DISCUSSION PAPER NO. 173

THE FORMATION OF FIRMS IN LABOUR-MANAGED ECONOMIES */

by

J. Greenberg

October 1975

*/ I gratefully acknowledge T. Ichiishi for many fruitful discussions. In particular, his idea for the proof of existence [8], made the existence of the present paper possible. I am also much indebted to J. Dreze for introducing me to this subject and the interesting discussions with him gave rise to the main definition in the paper.

ABSTRACT

The formation of firms is explained due to the existence of some commodities (like iniative, skill, imagination, knowledge, connections etc.), which have the three properties:

-- cannot be transfered (i.e., individually specific)

-- cannot be produced (but, rather, serve as inputs)

-- cannot be marketed, (i.e., the firm does not face a price per unit in which it can acquire these commodities.

We refer to such commodities as types of labour.

When a group of people (coalition) forms a firm, it faces a technology set which uses only the amount of labour its members have. For example, the shares in Arrow-Debreu's economy are, actually, types of labour. We prove that under the standard A-D assumptions, there exists a firm structure in which no coalition has the incentive to withdraw and form a new firm operating in the market with the existing prices (for the marketed goods).

We consider also the Replica Economy, and prove the equivalence of the core and equilibria allocations.

I. INTRODUCTION

There is recently a growing interest in labour managed economies [4, 7, 8, 9, 12] , the main feature of which is the existence of types of labour which are not marketed, hence labour has no price ("wage"), but rather a share in the value added. (A detailed description of the labour-managed economy is given in [4]). In fact, we shall argue that the standard Arrow-Debreu economy is an economy of this type where the non-marketed goods are the shares $\{\theta_{ij}\}$.

Since labour is not marketed it is only natural to use the production coalition economy, (defined by Hildenbrand [6]), in this context. This was recognized by D. Sondermann [11], and was further discussed by T. Ichiishi [7]. However, none of the above gave a satisfactory solution as to the formation of firms in labour managed economies. Such a solution is proposed in this paper.

In a forthcoming paper by J. Dreze and the author, an analogy between local public goods and labour managed economies has been established. Thus the definition we use here for an equilibrium is closely related to that of a structural equilibrium in [5]. We then prove the existence of a stable firm structure, in which no group of workers have an incentive to withdraw from the existing structure and form a new firm, operating in the market with the existing prices (for the marketed goods).

The technique for proving the existence of such a structure is by defining for every price vector p, a game without side payments v_p, for which the core is nonempty. The excess demand set-function is shown to be convex and upper semi continuous, enabling us to use Debreu's lemma [3].

This very elegant proof, which is a generalization of Bohem's paper [1], is due to T. Ichiishi, who uses this method in his current paper [8]. However, the economic model, the assumptions, the techniques of the existence proof within the above general framework and the results are quite different. It is also shown that Ichiishi's model is a special case of the one presented in this paper.

In Section II we present the model, the definitions and the assumptions. We show in Section III that our model is a generalization of the Arrow-Debreu economy, which is, in fact a special case of a non-marketed goods economy. We then define in Section IV the super-additive technology in which the firm is allowed to form independent autonomous "sub-firms" (departments). With the use of this technology we prove in Section V the existence of a stable firm structure. Surprisingly, the assumptions we use are only the standard assumptions which assure the existence of a competitive equilibrium. In Section VI we discuss the formation of a coalition structure in which each individual works in one and only one firm. We give an example which implies that usually such a structure will never be a stable one. (In fact, it is usually not pareto-efficient). However, adding a balancedness assumption on the technology, we are able to prove the existence of a stable coalition structure. In Section VII, we prove that under very plausible assumptions, the core of the replica economy coincides with the set of structural competitive equilibria. Moreover, in this case each type of labour has its price, giving rise to the conjecture that goods are non-marketed due to their scarcity. The proof is a corollary of Bohem's paper [2] using the concept of the super-additive technology.

II. THE MODEL

There are ℓ_1 marketed goods, ℓ_2 non-marketed commodities (which will be referred to as types of labour). $M = \{1,\ldots,m\}$ denotes the set of individuals each endowed with $w^i = (w_c^i, w_L^i) \in R_+^{\ell_1+\ell_2}$, (i.e., w_c^i is the initial endowment of the marketed goods and w_L^i is the amount of labour individual i is endowed with)[*/] and a utility function u^i defined over his consumption set $\underline{\bar{X}}^i$. Non-marketed goods cannot be produced.

As for the technology sets, we consider the coalition production approach, modified to our case of non-marketed commodities. Namely, each coalition $S, (S \subset M)$, faces a production set $Y(S) \subset R^{\ell_1} \times R^{\ell_2}$ in which $y \in Y(S)$ implies that for all $i \in S$ there exist $\tilde{x}^i = (\tilde{x}_c^i, \tilde{x}_L^i) \in \underline{\bar{X}}_c^i$, $\tilde{x}_L^i \leq w_L^i$, such that $y = (y_c, y_L)$ with $y_L \geq \sum_{i \in S} \tilde{x}_L^i - \sum_{i \in S} w_L^i$. (We use the usual sign convention where $y_j < 0$ implies j is used as a production factor.) This modification captures the idea of non-marketed goods. Each S, if and when formed, faces a fixed maximum amount of the non-marketed commodities which is available to it. If S desires to acquire more of these goods, it must be by adjoining to it some non-members, $\tilde{S}$, of S, and form the new coalition $S \cup \tilde{S}$. Moreover, since non-marketed goods cannot be transferred among individuals, $\tilde{x}_L^i \leq w_L^i$, for all $i \in S$. Note, that although S faces a fixed amount of non-marketed goods that its members have, it will usually use less than that maximum amount as a result of the preferences of the individuals in S, who will decide upon the type and the amount of work they will perform.

[*/] We shall use the subscript c to denote the marketed goods, and the subscript L to denote the labour.

Denote:

C = the set of all nonempty subsets of M, (i.e., C is the set of all possible coalitions, $|C| = 2^m - 1$).

D = the set of all non-empty subsets of C.

$P = [p \in R_+^{\ell_1} \mid \sum_{j=1}^{\ell_1} p_j = 1]$, i.e., P is the price simplex.

Definition 1: A firm structure is an element of D.

Definition 2: A Structural Competitive Equilibrium consists of a price vector $p \in P$ (for the marketed goods, only), consumption bundles $x^i \in \underline{\bar{X}}^i$, $i \in M$, a firm structure B, production planes $y(S) \in Y(S)$, $S \in B$, a profit distribution $\{t_{is}\}$, $i \in M$, $S \in B$, such that:

(i) $\sum_{i \in M} x^i \leq \sum_{i \in M} w^i + \sum_{S \in B} y(S)$, $x_L^i \leq w_L^i \ \forall \ i \in M$

(ii) There is no:

$S_\circ \subset M$, $\bar{x}^i \in \underline{X}^i$, $\bar{x}_L^i \leq w_L^i$, $i \in S_\circ$, $y(S_\circ) \in Y(S_\circ)$

such that $\sum_{i \in S_\circ} p\bar{x}_c^i \leq p \sum_{i \in S_\circ} w_c^i + p\, y_c(S_\circ)$, $\sum_{i \in S_\circ} \bar{x}_L^i - \sum_{i \in S_\circ} w_L^i \leq y_L(S_\circ)$,

and $u^i(\bar{x}^i) > u^i(x^i) \ \forall \ i \in S_\circ$

(iii) $p\,x_c^i \leq p\,w_c^i + \sum_{\substack{S \in B \\ i \in S}} t_{is}$

(iv) For all $S \in B$, $\sum_{i \in S} t_{is} \leq p\,y_c(S)$,

(v) For all $S \in B$, $p\tilde{y}_c(S) \leq py_c(S)$, $\forall\ (\tilde{y}_c(S), y_L(S)) \in Y(S)$

Conditions (i), (iii), and (iv) need no comments. Condition (v) is the equivalent of profit maximization. It states that in each firm S the value added, given the labour vector $y_L(S)$, is maximized. Condition (ii) assures that no mobility will occur. There is no incentive to any coalition, $S_\circ$ to withdraw from B, produce by itself $y(S_\circ)$, distribute the value added $py_c(S_\circ)$ in such a way that every member of $S_\circ$ can purchase a vector $\bar{x}^i_c$ in the market, whereby his utility is increased.

For the following we need only the standard assumptions on our economy (compare for analogy, [3] p. 83), namely,

For every $i \in M$:

(a) $\underline{\bar{X}}^i$ is a closed convex subset of $R_+^{\ell_1+\ell_2}$

(b.1) there is no satiation in consumption in $\underline{\bar{X}}^i$:

i.e., for any $x \in \underline{\bar{X}}^i$ and for any neighborhood U of x in $\underline{\bar{X}}^i$, there exists an $\tilde{x} \in U$, $\tilde{x}_L = x_L$ such that $u^i(\tilde{x}) > u^i(x)$.

(b.2) u^i is a continuous function on $\underline{\bar{X}}^i$

(b.3) u^i is quasi concave.

(c) $w^i_c >> 0$, $w^i_L \geqq 0$, $w^i\ \underline{\bar{X}}^i$

For every $S \in C$:

(d.1) $Y(S) \subset R^{\ell_1} \times R_-^{\ell_2}$ with $0 \in Y(S)$

For every $d \in D$:

(d.2) $Y(d) \equiv \sum_{S \in d} Y(S)$ is closed and convex.

(d.3) $Y(d) \cap -Y(d) \subset \{0\}$

(d.4) $Y(d) \supset -R_+^{\ell_1+\ell_2}$

Note that in view of (d.1), (d.2) - (d.4) could be replaced by assuming (d.2) - (d.4) only for $d^* \equiv C$.

Denote the above set of assumptions by A. Our main purpose is to prove:

<u>Theorem 1:</u> Under assumptions A there exists a structural competitive equilibrium.

III. THE ARROW-DEBREU ECONOMY

A good example of a non-marketed goods economy is provided by the standard A - D economy, where there are M individuals each endowed with $\tilde{w}^i \in E_+^{\ell_1}$. The consumption set is $\underline{\tilde{X}}^i \subset E_+^{\ell_1}$, over which the utility function $\tilde{u}^i$ is defined. There are J firms with the technology sets $\tilde{Y}^j \subset E^{\ell_1}$, $j = 1,\ldots,J$. Each firm j is owned by the individuals who hold its shares. The shares θ_{ij}, $j = 1,\ldots,J$ individual $i \in M$ holds are a-priori given and are not marketed. For all j, $\sum_{i \in M} \theta_{ij} = 1$, with $\theta_{ij} \geq 0$ for all $i \in M$.

A competitive equilibrium consists of a price vector $p \in P$, consumption bundles $\tilde{x}^i \in \underline{\tilde{X}}^i$, $i \in M$, production planes $\tilde{y}^j \in \tilde{Y}^j$, $j = 1,\ldots,J$ such that

(α) $p\tilde{x}^i \leq p\tilde{w}^i + \sum_{j=1}^{J} \theta_{ij} p\tilde{y}^j$ and $\tilde{u}^i(\hat{x}^i) > \tilde{u}^i(\tilde{x}^i)$ implies

$p\hat{x}^i > p\tilde{x}^i$, $i = 1,\ldots,m$

(β) $p\tilde{y}^j = \text{Max } p\tilde{Y}^j$

(γ) $\sum_{i \in M} \tilde{x}^i \leq \sum \tilde{w}^i + \sum_{j=1}^{J} \tilde{y}^j$

The natural way to embed this economy in our model is by defining the following:

$$w^i \equiv (\tilde{w}^i, \theta_{i1}, \ldots, \theta_{iJ}) \in E^{\ell_1 + \ell_2}, \quad (\ell_2 = J)$$

$$\underline{X}^i = \underline{\tilde{X}}^i \times E_+^{\ell_2}$$

$u^i(x^i) = \tilde{u}^i(x_c^i)$ (i.e., the utility is independent of the non-marketed goods).

$$Y^j = [y \in E^{\ell_1} \times E_-^{\ell_2} \mid y = t\tilde{y},\ \tilde{y}_c \in \tilde{Y}^j, \tilde{y}_L \leq -e^j, 0 \leq t \leq 1, \quad j = 1,\ldots,J]$$

$$Y = [y \mid y = \sum_{j=1}^{J} y^j,\ y^j \in Y^j]$$

Then, the coalition production technology is given by:

$$Y(S) = [y \in Y \mid y_L \geq - \sum_{i \in S} w_L^i\ ]$$

$$S_j = [i \in M \mid \theta_{ij} > 0]$$

$$B = [S_1, \ldots, S_J]$$

When the J firms form (as assumed in A-D model), the firm structure B is realized. The following theorem proves that the A-D economy is a special case of our model:

Theorem 2: $(p, \{\tilde{x}^i\}_{i \in M}, \{\tilde{y}^j\}_{j=1}^{J})$ is a competitive equilibrium if and only if $(p, \{x^i\}_{i \in M}, \{y(S_j)\}_{j=1}^{J}, B, \{t_{is}\})$ is a structural competitive equilibrium with $x_c^i = \tilde{x}^i$, $\tilde{x}_L^i = 0$, $i \in M$, $y_c(S_j) = \tilde{y}^j$, $j = 1, \ldots, J$, $t_{is_j} = \theta_{ij} p \tilde{y}^j$, $i \in M$, $j = 1, \ldots, J$.

Proof: Let $(p, \{\tilde{x}^i\}_{i \in M}, \{\tilde{y}^j\}_{j=1}^{J})$ be a competitive equilibrium. Define: $x_c^i \equiv \tilde{x}^i$ $i \in M$, $x_L^i \equiv 0$, $i \in M$, $y(S_j) = (\tilde{y}^j, - e^j)$ $j = 1, \ldots, J$. By (γ),

$\sum_{i \in M} x_c^i \leq \sum_{i \in M} w_c^i + \sum_{S \in B} y_c(S)$, and by the above definitions:

$\sum_{i \in M} x_L^i \leq \sum_{i \in M} w_L^i + \sum_{S \in B} y_L(S)$. Hence, (i) is fulfilled.

It is easily verified that for all $S \subset M$, $Y(S) = \sum_{i \in S} Y(\{i\})$, and for all $i \in M, Y(\{i\}) = \sum_{j=1}^{J} \theta_{ij} Y^j$. (In fact, we could use this formulation to represent the production coalition technology sets). By (β), for all $y(S) \in Y(S)$, $S \subset M$, $py_c(S) \leq \sum_{i \in S} \sum_{j=1}^{J} \theta_{ij} p\tilde{y}^j$, which establishes (v). (iii) and (iv) follow immediately from the definition of t_{is}. Suppose $u^i(\bar{x}^i) > u^i(x^i)$, $\forall\ i \in S_\circ \subset M$. By ($\alpha$) $p\bar{x}_c^i > px_c^i = pw_c^i + \sum_{j=1}^{J} \theta_{ij} p\tilde{y}^j$.

By the previous inequality we get

$\sum_{i \in S} p\bar{x}_c^i > \sum_{i \in S} pw_c^i + py_c(S)\ \forall\ y(S) \in Y(S)$, thus (ii) is also fulfilled.

Let $(p, \{x^i\}_{i \in M}, \{y(S_j)\}_{j=1}^{J}, B, \{t_{is}\})$ be a structural competitive equilibrium.

Define $\tilde{x}^i = x_c^i$, $i \in M$, $\tilde{y}^j = y_c(S_j)$, $j = 1, \ldots, J$. (i) implies (γ), and (v) implies (β). By (ii) and (iii), since $\sum_{j=1}^{J} \theta_{ij} y_c(s_j) \in \tilde{Y}(\{i\})$ the non-satiation assumption implies that $\sum_{\substack{S \in B \\ i \in S}} t_{is} \geq \sum_{j=1}^{J} \theta_{ij} py_c(S_j)$. Summing (iv) over all $S \in B$ yields: $\sum_{\substack{S \in B \\ i \in S}} t_{is} = \sum_{j=1}^{J} \theta_{ij} py_c(S_j)$. The first part of ($\alpha$) is implied by (iii), and the second part is derived by (ii) and (β).

Q.E.D.

IV. THE SUPER ADDITIVE TECHNOLOGY

Definition 3: The super-additive technology of coalition S, denoted by $Y^*(S)$ is given by:

$$Y^*(S) = [y \in R^{\ell_1+\ell_2} \mid y = \sum_{j=1}^{k} y_j, y_j \in Y(S_j),\ S_j \subset S,\ j = 1,\dots,k,$$

$$y_L \geq \sum_{i \in S} \tilde{x}^i_L - \sum_{i \in S} w^i_L \quad \text{for some} \quad \tilde{x}^i \in \underline{\overline{X}}^i, \quad \text{with} \quad \tilde{x}^i_L \leq w^i_L,\ i \in S].$$

In words, the people of S can decide, instead of forming one firm S, to form k "sub-firms", $S_j, S_j \subset S$, $j = 1,\dots,k$, and to produce separately in each S_j. The only constraint is that the overall amount of labour used (y_L) will not exceed the amount of labour S faces. Obviously, by choosing $k = 1$, $S_1 = S$, we get $Y^*(S) \supset Y(S)$. In fact Y^* is a super additive function, if (d.1) is assumed, (specifically, $0 \in Y(T)$ for all $T \subset M$), i.e.,

$$\text{if} \quad S_1 \cap S_2 = \emptyset, S_1 \cup S_2 \subset S \text{ then } Y^*(S) \supset Y^*(S_1) + Y^*(S_2).$$

It can be easily verified that we can define $Y^*(S)$ as:

$$Y^*(S) = [y \mid y = \sum_{j=1}^{k} y_j, y_j \in Y^*(S_j), S_j \subset S,\ j = 1,\dots,k,\ y_L \geq \sum_{i \in S} \tilde{x}^i_L - \sum_{i \in S} w^i_L$$

$$\text{for some } \tilde{x}^i \in \underline{\overline{X}}^i, \tilde{x}^i_L \leq w^i_L,\ i \in S].$$

Theorem 3: Let $p \in P$ be a price vector, $x^i \in \underline{\overline{X}}^i$, $i \in M$ consumption bundles, B a firm structure, $y(S) \in Y(S)$, $S \in B$ production planes, such that:

(i') $\sum_{i \in M} x^i \leq \sum_{i \in M} w^i + \sum_{S \in B} y(S)$, $x^i_L \leq w^i_L$, $i \in M$

(ii') There is no:

$$S_\circ \subset M,\ \overline{x}^i \in \underline{\overline{X}}^i, \overline{x}^i_L \leq w^i_L,\ i \in S_\circ, y(S_\circ) \in Y^*(S_\circ) \quad \text{such that}$$

$$\sum_{i \in S_\circ} p\bar{x}_c^i \leq p \sum_{i \in S_\circ} w_c^i + py_c(S_\circ),\ \sum_{i \in S_\circ} \bar{x}_L^i - \sum_{i \in S_\circ} w_L^i \leq y_L(S_\circ) \quad \text{and for all} \quad i \in S_\circ$$

$$u^i(\bar{x}^i) > u^i(x^i).$$

Assuming non-satiation, (b.1), there exists profit distributions $\{t_{is}\}$ $i \in M$, $S \in B$ such that $p, \{x^i\}_{i \in M}, \{y(S)\}_{S \in B} \{t_{is}\} i \in M, S \in B$, is a structural competitive equilibrium.

<u>Proof:</u> Since $p \in P$, $p \geqq 0$, we have by (i')

$$(1) \quad p \sum_{i \in M} x_c^i \leq p \sum_{i \in M} w_c^i + p \sum_{S \in B} y_c(S).$$

Define:

$$(2) \quad \sum_{\substack{S \in B \\ i \in S}} t_{is} \equiv px_c^i - pw_c^i,\ i \in M$$

We shall first show that (iv) of Definition 2 is fulfilled. Suppose not, i.e., $\exists\ \hat{S} \in B$ such that

$$(3) \quad \sum_{i \in S} t_{is} > py_c(\hat{S})$$

By (1), (2) and (3):

$$\sum_{S \in B} py_c(S) \geq \sum_{i \in M} (px_c^i - pw_c^i) = \sum_{i \in M} \sum_{\substack{S \in B \\ i \in S}} t_{is} = \sum_{S \in B} \sum_{i \in S} t_{is} > \sum_{\substack{S \in B \\ S \neq \hat{S}}} \sum_{i \in S} t_{is} + py_c(\hat{S}).$$

Hence, $\{\hat{S}\} \neq B$. Therefore, there exists $\bar{S} \in B$, such that

$$(4) \quad py_c(\bar{S}) > \sum_{i \in \bar{S}} t_{i\bar{S}}$$

Define: $$\hat{B} = [S \in B \mid \sum_{i \in S} t_{is} > py_c(S)]$$

$$\bar{B} = [S \in B \mid \sum_{i \in S} t_{is} < py_c(S)]$$

We proved that $\hat{B} \neq \emptyset$ implies $\bar{B} \neq \emptyset$. To complete the proof we need the following definitions:

<u>Definition 4:</u> We shall say that <u>i is connected to j</u> if there are individuals $i_\circ, i_1, \ldots, i_k$ and coalitions $S_\circ, \ldots, S_k, S_{k+1}$, with $i_\circ = i$, $i_k = j$, such that

$$(i_{t-1}, i_t) \in S_t,\ S_t \in B,\ t = 1, \ldots, k,\ i_\circ \in S_\circ \in B\ ,\ i_k \in S_{k+1} \in B.$$

<u>Definition 5:</u> A <u>coalition G is connected to a coalition H</u>, if there exists $i \in G$, $j \in H$ such that i is connected to j.

<u>Definition 6:</u> $\{S_1, \ldots, S_{k,k+1}\}\{i_\circ, \ldots, i_k\}$ is called a <u>minimal connection for</u> H and G if it connects H and G, and there is no other connection of G and H with less than $k+1$ coalitions.

Since B is a finite collection of coalitions, each has a finite number of members, it follows that minimal connections exist.

For each $i \in M$, define:

$$K_i = [j \in M \mid j \text{ is connected to } i].$$

Denote:

$$\hat{K} = \bigcup_{\substack{i \in S \\ S \in \hat{B}}} K_i, \quad \overline{K} = \bigcup_{\substack{i \in S \\ S \in \overline{B}}} K_i$$

By (3) $\hat{K} \neq \emptyset$ and by (4) $\overline{K} \neq \emptyset$.

Distinguish between the two cases:

I. $\overline{K} \cap \hat{K} = \emptyset$.

In this case we shall show that $\overline{K}$ violates (ii'). Let $G = [S \in B \mid S \subset \overline{K}]$. $G \neq \emptyset$ since $\overline{S} \in G$. Moreover, $S \in G$ implies $S \cap \hat{K} = \emptyset$, hence,

for all $i \in S \in G, \{S \in B \mid i \in S\} \cap \hat{B} = \emptyset$.

By definition of $\hat{B}$;

For all $S \in G$, $\sum_{i \in S} t_{is} \leq py_c(S)$

Since $\overline{S} \in G$, we get:

$$(5) \quad \sum_{S \in G} py_c(S) > \sum_{S \in G} \sum_{i \in S} t_{is} = \sum_{i \in \overline{K}} \sum_{\substack{S \in G \\ i \in S}} t_{is} = \sum_{i \in \overline{K}} (px_c^i - pw_c^i)$$

Clearly, $\sum_{S \in G} y(S) \in Y^*(\overline{K})$, $x_L^i \leq w_L^i$, $i \in \overline{K}$, and $\sum_{S \in G} y_L(S) \geq \sum_{i \in \overline{K}} x_L^i - \sum_{i \in \overline{K}} w_L^i$.

By the local non-satiation, there exists for all $i \in \overline{K}$, $\tilde{x}^i$ with $\tilde{x}_L^i = x_L^i$

$u^i(\tilde{x}^i) > u^i(x^i)$ and by (5)

$$\sum_{i \in \overline{K}} (p\tilde{x}_c^i - pw^i) \leq \sum_{S \in G} py_c(S).$$

Hence $\overline{K}$ violates (ii'). Contradiction.

II. $\bar{K} \cap \hat{K} \neq \emptyset$. Let $\{S_\circ, S_1, \ldots, S_{k+1}\}\{i_\circ, i_1, \ldots, i_k\}$ be a minimal connection of $i_\circ \in S_\circ \in \hat{B}$ to $i_k \in S_{k+1} \in \bar{B}$. Define:

$$\tilde{t}_{i_t S_t} = t_{i_t S_t} - \delta \qquad t = 0, \ldots, k$$

$$\tilde{t}_{i_t S_{t+1}} = t_{i_t S_{t+1}} + \delta \qquad t = 0, \ldots, k$$

$$\tilde{t}_{iS} = t_{iS} \quad \text{otherwise.}$$

Since $i_t \in S_t \cap S_{t+1}$, $t = 0, \ldots, k$, $\sum\limits_{\substack{S \in B \\ i \in S}} \tilde{t}_{iS} = \sum\limits_{\substack{S \in B \\ i \in S}} t_{iS}$ for all $i \in M$.

Thus, by (2)

$$(6) \quad \sum_{\substack{S \in B \\ i \in S}} \tilde{t}_{iS} = px^i_c - pw^i_c, \; i \in M$$

Moreover, for all $S \in B$, $S \neq S_\circ$, $S \neq S_{k+1}$

$$(7) \quad \sum_{i \in S} \tilde{t}_{iS} = \sum_{i \in S} t_{iS} \; .$$

Choose $\delta = \text{Min}\,[(\sum\limits_{i \in S_\circ} t_{iS_\circ} - py_c(S_\circ)), (py_c(S_{k+1}) - \sum\limits_{i \in S_{k+1}} t_{iS_{k+1}})]$

As $S_\circ \in \hat{B}, S_{k+1} \in \bar{B}$, $\delta > 0$. With $\tilde{t}_{iS}$ replacing t_{iS}, in view of (7), the number of coalitions in either $\hat{B}$ or $\bar{B}$ reduces. Repeating this process for $\tilde{t}_{iS}$, (since, by (6), (iii) holds), after a finite number of steps either $\bar{K} \cap \hat{K} = \emptyset$, which is case I or else $\bar{B}$ or $\hat{B}$ is empty. If $\hat{B}$ is empty, (iv) is satisfied. If $\bar{B}$ is empty, by (1), $\hat{B}$ is empty. Hence, in any case we proved that (i'), (ii') imply (iii) and (iv). In order to complete the proof of the theorem, we have to show that (v) & (vi) hold. Suppose not, i.e., $\exists (\tilde{y}_c(S_o), \tilde{y}_L(S_o)) \in Y(S_o)$ with $py_c(S_o) < p\tilde{y}_c(S_o), S_o \in B$. Then M violates (ii'), since $y \equiv \sum_{\substack{S \in B \\ S \neq S_o}} y(S) + \tilde{y}(S_o) \in Y^*(M)$.

By (b.1), there exists $\bar{x}^i \in \underline{X}^i, u^i(\bar{x}^i) > u^i(x^i)$ and $\bar{x}^i_L = x^i_L$ which using (i') implies that $\sum_{i \in M} \bar{x}^i_L - \sum_{i \in M} w^i_L \leq y_L$. Moreover, we can choose $\bar{x}^i$ close enough to x^i, such that by (1):

$$p \sum_{i \in M} \bar{x}^i_c \leq p \sum_{i \in M} w^i_c + py_c$$

Hence, (ii') is not fulfilled, a contradiction which establishes that (v) holds, as well.

Q.E.D.

Theorem 3 enables us to concentrate only on the allocation of the structural competitive equilibrium without considering the distribution of the value added. For the proof of Theorem 1, we shall need also the following lemma.

Lemma 1: Under assumptions (a), (d.1), (d.2), $\{Y^*(s)\}_{S \subset M}$ is balanced, i.e., if $\{S_k\}_{k=1}^t$ is a balanced collection with weights $\{\gamma_k\}_{k=1}^t$ */, then $Y^*(M) \supset \sum_{k=1}^t \gamma_k Y^*(S_k)$.

*/ A set of coalitions $\{S_k\}_{k=1}^t$ is called balanced, if there exist "weights" $\{\gamma_k\}_{k=1}^t$ such that for all $k, \gamma_k \geq 0$ and for all $i \in M$,

$$\sum_{\substack{k \\ i \in S_k}} \gamma_k = 1.$$

Proof: Let $y(S_k) \in Y^*(S_k)$, $k = 1,\ldots,t$. Define $y \equiv \sum_k \gamma_k y(S_k)$. By (d.1) $0 \in Y^*(S_k)$ for all k, and by (d.2) $Y^*(S_k)$ is convex. Hence, since $\gamma_k \leq 1$ for all k, $\gamma_k y(S_k) \in Y^*(S_k)$. Thus, to prove that $y \in Y^*(M)$ it is left to be shown that $y_L \geq \sum_{i \in M} \tilde{x}^i_L - \sum_{i \in M} w^i_L$ for $\tilde{x}^i \in \underline{\bar{X}}^i$.

As $y(S_k) \in Y^*(S_k)$,

$$(8) \quad y_L(S_k) \geq \sum_{i \in S_k} \tilde{x}^i_L(S_k) - \sum_{i \in S_k} w^i_L, \ \tilde{x}^i_L(S_k) \leq w^i_L, \tilde{x}^i(S_k) \in \underline{\bar{X}}^i.$$

Define $\tilde{x}^i \equiv \sum_{\substack{k \\ i \in S_k}} \gamma_k \tilde{x}^i(S_k)$. By the convexity of $\underline{\bar{X}}^i$, $\tilde{x}^i \in \underline{\bar{X}}^i$. By (8)

$$y_L = \sum_k \gamma_k y_L(S_k) \geq \sum_k \gamma_k \left(\sum_{i \in S_k} \tilde{x}^i_L(S_k) - \sum_{i \in S_k} w^i_L \right) =$$

$$= \sum_{i \in M} \sum_{\substack{k \\ i \in S_k}} \gamma_k \tilde{x}^i_L(S_k) - \sum_{i \in M} \sum_{\substack{k \\ i \in S_k}} \gamma_k w^i_L =$$

$$= \sum_{i \in M} \tilde{x}^i_L - \sum_{i \in M} w^i_L.$$

Since $\tilde{x}^i_L(S_k) \leq w^i_L \ \forall \ i \in M, \forall \ S_k$, $\tilde{x}^i_L = \sum_{\substack{k \\ i \in S_k}} \gamma_k \tilde{x}^i_L(S_k) \leq \sum_{\substack{k \\ i \in S_k}} \gamma_k w^i_L = w^i_L$.

Q.E.D.

<u>Remark 1:</u> Note that conditions (i) - (v) in Definition 2 do not imply (i') - (ii') of Theorem 3. (Hence, we cannot define a structural competitive equilibrium using Theorem 3). To prove this consider the following example:

There are three individuals, each endowed with one type of non-marketed commodity. The utility of the individual depends solely upon the amount of the (one) marketed good he consumes, (x_1^i), and is strictly monotonic in this good.

Let: $w^1 = (0,1,0,0)$, $w^2 = (0,0,2,0)$, $w^3 = (0,0,0,1)$, and let the technology sets, for producing the marketed good be given by:

$$Y(\{1,2\}) = [y \in R^4 \mid y = t(2,-1,-b,0),\ 1 \leq b \leq 2,\ 0 \leq t \leq 1]$$

$$Y(\{2,3\}) = [y \in R^4 \mid y = t(1,0,-b,-1),\ 1 \leq b \leq 2,\ 0 \leq t \leq 1].$$

$$Y(\{1,2,3\}) =]y \in R^4 \mid y = t(2.6,\ -1,-2,-1),\ 0 \leq t \leq 1].$$

Obviously, $Y(\cdot)$ must be different from $Y^*(\cdot)$, in order for (ii) and (ii') not to be equivalent. Indeed, in our case, $Y(M) \neq Y^*(M)$.

Define: $x^1 = x^2 = (1.2,0,0,0)$, $x^3 = (0.2,0,0,0)$.

Clearly, $\{x^i\}_{i \in M}$ is a structural competitive allocation, for $B = \{1,2,3\}$, $p = 1$, $t_{iM} = x_1^i$, $i \in M$.

In particular, there is no coalition $S \subset M$, such that $y \in Y(S)$ with $y_1 \geq \sum_{i \in S} x_1^i$. Hence, (ii) is fulfilled. However, (ii') is not satisfied, since there exists $y \in Y^*(M)$ with $y_1 = 3 > \sum_{i \in M} x_1^i = 2.6$. This

y is achieved by M forming the "sub-firms" $\{1,2\}$ and $\{2,3\}$.

Note also that this example shows that an inefficient **firm structure** */ can belong to a structural competitive equilibirum. In the case in which (ii') is satisfied, since $Y^*(M)$ considers, in fact, all possible firm structures, the firm structure has to be an efficient one.

*/ A firm structure B is <u>efficient</u> if there exists a B-feasible allocation $\{x^i\}_{i \in M}$ such that for all other firm structure $\tilde{B}$, there is no $\tilde{B}$ - feasible allocation $\{y^i\}_{i \in M}$ with $u^i(y^i) \geq u^i(x^i), i \in M$ and $u^i(y^i) > u^i(x^i)$ for all $i \in S_j \in \tilde{B}$.

V. THE PROOF OF EXISTENCE

Definition 7: For each price vector $p \in P$, the super additive co-operative game without side payments (M, v_p^*) is defined by:

$$v_p^*(S) = [\{u^i\}_{i \in M} \in E_+^m \mid \forall\, i \notin S\ u^i = 0.\ \forall\, i \in S, \exists\, x^i \in \underline{\bar{X}}^i,\ x_L^i \leq w_L^i, \text{ with}$$
$$u^i \leq u^i(x^i), \sum_{i \in S} px_c^i \leq \sum_{i \in S} pw_c^i + py_c(S), y(s) \in Y^*(S), y_L(S) \geq \sum_{i \in S} x_L^i - \sum_{i \in S} w_L^i].$$

In other words, $v_p^*(S)$ is the set of all attainable utilities to members of S, which can be achieved by producing in their super-additive technology $Y^*(S)$, and purchasing the marketed goods in the prices p. Note that there is no feasibility constraint, such as: $y(S) \geq \sum_{i \in S} x^i - \sum_{i \in S} w^i$.

For (M, v_p^*) to be a game, v_p^* has to be a compact set. We shall follow Debreu's idea [3 ,p. 85] and intersect v_p^* with a cube K, which contains in its interior the set of attainable utilities R, where
$R = [\{u^i\}_{i \in M} \in E^m \mid$ For all $i \in M$ there exists $x^i \in \underline{\bar{X}}^i, x_L^i \leq w_L^i$, with $u^i \leq u^i(x^i)$ and $\sum_{i \in M} x^i \leq \sum_{i \in M} w^i + y^*(M), y^*(M) \in Y^*(M)]$.

Under assumptions (a), (d.2), (d.3), (d.4) R is bounded (see [3], p. 77).

Lemma 2: Under assumptions (a), (b.3),(d.1) - (d.4), for every price vector $p \in P$, the core of $v_p^* \cap K$, denoted by $C^*(p)$, is nonempty.

Proof: The nonemptiness of the core will follow if we prove that v_p^* (hence $v_p^* \cap K$) is balanced for all p [10]. Let $\{S_k\}_{k=1}^t$ be a balanced collection with $\{\gamma_k\}_{k=1}^t$ serving as weights. Let $u(S_k) \in v_p^*(S_k)$, i.e.,

$$u^i(S_k) \leq u^i(x^i(k)) \quad i \in S_k,\ u^i(S_k) = 0,\ i \notin S_k$$

Define: $x^i \equiv \sum_{\substack{k \\ i \in S_k}} \gamma_k x^i(k),\ i \in M.$

By (a) $x^i \in \bar{X}^i$, $\forall\ i \in M$. By (b.3), for all $i \in M$, $u^i(x^i) \geq \sum_k \gamma_k u^i(S_k) \geq u^i(S_k)$.

Since $u(S_k) \in v_p^*(S_k)$, $\sum_{i \in S_k} px_c^i(k) \leq \sum_{i \in S_k} pw_c^i + py_c(S_k)$.

hence:

$$\sum_k \gamma_k \sum_{i \in S_k} px_c^i(k) \leq \sum_k \gamma_k \sum_{i \in S_k} pw_c^i + \sum \gamma_k py_c(S_k).$$

$$\Rightarrow \sum_{i \in M} p \sum_{\substack{k \\ i \in S_k}} \gamma_k x_c^i(k) \leq \sum_{i \in M} p \sum_{\substack{k \\ i \in S_k}} \gamma_k w_c^i + p \sum_k \gamma_k y_c(S_k).$$

Define $y = \sum_k \gamma_k y\ (S_k)$, then we have $\sum_{i \in M} px_c^i \leq \sum_{i \in M} pw_c^i + py_c$. By Lemma 1, $y \in Y^*(M)$. It is left to be shown that $y_L \geq \sum_{i \in M} x_L^i - \sum_{i \in M} w_L^i$, and $x_L^i \leq w_L^i$. This follows from the facts that $y_L(S_k) \geq \sum_{i \in S_k} x_L^i(k) - \sum_{i \in S_k} w_L^i$ and $x_L^i(k) \leq w_L^i$. Multiplying both sides by γ_k and summing over k, we get the desired inequalities.

Q.E.D.

For each $p \in P$, $c^*(p) \neq \emptyset$. Thus, there are allocations $x^i(p)$, $i \in M$, such that $u^i(p) = u^i(x^i(p))$, $\{u^i(p)\}_{i \in M} \in c^*(p)$, with $y(p) \in Y^*(M)$ being the corresponding production. That is, each $p \in P$ defines a set of points $E(p)$(of "core" allocations), in $\prod_{i \in M} \bar{X}^i \times Y^*(M)$, such that

(9) $p \sum_{i \in M} (x_c^i(p) - w_c^i) - py_c(p) \leq 0$. Consider the correspondence

$$\theta: P \to R^{\ell_1 + \ell_2} \quad \text{given by:}$$

$$\theta(p) = \sum_{i \in M} x_c^i(p) - \sum_{i \in M} w_c^i - y_c(p), (\prod_{i \in M} x^i(p), y(p)) \in E(p).$$

By (9) $p\theta(p) \leq 0$ for all $p \in P$.

<u>Lemma 3:</u> Under assumptions A, $\theta(p)$ is a convex set for every $p \in P$.

<u>Proof</u>: Let $(\{\tilde{x}^i\}_{i \in M}, \tilde{y}), (\{\bar{x}^i\}_{i \in M}, \bar{y}) \in E(p)$.

Define: $x^i = \lambda \tilde{x}^i + (1-\lambda)\bar{x}^i$, $i \in M$; $y = \lambda \tilde{y} + (1-\lambda)\bar{y}$, $0 \leq \lambda \leq 1$.

By (a), $x^i \in \underline{\bar{X}}^i$ $i \in M$. As $\sum_{i \in M} p\tilde{x}_c^i \leq \sum_{i \in M} pw_c^i + p\tilde{y}$ and

$\sum_{i \in M} p\bar{x}_c^i \leq \sum_{i \in M} pw_c^i + p\bar{y}_c$, we have $\sum_{i \in M} px_c^i \leq \sum_{i \in M} pw_c^i + py$. Similarly,

$\tilde{y}_L \geq \sum_{i \in M} \tilde{x}_L^i - \sum_{i \in M} w_L^i$ and $\bar{y}_L \geq \sum_{i \in M} \bar{x}_L^i - \sum_{i \in M} w_L^i$ implies $y_L \geq \sum_{i \in M} x_L^i - \sum_{i \in M} w_L^i$.

Since $\bar{y}, \tilde{y} \in Y^*(M)$, $\bar{y} = \sum_{S_j \in \bar{B}} y(S_j)$, $\tilde{y} = \sum_{T_j \in \tilde{B}} y(T_j)$. By (d.1) and (d.2),

$\lambda \tilde{y} \in \sum_{S_j \in \bar{B}} Y(S_j)$, $(1-\lambda)\bar{y} \in \sum_{T_j \in \tilde{B}} y(T_j)$. Let $B = (\bar{B}, \tilde{B})$. Hence, $y \in \sum_{S_j \in B} Y(S_j)$.

As shown above, $y_L \geq \sum_{i \in M} x_L^i - \sum_{i \in M} w_L^i$, $x^i \in \underline{\bar{X}}^i$, $x_L^i \leq w_L^i$, which imply

$y \in Y^*(M)$. Hence, $\{u^i(x^i)\}_{i \in M}$ is in $v_p^*(M)$. To complete the proof, we have to show that $\{u^i(x^i)\}_{i \in M}$ is in $C^*(p)$. This follows from the quasi concavity of u^i, and the fact that $\{u^i(\tilde{x}^i)\}_{i \in M}$ and $\{u^i(\bar{x}^i)\}_{i \in M}$ are in $^*(p)$.

Q.E.D.

<u>Lemma 4:</u> Under assumption A, $\theta(p)$ is an upper semi-continuous function of p.

<u>Proof:</u> Let $p^k \underset{k\to\infty}{\to} p^\circ$, $(y(p^k),\{x^i(p^k)\}) \in E(p^k)$, $k = 1,2,\ldots$.

$(y(p^k),\{x^i(p^k)\}) \to (y(p^\circ),\{x^i(p^\circ)\})$. We have to show that $(y(p^\circ),\{x^i(p^\circ)\}) \in E(p^\circ)$. By definition of the super additive technology, each p^k defines a set $D(p^k)$ which is a subset of D. Since D is a finite set, there exists at least one firm structure $d \in D$, such that $d \in D(p^{k_j})$, $j = 1,2,\ldots$ where $\{k_j\}$ is a subsequence of $\{k\}$. Without loss of generality $\{k_j\} = \{k\}$. Hence, $y(p^k) \in \sum_{S\in d} Y(S)$, which by (d.2) is a closed set. Thus, $y(p^k) \to y(p^\circ)$ implies $y(p^\circ) \in \sum_{S\in d} Y(S) \subset Y^*(M)$. Since $\underline{\overline{X}}^i$, $i \in M$, are closed sets, $x^i(p^k) \in \underline{\overline{X}}^i$ implies $x^i(p^\circ) \in \underline{\overline{X}}^i$, $i \in M$.

$$\sum_{i\in M} p^k x_c^i(p^k) \le \sum_{i\in M} p^k w_c^i + p^k y_c(p^k). \quad \text{Hence,}$$

$\sum_{i\in M} p^\circ x_c^i(p^\circ) \le \sum_{i\in M} p^\circ w_c^i + p^\circ y_c(p^\circ)$. Similarily, $y_L(p^k) \ge \sum_{i\in M} x_L^i(p^k) - \sum_{i\in p^k} w_L^i$ implies $y_L(p^\circ) \ge \sum_{i\in M} x_L^i(p^\circ) - \sum_{i\in M^\circ} w_L^i$, and $x_L^i(p^k) \le w_L^i$ implies $x_L^i(p^\circ) \le w_L^i$. Hence,

$$\{u^i(p^\circ)\}_{i\in M} \equiv \{u^i(x^i(p^\circ))\}_{i\in M} \in v^*_{p^\circ}(M).$$

It is left to be shown that $\{u^i(p^\circ)\}_{i\in M} \in C^*(p^\circ)$. Suppose not, i.e. $\exists S_\circ$ and $\bar{u} \in v^*_{p^\circ}(S_\circ)$ with $\bar{u}^i > u^i(p^\circ)\ \forall\ i \in S_\circ$. That is, for all $i \in S_\circ$ there exists $\bar{x}^i \in \underline{\overline{X}}^i$, $\bar{x}^i_L \le w^i_L$, $\bar{u}^i \le u^i(\bar{x}^i)$ such that

$$(10) \quad \sum_{i\in S_\circ} p_c^\circ \bar{x}^i \le \sum_{i\in S_\circ} p_c^\circ w_c^i + p_c^\circ \bar{y}(S_\circ),\ \bar{y}(S_\circ) \in Y^*(S_\circ)$$

$$(11) \quad \bar{y}_L(S_\circ) \geq \sum_{i \in S_\circ} \bar{x}^i - \sum_{i \in S_\circ} w_L^i .$$

By (b.2), there exists $\lambda < 1$, large enough such that $u^i(\lambda\bar{x}^i) > u^i(p^\circ)$ $\forall\, i \in S_\circ$. Since $\bar{y}(S_\circ) \in Y^*(S_\circ)$, $0 \in Y(S)$ for all $S \subset M$, by (d.2), $\lambda\bar{y}(S_\circ) \in Y^*(S_\circ)$. From (10), (c) and the fact that $p^\circ \in P$;

$$(12) \quad \sum_{i \in S_\circ} \lambda p^\circ \bar{x}_c^i < \sum_{i \in S_\circ} p^\circ w_c^i + \lambda p^\circ \bar{y}_c(S_\circ)$$

and by (c) and (11):

$$(13) \quad \lambda\bar{y}_L(S_\circ) \geq \sum_{i \in S_\circ} \lambda\bar{x}_L^i - \sum_{i \in S_\circ} w_L^i$$

For all k large enough, by (12)

$$(14) \quad \sum_{i \in S_\circ} \lambda p^k \bar{x}_c^i < \sum_{i \in S_\circ} p^k w_c^i + \lambda p^k \bar{y}_c(S_\circ)$$

By (13), (14), and as $\lambda\bar{x}_L^i \leq \bar{x}_L^i \leq w_L^i$, $\{u^i(\lambda\bar{x}^i)\}_{i \in S_\circ} \in V^*_{p^k}(S_\circ)$. Since $u^i(\lambda\bar{x}^i) > u^i(p^\circ)$, by continuity of u^i we have, for k large enough: $u^i(\lambda\bar{x}^i) > u^i(x^i(p^k))$, $i \in S_\circ$. This contradicts the fact that $u^i(x^i(p^k)) \in C^*(p^k)$. Hence, $\{u^i(p^\circ)\}_{i \in M} \in C^*(p^\circ)$.

Q.E.D.

<u>Theorem 1:</u> Under assumptions A, there exists a structural competitive equilibrium.

Proof: By lemmata 3 and 4, $\theta(p)$ is convexed valued and upper semi-continuous and for each $p \in P$, $p \cdot \theta(p) \leq 0$. We can, therefore, use Debreu's lemma ([3], p. 82), and conclude that there exists a $p^* \in P$, $(\{x^{i*}\}_{i \in M}, y^*) \in E(p^*)$ with

$$\sum_{i \in M} x^{i*} - \sum_{i \in M} w^i \leq y^*, x_L^i \leq w_L^i, \; i \in M$$

i.e., (i') of Theorem 3 is fulfilled, where B is any firm structure which gives rise to $y^* \in Y^*(M)$, i.e.,

$$y^* = \sum_{S_j \in B} y^*(S_j). \quad \text{Moreoever, since } \{u^i(x^{i*})\}_{i \in M} \in C^*(p^*)$$

(ii') is also fulfilled. Using Theorem 3, there exists $\{t_{is}^*\}$ $i \in M$, $S \in B$, such that $(p^*, \{x^{i*}\}_{i \in M}, y^*, t_{is}^*)$ is a structural competitive equilibrium.

Q.E.D.

Remark 2: In Definition 2 there was no requirement that whenever individual i contributes some labour to firm S, he must belong to S. Moreover, it may be the case that i "contributes a negative amount of labour" to firm S, (even though the overall amount of labour supplied by its members, $-y_L(S)$, is positive). Obviously, one way to solve this problem is simply by enlarging the commodity space, i.e., types of labour are named commodities. We, instead, choose to modify our definition of an equilibrium, leaving the commodity space, $R^{\ell_1+\ell_2}$, constant, which enables us to consider the replica economy. (Otherwise, the number of commodities tends to infinity). Moreover, the above theorems and their proofs carry over, with the required modifications.

Definitions:

I. Let $T \subset M$. A T-firm structure is a firm structure where T replaces M. (i.e., it is a non-empty collection of subsets of T).

II. Let Z be a T-firm structure. The pair $(\{x^i\}_{i\in T}, \{y(S)\}_{S\in Z})$ is called a

T - feasible program realized via Z, if for all $S\in Z$ there exist vectors $\{y_L^i(S)\}_{i\in S}, \in E^{\ell_2}$ such that:

(1) For all $i \in T$: $x^i \in \underline{\overline{X}}^i$ and $\sum_{\substack{S\in Z\\ i\in S}} y_L^i(S) \geq x_L^i - w_L^i$.

(2) For all $S \in Z$: $y(S) \in Y(S)$ and $\sum_{i\in S} y_L^i(S) = y_L(S)$.

(3) For all $i \in T$ and for all $S \in Z$: $y_L^i(S) \leqq 0$.

$y_L^i(S)$ denotes the amount of labour individual i contributes to coalition S, in the production of $y(S)$. Since labour cannot be transferred

$x_L^i - w_L^i \leqq \sum_{\substack{S \in Z \\ i \in S}} y_L^i(S)$, and $y_L^i(S) \leqq 0$. Moreover, if individual i contributes some labour to S, he must belong to S. Indeed, $y_L(S) = \sum_{i \in S} y_L^i(S)$ and $y_L^i(S) \leqq 0$ imply that for $i \notin S$, $y_L^i(S) = 0$.

Note that we do not exclude the possibility that $i \in S$ and $y_L^i(S) = 0$. However, it seems to be the case that if ℓ_2 represents <u>all</u> types of labour, $y \in Y(S)$, $i \in S$ with $y_L^i(S) = 0$ imply $y \in Y(S \setminus \{i\})$. (By convention $Y(\emptyset) = \{0\}$).

We can now state the modification in Definition 2:

(i) is replaced by:

(i^*) $(\{x^i\}_{i \in M}, \{y(S)\}_{S \in B})$ is an M-feasible program realized via B, and $\sum_{i \in M} x^i \leq \sum_{i \in M} w^i + \sum_{S \in B} y(S)$. Clearly, ($i^*$) implies (i) since (1) and (3) imply that $x_L^i \leq w_L^i$ for all $i \in M$.

Similarily in Theorem 3, (i') is replaced by (i^*) and (ii') by:

(ii^*) There is no:

$S_o \subset M$, S_o - firm structure Z, such that

$(\{\bar{x}^i\}_{i \in S_o}, \{y(S)\}_{S \in Z})$ is an S_o - feasible program realized via Z, $\sum_{i \in S_o} p\bar{x}_c^i \leq \sum_{i \in S_o} pw_c^i + \sum_{S \in Z} py_c(S)$, and for all $i \in S_o, u^i(\bar{x}^i) > u^i(x^i)$.

Note that (ii^*) implies (ii'), by defining $y(S_o) = \sum_{S \in Z} y(S)$, $y_L^i(S_o) = \sum_{S \in Z} y_L^i(S)$. By (1) and (3), $x_L^i \leq w_L^i$, by (1) and (2) $y_L(S_o) \geq \sum_{i \in S_o} x_L^i - \sum_{i \in S_o} w_L^i$, and clearly, $y(S_o) \in Y^*(S_o)$. In particular, (ii^*) implies (ii).

To complete the proof of Theorem 1, define:

$$v_p^*(S) = [\{u^i\}_{i\in M} \in E_+^m \mid \forall\ i \notin S,\ u^i = 0.$$

$$\forall\ i \in S,\ u^i \leq u^i(x^i),\ \sum_{i\in S} px_c^i \leq \sum_{i\in S} pw_c^i + py_c(S) \quad \text{where}$$

$y(S) = \sum_{j=1}^{k} y(S_j) \in Y^*(S)$, and $(\{x^i\}_{i\in S}, \{y(S_j)\}_{j=1}^{k})$ is an S-feasible program realized via $\{S_j\}_{j=1}^{k}]$.

The balancedness of V_p^* is proved in the same way as in lemmas 1 and 2 (where $y_L^i \equiv \sum_{k\mid i\in S_k} \gamma_k y_L^i(S_k))$. Similarily, in lemma 3, define for all $i \in M$, $i \in S \in B$, $y_L^i(S) \equiv \lambda\bar{y}_L^i(S) + (1-\lambda)\tilde{y}_L^i(S)$. Hence, $(\{x^i\}_{i\in M}, \{y(S)\}_{S\in B})$ is an M-feasible program realized via B. Lemma 4 follows from the fact that feasible programs are defined by weak inequalities.

Clearly, in the case of a coalition structure, no modification is required. (Simply, define in $\tilde{v}_p(S)$ of Section VI, $y_L^i(S) \equiv x_L^i - w_L^i$, for $i \in S$). In Section VII, where the replica economy is considered, $v(S)$ should be replaced by:

$$v(S) = [\{u^{i,g}\}_{(i,g)\in M_r} \in E_+^{mr} \mid \forall (i,g) \notin S,\ u^{i,g} = 0.$$

$$\forall\ (i,g) \in S,\ u^{i,g} \leq u^i(x^{i,g}),\ \sum_{(i,g)\in S} x^{i,g} \leq \sum_{(i,g)\in S} w^{i,g} + y(S), \text{ where}$$

$y(S) = \sum_{j=1}^{k} y(S_j) \in Y^*(S)$, and $(\{x^{i,g}\}_{(i,g)\in S}, \{y(S_j)\}_{j=1}^{k})$ is an S-feasible program realized via $\{S_j\}_{j=1}^{k}]$.

All other ("feasibility") modifications are obvious.

VI. COALITION STRUCTURE

Theorem 1 proves the existence of a firm structure which brings about structural competitive equilibrium. In particular, individuals may exist who participate in more than one firm. Though as shown above, in the standard A - D economy this type of structure is formed, one may desire to get a coalition structure rather than a firm structure [8], where:

Definition 8: A coalition structure is a firm structure, $d \in D$, such that for all $S_i, S_j \in d$, $S_i \cap S_j = \emptyset$.

Denote the set of all coalition structures by F, i.e.,

$$F = [d \in D \mid S_i, S_j \in d \Rightarrow S_i \cap S_j = \emptyset].$$

Note that we could define a coalition structure as a partition of the set of individuals (which is the usual definition in game-theory for a coalition structure), if for $f \in F$, we "add" the firm $S(f) \equiv M \setminus \bigcup_{S \in f} S$, and define $y(S(f)) \equiv \{0\}$ (which, by (d.1) is feasible). That is, we add the "firm" of all individuals who are "unemployed".

The question that naturally arises is whether for every economy that fulfills assumptions A, there exists a structural competitive equilibrium, the firm structure of whose is a coalition structure? The following example will shed some light upon this problem:

Consider an economy with three individuals, three non-marketed goods and one marketed good.

$$w^i = (0, e^i) \quad \text{where } e^i \text{ is the } i\underline{th} \text{ unit vector in } R^3.$$

That is, each individual is endowed with one unit of one type of labour. (It can be interpreted as one unit of person type i). $\underline{\bar{X}}^i = R^4_+$, $i = 1,2,3$.

For all i, and all $x^i \in \underline{\bar{X}}^i$, $u^i(x^i) = x^i_1$. That is, each agent receives benefits only from the marketed commodity x_1. The technology sets $Y(S)$ are given by:

$$Y(\{i,j\}) = \{y \in R^4 \mid y = t(1, -e^i - e^j) \quad 0 \leq t \leq 1\},$$

$$Y(S) = \{0\} \text{ otherwise.}$$

In essence, this is the "game of pairs", where every two individuals can produce one unit of x_1 if they work full-time, or any convex combination of this production.

If $\{x^i\}_{i=1}^3$ is a structural competitive allocation, we must have $\sum_{i=1}^3 x^i_1 = 1\frac{1}{2}$, which is realized by forming the firm structure

$B = (\{1,2\},\{2,3\},\{1,3\})$, and $y(S) = (\frac{1}{2}, -\frac{1}{2}(e^i + e^j))$ for all $\{i,j\} \in B$. No coalition structure can give rise to such an allocation, since for any coalition structure $f \in F$, $y \in \sum_{S \in f} Y(S)$ implies $y_1 \leq 1$, hence $\sum_{i \in M} x^i_1 \leq 1$.

Most general models will include this type of economy as a special case, since, in particular, the utilities here do not depend upon the non-marketed goods. It therefore follows, that if we want to get a positive result as to the formation of coalition structures rather than firm structures we shall have to impose rather strong restrictions either on the utilities of the individuals (e.g., if i works for more than one firm, $u^i(x^i) < u^i(w^i)$, for all attainable x^i's), or on the technology sets. This last approach was followed in a current paper by T. Ichiishi [8], where he assumed some sort of balancedness of the technology. We shall show that we derive Ichiishi's result as a corollary of our model. The main advantage of our proof is by its relatively simple mathematics techniques, its connection with the above firm structure model, and in the fact that we need less assumptions than are needed in [8]. In particular, we do not assume that for every $i \in M$, $x, \tilde{x} \in \underline{\bar{X}}^i$ implies $(x_c, \tilde{x}_L) \in \underline{\bar{X}}^i$. It seems to us that the amount (or type) of labour an individual can offer, depends heavily upon his consumption bundle of the marketed goods (e.g., food).

Define: $$\tilde{Y}(S) = [y \mid y = \sum_{k=1}^{t} y(S_k), y(S_k) \in Y(S_k),$$

$$S_k \subset S,\ S_i \cap S_j = \emptyset]$$

Equivalently: $$\tilde{Y}(S) = [y \mid y = \sum_{k=1}^{t} y(S_k), y(S_k) \in \tilde{Y}(S_k),$$

$$S_k \subset S,\ S_i \cap S_j = \emptyset].$$

By definition, $y(S_k) \in Y(S_k)$ implies $y_L(S_k) = \sum_{i \in S_k} y_L^i(S_k)$ where $y_L^i(S_k)$ is the labour vector i puts in S_k when $y(S_k)$ is produced, and $\sum_{S_k | i \in S_k} y_L^i(S_k) \geq x_L^i - w_L^i$, for some $x^i \in \underline{\bar{X}}^i$.

<u>Assumption (e):</u> Let $\{S_k\}$ be a balanced collection with $\{\gamma_k\}$ as weights. Let $y(S_k) \in \tilde{Y}(S_k)$. Then, $y = \sum_k \gamma_k y(S_k) \in \tilde{Y}(M)$ and $y_L^i \geq \sum_{\substack{k \\ i \in S_k}} \gamma_k y_L^i(S_k)$.

Assumption (e) is equivalent to the balancedness assumption in [8].

<u>Theorem 4:</u> Under Assumptions A and (e), there exists a structural competitive equilibrium whose firm structure is a coalition structure.

<u>Proof:</u> The proof follows exactly the pattern of Theorem 1. First, note that Theorem 2 holds if $\tilde{Y}(S)$ replaces $Y^*(S)$ in (ii') (and B is, therefore, a coalition structure). In fact, the proof is much easier, since $\bar{S}$ serves as the blocking coalition. Hence, (4) and (5) in the proof of Theorem 3 above, coincide, thus proving the Theorem. Define:

$$\tilde{v}_p(S) = [\{u^i\}_{i \in M} \in E_+^m \mid \forall\, i \notin S,\ u^i = 0.\ \ \forall\, i \in S,$$

$$\exists\, x^i \in \underline{\bar{X}}^i,\ x_L^i \leq w_L^i \ \text{ with } \ u^i \leq u^i(x^i),$$

$$\sum_{i \in S} px_c^i \leq \sum_{i \in S} pw_c^i + py_c(S), y(S) = \sum_{j=1}^{k} y(S_j) \in \tilde{Y}(S)$$

$$S_j \cap S_i = \emptyset, \ \text{ and for all } \ j,\ j = 1,\ldots,k$$

$$y_L(S_j) \geq \sum_{i \in S_j} x_L^i - \sum_{i \in S_j} w_L^i]$$

By assumption (e), the analogue of Lemma 2 is easily proved. We then define $\tilde{\theta}(p)$ in the same way we defined $\theta(p)$. The proof that $\tilde{\theta}(p)$ is convex for every $p \in P$ follows from the fact that $\tilde{Y}(M)$ is a convex set, due to its balancedness, using the same arguments as in Lemma 3. With the required notational modifications, ($\sim$ replaces $*$), Lemma 4 and hence the proof of Theorem 1 carry over word for word.

Q.E.D.

VII. THE REPLICA ECONOMY

Consider an economy in which there exists a general technology set Y, $Y \subset R^{\ell_1} \times R_-^{\ell_2}$, which exhibits the general accumulated knowledge of the society. Each coalition S faces the technology $Y(S)$. Moreover, we assume that the super-additive technology $Y^*(S)$, is given by:

$$(d.5)\quad Y^*(S) = [y \in Y \mid y_L(S) \geq \sum_{i\in S} \tilde{x}_L^i - \sum_{i\in S} w_L^i \text{ for some}$$

$$\tilde{x}^i \in \underline{\overline{X}}^i,\ \tilde{x}_L^i \leqq w_L^i,\ i \in S].$$

We also assume:

(b.4) For each individidual i, the utility function u^i is strictly concave.

(c') $w^i \gg 0$

(d.3') $Y \cap - Y \subset \{0\}$

(d.4') $Y \supset R_-^{\ell_1+\ell_2}$

(d.6) Y is a closed convex set of $R^{\ell_1+\ell_2}$

Let M_r denote the r-fold replica of M. M_r consists of $m \cdot r$ consumers which will be indexed by the pair (i,g), $i = 1,\ldots,m, g = 1,\ldots,r$.

By (d.5), $Y^*(M_r) \supset r \cdot Y^*(M)$. Let $y^r \in Y^*(M_r)$, i.e. $y^r \in Y$,

$$y_L^r \geq \sum_{(i,g)\in M_r} \tilde{x}_L^{(i,g)} - \sum_{(i,g)\in M_r} w_L^i,\ \tilde{x}^{i,g} \in \underline{\overline{X}}^i,\ \tilde{x}_L^{i,g} \leq w_L^i,\ i = 1,\ldots,m, g = 1,\ldots,r.$$

By (a), $\frac{1}{r}\sum_{g=1}^{r} \tilde{x}^{i,g} \in \underline{\overline{X}}^i$, $\frac{1}{r}\sum_{g=1}^{r} \tilde{x}_L^{i,g} \leq w_L^i$. By (d.4') and (d.6), Y is convex with

$0 \in Y$, hence $\frac{1}{r} y^r \in Y$. Therefore, $\frac{1}{r} y^r \in Y^*(M)$. Thus:

(15) $Y_r^* \equiv Y^*(M_r) = r \cdot Y_1^* \equiv r \cdot Y^*(M)$.

By similar arguments, using (d.4'), (d.5) and (d.6) we get:

(16) $S \subset M_r$ implies $Y^*(S) \subset Y_r^*$

(17) For any coalition S with $k_i(S)$ members of type $i, k_i(S) \geq 1$ for all $i \in M$,

$$\text{Min } [k_i(S) \mid i \in M] \; Y_1^* \subset Y^*(S).$$

The game generated by the economy with M_r consumers, is given by (M_r, v) where:

$$v(S) = [\{u^{i,g}\}_{(i,g) \in M_r} \in E_+^{m \cdot r} \mid \forall \, (i,g) \notin S, \; u^{i,g} = 0$$

$$\forall (i,g) \in S, \; \exists \; x^{i,g} \in \bar{\underline{X}}^i, x_L^{i,g} \leq w_L^i, \text{ with } u^{i,g} \leq u^i(x^{i,g}),$$

$$\sum_{(i,g) \in S} x^{i,g} \leq \sum_{(i,g) \in S} w^{i,g} + y(S),$$

$$y(S) \in Y^*(S)].$$

Obviously, for all $p \in P$, $v(S) \subset v_p^*(S)$.

Denote the core of the economy E_r by C_r.

<u>Lemma 5:</u> Under assumptions (a), (b.1), (b.4), (c'), (d.1), (d.3'), (d.4'), (d.5) and (d.6), for every $(x^{i,g}) \in \bigcap_{r=1}^{\infty} C_r$ there exists prices $\varphi \in R^{\ell_1 + \ell_2}$, $\varphi \neq 0$, and profit payments $(t_{i,g}), (i,g) \in M_r$ such that for

all E_r:

(18) $$\sum_{(i,g)\in M_r} x^{i,g} = r \sum_{i\in M} w^i + y^r,\ y^r \in Y^*(M_r),\ x_L^{i,g} \leq w_L^{i,g},\ (i,g) \in M_r.$$

(19) $$t_{i,g} \geq 0\ ,\ \forall (i,g) \in M_r$$

(20) $$\varphi x^{i,g} \leq \varphi w^{i,g} + t_{i,g}, \text{ and } u^i(\bar{x}^{i,g}) > u^i(x^{i,g}),\ \bar{x}_L^{i,g} \leq w_L^{i,g}$$

$$\text{implies } \varphi \bar{x}^{i,g} > \varphi w^{i,g} + t_{i,g}.$$

(21) $$\sum_{(i,g)\in M_r} t_{i,g} = \varphi y^r$$

(22) $$\sum_{(i,g)\in S} t_{i,g} \geq \text{Sup } pY^*(S) \quad \text{for all} \quad S \subset M_r$$

Proof: Since (15), (16), and (17) are implied by our assumptions, we apply the theorem in [2], replacing $\Gamma^i(x^i)$ by:

$$\bar{\Gamma}^i(x^i) \equiv [z^i | z_L^i \leqq 0,\ u^i(z^i + w^i) > u^i(x^i)]$$

Theorem 5: Under the assumptions of Lemma 5, for every $x \in \bigcap_{r=1}^{\infty} C_r$, there exists for each E_r, a structural competitive equilibrium, whose allocation is $(x^{i,g})$.

Proof: Define: $p_j = \varphi_j$, $j=1,\ldots,\ell_1$. By (d.4'), $\varphi \geq 0$, using (20) and (b.1) we derive that $p \in E_+^{\ell_1}$, $p \neq 0$. Since $y^r \in Y^*(M_r)$, there exists a

B, which is the firm structure that gives rise to y. By (15), B is fixed overall r. We shall prove that $(p,(x^{i,g}),B,y^r)$ fulfills (i'),(ii') of Theorem 3 above, thus it is a structural competitive equilibrium. (i') follows directly from (18). Let $u^i(\bar{x}^{i,g}) > u^i(x^{i,g})$, $\bar{x}_L^{i,g} \leqq w_L^{i,g}$, $(i.g) \in S \subset M_r$, and let $\bar{y}(S) \in Y^*(S)$ with $\bar{y}_L(S) \geq \sum_{(i,g)\in S} \bar{x}_L^{i,g} - \sum_{(i,g)\in S} w_L^{i,g}$.

By (20), $\varphi \bar{x}^{i,g} > \varphi w^{i,g} + t_{i,g}$. Summing over $(i,g) \in S$, and using (22) we get:

$$(23) \qquad \sum_{(i,g)\in S} \varphi \bar{x}^{i,g} > \sum_{(i,g)\in S} \varphi w^{i,g} + \sum_{(i,g)\in S} t_{i,g} \geq \sum_{(i,g)\in S} \varphi w^{i,g} + \varphi \bar{y}(S).$$

Since $\varphi \geqq 0$, $\varphi_L \equiv (\varphi_j)_{j=\ell_1+1}^{\ell_1+\ell_2} \geqq 0$. Hence, $\varphi_L \bar{y}_L(S) \geq \varphi_L \sum_{(i,g)\in S} \bar{x}_L^{i,g} - \varphi_L \sum_{(i,g)\in S} w_L^{i,g}$. By (23): $\sum_{(i,g)\in S} p\bar{x}_c^{i,g} > \sum_{(i,g)\in S} pw_c^{i,g} + p\bar{y}_c(S)$. Hence, (ii') holds.

Q.E.D.

Note that φ may serve as competitive prices for all goods, marketed as well as non-marketed. This gives rise to the hypothesis that non-marketed goods have to be scarce, which fits within the intuitive notion of non-marketed goods. Theorem 5 proves this conjecture since in Lemma 5 we get $(t_{i,g}) \equiv 0$, $(i,g) \in M_r$. This result, generally, does not hold in the finite case.