I gratefully acknowledge T. Ichiishi for many fruitful discussions. In particular, his idea for the proof of existence [8], made the existence of the present paper possible. I am also much indebted to J. Drez for introducing me to this subject and the interesting discussions with him gave rise to the main definition in the paper.
The formation of firms is explained due to the existence of some commodities (like initiative, skill, imagination, knowledge, connections etc.), which have the three properties:

-- cannot be transferred (i.e., individually specific)
-- cannot be produced (but, rather, serve as inputs)
-- cannot be marketed, (i.e., the firm does not face a price per unit in which it can acquire these commodities.

We refer to such commodities as types of labour.

When a group of people (coalition) forms a firm, it faces a technology set which uses only the amount of labour its members have. For example, the goods in Arrow-Debreu's economy are, actually, types of labour. We prove that under the standard A-D assumptions, there exists a firm structure in which no coalition has the incentive to withdraw and form a new firm operating in the market with the existing prices (for the marketed goods).

We consider also the replica economy, and prove the equivalence of the core and equilibria allocations.
I. INTRODUCTION

There is recently a growing interest in labour managed economies [4, 7, 8, 9, 12], the main feature of which is the existence of types of labour which are not marketed, hence labour has no price ("wage"), but rather a share in the value added. (A detailed description of the labour-managed economy is given in [4]). In fact, we shall argue that the standard Arrow-Debreu economy is an economy of this type where the non-marketed goods are the shares [9].

Since labour is not marketed it is only natural to use the production coalition economy, (defined by Hildenbrand [6]), in this context. This was recognized by D. Sondermann [11], and was further discussed by T. Ichishi [7]. However, none of the above gave a satisfactory solution as to the formation of firms in labour managed economies. Such a solution is proposed in this paper.

In a forthcoming paper by J. Dreze and the author, an analogy between local public goods and labour managed economies has been established. Thus the definition we use here for an equilibrium is closely related to that of a structural equilibrium in [5]. We then prove the existence of a stable firm structure, in which no group of workers have an incentive to withdraw from the existing structure and form a new firm, operating in the market with the existing prices (for the marketed goods).

The technique for proving the existence of such a structure is by defining for every price vector \( p \), a game without side payments \( v_p \), for which the core is nonempty. The excess demand set-function is shown to be convex and upper semi continuous, enabling us to use Debreu's lemma [3].
This very elegant proof, which is a generalisation of Bohem's paper [1], is due to T. Ichiishi, who uses this method in his current paper [8]. However, the economic model, the assumptions, the techniques of the existence proof within the above general framework and the results are quite different. It is also shown that Ichiishi's model is a special case of the one presented in this paper.

In Section II we present the model, the definitions and the assumptions. We show in Section III that our model is a generalisation of the Arrow-Debreu economy, which is, in fact a special case of a non-marketed goods economy. We then define in Section IV the super-additive technology in which the firm is allowed to form independent autonomous "sub-firms" (departments). With the use of this technology we prove in Section V the existence of a stable firm structure. Surprisingly, the assumptions we use are only the standard assumptions which assure the existence of a competitive equilibrium. In Section VI we discuss the formation of a coalition structure in which each individual works in one and only one firm. We give an example which implies that usually such a structure will never be a stable one. (In fact, it is usually not pareto-efficient). However, adding a balancedness assumption on the technology, we are able to prove the existence of a stable coalition structure. In Section VII, we prove that under very plausible assumptions, the core of the replica economy coincides with the set of structural competitive equilibria. Moreover, in this case each type of labour has its price, giving rise to the conjecture that goods are non-marketed due to their scarcity. The proof is a corollary of Bohem's paper [2] using the concept of the super-additive technology.
II. THE MODEL

There are \( E_1 \) marketed goods, \( E_2 \) non-marketed commodities (which will be referred to as types of labour). \( N = \{1, \ldots, m\} \) denotes the set of individuals each endowed with \( v^i = (v_{1}^{i}, v_{2}^{i}) \in E_1 \times E_2 \), (i.e., \( v_{1}^{i} \) is the initial endowment of the marketed goods and \( v_{2}^{i} \) is the amount of labour individual \( i \) is endowed with)\(^7\) and a utility function \( u^i \) defined over his consumption set \( \bar{E}^1 \). Non-marketed goods cannot be produced.

As for the technology sets, we consider the coalition production approach, modified to our case of non-marketed commodities. Namely, each coalition \( S \), \( S \subset N \), faces a production set \( Y(S) = \mathbb{R} \times \mathbb{R} \) in which \( y \in Y(S) \) implies that for all \( i \in S \) there exist \( \bar{u}_{i} = (\bar{u}_{1}^{i}, \bar{u}_{2}^{i}) \in \mathbb{N}^2 \), \( \bar{v}_{i}^{1} \leq \bar{u}_{i}^{1} \), such that \( y = (y_{1}, y_{2}) \) with \( y_{1} \in \sum_{i \in S} \bar{u}_{i}^{1} - \sum_{i \in S} \bar{v}_{i}^{1} \). (We use the usual sign convention where \( y_{j} \leq 0 \) implies \( j \) is used as a production factor.) This modification captures the idea of non-marketed goods. Each \( S \), if and when formed, faces a fixed maximum amount of the non-marketed commodities which is available to it. If \( S \) desires to acquire more of these goods, it must be by adjoining to it some non-members, \( \tilde{S} \), of \( S \), and form the new coalition \( S \cup \tilde{S} \). Moreover, since non-marketed goods cannot be transferred among individuals, \( \bar{u}_{i}^{1} \leq \bar{u}_{S}^{1} \), for all \( i \in S \). Note, that although \( S \) faces a fixed amount of non-marketed goods that its members have, it will usually use less than that maximum amount as a result of the preferences of the individuals in \( S \), who will decide upon the type and the amount of work they will perform.

\(^7\) We shall use the subscript \( c \) to denote the marketed goods, and the subscript \( l \) to denote the labour.
Denote:

\[ C = \text{the set of all nonempty subsets of } M, \text{ (i.e., } C \text{ is the set of all possible coalitions, } |C| = 2^M - 1). \]

\[ D = \text{the set of all non-empty subsets of } C. \]

\[ P = \{ p \in \mathbb{R}_+^M \mid \sum_{j=1}^M p_j = 1 \}, \text{i.e., } P \text{ is the price simplex.} \]

**Definition 1:** A *firm structure* is an element of $D$.

**Definition 2:** A *Structural Competitive Equilibrium* consists of a price vector $p \in P$ (for the marketed goods, only), consumption bundles $x^i \in \mathbb{R}^M_+$, $i \in N$, a firm structure $B$, production planes $y(S) \in Y(S)$, $S \in B$, a profit distribution $[\tau_{IS}]$, $i \in N$, $S \in B$, such that:

1. \[ \sum_{i \in M} x^i \leq \sum_{i \in M} w^i + \sum_{S \in B} y(S), \quad x^i \leq w^i \quad i \in M \]

2. (1) There is no:

\[ S_0 \subset M, \quad \exists i \in \mathbb{R}^M_+, \quad \exists \lambda \leq \sum_{i \in S_0} y(S_i) \subset Y(S_0) \]

such that:

\[ \forall i \in S_0, \quad x_i^1 \leq p \sum_{i \in S_0} x_i^1 + py_i(S_i), \quad \sum_{i \in S_0} y_i(S_i) \leq \sum_{i \in S_0} y_i^1 \leq y_i(S_i), \]

and $u^i(x^2) > u^i(x^1) \forall i \in S_0$

3. (iii) \[ p_0 x_0^1 \leq p y_0^1 \quad \sum_{S \in B} \tau_{IS} \]

4. (iv) For all $S \in B$, $\sum_{i \in S} \tau_{IS} \leq y_i(S), \quad i \in S$
(v) For all $S \in B$, $\tilde{y}_C(S) \leq y_C(S)$, and $(\tilde{y}_C(S), y_L(S)) \not\in Y(S)$.

Conditions (i), (iii), and (iv) need no comment. Condition (v) is the equivalent of profit maximization. It states that in each firm $S$ the value added, given the labor vector $y_L(S)$, is maximized. Condition (ii) assures that no mobility will occur. There is no incentive to any coalition, $S_a$, to withdraw from $B$, produce by itself $y(S_a)$, distribute the value added $p_C(S_a)$ in such a way that every member of $S_a$ can purchase a vector $x_C$ in the market, whereby its utility is increased.

For the following we need only the standard assumptions on our economy (compare for analogy, [ ] p. 83), namely,

For every $i \in M$:
(a) $\tilde{A}^i$ is a closed convex subset of $\mathbb{R}^{l_1 + l_2}$
(b.1) there is no satiation in consumption in $\tilde{A}^i$:
   i.e., for any $x \in \tilde{A}^i$ and for any neighborhood $U$ of $x$ in $\tilde{A}^i$, there exists an $\bar{x} \in U$, $\bar{x}_L = x_L$ such that $u^i(\bar{x}) > u^i(x)$.
(b.2) $u^i$ is a continuous function on $\tilde{A}^i$
(b.3) $u^i$ is quasi concave.
(c) $w^i > 0$, $w^i = 0$, $w^i \tilde{A}^i$

For every $S \in C$:
(d.) $Y(S) \subset \mathbb{R}^{l_1 + l_2}_+$ with $0 \not\in Y(S)$

For every $d \in D$:
(d.2) $Y(d) : S \in d$ is closed and convex.
(d.3) $Y(d) \cap -Y(d) \subset \mathbb{R}^{l_1 + l_2}_+$
(d.4) $Y(d) \subset -\mathbb{R}^{l_1 + l_2}_+$
Note that in view of (d.1), (d.2) - (d.4) could be replaced by assuming (d.2) - (d.4) only for \( d^m = C \).

Denote the above set of assumptions by \( A \). Our main purpose is to prove:

**Theorem 1:** Under assumptions \( A \) there exists a structural competitive equilibrium.
III. THE ARROW-DEBREU ECONOMY

A good example of a non-marketed goods economy is provided by the standard A-D economy, where there are $M$ individuals each endowed with $w^i \in \mathbb{R}_+^{L_1}$. The consumption set is $\bar{X}^i \subset \mathbb{R}_+^{L_1}$, over which the utility function $u^i(\cdot)$ is defined. There are $J$ firms with the technology sets $\bar{Y}^j \subset \mathbb{R}_+^{L_1}$, $j = 1, \ldots, J$. Each firm $j$ is owned by the individuals who hold its shares. The share $\theta_{ij}$, $j = 1, \ldots, J$ individual $i \in M$ holds are a-priori given and are not marketed. For all $j$, $\sum_{i \in M} \theta_{ij} = 1$, with $\theta_{ij} \geq 0$ for all $j \in M$.

A competitive equilibrium consists of a price vector $p \in \mathbb{R}_+$, consumption bundles $x^i \in \bar{X}^i$, $i \in M$, production planes $\bar{y}^j \in \bar{Y}^j$, $j = 1, \ldots, J$ such that

(a) $px^i \leq \sum_{j=1}^J \theta_{ij} p\bar{y}^j$ and $u^i(x^i) > u^i(x^i')$ implies $px^i > px^i'$, $i = 1, \ldots, M$

(b) $p\bar{y}^j = \max p\bar{y}^j$

(c) $\sum_{i \in M} x^i \leq \sum_{j=1}^J \sum_{j=1}^J \bar{y}^j$, $i = 1, \ldots, J$

The natural way to embed this economy in our model is by defining the following:

$w^i = (w^1, \ldots, \theta_{i1}, \ldots, \theta_{iJ}) \in \mathbb{R}_+^{L_1+L_2}$, \quad (L_2 = J)$

$x^i = x^1 \times \mathbb{R}_+^{L_2}$

$u^i(x^i) = u^i(x^i_1)$ (i.e., the utility is independent of the non-marketed goods).

$\bar{y}^j = \{y \in \mathbb{R}_+^{L_1} \times \mathbb{R}_+^{L_2} | y = ty, y \in \bar{Y}^j, 0 \leq t \leq 1, j = 1, \ldots, J\}$
\[ Y = \{ y \mid y = \bigoplus_{j=1}^{J} y_j, y_j \in Y_j \} \]

Then, the coalition production technology is given by:

\[ Y(S) = \{ y \in Y \mid y_{L} \geq \sum_{i \in S} w_i^L \} \]

\[ S_j = \{ i \in M \mid \theta_{ij} > 0 \} \]

\[ B = \{ S_1, \ldots, S_J \} \]

when the \( J \) firms form (as assumed in A-D model), the firm structure \( B \) is realized. The following theorem proves that the A-D economy is a special case of our model:

**Theorem 2:** \( (p, \{x_i^L\}_{i \in M}^{x^L}, \{y_j^L\}_{j=1}^{y^L}) \) is a competitive equilibrium if and only if \( (p, \{x_i^L\}_{i \in M}^{x^L}, \{y_j^L\}_{j=1}^{y^L}) \) is a structural competitive equilibrium with \( x_i^L - x^L, x_j^L = 0, i \in M, j = 1, \ldots, J, \) \( t_{ij} = 0 \) \( i \neq j \), \( i \in M, j = 1, \ldots, J. \)

**Proof:** Let \( (p, \{x_i^L\}_{i \in M}^{x^L}, \{y_j^L\}_{j=1}^{y^L}) \) be a competitive equilibrium. Define:

\[ x_i^L = x^L, \quad i \in M, \quad x_j^L = 0, \quad i \in M, \quad y_j^L = y^L, \quad j = 1, \ldots, J. \]  

By (γ),

\[ \sum_{i \in M}^{x^L} w_i^L \leq \sum_{i \in M}^{x^L} w_i^L + \sum_{S \in B}^{S} v_i(S), \]

and by the above definitions:

\[ \sum_{i \in M}^{x_i^L} w_i^L \leq \sum_{i \in M}^{x_i^L} w_i^L + \sum_{S \in B}^{S} v_i(S). \]

Hence, (1) is fulfilled.

It is easily verified that for all \( S \subset M, Y(S) = \sum_{i \in S} Y(\{i\}), \) and for \( i \in S \)
all $i \in N, Y((i)) = \sum_{j=1}^{J} i_j \gamma_j$. (In fact, we could use this formulation to represent the production coalition technology sets). By (B), for all $\gamma(S) \leq Y(S), S \subseteq N, \gamma(S) \leq \sum_{i \in S} \sum_{j=1}^{J} i_j \gamma_j$, which establishes (v), (iii) and (iv) follow immediately from the definition of $t_i$. Suppose $u^1(\tilde{x}^1) > u^1(x^1), \forall i \in S, x \subseteq N$. By (c) $x^1_i > x^1_c = p_x^1 = p_x^1 + \sum_{j=1}^{J} i_j \gamma_j$.

By the previous inequality we get

$\sum_{i \in S} i_j \gamma_j > \sum_{i \in S} i_j \gamma_j + p_x^1 \forall \gamma(S) \leq Y(S)$, thus (ii) is also fulfilled.

Let $(p_c, x^1, x^1_i \in Y(S)^{i=1, B}(x^1), B, t_i)$ be a structural competitive equilibrium.

Define $x^1_i = x^1_c, i \in N, y^1 = y_c, j = 1, \ldots, J$. (i) implies (y), and (v) implies (B). By (ii) and (iii), since $\sum_{j=1}^{J} i_j \gamma_j(S) \in \tilde{Y}(1) = \gamma(S)$ the non-satisfaction assumption implies that $\sum_{i \in S} t_i \geq \sum_{i \in S} i_j \gamma_j(S)$. Summing $\sum_{i \in S} t_i \geq \sum_{i \in S} i_j \gamma_j(S)$. The first part $S \subseteq B$ yields: $\sum_{i \in S} t_i \geq S \subseteq t_i$. The second part is derived by (ii) and (B).

Q.E.D.
IV. THE SUPER ADDITIVE TECHNOLOGY

Definition 3: The super-additive technology of coalition $S$, denoted by $\gamma^S(S)$ is given by:

$$\gamma^S(S) = \{y \in \mathbb{R}^{I_1 + \ldots + I_k} \mid y = \sum_{j=1}^{k} y_j, y_j \in \gamma(S_j), S_j \subseteq S, j = 1, \ldots, k, \gamma_L \geq \sum_{i \in S_k} x_i^1 - \sum_{i \in S_k} w_i^1 \text{ for some } x_i^1, w_i^1, i \in S\}.$$

In words, the people of $S$ can decide, instead of forming one firm $S$, to form $k$ "sub-firms", $S_j, S_j \subseteq S, j = 1, \ldots, k,$ and to produce separately in each $S_j$. The only constraint is that the total amount of labour used $(\gamma_L)$ will not exceed the amount of labour $S$ faces. Obviously, by choosing $k = 1, S_1 = S$, we get $\gamma^S(S) \supseteq \gamma(S)$. In fact $\gamma^S$ is a super additive function, if (4.1) is assumed, (specifically, $0 \in \gamma(T)$ for all $T \subseteq N$, i.e.,

$$\text{if } S_1 \cap S_2 = \emptyset, S_1 \cup S_2 \subseteq S \text{ then } \gamma^S(S) \supseteq \gamma^S(S_1) + \gamma^S(S_2).$$

It can be easily verified that we can define $\gamma^S(S)$ as:

$$\gamma^S(S) = \{y \mid y = \sum_{j=1}^{k} y_j, y_j \in \gamma(S_j), S_j \subseteq S, j = 1, \ldots, k, \gamma_L \geq \sum_{i \in S_k} x_i^1 - \sum_{i \in S_k} w_i^1 \text{ for some } x_i^1 \in X_i^1, w_i^1 \in L_i, i \in S\}.$$

Theorem 3: Let $p \in \mathbb{P}$ be a price vector, $x_i^1 \in X_i^1, i \in M$ consumption bundles, $B$ a firm structure, $y(S) \in \gamma(S), i \in B$ production planes, such that:

\[(i') \Sigma_{i \in M} x_i^1 \leq \Sigma_{i \in \mathbb{M}} u_i^1 + \Sigma_{i \in \mathbb{B}} y(S), x_i^1 \leq u_i^1, i \in M, S \subseteq B\]

\[(i'') \text{ There is no: } S, i \in i \in S_n, \tau(S_i) \in \gamma^S(S_i) \text{ such that: }\]

\[S = M, x_i \in S_i, x_i^1 \leq u_i^1, i \in S_n, \tau(S_i) \in \gamma^S(S_i)\]
\[ \sum \gamma_c^i \leq p \sum \gamma_c^i + py_c(S_i), \quad \forall i \in S_i \] for all \( i \in S_i \).

\[ u_i(x^i) > u_i'(x^i). \]

Assuming non-satiation, (5.1), there exists profit distributions
\[ \{t_{is}\}_{i \in S_i}, \quad s \in S \] such that
\[ p, \{x^i_{s}\}_{i \in M, \{y(s)\}_{s \in S}, \{t_{is}\}_{i \in M, s \in S}, \] is a structural competitive equilibrium.

**Proof:** Since \( p \in P, \ p \geq 0 \), we have by (1')

\[ (1) \quad p \sum_{i \in M} \gamma_c^i \leq p \sum_{i \in M} \gamma_c^i + p \sum_{s \in S} \gamma_c(s). \]

**Define:**

\[ (2) \quad \sum_{s \in S} t_{is} = px_c^i - pw_c^i, \quad i \in M \]
\[ = \sum_{i \in S} t_{is} \]

We shall first show that (iv) of Definition 2 is fulfilled. Suppose not,
\( i.e., \sum_{s \in S} t_{is} \geq py_c(S) \)

by (1), (2) and (3):

\[ \sum_{s \in S} py_c(S) \geq \sum_{i \in M} (px_c^i - pw_c^i) = \sum_{i \in M} t_{is} \leq \sum_{s \in S} t_{is} > \sum_{s \in S} t_{is} + py_c(S). \]

Hence, \( \exists \) \( S \in L \). Therefore, there exists \( S \in L \), such that

\[ (4) \quad py_c(S) > \sum_{i \in S} t_{is} \]
Define: \[ \hat{\mathcal{B}} = \{ S \in \mathcal{B} \mid \sum_{i \in S} t_{ij} > py_c(S) \} \]
\[ \overline{\mathcal{B}} = \{ S \in \mathcal{B} \mid \sum_{i \in S} t_{ij} < py_c(S) \} \]

We proved that \( \hat{\mathcal{B}} \neq \emptyset \) implies \( \overline{\mathcal{B}} \neq \emptyset \). To complete the proof we need the following definitions:

**Definition 4:** We shall say that \( i \) is connected to \( j \) if there are individuals \( i_s, i_{t+1}, \ldots, i_k \) and coalitions \( S, S_t, S_{t+1}, \ldots, S_k \), with \( i_s = i, i_k = j \), such that
\[ (i_{t+1}, i_t) \in S_t, S_t \in \mathcal{B}, t = 1, \ldots, k, i_s \in S, i_t \in \mathcal{B} \setminus S_t, i_k \in S_{k+1} \subseteq \mathcal{B}. \]

**Definition 5:** A coalition \( G \) is connected to a coalition \( H \), if there exists \( i \in G, j \in H \) such that \( i \) is connected to \( j \).

**Definition 6:** \( \{ S_1, \ldots, S_{k+1}, i_s, \ldots, i_k \} \) is called a minimal connection for \( H \) and \( G \) if it connects \( H \) and \( G \), and there is no other connection of \( G \) and \( H \) with less than \( k+1 \) coalitions.

Since \( \mathcal{B} \) is a finite collection of coalitions, each has a finite number of members, it follows that minimal connections exist.

For each \( i \in M \), define:
\[ K_i = \{ j \in M \mid j \text{ is connected to } i \} \]
Denote:
\[ \hat{K} = \bigcup_{i \in \mathbb{K}} K_i, \quad \bar{K} = \bigcup_{i \in \mathbb{K}} K_i \]
\[ S \subseteq B \quad S \subseteq B \]

By (3) \( \hat{K} \neq \emptyset \) and by (4) \( \bar{K} \neq \emptyset \).

Distinguish between the two cases:

I. \( \bar{K} \cap \hat{K} = \emptyset \).

In this case we shall show that \( \bar{K} \) violates (ii'). Let \( G = \{ S \subseteq B \mid S \subseteq \bar{K} \} \). \( G \neq \emptyset \) since \( \bar{S} \subseteq G \). Moreover, \( S \subseteq G \) implies \( S \cap \bar{K} = \emptyset \), hence,

for all \( i \in S \subseteq G, \{ S \subseteq B \mid i \in S \} \cap \hat{K} = \emptyset \).

By definition of \( \bar{B} \):

For all \( S \subseteq G, \sum_{i \in S} t_{is} \leq p_{c}(S) \)

Since \( \bar{S} \subseteq G \), we get:

\[ \sum_{S \subseteq G} p_{c}(S) \geq \sum_{i \in \hat{K}} \sum_{S \subseteq G} t_{is} = \sum_{i \in \hat{K}} \sum_{S \subseteq G} t_{is} = \sum_{i \in \hat{K}} \sum_{S \subseteq G} (p_{x_{i}^{c}}^{i} - p_{w_{i}^{c}}^{i}) \]

Clearly, \( \sum_{S \subseteq G} y(S) \in Y(\bar{K}), \quad x_{i}^{1} \leq w_{L}^{i}, \quad i \in \mathbb{K} \), and \( \sum_{S \subseteq G} y_{L}(S) \geq \sum_{i \in \bar{K}} x_{L}^{i} \geq \sum_{i \in \hat{K}} x_{L}^{i} \).

By the local non-satisfaction, there exists for all \( i \in \mathbb{K} \), \( x_{i}^{1} \) with \( x_{L}^{i} = x_{L}^{i} \)

\[ u^{i}(x_{i}^{1}) > u^{i}(x_{i}^{1}) \] and by (5)

\[ \sum_{i \in K} (p_{x_{i}^{c}}^{i} - p_{w_{i}^{c}}^{i}) \leq \sum_{S \subseteq G} p_{c}(S) \]

Hence \( \bar{K} \) violates (ii'). Contradiction.
II. \( \hat{\mathbb{K}} \cap \hat{\mathbb{B}} \neq \emptyset \). Let \( \{S_0, S_1, \ldots, S_{k+1}\} \) be a minimal connection of \( i_1 \in S_1, \ldots, i_k \in S_{k+1} \in \hat{\mathbb{B}} \). Define:

\[
\tilde{t}_{i, s} = \begin{cases} 
    t_{i, s} & t = 0, \ldots, k \\
    t_{i, s} + \epsilon & t = 0, \ldots, k
\end{cases}
\]

\[
\tilde{t}_{s, t+1} = \tilde{t}_{i, s} + \epsilon 
\]

\[
\tilde{t}_{i, s} = \tilde{t}_{i, s} \quad \text{otherwise.}
\]

Since \( i_1 \in S_1 \cap S_{k+1}, t = 0, \ldots, k \), \( \sum_{S \in \hat{\mathbb{S}}} \tilde{t}_{i, s} = \sum_{S \in \hat{\mathbb{S}}} t_{i, s} \) for all \( i \in M \).

Thus, by (2)

\[
\sum_{S \in \hat{\mathbb{S}}} \tilde{t}_{i, s} = \sum_{S \in \hat{\mathbb{S}}} t_{i, s} = \sum_{S \in \hat{\mathbb{S}}} \tilde{t}_{i, s}
\]

Moreover, for all \( S \in \hat{\mathbb{B}}, S \neq \hat{\mathbb{B}} \), \( S \neq S_{k+1} \)

\[
\sum_{i \in \hat{\mathbb{S}}} \tilde{t}_{i, s} = \sum_{i \in \hat{\mathbb{S}}} t_{i, s} = \sum_{i \in \hat{\mathbb{S}}} \tilde{t}_{i, s}
\]

Choose \( \varepsilon = \min \{ \sum_{i \in \hat{\mathbb{S}}} t_{i, s} - \sum_{i \in \hat{\mathbb{S}}} t_{i, s} \} \)

As \( S_1 \in \hat{\mathbb{B}}, S_{k+1} \in \hat{\mathbb{B}}, \varepsilon > 0 \). With \( \tilde{t}_{i, s} \) replacing \( t_{i, s} \) in view of (7),

the number of coalitions in either \( \hat{\mathbb{B}} \) or \( \hat{\mathbb{B}} \) reduces. Repeating this

process for \( \tilde{t}_{i, s} \) (since, by (6), (iii) holds), after a finite number of
steps either \( \hat{\mathbb{K}} \cap \hat{\mathbb{K}} = \emptyset \), which is case (i) or else \( \hat{\mathbb{K}} \) or \( \hat{\mathbb{B}} \) is empty. If

\( \hat{\mathbb{B}} \) is empty, (iv) is satisfied. If \( \hat{\mathbb{B}} \) is empty, by (1), \( \hat{\mathbb{B}} \) is empty.

Hence, in any case we proved that (i'), (ii') imply (iii) and (iv). In order
to complete the proof of the theorem, we have to show that (v) and (vi) hold. Suppose not, i.e., \( y(y_c(S_\alpha), y'_c(S_\alpha)) \in Y(S) \) with \( \bar{p}_c(S_\alpha) < \bar{p}_c(S_\alpha), S_\alpha \in B \). Then \( M \) violates (ii'), since \( y = y(S) + \bar{y}(S) \in Y^*(M) \).

By (b.1), there exists \( \bar{x}^i \in x^i, u_i^A(\bar{x}^i) > u_i(x^i) \) and \( x^i_L = x^i \) which using \( i' \) implies that \( \sum_{k \in M} x^i_k \geq \sum_{k \in M} x^i_k \leq y^L \). Moreover, we can choose \( \bar{x}^i \) close enough to \( x^i \), such that by (1):

\[
\bar{p} \cdot \bar{x}^i \leq \bar{p} \cdot x^i + \bar{p}_c (4)
\]

Hence, (ii') is not fulfilled, a contradiction which establishes that (v) holds, as well.

Q.E.D.

Theorem 3 enables us to concentrate only on the allocation of the structural competitive equilibrium without considering the distribution of the value added. For the proof of Theorem 1, we shall need also the following lemma.

**Lemma 1:** Under assumptions (a), (d.1), (d.2), \( \{y^A(s)\}_{s \in CM} \) is balanced, i.e., if \( [S_k]_{k=1}^t \) is a balanced collection with weights \( (\gamma_k)_{k=1}^t \), then

\[
y^A(s) = \sum_{k=1}^t \gamma_k y^A(S_k).
\]

\( ^{2/} \) A set of coalitions \([S_k]_{k=1}^t\) is called balanced, if there exist "weights" \( \{\gamma_k\}_{k=1}^t \) such that for all \( k, \gamma_k \geq 0 \) and for all \( i \in M, \sum_{k \in S_k} \gamma_k = 1 \).
Proof: Let $y(S_k) \in Y^k(S_k)$, $k = 1, \ldots, z$. Define $y = \sum_k y(S_k)$. By

(4.1) $y \in Y^k(S_k)$ for all $k$, and by (4.2) $Y^k(S_k)$ is convex. Hence, since $y_k \leq 1$ for all $k$, $y_k y(S_k) \in Y^k(S_k)$. Thus, to prove that $y \in Y^k(0)$ it is left to be shown that $y_L \leq \sum_{i \in M} w_i^L$, $\sum_{i \in M} y_i L(S_k) \in x^L$. As $y(S_k) \in Y^k(S_k)$,

(5) $y_L(S_k) \geq \sum_{i \in S_k} x_i^L(S_k) - \sum_{i \in S_k} w_i^L$, $x^L(S_k) \leq w_i^L x^L(S_k) \in x^L$.

Define $x^L = \sum_k y_k x^L(S_k)$. By the convexity of $x^L$, $x^L \in x^L$. By (5)

$y_L = \sum_{k \in S_k} y_k L(S_k) \geq \sum_{k \in S_k} \left( \sum_{i \in S_k} x_i^L(S_k) - \sum_{i \in S_k} w_i^L \right) = \sum_{i \in M} w_i^L - \sum_{i \in S_k} y_i L(S_k) = \sum_{i \in M} w_i^L - \sum_{i \in S_k} y_i L(S_k) = \sum_{i \in M} w_i^L - \sum_{i \in S_k} y_i L(S_k) \leq \sum_{i \in M} w_i^L$. Since $y_i L(S_k) \leq w_i^L$ for $i \in M$, $w_i^L \geq x_i^L(S_k) \leq y_i w_i^L \leq w_i^L$. Q.E.D.
Remark 1: Note that conditions (i) - (v) in Definition 2 do not imply (i') - (ii') of Theorem 3. (Hence, we cannot define a structural competitive equilibrium using Theorem 1.) To prove this consider the following example:

There are three individuals, each endowed with one type of non-marketed commodity. The utility of the individual depends solely upon the amount of the (one) marketed good he consumes, \( x_1 \), and is strictly monotonic in this good.

Let: \( w^1 = (0, 1, 0, 0) \), \( w^2 = (0, 0, 2, 0) \), \( w^3 = (0, 0, 0, 1) \), and let the technology sets, for producing the marketed good be given by:

\[
Y((1, 2)) = \{ y \in \mathbb{R}^4 \mid y = t(2, -1, -b, 0), 1 \leq b \leq 2, 0 \leq t \leq 1 \}
\]

\[
Y((2, 3)) = \{ y \in \mathbb{R}^4 \mid y = t(1, 0, -b, -1), 1 \leq b \leq 2, 0 \leq t \leq 1 \}
\]

\[
Y((1, 2, 3)) = \{ y \in \mathbb{R}^4 \mid y = t(2, 1, -b, -1), 0 \leq t \leq 1 \}.
\]

Obviously, \( Y(\cdot) \) must be different from \( \tilde{Y}(\cdot) \), in order for (ii) and (ii') not to be equivalent. Indeed, in our case, \( Y(0) \neq \tilde{Y}(0) \).

Define: \( x^1 = x^2 = (1, 2, 0, 0, 0), x^3 = (0, 2, 0, 0, 0) \).

Clearly, \( \{ x_i \}_{i \in M} \) is a structural competitive allocation, for \( B = \{ 1, 2, 3 \}, p = 1, t_{1M} = x^1_1, 1 \in M \).

In particular, there is no coalition \( S \subset M \), such that \( y \in Y(S) \) with \( y_1 = x^1_1 \). Hence, (ii) is fulfilled. However, (ii') is not satisfied, since there exists \( y \in Y(M) \) with \( y_1 = 3 > x^1_1 = 2.6 \).

This completes the example.
$y$ is achieved by $M$ forming the "sub-firms" $\{1,2\}$ and $\{2,3\}$.

Note also that this example shows that an inefficient firm structure 3 can belong to a structural competitive equilibrium. In the case in which (iv') is satisfied, since $Y(\tilde{M})$ considers, in fact, all possible firm structures, the firm structure has to be an efficient one.

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A firm structure 3 is efficient if there exists a 3-feasible allocation $\{x^3\}_{i \in \tilde{M}}$ such that for all other firm structure 3, there is no 3-feasible allocation $\{y^3\}_{i \in \tilde{M}}$ with $u^f(y^3) \geq u^f(x^3), i \in \tilde{M}$ and $u^f(y^3) > u^f(x^3)$ for all $i \in \tilde{M}, j \in \tilde{S}$. 
V. THE PROOF OF EXISTENCE

Definition 7: For each price vector $p \in P$, the super additive co-operative game without side payments $(M, v^w_p)$ is defined by:

$$v^w_p(S) = \{ [u^i]_{i \in M} : \sum_{i \in S} u^i = 0, \sum_{i \in S} x^i + \sum_{i \in \bar{S}} x^i \leq u^i, \text{ with}$$

$$u^i \leq u^i(x^i), \sum_{i \in S} p^i x^i \leq \sum_{i \in \bar{S}} p^i x^i + p^p(S), y(s) \in Y^w(S), y_p(S) \geq \sum_{i \in S} x^i - \sum_{i \in \bar{S}} x^i \}.$$  

In other words, $v^w_p(S)$ is the set of all attainable utilities to members of $S$, which can be achieved by producing in their super-additive technology $Y^w(S)$, and purchasing the market goods in the prices $p$.

Note that there is no feasibility constraint, such as: $y(S) \geq \sum_{i \in S} x^i - \sum_{i \in \bar{S}} x^i$.  

For $(M, v^w_p)$ to be a game, $v^w_p$ has to be a compact set. We shall follow Debreu's idea [3, p. 85] and intersect $v^w_p$ with a cube $K$, which contains in its interior the set of attainable utilities $x$, where

$$n = \{ [u^i]_{i \in M} : \text{for all } i \in M \text{ there exists } x^i \in x^i, x^i \leq u^i, \text{ with}$$

$$u^i \leq u^i(x^i) \text{ and } \sum_{i \in M} x^i \leq u^i + y^w(M), y^w(M) \in Y^w(M) \}.$$  

Under assumptions (a), (d.2), (d.3), (d.4) $R$ is bounded (see [3], p. 77).

Lemma 2: Under assumptions (a), (b.3), (d.1) - (d.4), for every price vector $p \in P$, the core of $v^w_p \cap K$, denoted by $C^w(p)$, is nonempty.

Proof. The nonemptiness of the core will follow if we prove that $v^w_p$ (hence $v^w_p \cap K$) is balanced for all $p \in P$. Let $\{ s_k^\varepsilon \}_{k=1}^\varepsilon$ be a balanced collection with $\{ s_k^\varepsilon \}_{k=1}^\varepsilon$ serving as weights. Let $u(s_k^\varepsilon) \in v^w_p(s_k^\varepsilon)$, i.e.,
\[ u^i(S_k) \leq u^i(\kappa^i(k)) \quad i \in S_k, \quad u^i(S_k) = 0, \quad i \notin S_k \]

Define: \[ x^i = \sum_{k \in S_k} v_k x^i(k), \quad i \in M. \]

By (a), \( x^i \in \kappa^i, \) \( i \in M. \) By (b,3), for all \( i \in M, \)

\[ u^i(\kappa^i) \geq \sum_{k \in S_k} v_k u^i(S_k) \geq u^i(S_k). \]

Since \( u(S_k) \in \kappa^0(S_k), \)

\[ \sum_{k \in S_k} p^0 x^i(k) \leq \sum_{k \in S_k} p^0 v_k^i + p y^i(S_k). \]

hence:

\[ \sum_{k \in S_k} p^0 v^i_k x^i(k) \leq \sum_{k \in S_k} v^i_k \sum_{k \in S_k} p^0 v^i_k + \sum_{k \in S_k} y^i_k p y^i(S_k). \]

Define \( y = \sum_{k \in M} v_k y_k^0(S_k), \) then we have \( \sum_{i \in M} x^i \leq \sum_{k \in S_k} p^0 v^i_k + p y^i_k. \) By Lemma 1,

\[ \gamma_y \in \gamma_y^{\kappa_0}(0). \] It is left to be shown that \( \gamma_y \geq \sum_{i \in M} x^i \leq \sum_{k \in S_k} u^i_k, \) and

\[ x^0_L \leq u^0_L. \] This follows from the facts that \( \gamma_y(S_k) \geq \sum_{i \in S_k} x^i(k) = \sum_{i \in S_k} u^i_k \)

and \( x^0_L \leq u^0_L. \) Multiplying both sides by \( v_k \) and summing over \( k, \) we get the desired inequalities.

Q.E.D.

For each \( p \in P, \) \( c^0(p) \neq 0. \) Thus, there are allocations \( x^i(p), i \in M, \)

such that \( u^i(p) = u^i(\kappa^i(p)), \) \( \{u^i(p)\}_{i \in M} \in c^0(p), \) with \( y(p) \in Y^0(M) \)

being the corresponding production. That is, each \( p \in P \) defines a set

of points \( E(p) \) (of "core" allocations), in \( \prod_{i \in M} X^0_i \times Y^0(M), \) such that
Consider the correspondence
\[ \theta(p) = \sum_{i \in M} x_i^p - y(p)_i. \]
By (9) \( p \theta(p) \leq 0 \) for all \( p \in P \).

**Lemma 3:** Under assumptions A, \( \theta(p) \) is a convex set for every \( p \in P \).

**Proof:** Let \( ((x_i^p)_i \in M, y_i)_i \in M, \bar{y}_i) \in \bar{E}(p) \).

Define: \( x_i^p = \lambda x_i^q + (1 - \lambda) x_i^q, i \in M \); \( y_i = \lambda y_i + (1 - \lambda) \bar{y}_i \), \( 0 \leq \lambda \leq 1 \).

By (a), \( x_i^p \in \bar{X}(i) \). As \( \sum_{i \in M} x_i^p \leq \sum_{i \in M} x_i^q + py_i \) and
\( \sum_{i \in M} x_i^q \leq \sum_{i \in M} x_i^q + py_i \), we have \( \sum_{i \in M} x_i^p \leq \sum_{i \in M} x_i^q + py_i \). Similarly,
\( y_i \leq y_i^q \) and \( \bar{y}_i \leq \bar{y}_i^q \) implies \( y_i \leq y_i^q \).

Since \( \gamma, y \in Y^p(M) \), \( \bar{y} \leq \sum_{j \in B} y_j^p \). By (d.1) and (d.2),
\( \lambda \gamma \leq \sum_{j \in B} y_j^p \). Let \( B = (E, \bar{B}) \). Hence, \( y \in Y^p(B) \).

As shown above, \( y \in Y^p(B) \). Hence, \( \{u_i^p(x_i^q)\}_{i \in M} \) is in \( v_i^p(M) \). To complete the proof, we have to show that \( \{u_i^p(x_i^q)\}_{i \in M} \) is in \( v_i^p(M) \). This follows from the quasi concavity of \( u_i^p \), and the fact that \( \{u_i^p(x_i^q)\}_{i \in M} \) and \( \{u_i^p(x_i^q)\}_{i \in M} \) are in \( v_i^p(M) \).

Q.E.D.
Lemma 4: Under assumption A, \( y(p) \) is an upper semi-continuous function of \( p \).

Proof: Let \( p^k \cdot p^s \), \((y(p^k), [x^k(p^k)]) \in E(p^k), k = 1,2, \ldots \)

\((y(p^k), [x^k(p^k)]) \rightarrow (y(p^s), [x^s(p^s)])\). We have to show that \((y(p^s), [x^s(p^s)]) \in E(p^s)\).

By definition of the super additive technology, each \( y^k \) defines a set \( D(p^k) \) which is a subset of \( D \). Since \( D \) is a finite set, there exists at least one firm structure \( d \in D \), such that \( d \in D(p^j), j = 1,2, \ldots \) where \([k_j] \) is a subsequence of \([k]\). Without loss of generality \([k_j] = [k]\).

Hence, \( y(p^k) \in \Sigma Y(S) \), which by (d.2) is a closed set. Thus,

\( y(p^k) \rightarrow y(p^s) \) implies \( y(p^s) \in \Sigma Y(S) \subset Y^s(M) \). Since \( \frac{x^i}{x^i} \), \( i \in M \), are closed sets, \( x^i(p^k) \in \frac{x^i}{x^i} \) implies \( x^i(p^s) \in \frac{x^i}{x^i} \), \( i \in M \).

\[ \sum_{i \in M} p^k_{i} x^i_c(p^k) \leq \sum_{i \in M} p^k_{i} x^i_c(p^s) + p^k_{i} y_c(p^k). \] Hence,

\[ \sum_{i \in M} p^s x^i_c(p^s) \leq \sum_{i \in M} p^s x^i_c(p^s) + p^s y_c(p^s). \] Similarly, \( y_L(p^k) \geq \sum_{i \in M} x^i_L(p^k) - \sum_{i \in M} u^i_L \).

implies \( y_L(p^s) \geq \sum_{i \in M} x^i_L(p^s) - \sum_{i \in M} u^i_L \), and \( x^i_L(p^k) \leq u^i_L \) implies \( x^i_L(p^s) \leq u^i_L \). Hence,

\[ \{ u^i(p^s) \}_{i \in M} = \{ u^i(\frac{x^s}{x^s}(p^s)) \}_{i \in M} \in Y^s(M). \]

It is left to be shown that \( \{ u^i(p^s) \}_{i \in M} \in C^e(p^s) \). Suppose not, i.e.

\( \exists S_0 \) and \( \hat{u} \in V^s(S) \) with \( u^s > u^i(p^s) \forall i \in S_0 \). That is, for all \( i \in S_0 \),

there exists \( x^i \in \frac{x^i}{x^i} \), \( x^i_L \leq u^i_L \), \( u^i \leq u^i(\frac{x^i}{x^i}) \) such that

\( \sum_{i \in S_0} p^s_{i} x^i \leq \sum_{i \in S_0} p^s_{i} x^i + p^s \hat{u}(S_0), \bar{y}(S_0) \in Y^s(S_0) \).
(11) \( \bar{\varphi}_L(S_n) \geq \sum_{i \in S_n} \varphi^i - \sum_{i \in S_n} \psi^i \).

By (b.2), there exists \( \lambda < 1 \), large enough such that \( u^i(\lambda \varphi^i) > u^i(p^i) \) \( \forall i \in S_n \). Since \( \bar{\varphi}(S_n) \in Y^+(S_n), \) \( 0 \in Y(\theta) \) for all \( S \subset M \), by (d.2), \( \lambda \bar{\varphi}(S_n) \in Y^+(S_n) \). From (10), (c) and the fact that \( p^i \in P; \)

(12) \( \sum_{i \in S_n} \lambda \varphi^i \psi^i < \sum_{i \in S_n} p^i \varphi^i + \lambda \varphi^i \psi^i(S_n) \)

and by (c) and (11):

(13) \( \lambda \bar{\varphi}_L(S_n) \geq \sum_{i \in S_n} \lambda \varphi^i \psi^i - \sum_{i \in S_n} \psi^i \)

For all \( k \) large enough, by (12)

(14) \( \sum_{i \in S_n} \lambda \varphi^i \psi^i < \sum_{i \in S_n} k \varphi^i + \lambda \varphi^i \psi^i(S_n) \)

By (13), (14), and as \( \lambda \varphi^i \psi^i \leq \lambda \varphi^i \psi^i(S_n) \leq \psi^i \psi^i(S_n) \in Y^+(S_n) \). Since \( u^i(\lambda \varphi^i) > u^i(p^i) \), by continuity of \( u^i \) we have, for \( k \) large enough:

\( u^i(\lambda \varphi^i) > u^i(x^i(p^i)), i \in S_n \). This contradicts the fact that \( u^i(x^i(p^k)) \in C^k(p^k) \). Hence, \( \{u^i(p^k)\}_{i \in M} \in C^k(p^k) \).

Q.E.D.

Theorem 1: Under assumptions A, there exists a structural competitive equilibrium.
Proof: By lemmas 3 and 4, \( \theta(p) \) is convexly valued and upper semi-
continuous and for each \( p \in P \), \( p \cdot \theta(p) \leq 0 \). We can, therefore, use Debreu's
lemma ([3], p. 82), and conclude that there exists a
\( p^* \in T, \{ (x^*_{iM})_{i \in M} \} \subseteq T(p^*) \) with
\[
\sum_{i \in M} x^*_{iM} = \sum_{i \in M} y^*_{iM} \leq y^*_{iN} \leq \sum_{i \in M} y^*_{iL}, \quad i \in M
\]
i.e., (I') of Theorem 3 is fulfilled, where \( B \) is any firm structure
which gives rise to \( y^* \in Y^*(\theta) \), i.e.,
\[
y^* = \sum_{S \in B} y^*(S). \quad \text{Moreover, since} \quad \{ c^*_{iS} \}_{i \in M} \subseteq C^*(p^*)
\]
(I''') is also fulfilled. Using Theorem 1, there exists \( \{ c^*_{iS} \} \}_{i \in M}, S \in B,
such that \( (p^*, x^*_{iM}, y^*, t^*_{iS}) \) is a structural competitive equilibrium.

Q.E.D.
Remark 2: In Definition 2 there was no requirement that whenever individual $i$ contributes some labour to firm $S$, he must belong to $S$. Moreover, it may be the case that $i$ "contributes a negative amount of labour" to firm $S$, (even though the overall amount of labour supplied by its members, $\gamma^L(S)$, is positive). Obviously, one way to solve this problem is simply by enlarging the commodity space, i.e., types of labour are named commodities. We, instead, choose to modify our definition of an equilibrium, leaving the commodity space, $\mathbb{R}^{1+Tz}$, constant, which enables us to consider the replica economy. (Otherwise, the number of commodities tends to infinity.) Moreover, the above theorems and their proofs carry over, with the required modifications.

Definitions:

I. Let $T \subseteq M$. A $T$-firm structure is a firm structure where $T$ replaces $M$. (i.e., it is a non-empty collection of subsets of $T$).

II. Let $Z$ be a $T$-firm structure. The pair $\{(x^T)_{i \in T}, (y(S))_{S \in Z}\}$ is called a $T$-feasible progress realized via $Z$, if for all $S \in Z$ there exist vectors $\{(y^1_L(S))_{L \in S}\}_{i \in T} \in \mathbb{R}^{Tz}$ such that:

1. For all $i \in T$: $x^i \in \mathbb{Z}^T$ and $\forall S \subseteq S$, $y^i_L(S) \geq x^i_L - \omega^i_L$.

2. For all $S \subseteq Z$: $y(S) \in \mathbb{R}^T$ and $\forall L \subseteq S$, $y^i_L(S) = \gamma^L(S)$.

3. For all $i \in T$ and for all $S \in Z$: $y^i_L(S) \leq 0$.

$\gamma^L(S)$ denotes the amount of labour individual $i$ contributes to coalition $S$, in the production of $y(S)$. Since labour cannot be transferred
\[ x^i_L - w^i_L \leq \gamma y^i_L(S), \text{ and } y^i_L(S) \leq 0. \] Moreover, if individual \( i \) contributes some labour to \( S \), he must belong to \( S \). Indeed, \( y^i_L(S) = \gamma \sum_{1 \in S} y^i_L(S) \) and \( y^i_L(S) \leq 0 \) imply that for \( i \not\in S \), \( y^i_L(S) = 0 \).

Note that we do not exclude the possibility that \( i \in S \) and \( y^i_L(S) = 0 \). However, it seems to be the case that if \( L_2 \) represents all types of labour, \( y \in Y(S), i \in S \) with \( y^i_L(S) = 0 \) imply \( y \not\in Y(S \setminus \{i\}) \).

(By convention \( Y(\emptyset) = \{0\} \).)

We can now state the modification in Definition 2:

(1) is replaced by:

(1') \( \left( x^i_{1 \in M} (y(S)) \right)_{1 \in S} \) is an \( N \)-feasible program realized via \( B \), and \( x^i_L \leq x^i_{\gamma} W^i + \sum y(S) \). Clearly, (1') implies (i) since (1) and (3) imply that \( x^i_L \leq u^i_L \) for all \( i \in M \).

Similarly, Theorem 3, (i') is replaced by (i') and (ii') by:

(ii') There is no:

\( S_0 \subset M, S_0 \) - firm structure \( Z \), such that

\( \left( x^i_{1 \in S_0} (y(S) \in S) \right) \) is an \( S_0 \) - feasible program realized via

\( Z, E \sum_{c, c}^i E \gamma c \cdot E \rho_{c} + \sum p y_{c} \in S_0 \), and for all \( i \in S_0 \), \( x^i_{1 \in S_0} (x^i) > u^i(x^i) \).

Note that (ii') implies (ii'), by defining \( y(S_0) = \sum (y(S)), y^i_{1 \in S_0} \) -

\( \sum y^i_L(S). \) By (1) and (3), \( x^i_L \leq u^i_L \), by (1) and (2) \( y^i_L(S_0) \geq \sum_{1 \in S_0} x^i_L - \sum_{1 \in S_0} W^i_L \), and clearly, \( y(S_0) \in Y(S_0). \) In particular, (ii') implies (ii).
To complete the proof of Theorem 1, define:

\[ v^*_p(S) = \{ (u^{i'})_{i \in S} \in E^+_p | \forall i \notin S, u^i = 0 \} \]

\[ \forall i \in S, u^i \leq \beta_{(x^i)} \sum_{s \in S} \sum_{i \in S} p^i_{(x)} + p_y(S) \text{ where} \]

\[ y(S) = \sum_{j=1}^k y(S_j) \in Y^o(S), \text{ and } \{(x^i_{i \in S})_j, (y(S_j))_{j=1}^k \text{ is an } S\text{-feasible program realized via } [S_j]_{j=1}^k \}. \]

The balancedness of \( v^*_p \) is proved in the same way as in lemmas 1 and 2 (where \( y_l = \sum_{k \in S} y^l_{k}(S_k) \)). Similarly, in lemma 3, define for all \( i \in N, l \in S \in B, y^l_i(S) = \alpha_{(x^l_i)} + (1 - \gamma^l_{(x^l_i)}(S_i) \text{ Hence, } \{(x^l_{i \in N})_j, (y(S))_{j \in S} \} \text{ is an } N\text{-feasible program realized via } B. \text{ Lemma 4 follows from the fact that feasible programs are defined by weak inequalities.} \]

Clearly, in the case of a coalition structure, no modification is required. (Simply, define in \( v^*_p(S) \) of Section VI, \( y^l_i(S) = x^l_i - w^l_i \), for \( i \in S \).) In Section VII, where the replica economy is considered, \( v(S) \) should be replaced by:

\[ v(S) = \{ (u^{i'}, S)_{(i, S) \in E^+_S} | (1, S) \notin S, u^{i'}, S = 0 \} \]

\[ \forall (i, S) \in S, u^{i', S} \leq u^i(x^{i', S}), \sum_{(i, S) \in S} u^{i', S} + y(S), \text{ where} \]

\[ y(S) = \sum_{j=1}^k y(S_j) \in Y^o(S), \text{ and } \{(x^{i', S})_{(i, S) \in S}, (y(S_j))_{j=1}^k \text{ is an } S\text{-feasible program realized via } [S_j]_{j=1}^k \}. \]

All other ("feasibility") modifications are obvious.
VI. COALITION STRUCTURE

Theorem 1 proves the existence of a firm structure which brings about structural competitive equilibrium. In particular, individuals may exist who participate in more than one firm. Though as shown above, in the standard A-D economy this type of structure is formed, one may desire to get a coalition structure rather than a firm structure \([8]\), where:

**Definition 8:** A coalition structure is a firm structure, \(d \in D\), such that for all \(S_i, S_j \in d\), \(S_i \cap S_j = \emptyset\).

Denote the set of all coalition structures by \(F\), i.e.,

\[ F = \{ d \in D \mid S_i, S_j \in d \Rightarrow S_i \cap S_j = \emptyset \} \]

Note that we could define a coalition structure as a partition of the set of individuals (which is the usual definition in game-theory for a coalition structure). If for \(f \in F\), we "add" the firm \(S(f) = N \setminus \cup S\), and define \(y(S(f)) = 0\) (which, by (6.1) is feasible). That is, we add the "firm" of all individuals who are "unemployed".

The question that naturally arises is whether for every economy that fulfills assumptions A, there exists a structural competitive equilibrium, the firm structure of whose is a coalition structure? The following example will shed some light upon this problem:

Consider an economy with three individuals, three non-marketed goods
and one marketed good, \( u^i = (0, e^i) \) where \( e^i \) is the \( i \)th unit vector in \( \mathbb{R}^3 \).

That is, each individual is endowed with one unit of one type of labour. (It can be interpreted as one unit of person type \( i \)). \( \alpha_i = \alpha_{i}', i = 1, 2, 3. \)

For all \( i \) and all \( x^i \in \mathbb{R}^3 \), \( u^i(x^i) = x^i_1 \). That is, each agent receives benefits only from the marketed commodity \( x^i_1 \). The technology sets \( Y(S) \) are given by:

\[
Y([i,j]) = \{ y \in \mathbb{R}^4 \mid y = t(1, -e^i - e^j) \quad 0 \leq t \leq 1 \},
\]

\[
Y(S) = \{0\} \text{ otherwise.}
\]

In essence, this is the "game of pairs", where every two individuals can produce one unit of \( x^i_1 \) if they work full-time, or any convex combination of this production.

If \( [x^i]_{i=1}^3 \) is a structural competitive allocation, we must have \( \sum_{i=1}^3 x^i_1 = 1 \), which is realized by forming the firm structure \( \mathcal{B} = ([1,2],[2,3],[1,3]), \) and \( y(S) = (\frac{1}{3}, -\frac{1}{3}(e^1 + e^3)) \) for all \( [i,j] \in \mathcal{B} \).

No coalition structure can give rise to such an allocation, since for any coalition structure \( \mathcal{E} \subset \mathcal{F}, y \in \sum_{\mathcal{E}} Y(S) \) implies \( y_1 \leq 1 \), hence \( \sum_{i \in \mathcal{F}} x^i_1 \leq 1. \)

Most general models will include this type of economy as a special case, since, in particular, the utilities here do not depend upon the non-marketed
goods. It therefore follows, that if we want to get a positive result as to the formation of coalition structures rather than firm structures we shall have to impose rather strong restrictions either on the utilities of the individuals (e.g., if \( i \) works for more than one firm, \( u'_i(x_i) < u'_i(x'_i) \), for all attainable \( x'_i\)'s), or on the technology sets. This last approach was followed in a current paper by T. Ichihashi [7], where he assumed some sort of balancedness of the technology. We shall show that we derive Ichihashi's result as a corollary of our model. The main advantage of our proof is by its relatively simple mathematics techniques, its connection with the above firm structure model, and in the fact that we need less assumptions than are needed in [8]. In particular, we do not assume that for every \( l \in N, x, x' \in \mathcal{X} \) implies \((x'_l, x'_l) \in \mathcal{X}'\). It seems to us that the amount (or type) of labour an individual can offer, depends heavily upon his consumption bundle of the marketed goods (e.g., food).

Define: 

\[
Y(S) = \{ y \mid y = \sum_{k=1}^{t} y(S_k), y(S_k) \in Y(S_k) \},
\]

where \( S_k \subseteq S, S \cap S_j = \emptyset \).

Equivalently: 

\[
\tilde{Y}(S) = \{ y \mid y = \sum_{k=1}^{t} y(S_k), y(S_k) \in \tilde{Y}(S_k) \},
\]

where \( S_k \subseteq S, S \cap S_j = \emptyset \).

By definition, \( y(S_k) \in Y(S_k) \) implies \( y_L(S_k) = \sum_{l \in S_k} y'_l(S_k) \) where \( y'_l(S_k) \) is the labour vector \( l \) puts in \( S_k \) when \( y(S_k) \) is produced.
and \[ \sum_{S_k \subseteq \mathcal{S}_k} y^i_{L(S_k)} \geq u^i_L - u^i_L, \] for some \( x^i \in \mathcal{X}_k \).

**Assumption (e):** Let \( [S_k] \) be a balanced collection with \( [Y_k] \) as weights. Let \( y(S_k) \in \overline{Y}(S_k) \). Then, \( y = \sum_{k} y_k y(S_k) \in \overline{Y}(S) \) and
\[ y^i \geq \frac{1}{k} \sum_{k} y_k y^i_{L(S_k)}. \]

Assumption (e) is equivalent to the balancedness assumption in [8].

**Theorem 4:** Under Assumptions A and (e), there exists a structural competitive equilibrium whose firm structure is a coalition structure.

**Proof:** The proof follows exactly the pattern of Theorem 1. First, note that Theorem 2 holds if \( \overline{Y}(S) \) replaces \( Y^0(S) \) in (11') (and \( S \) is, therefore, a coalition structure). In fact, the proof is much easier, since \( \overline{S} \) serves as the blocking coalition. Hence, (4) and (5) in the proof of Theorem 3 above, coincide, thus proving the Theorem. Define:

\[ v^i(S) = \left\{ [u^i]_{1 \leq i \leq n} \mid u^i \in \mathcal{X}_k, u^i = 0, \forall i \in S, \right\} \]

\[ x^i \in \mathcal{X}_k, x^i_L \leq u^i_L \text{ with } u^i \leq u^i(x^i), \]

\[ \sum_{i \in S_j} p_y^i \leq \sum_{i \in S_j} p_y^i + p_y(S), \text{ for all } j = 1, \ldots, k. \]

\[ S_j \cap S_i = \emptyset, \text{ for all } j, \text{ for all } j = 1, \ldots, k. \]

\[ y^i_{L(S_j)} \geq \sum_{i \in S_j} x^i_L - \sum_{i \in S_j} u^i_L. \]
By assumption (i), the analogue of Lemma 2 is easily proved. We then define \( \tilde{\Theta}(p) \) in the same way we defined \( \Theta(p) \). The proof that \( \tilde{\Theta}(p) \) is convex for every \( p \in \mathcal{P} \) follows from the fact that \( \tilde{\mathcal{Y}}(M) \) is a convex set, due to its balancedness, using the same arguments as in Lemma 3. With the required notational modifications, ( \( - \) replaces \( * \)), Lemma 4 and hence the proof of Theorem 1 carry over word for word.

Q.E.D.
VII. THE REPLICA ECONOMY

Consider an economy in which there exists a general technology set

\[ Y, Y \subseteq \mathbb{R}^+ \times \mathbb{R}^+ \]

which exhibits the general accumulated knowledge of the society. Each coalition \( S \) faces the technology \( Y(S) \). Moreover, we assume that the super-additive technology \( Y(\hat{S}) \), is given by:

\[ Y(\hat{S}) = \{ y \in Y \mid \forall L \subseteq S, \sum_{i \in L} x_i \leq \sum_{i \in S} w_i \} \]

We also assume:

(b.4) For each individual \( i \), the utility function \( u_i \) is strictly concave.

(c') \( u_i > 0 \)

(d.3') \( Y \cap -Y \subseteq \{0\} \)

(d.4') \( Y \supseteq \mathbb{R}^+_1 \times \mathbb{R}^+_2 \)

(d.6) \( Y \) is a closed convex set of \( \mathbb{R}^+_1 + \mathbb{R}^+_2 \)

Let \( M_r \) denote the \( r \)-fold replica of \( M \). \( M_r \) consists of \( m \cdot r \) consumers which will be indexed by the pair \((i,g)\), \( i = 1, \ldots, m; g = 1, \ldots, r \).

By (d.5), \( Y(\hat{M}_r) \supseteq Y \circ Y(\hat{M}) \). Let \( Y^r \in Y(\hat{M}_r) \), i.e., \( Y^r \subseteq Y \).

By (a), \( \frac{1}{r} \sum_{r=1}^{r} x_i g \leq x_i g \). By (d.4') and (d.6), \( Y \) is convex with
0 \not\in Y$, hence $\frac{1}{x} y^x \in Y$. Therefore, $\frac{1}{x} y^x \in Y^0(M)$. Thus:

$$(15) \quad Y^0_{Mx} = Y^0(Mx) = x \cdot Y^0_M = x \cdot Y^0(M).$$

By similar arguments, using (d.4'), (d.5) and (d.6) we get:

$$(16) \quad S \subseteq Mx \implies Y^0(S) \subseteq Y^0_M$$

$$(17) \quad \text{For any coalition } S\text{ with } k_1(S) \text{ members of type } i, k_1(S) \geq 1 \text{ for all } i \in M,$$

$$\min \{ k_1(S) | 1 \in M, Y^0_i \subseteq Y^0(S) \}.$$  

The game generated by the economy with $Mx$ consumers, is given by $(Mx, v)$ where:

$$v(S) = \left\{ \left[ u^x_i, x^x_i \right] \left| \left( i, x^x_i \right) \in Mx \subseteq E^x_m \right. \right\} \cup \left\{ \left( 1, x^1 \right) \right\} S, x^1 = 0$$

$$\omega(1, x) \in S \subseteq x^1 + \sum_{i \in S} x^1_i \leq x^1, \text{ with } u^i, x \leq u^i(x^1) \omega,$$

$$\sum_{(1, x) \in S} x^1_i \omega \leq \sum_{(1, x) \in S} u^i(x^1) \omega + v(S),$$

$$\gamma(S) \in Y^0(S).$$

Obviously, for all $p \in P, v(S) \subseteq v^0_p(S).$

Denote the core of the economy $E^x_m$ by $C^x_m$.

**Lemma 5:** Under assumptions (a), (b.1), (b.4), (c'), (d.1), (d.3'), (d.4'), (d.5) and (d.6), for every $(x^*, \omega) \in \cap C^x_m$ there exists prices\( i \in \mathbb{R}^m \), \( \omega \neq 0 \), and profit payments \( \{ t_i, x^i \}, (1, x) \in E^x_m \) such that for...
all \( E_r \):

\[
\begin{align*}
\sum_{(i,g) \in \mathcal{E} r} x_{i,g}^r &= x_{i,r}^g + y_{i,r}^g, \\
\sum_{(i,g) \in \mathcal{E} r} y_{i,r}^g &= y_{i,r}^g, \\
\sum_{(i,g) \in \mathcal{E} r} \delta_{i,r}^g &= \delta_{i,r}^g,
\end{align*}
\]

(19) \( t_{i,g} \geq 0, \forall (i,g) \in \mathcal{E} r \)

(20) \( \omega x_{i,g}^r \leq \omega x_{i,g}^r + t_{i,g} \), and \( u^i(x_{i,g}^r) > u^i(x_{i,g}^r), \quad x_{i,g}^r \leq u_{i,r} \)

implies \( \omega x_{i,g}^r > \omega x_{i,g}^r + t_{i,g} \).

(21) \( \sum_{(i,g) \in \mathcal{E} r} t_{i,g} \geq \varphi^r(S) \) for all \( S \in \mathcal{M} r \)

Proof: Since (15), (16), and (17) are implied by our assumptions, we apply the theorem in \( \mathcal{I} \), replacing \( \gamma^r(x_{i,r}^g) \) by:

\[ \gamma^r(x_{i,r}^g) = [z_{i,r}^g x_{i,r}^g, \omega x_{i,r}^g, u^i(x_{i,r}^g)] \]

Theorem 5: Under the assumptions of Lemma 5, for every \( x \in \bigcap_{r=1}^n C_{r,e} \) there exists for each \( E_r \), a structural competitive equilibrium, whose allocation is \( (x_{i,g}^r) \).

Proof: Define: \( p_j = \eta_j, j = 1, \ldots, L_1 \). By (6.4'), \( \gamma_{\omega} \geq 0 \), using (20) and (b.1) we derive that \( p \geq L_1 \), \( p \neq 0 \). Since \( y^r \in \mathcal{Y}(M_r) \), there exists a
\( B \), which is the firm structure that gives rise to \( y \). By (15), \( B \) is fixed overall. We shall prove that \( (P, (x^1, y^1), B, y^v) \) fulfills \((1'),(11')\) of Theorem 3 above, thus it is a structural competitive equilibrium. \((1')\) follows directly from (14). Let \( u_i^1(x^1, y^1) > u_i^1(x^0, y^1) \), \( \omega_i \leq \omega_{i_L} \), (1, g) \( \in S \subset N_4 \), and let \( y(S) \in Y(S) \) with \( y(S) \geq \sum_{(i, g) \in S} \omega_{i_L} y^i_{L} \). By (20), \( \omega_i x^i_{L} > \omega_i y^i_{L} + \tau_{4, L} \). Summing over \( (1, g) \in S \), and using (22) we get:

\[
\sum_{(1, g) \in S} \omega_i x^i_{L} > \sum_{(1, g) \in S} \omega_i y^i_{L} + \sum_{(1, g) \in S} \tau_{4, L} \geq \sum_{(1, g) \in S} \omega_i y^i_{L} + \sum_{(1, g) \in S} \omega_i y^i_{L} + \sum_{(1, g) \in S} \tau_{4, L} \sum_{(1, g) \in S} y(S).
\]

Since \( \omega \geq 0 \), \( \omega_{i_L} \geq \sum_{j=1}^{k} \frac{1}{k} \frac{1}{i_L} \). Hence, \( \omega \sum_{(1, g) \in S} y(S) \geq \sum_{(1, g) \in S} \omega_i y^i_{L} + \sum_{(1, g) \in S} \tau_{4, L} \sum_{(1, g) \in S} y(S).
\]

By (23)\( \frac{1}{p_{i, c}} y^i_{L} > \sum_{(1, g) \in S} \omega_i y^i_{L} + \sum_{(1, g) \in S} \tau_{4, L} \sum_{(1, g) \in S} y(S) \). Hence, (11') holds.

Q.E.D.

Note that \( \omega \) may serve as competitive prices for all goods, marketed as well as non-marketed. This gives rise to the hypothesis that non-marketed goods have to be scarce, which fits within the intuitive notion of non-marketed goods. Theorem 5 proves this conjecture since in Lemma 5 we get \( (\tau_{4, L}) = 0 \), (1, g) \( \in N_L \). This result, generally, does not hold in the finite case.