

(iv) Using (16) and the fact that

$$\lambda d = \sum_{j=0}^{\infty} jp_j(d) = \sum_{k=0}^{\infty} Q_k(d)$$

equation (18) follows.

## 2. CAPACITY DESIGN

The determination of optimal system capacity is an important design problem in many queuing systems. In this section we consider the effect of varying the system capacity on the optimal gain rate and the relative values. Through this analysis we are able to identify the maximum system capacity $\overline{N}$, beyond which the gain rate decreases as the capacity is increased. In the next section we show the form of the optimal control policy for any capacity less than $\overline{N}$.

For any given capacity of N+1 we can solve the functional equations (8) and (9) with N+1 replacing N, for the values $h_n^{N+1}$, $n = 0,1,2,\ldots,N$ and $g^{N+1}$. Now either $h_N^{N+1} \leq h_{N-1}^{N+1}$ or $h_N^{N+1} > h_{N-1}^{N+1}$. The next lemma shows that, if the former holds, then it is optimal to reduce the capacity since the optimal average gain rate increases.

Lemma 2: If $h_N^{N+1} \leq h_{N-1}^{N+1}$, then $g^{N+1} < g^N$.

Proof: Given that $h_N^{N+1} \leq h_{N-1}^{N+1}$, assume to the contrary that $g^{N+1} \geq g^N$. Consider equation (12), with N+1 replacing N and $n = N+1$,

$$P_0(d_{N-1}^{N+1})[h_{N-1}^{N+1} - h_{N-2}^{N+1}] = R(d_{N-1}^{N+1}) - C^{N+1}(N-1,d_{N-1}^{N+1})$$
$$- g^{N+1}\, d_{N-1}^{N+1} + Q_1(d_{N-1}^{N+1})[h_N^{N+1} - h_{N-1}^{N+1}] \quad (20)$$

while equation (15) with $d = d_{N-1}^{N+1}$ yields

$$P_0(d_{N-1}^{N+1})[h_{N-1}^{N} - h_{N-2}^{N}] \geq R(d_{N-1}^{N+1}) - C^N(N-1,d_{N-1}^{N+1}) - g^N\, d_{N-1}^{N+1} \quad (21)$$

Subtracting (21) from (20) and using (18) yields

$$P_0(d_{N-1}^{N+1})[(h_{N-1}^{N+1} - h_{N-2}^{N+1}) - (h_{N-1}^{N} - h_{N-2}^{N})]$$

$$\leq - \frac{c}{\lambda} \sum_{k=2}^{\infty} Q_k(d_{N-1}^{N+1}) - (g^{N+1}-g^N)d_{N-1}^{N+1} + Q_1(d_{N-1}^{N+1})[h_N^{N+1} - h_{N-1}^{N+1}] < 0$$

by assumption and the hyptoehseis. Since $P_0\ (d_{N-1}^{N+1}) \geq 0$, we have $h_{N-1}^{N+1} - h_{N-2}^{N+1} < h_{N-1}^{N} - h_{N-2}^{N}$. Suppose $h_{n+k}^{N+1} - h_{n+k-1}^{N+1} < h_{n+k}^{N} - h_{n+k-1}^{N}$ for $k = 1,2,\ldots,N-n-1$. To show that this inequality holds for $k=0$ write (12) with $(N+1)$ replacing $N$ as

$$P_0(d_n^{N+1})[h_n^{N+1} - h_{n-1}^{N+1}] = R(d_n^{N+1}) - C^{N+1}(n,d_n^{N+1}) - g^{N+1}\ d_n^{N+1} + \sum_{k=1}^{N-n} Q_k(d_n^{N+1})[h_{n+k}^{N+1} - h_{n+k-1}^{N+1}] \qquad (22)$$

while (14) with $d = d_n^{N+1}$ becomes

$$P_0(d_n^{N+1})[h_n^{N} - h_{n-1}^{N}] \geq R(d_n^{N+1}) - C^{N}(n,d_n^{N+1}) - g^{N}\ d_n^{N+1} + \sum_{k=1}^{N-n-1} Q_k(d_n^{N+1})[h_{n+k}^{N} - h_{n+k-1}^{N}] \qquad (23)$$

and subtracting (23) from (22) yields

$$P_0(d_n^{N+1})[(h_n^{N+1} - h_{n-1}^{N+1}) - (h_n^{N} - h_{n-1}^{N})]$$

$$\leq - \frac{c}{\lambda} \sum_{k=N-n+1}^{\infty} Q_k(d_n^{N+1}) - (g^{N+1} - g^{N})\ d_n^{N+1} + Q_{N-n}\ (d_n^{N+1})[h_N^{N+1}-h_{N-1}^{N+1}]$$

$$+ \sum_{k=1}^{N-n-1} Q_k(d_n^{N+1})[(h_{n+k}^{N+1} - h_{n+k-1}^{N+1}) - (h_{n+k}^{N} - h_{n+k-1}^{N})]$$

$< 0$, by the induction hypothesis and the assumptions.

Therefore, $h_n^{N+1} - h_{n-1}^{N+1} < h_n^N - h_{n-1}^N$ for all $n = 1,2,\ldots,N-1$ and hence $\frac{g^{N+1}}{\lambda} = h_1^{N+1} - h_0^{N+1} < h_1^N - h_0^N = \frac{g^N}{\lambda}$, which contradicts the assumption that $g^{N+1} \geq g^N$.

Q.E.D.

Thus, a capacity N is of interest (i.e. is a candidate for the optimal capacity) only if $h_{N-1}^N > h_{N-2}^N$. Lemma 5 below shows that the region of the capacities of interest is convex. However, its proof requires the following results, which are of interest in themselves.

<u>Lemma 3</u>: If $h_{N-1}^N \geq h_{N-2}^N$ then $h_n^N$ is strictly concave and increasing in n, i.e.

$$h_1^N - h_0^N > \ldots > h_{N-2}^N - h_{N-3}^N > h_{N-1}^N - h_{N-2}^N \geq 0$$

<u>Proof</u>: Equation (14) with n=N-2 and $d=d_{N-1}^N$ yields

$$P_0(d_{N-1}^N)[h_{N-2}^N - h_{N-3}^N] \geq R(d_{N-1}^N) - C^N(N-2,d_{N-1}^N) - g^N d_{N-1}^N + Q_1(d_{N-1}^N)[h_{N-1}^N - h_{N-2}^N) \tag{24}$$

Subtracting equation (13) from (24) yields

$$P_0(d_{N-1}^N)[(h_{N-2}^N - h_{N-3}^N) - (h_{N-1}^N - h_{N-2}^N)] \geq C^N(N-1,d_{N-1}^N) - C^N(N-2,d_{N-1}^N) + Q_1(d_{N-1}^N)[h_{N-1}^N - h_{N-2}^N] > 0,$$

since $C^N(n,d) > C^N(n-1,d)$ and $h_{N-1}^N - h_{N-2}^N \geq 0$ by hypothesis.

Now suppose

$$h^N_{n+k} - h^N_{n+k-1} \geq h^N_{n+k+1} - h^N_{n+k}$$

for $k = 1,2,\ldots,N-n-2$. To show that this inequality holds for $k=0$, equation (14) with $d = d^N_{n+1}$ yields

$$P_0(d^N_{n+1})[h^N_n - h^N_{n-1}] \geq R(d^N_{n+1}) - C^N(n,d^N_{n+1}) - g^N d^N_{n+1} + \sum_{k=1}^{N-n-1} Q_k(d^N_{n+1})[h^N_{n+k} - h^N_{n+k-1}] \qquad (25)$$

while (12) with n replaced by (n+1) yields

$$P_0(d^N_{n+1})[h^N_{n+1} - h^N_n] = R(d^N_{n+1}) - C^N(n+1,d^N_{n+1}) - g^N d^N_{n+1} + \sum_{k=1}^{N-n-2} Q_k(d^N_{n+1})[h^N_{n+k+1} - h^N_{n+k}] \qquad (26)$$

Therefore,

$$P_0(d^N_{n+1})[(h^N_n - h^N_{n-1}) - (h^N_{n+1} - h^N_n)] \geq C^N(n+1,d^N_{n+1}) - C^N(n,d^N_{n+1})$$

$$+ \sum_{k=1}^{N-n-2} Q_k(d^N_{n+1})[(h^N_{n+k} - h^N_{n+k-1}) - (h^N_{n+k+1} - h^N_{n+k})]$$

$$+ Q_{N-n-1}(d^N_{n+1})[h^N_{N-1} - h^N_{N-2}] \geq 0,$$

using the induction hyptohesis, monotonicity of $C^N(n,d)$ in n and the hypothesis that $h^N_{N-1} \geq h^N_{N-2}$. This completes the induction arguement.

Q.E.D.

Since $\frac{g^N}{\lambda} = h^N_1 - h^N_0$, the above lemma implies the following

<u>Corollary</u>: If $h^N_{N-1} \geq h^N_{N-2}$ then $g^N > 0$.

Thus, if the system capacity N is such that $h^N_{N-1} \geq h^N_{N-2}$, then following an optimal service policy yields a positive average gain per unit time and the relative value of starting with more customers in the system is greater though the advantage decreases as the system gets filled up.

In the next lemma we show that, if $h^N_{N-1} \geq h^N_{N-2}$, then increasing the capacity by 1 cannot decrease the optimal average gain by more than the maximum additional waiting cost.

<u>Lemma 4</u>: If $h^{N+1}_N \geq h^{N+1}_{N-1}$, then $g^N - g^{N+1} \leq c$.

<u>Proof</u>: Given $h^{N+1}_N \geq h^{N+1}_{N-1}$, assume to the contrary that $g^N - g^{N+1} > c > 0$. Equation (15) with N replaced by N+1 and $d = d^N_{N-1}$ yields

$$P_0(d^N_{N-1})[h^{N+1}_N - h^{N+1}_{N-1}] \geq R(d^N_{N-1}) - C^{N+1}(N,d^N_{N-1}) - g^{N+1}d^N_{N-1} \qquad (27)$$

and equation (13) is

$$P_0(d^N_{N-1})[h^N_{N-1}-h^N_{N-2}] = R(d^N_{N-1}) - C^N(N-1,d^N_{N-1}) - g^N d^N_{N-1}$$

Subtracting and using the fact that $C^{N+1}(n,d) - C^N(n-1,d) = cd$ (by Lemma 1) we get

$$P_0(d^N_{N-1})[(h^{N+1}_N - h^{N+1}_{N-1}) - (h^N_{N-1} - h^N_{N-2})] \geq (g^N - g^{N+1}-c)d^N_{N-1} > 0$$

by the assumption. Thus,

$$[h^{N+1}_N - h^{N+1}_{N-1} > [h^N_{N-1} - h^N_{N-2}] \ .$$

Suppose $[h^{N+1}_{n+k}-h^{N+1}_{n+k-1}] > [h^{N}_{n+k-1}-h^{N}_{N+k-2}]$ for $k = 1,2,\ldots,N-n.$

Now (14) with (N+1) replacing N and $d = d^N_n$ yields

$$P_0(d^N_{n-1})[h^{N+1}_{n}-h^{N+1}_{n-1}] \geq R(d^N_{n-1})-C^{N+1}(n,d^N_{n-1}) - g^{N+1}\, d^N_{n-1}$$

$$+ \sum_{k=1}^{N-n} Q_k(d^N_{n-1})[h^{N+1}_{n+k} - h^{N+1}_{n+1-1}] \qquad (28)$$

while (12) with (n-1) replacing n yields

$$P_0(d^N_{n-1})[h^N_{n-1} - h^N_{n-2}] = R(d^N_{n-1}) - C^N(n-1,d^N_{n-1}) - g^N\, d^N_{n-1}$$

$$+ \sum_{k=1}^{N-n} Q_k(d^N_{n-1})[h^N_{n+k-1} - h^N_{n+k-2}] \qquad (29)$$

so that upon subtracting

$$P_0(d^N_{n-1})[(h^{N+1}_{n}-h^{N+1}_{n-1}) - (h^N_{n-1}-h^N_{n-2})]$$

$$\geq (g^N-g^{N+1}-c)d^N_{n-1} + \sum_{k=1}^{N-n} Q_k(d^N_{n-1})[(h^{N+1}_{n+k} \;\; h^{N+1}_{n+k-1}) - (h^N_{n+k-1} - h^N_{n+k-2})]$$

$$> 0,$$

by the assumption and the induction hypothesis. Thus,

$$h^{N+1}_{n} - h^{N+1}_{n-1} > h^N_{n-1} - h^N_{n-2} \text{ for all } n \geq 2$$

so that $h^{N+1}_2 - h^{N+1}_1 > h^N_1 - h^N_0$.

Now, $h^{N+1}_N \geq h^{N+1}_{N-1}$ implies, by Lemma 3, that $h^{N+1}_1 - h^{N+1}_0 \geq h^{N+1}_2 - h^{N+1}_1$, so that $h^{N+1}_1 - h^{N+1}_0 > h^N_1 - h^N_0$. Hence $\frac{g^{N+1}}{\lambda} > \frac{g^N}{\lambda}$, contradicting the assumption. Q.E.D.

The following result implies that if the system capacity (N+1) is in the region of interest, then so is the capacity N, i.e. the region of interest is convex.

Lemma 5: If $h_N^{N+1} \geq h_{N-1}^{N+1}$ then $h_{n-1}^N - h_{n-2}^N \geq h_n^{N+1} - h_{n-1}^{N+1}$, $1 \leq n \leq N$, so that $h_{N-1}^N > h_{N-2}^N$.

Proof: Equation (13) with (N+1) replacing N yields

$$P_0(d_N^{N+1})[h_N^{N+1}-h_{N-1}^{N+1}] = R(d_N^{N+1}) - C^{N+1}(N,d_N^{N+1}) - g^{N+1}\, d_N^{N+1} \;. \tag{30}$$

while (15) with $d = d_N^{N+1}$ yields

$$P_0(d_N^{N+1})[h_{N-1}^N - h_{N-2}^N] \geq R(d_N^{N+1}) - C^N(N-1,d_N^{N+1}) - g^N\, d_N^{N+1} \tag{31}$$

so that, taking a difference yields

$$P_0(d_N^{N+1})[(h_N^{N+1}-h_{N-1}^{N+1}) - (h_{N-1}^N-h_{N-2}^N)] \leq (g^N - g^{N+1} - c)\, d_N^{N+1} \leq 0$$

by Lemma 4, since $h_N^{N+1} \geq h_{N-1}^{N+1}$, so that the result holds with $n = N$. Suppose

$$h_{n+k}^{N+1} - h_{n+k-1}^{N+1} \leq h_{n+k-1}^N - h_{n+k-2}^N, \text{ for } k = 1,2,\ldots,N-n.$$

Now (12) with (N+1) replacing N yields

$$P_0(d_n^{N+1})[h_n^{N+1}-h_{n-1}^{N+1}] = R(d_n^{N+1}) - C^{N+1}(n,d_n^{N+1}) - g^{N+1}\, d_n^{N+1} + \sum_{k=1}^{N-n} Q_k(d_n^{N+1})[h_{n+k}^{N+1} - h_{n+k-1}^{N+1}] \tag{32}$$

while (14) with (n-1) replacing n and $d = d_n^{N+1}$ yields

$$P_0(d_n^{N+1})[h_{n-1}^N - h_{n-2}^N] \geq R(d_n^{N+1}) - C^N(n-1, d_n^{N+1}) - g^N d_n^{N+1} + \sum_{k=1}^{N-n} Q_k(d_n^{N+1})[h_{n+k-1}^N - h_{n+k-2}^N] \qquad (33)$$

so that a subtraction yields

$$\begin{aligned} &P_0(d_n^{N+1})[(h_n^{N+1} - h_{n-1}^{N+1}) - (h_{n-1}^N - h_{n-2}^N)] \\ \leq\ & (g^N - g^{N+1} - c)\, d_n^{N+1} + \sum_{k=1}^{N-n} Q_k(d_n^{N+1})[(h_{n+k}^{N+1} - h_{n+k-1}^{N+1}) - (h_{n+k-1}^N - h_{n+k-2}^N)] \qquad (34) \\ <\ & 0, \end{aligned}$$

by the induction hypothesis and Lemma 4.

Q.E.D.

Define $\bar{N} = \text{Sup}\{N : h_{N-1}^N \geq h_{N-2}^N\}$ and $N^*$ be such that $g^{N^*} = \underset{N}{\text{Sup}}\, g^N$. (Both $N^*$ and $\bar{N}$ may be interpreted to be $\infty$ if the corresponding sup is not attained.) Then $N^*$ is the optimum system capacity and we have the following proposition.

<u>Proposition 1</u>. $N^* \leq \bar{N} < \infty$.

<u>Proof</u>: We have

$$\begin{aligned} &P_0(d_{N-1}^N)[h_{N-1}^N - h_{N-2}^N] = R(d_{N-1}^N) - C^N(N-1, d_{N-1}^N) - g^N d_{N-1}^N \\ \leq\ & R(D_2) - C^N(N-1, D_1), \end{aligned}$$

since $g^N > 0$ in the region of interest (by the corrollary to Lemma 2) and $R(\cdot)$ and $C^N(N-1,\cdot)$ are increasing in $d \in [D_1,D_2]$. Now $C^N(N-1,D_1)$ increases without bound as N increases (since $D_1 > 0$). Hence beyond some N the left hand side of the above inequality is negative, implying the finiteness of $\overline{N}$.

Next, if $N > \overline{N}$, i.e. if $h^N_{N-1} < h^N_{N-2}$, then by Lemma 2, $g^N < g^{N-1}$. Continuing in this manner we get $g^N < g^{N-1} < \ldots < g^{\overline{N}}$. Further, since $h^N_{N-1} < h^N_{N-2}$, from Lemma 5 we know that $h^{N+1}_N < h^{N+1}_{N-1}$, so that by Lemma 2 we have $g^{N+1} < g^N$. Continuing, we have $g^{\overline{N}} > g^{\overline{N}-1} > \ldots > g^N > g^{N+1} \ldots$. Hence $N^* \leq \overline{N}$.

Q.E.D.

Thus, the optimal capacity $N^*$ which maximizes $g^N$ lies in the region of interest. It is reasonable to expect $g^N$ to be unimodal in N, in which case the optimal capacity $N^*$ can be determined by some one dimensional search technique. However, it seems difficult to prove unimodularity of $g^N$. In any case, since the region of capacities of interest is finite, $N^*$ can be determined by exhaustive search.

## 3. SERVICE CONTROL POLICIES

In section 1 we established the existence of an optimal stationary policy for any given system capacity. In section 2 we identified system capacities of economic interest. In this section we consider the qualitative structure of the optimal control service policy for any given system capacity in the region of interest. We also examine the behavior of the optimal service control policy as the system capacity is varied.

Proposition 2. If $N \leq \overline{N}$, then

(i) $d_n^N \geq d_{n+1}^N \qquad n = 1,2,\ldots,N-2, \quad N \geq 3$

(ii) $d_n^N \leq d_{n-1}^{N-1} \qquad n = 2,3,\ldots,N-2, \quad N \geq 4.$

Proof:

(i) We know that $d_n^N$ maximizes the right hand side of the functional equation

$$h_n^N = \max_{D_1 \leq d \leq D_2} \{R(d) - C^N(n,d) - g^N d + h_{N-1}^N - \sum_{k=0}^{N-n-1} P_k(d)[h_{n+k}^N - h_{n+k-1}^N]\}.$$

Suppose that $W^N(n,d)$ denotes the maximand on the right hand side of the above functional equation.

To show that $d_n^N$ is decreasing in n it suffices to show that $W^N(n,d)$ is "subadditive," i.e. for any $d_2 \geq d_1$, $[W^N(n,d_2) - W^N(n,d_1)]$ is monotone decreasing in n (see Stidham and Prabhu [11], Theorem 4.1). Now $C^N(n,d_1) - C^N(n,d_2)$ is decreasing in n, by Lemma 1 (iii). In addition,

$$P_k(d_1) - P_k(d_2) = \frac{1}{k!} \int_{\lambda d_1}^{\lambda d_2} e^{-t} t^k \, dt \geq 0.$$

Further, $h^N_{n+k} - h^N_{n+k-1}$ is nonnegative and decreasing in n for each k, (by Lemma 3, since $N \leq \overline{N}$. Hence,

$$\sum_{k=0}^{N-n-1} [P_k(d_1) - P_k(d_2)][h^N{n+k} - h^N_{N+k-1}]$$

is decreasing in n. Thus, $W^N(n,d)$ is subadditive and hence $d^N_n \geq d^N_{n+1}$.

(ii) Equation (14) with $d = d^{N-1}_{n-1}$ gives

$$P_0(d^{N-1}_{n-1})[h^N_n - h^N_{n-1}] \geq R(d^{N-1}_{n-1}) - C^N(n, d^{N-1}_{n-1}) - g^N\, d^{N-1}_{n-1}$$
$$+ \sum_{k=1}^{N-n-1} Q_k(d^{N-1}_{n-1})[h^N_{n+k} - h^N_{n+k-1}]$$

while (12) with n replaced by (n-1) and N replaced by (N-1) gives

$$P_0(d^{N-1}_{n-1})[h^{N-1}_{n-1} - h^{N-1}_{n-2}] = R(d^{N-1}_{n-1}) - C^{N-1}(n-1, d^{N-1}_{n-1}) - g^{N-1} d^{N-1}_{n-1}$$
$$+ \sum_{k=1}^{N-n-2} Q_k(d^{N-1}_{n-1})[h^{N-1}_{n+k-1} - h^{N-1}_{n+k-2}]$$

so that a subtraction yields

$$P_0(d_{n-1}^{N-1})[(h_n^N-h_{n-1}^N) - (h_{n-1}^{N-1}-h_{n-2}^{N-1})]$$

$$\geq (g^{N-1}-g^N-c)d_{n-1}^{N-1} + \sum_{k=1}^{N-n-2} Q_k(d_{n-1}^{N-1})[(h_{n+k}^N-h_{n+k-1}^N) - (h_{n+k-1}^{N-1} - h_{n+k-2}^{N-1})]$$

$$+ Q_{N-n-1}(d_{n-1}^{N-1})[h_{N-1}^N-h_{N-2}^N] \tag{35}$$

Dividing both sides of (34), with N replaced by N-1, by $P_0(d_n^N)$ and both sides of (35) by $P_0(d_{n-1}^{N-1})$ and comparing we get

$$(g^{N-1}-g^N-c)\frac{d_n^N}{P_0(d_n^N)} + \sum_{k=1}^{N-n-1}\left[\frac{Q_k(d_n^N)}{P_0(d_n^N)}\right][(h_{n+k}^N - h_{n+k-1}^N) - (h_{n+k-1}^{N-1} - h_{n+k-2}^{N-1})]$$

$$\geq (g^{N-1}-g^N-c)\left[\frac{d_{n-1}^{N-1}}{P_0(d_{n-1}^{N-1})}\right] + \sum_{k=1}^{N-n-2}\left[\frac{Q_k(d_{n-1}^{N-1})}{P_0(d_{n-1}^{N-1})}\right][(h_{n+k}^N-h_{n+k-1}^N) - (h_{n+k-1}^{N-1}-h_{n+k-2}^{N-1})]$$

$$+ \frac{Q_{N-n-1}(d_{n-1}^{N-1})}{P_0(d_{n-1}^{N-1})}[h_{N-1}^N-h_{N-2}^N]$$

Since $d/P_0(d)$ and $Q_k(d)/P_0(d)$ are both increasing in d, while $g^{N-1} - g^N - c \leq 0$ (by Lemma 4) and $h^{N+1}_{n+k} - h^{N+1}_{n+k-1} \leq h^N_{n+k-1} - h^N_{n+k-2}$ for all $k = 1,2,\ldots,N-n-1$ (by Lemma 5), we have $d^{N-1}_{n-1} \geq d^N_n$ .

Q.E.D.

Thus, for any system capacity in the region of interest, the optimal quality of service provided to a customer decreases as the number of customers waiting for service increases. Furthermore, when comparing optimal service policies for different system capacities, we have shown that, for tne same unfilled capacity, the quality of service provided to a customer increases as the system capacity decreases.

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