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CONSTRUCTION OF A CONTINUOUS UTILITY FUNCTION FOR A CLASS OF PREFERENCES

by

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ABSTRACT

We construct a continuous utility indicator for a subclass of continuous preference relations, including some with thick indifference classes, using a measure theoretic technique related to that of Neuefeind [11]. This indicator is not continuous on the full class of continuous preferences endowed with the closed convergence topology. It appears that no such indicator can be constructed, although Mas-Colell [7] has established that one exists. A finer topology for preferences seems appropriate.

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Section 1.

In certain economic models preferences of economic agents are allowed to vary. In such situations it is convenient to be able to represent preferences by a numerical utility indicator which is jointly continuous in preferences and commodities. This situation arises in the study of resource allocation mechanisms where one is interested in how a mechanism functions over a class of economic environments (see, for example [9]). If a suitably continuous representation of preferences is available, it is possible to represent a class of environments continuously by the set of utility allocations feasible for each economy. One important motivation for the study of resource allocation mechanisms is to find mechanisms that function well in economies in which preferences do not have the "classical" properties essential to the optimal functioning of price mechanisms. In particular, convexity, local non-satiation of preferences, or monotonicity assumptions may not be appropriate. An example of such a situation is provided by [5 p. 202] where in order to incorporate producers into the B-process in a decentralized way a producer is represented as a consumer whose admissible consumption set is his production possibility set, and whose preferences are "flat" over that set, i.e. he is indifferent between any two feasible input-output vectors. No continuous representation of preferences has been constructed for a class including such flat agents as well as, say, monotone ones.

Y. Kannai [6] first introduced a topology on the set of preference relations. He considered continuous and monotone preferences defined on the positive orthant of $\mathbb{R}^\ell$ (ℓ-dimensional euclidean space). He constructed a utility indicator for that class of preferences which is jointly continuous in preferences and commodities. G. Debreu [2] introduced the topology of Hausdorff distance on the space of the graphs of preference relations. W. Hildenbrand [4] introduced the topology of closed convergence, and constructed a utility function which represents the class of agents whose consumption sets are $\mathbb{R}^\ell_+$ (the non-negative orthant of $\mathbb{R}^\ell$) and whose preferences are continuous and monotone. His utility function is jointly continuous in preferences and commodities, using the topology of closed convergence for preferences and the usual euclidean metric for commodities [4, Appendix B. p. 183].

Subsequently, Neuefeind [11], constructed a utility function for the class of agents whose consumption set is a connected subset of R^ℓ and is the closure of its interior, and whose preferences are continuous, with indifference classes of Lebesgue measure 0. In an unpublished paper [10] the present authors constructed a jointly continuous utility for the class of agents whose preferences are continuous on a closed convex consumption set (in $\mathbb{R}^\ell$) and satisfy local non-satiation. [1/] We also showed that the existence of a continuous utility function for the full class of continuous

[1/] The original proof of this result contained an error. We are grateful to B. Grodal for pointing it out and suggesting a correction. We are also grateful to Y. Kannai for several useful comments on that paper and for stimulating discussions of problems of representing preferences.

preferences is equivalent to the lower hemi continuity of the correspondence ϕ^{-1} which associates with each preference relation the set of all its continuous utility representations. This equivalence follows from a theorem of Michaels on the existence of a continuous selection from a correspondence [8].

A. Mas-Colell [7] has made use of this theorem of Michael's to establish the existence of a continuous utility function for the class of continuous preferences on a second countable, locally compact topological space. Mas-Colell does not construct the utility function from the data of preference relations and commodities, but establishes the conditions which according to Michael's theorem ensure that there is a continuous selection from the correspondence mentioned above.

In this paper we present a method, related to that of Neuefeind in that it relies on measure, for constructing a utility function for a class of continuous preferences. We take the consumption set $\underline{\overline{X}}$ to be a convex subset of $\mathbb{R}^{\ell}$; we make use of certain measures, related to each other, on $\underline{\overline{X}}$, and $\underline{\overline{X}} \times \underline{\overline{X}}$. We define the utility of the point $x \in \underline{\overline{X}}$, when the preference relation is α, denoted $U(\alpha,x)$, by the measure of the set of points (u,v) in $\underline{\overline{X}} \times \underline{\overline{X}}$ such that $v \underset{\alpha}{\precsim} u \underset{\alpha}{\precsim} x$, less the measure of those pairs (u,v) in that set which lie over the interior of the indifference relation. This is related intuitively to the idea of taking the measure of the lower contour set of x (in the consumption set) acording to α, and subtracting the measure of all "thick" indifference classes below x in preference. A precise definition of this utility function is given in Definition 1 of Section 2. We then study the joint continuity of the function U on the class of continuous agents endowed with the topology of closed convergence. Our results are as follows. Theorem 1 of Section 2 establishes the equivalence between the joint continuity of U and certain properties given in Definition 2 of Section 2, involving the interior of the graph of the indifference correspondence. These conditions are discussed in Section 2. In Section 3 further conditions are given which characterize a class of agents on which U is a continuous utility. It is shown that, while there is no largest class of agents in which U is continuous, there is a class which includes the monotone agents and the flat agent. Theorem 2 in Section 3 establishes this result.

It is clear from Examples 1 and 2 given below, as well as from the theorems, that the closed convergence topology counts as neighboring agents, some who are widely different in terms of the utility function U. One may interpret this situation as raising the question whether the closed convergence topology is the natural one for the class of continuous agents. It is possible to give the class of agents a finer topology in which the function U is jointly continuous on the full class of continuous agents. First, note that the function F defined in Definition 2, Section 2 below, induces a psuedo-metric on the class of agents; which, in turn induces a topology. The common refinement of this topology and the closed convergence topology is a topology in which the function U is jointly continuous on the full class of continuous agents. Moreover, since it is a refinement of the closed convergence topology, all mappings which are continuous in that topology remain continuous in the finer one. [2/]

The question whether it is possible to construct a jointly continuous utility indicator for the full class of continuous preferences endowed with the topology of closed convergence with constant consumption set $\mathbb{R}_+^\ell$ remains. To pose this question in a precise way one must indicate what would be allowed as a method of construction. We discuss briefly one such notion, namely construction via a "measurement process", in Section 4 below and give examples of utility functions so constructed.

[2/] Chichilnisky has introduced a topology on preferences based on the order topology induced by the (set) inclusion relation applied to graphs of strict preference relations. It appears that the function U is jointly continuous on the full class of continuous agents in that topology.

Section 2.

We shall assume that the commodity space is R^ℓ where R^ℓ denotes euclidean ℓ-dimensional space, and that the consumption set $\underline{\bar{X}}$ is a convex subset of R^ℓ.[3/] We shall adopt the following conventions and notations:

$R^\ell_+ = \{(x_1, \ldots, x_\ell) \mid x_i > 0 \quad 1 \leq i \leq \ell\}$

$|\ |$ denotes the norm in R^ℓ (and in R)

$\|\ \|$ denotes the norm in $R^\ell \times R^\ell$

$d(.,.)$ will denote the Hausdorff metric on $R^\ell \times R^\ell \cup \{\infty\}$ which we shall assume is bounded above by 1. If $A, B \subset R^\ell \times R^\ell$, then $d(A, B) = d(A \cup \{\infty\}, B \cup \{\infty\})$.

ν will denote a measure on $\underline{\bar{X}}$ which assigns positive measure to sets open in $\underline{\bar{X}}$ and assigns a finite value to $\underline{\bar{X}}$.

$\mu = \nu \times \nu$ will denote the product measure on $\underline{\bar{X}} \times \underline{\bar{X}}$.

Δ will denote the diagonal of $\underline{\bar{X}} \times \underline{\bar{X}}$.

$\bar{\nu}$ will denote the measure on Δ which assigns to each subset of Δ the measure of its projection into $\underline{\bar{X}}$.

$\mathcal{S}$ will denote the class of agents with continuous preferences on $\underline{\bar{X}}$.

A^o will denote the interior of the set A, for $A \subset \underline{\bar{X}} \times \underline{\bar{X}}$ (or $\underline{\bar{X}}$).

$\bar{A}$ will denote the closure of A in $\underline{\bar{X}} \times \underline{\bar{X}}$ (or $\underline{\bar{X}}$).

A^c will denote the complement of A in $\underline{\bar{X}} \times \underline{\bar{X}}$ (or $\underline{\bar{X}}$).

$\Gamma(\alpha) \subset \underline{X} \times \underline{X}$ for $\alpha \in \mathcal{S}$, will denote the graph of the preference relation α, i.e. $\Gamma(\alpha) = \{(u, v) \mid u \succsim_\alpha v\}$.

[3/] It would be possible to allow $\underline{\bar{X}}$ to be infinite dimensional as long as there is a suitable measure defined on it; e.g., a Hilbert measure for $\underline{X}$ a convex subset of Hilbert space. Generalizations which allow $\underline{\bar{X}}$ to vary with the agent are also possible.

$I(\alpha)$ will denote the image of the indifference correspondence at α, i.e. $I(\alpha) = \{(u, v) \mid u \sim_\alpha v\}$.

$I(\alpha, y)$ will denote the set $\{z \mid z \sim_\alpha y\}$.

$\mathcal{J}(\alpha)$ will denote the set $\bigcup_{y \in \bar{X}} I(\alpha, y)^\circ$. (We write $I(\alpha,y)^\circ$ for $[I(\alpha,y)]^\circ$.)

$S(\alpha, x)$ will denote the set $\{(u, v) \mid v \precsim_\alpha u \precsim_\alpha x\}$.

$A \oplus B$ for sets $A, B \subset X \times \bar{X}$ will denote the symmetric difference $(A \cap B^c) \cup (A^c \cap B)$.

$\underline{\mathrm{Lim}}\, A_j$ denotes the topological limit infimum of a sequence of sets A_j, i.e. $x \in \underline{\mathrm{Lim}}\, A_j$ iff for each open neighborhood U of x, $A_j \cap U \neq \emptyset$ for all but a finite number of j. Similarly $\overline{\mathrm{Lim}}\, A_j$ denotes the topological limit supremum.

$\underline{\lim}\, A_j$ denotes the set theoretic limit infimum of the sequence of sets A_j.

<u>Definition 1</u>: $U(\alpha, x) = \mu[S(\alpha, x) \cap (\mathcal{J}(\alpha)^c \times \bar{X})] = \mu[S(\alpha,x)] - \mu[S(\alpha,x) \cap (\mathcal{J}(\alpha) \times \bar{X})]$

<u>Lemma 1</u>: $\mathcal{J}(\alpha) = pr_1\, I(\alpha)^\circ = pr_1(I(\alpha)^\circ \cap \Delta)$ where pr_1 denotes the function from $\bar{X} \times \bar{X}$ which carries (u, v) to u.

Proof: First $\mathcal{J}(\alpha) \subset pr_1\, I(\alpha)^\circ$, because if $u \in \mathcal{J}(\alpha)$, then $u \in I(\alpha,z)^\circ$ for some z, and hence $(u,z) \in I(\alpha,z)^\circ \times I(\alpha,z)^\circ \subset I(\alpha)^\circ$. Assume now that $u \in pr_1\, I(\alpha)^\circ$. There exists $v \in \bar{X}$ such that $(u, v) \in B \times B' \subset I(\alpha)^\circ$, where B and B' are nonempty open sets in $\bar{X}$. We claim that $B \subset \mathcal{J}(\alpha)$. It will suffice to show that if $u' \in B$, then $u' \sim_\alpha u$. Now (u, v) and (u', v) are both elements of $B \times B' \subset I(\alpha)$. Hence, by transitivity of indifference, $u \sim_\alpha v \sim_\alpha u'$; thus $u \sim_\alpha u'$. Finally, if $y \in \mathcal{J}(\alpha)$, then $y \in I(\alpha, y)^\circ$ and $(y, y) \in I(\alpha, y)^\circ \cap \Delta$. Thus $\mathcal{J}(\alpha) \subset pr_1(I(\alpha)^\circ \cap \Delta)$.

<u>Lemma 2</u>: If B and B' are subsets of $\bar{X}$ such that for $\alpha \in \mathcal{S}$, $B \times B' \subset I(\alpha)$, then $(B \cup B') \times (B \cup B') \subset I(\alpha)$.

Proof: If $(p, q) \in I(\alpha)$, then $p \sim_\alpha q$ and hence (p, p), (q, p) and (q, q) are also in $I(\alpha)$.

Lemma 3: If $\alpha, \beta \in \mathcal{B}$ such that $d(\Gamma(\alpha), \Gamma(\beta)) < \epsilon$, then $d(I(\alpha), I(\beta)) < \epsilon$.

Proof: Suppose $(u, v) \in I(\alpha)$. Because $d(\Gamma(\alpha), \Gamma(\beta)) < \epsilon$, there exist (u', v') and (u'', v'') in $\underline{\bar{X}} \times \underline{\bar{X}}$ such that $u' \succsim_\beta v'$ and $u'' \precsim_\beta v''$ where $d((u, v), (u', v')) < \epsilon$ and $d(u, v), (u'', v'')) < \epsilon$. The existence of such a point (u', v') is immediate. That there is a point (u'', v'') with the required properties can be shown as follows. Since $(u, v) \in I(\alpha)$, it follows that $(v, u) \in I(\alpha)$. Hence there exists $(v'', u'') \in \Gamma(\beta)$ such that $d((v, u), (v'', u'')) < \epsilon$. But $d((v, u), (v'', u'')) < \epsilon$ implies $d((u, v), (u'', v'')) < \epsilon$, while $(v'', u'') \in \Gamma(\beta)$ implies $u'' \precsim_\beta v''$. Now on the line segment [4/] from (u', v') to (u'', v'') there is a point (u^*, v^*) where $u^* \sim_\beta v^*$, by continuity of preference. Because the assertion is symmetric in α and β, this establishes the Lemma.

Definition 2: Denote by A the set of graphs of equivalence relations on $\underline{\bar{X}}$. Thus $I \in A$ iff $I = \{(u, v) \mid u \sim_\alpha v\}$ for some equivalence relation α on $\underline{\bar{X}}$. We define a real valued function F from $A \times A$ to R as follows: $F(C, D) = \bar{\nu}(C^\circ \cap \Delta \oplus D^\circ \cap \Delta)$. We consider A as a topological space using as a metric on A the metric $d(.,.)$. A subspace $B \subset A$ will be said to be an <u>admissible class of indifferences</u> iff:

(1) for each $I \in B$, $\mu(I) = \mu(I^\circ)$.

(2) **for each** $x \in \underline{\bar{X}}$, $\nu(I(\alpha, x) - I(\alpha, x)^\circ) \doteq 0$.

(3) if $\alpha_j, \alpha \in B$ and $\lim_j \alpha_j = \alpha$, then $\lim_j F(\alpha_j, \alpha) = 0$.

4/ Convexity of $\underline{\bar{X}}$ is used here .

Definition 3: If B is an admissible class of indifferences, then we shall denote by $\mathcal{E}(B)$ the subclass of agents in $\mathcal{E}$ which satisfy the following conditions:

(i) if $\alpha \in \mathcal{E}(B)$, then $I(\alpha) \in B$

(ii) (Cantor condition) for each open set V in $\bar{X}$ and $\alpha \in \mathcal{E}(B)$ such that $V \times V \not\subset I(\alpha)$, $\mu[\Gamma(\alpha)^c \cap (\mathcal{J}(\alpha)^c \times \bar{X}) \cap (V \times \bar{X})] \neq 0$.

Lemma 4: If $\alpha \in \mathcal{E}$ and if α satisfies the Cantor condition, then $U(\alpha, \cdot)$ is monotone with respect to $\precsim_\alpha$ on $\bar{X}$.

Proof: If $x \precsim_\alpha y$, then $S(\alpha, x) \subseteq S(\alpha, y)$, and hence $\mu[S(\alpha, x) \cap (\mathcal{J}(\alpha)^c \times \bar{X})] \leq \mu[S(\alpha, y) \cap (\mathcal{J}(\alpha)^c \times \bar{X})]$. To complete the proof of the Lemma we need only show that if $x \prec_\alpha y$, then $\mu[S(\alpha, x) \cap (\mathcal{J}(\alpha)^c \times \bar{X})] < \mu[S(\alpha, y) \cap (\mathcal{J}(\alpha)^c \times \bar{X})]$. However $S(\alpha, y) = S(\alpha, x) \cup \{(u, v) \mid v \precsim_\alpha u \precsim_\alpha y \text{ and } u \succ_\alpha x\}$, thus we need only show that

$$\mu[\{(u, v) \mid v \precsim_\alpha u \precsim_\alpha y \text{ and } u \succ_\alpha x\} \cap (\mathcal{J}(\alpha)^c \times \bar{X})] \neq 0.$$

Because $x \prec_\alpha y$, there are points p and q such that $x \prec_\alpha p \prec_\alpha q \prec_\alpha y$. Hence the set $V = \{u \mid x \prec_\alpha u \prec_\alpha y\}$ is both open and nonempty, and $V \times V \not\subset I(\alpha)$ because $(p, q) \in V \times V$ where $p \prec_\alpha q$. Hence $\mu[\Gamma(\alpha)^c \cap (\mathcal{J}(\alpha)^c \times \bar{X}) \cap (V \times \bar{X})] \neq 0$. However $\Gamma(\alpha)^c \cap (\mathcal{J}(\alpha)^c \times \bar{X}) \cap (V \times \bar{X}) = \{(u, v) \mid v \prec_\alpha u, x \prec_\alpha u \prec_\alpha y, u \notin \mathcal{J}(\alpha)\}$. This completes the proof.

Theorem 1: If B is an admissible class of indifferences, if $\{(\beta, y)\}$ is an infinite sequence in $\mathcal{E}(B) \times \bar{X}$ and $(\alpha, x) \in \mathcal{E}(B) \times \bar{X}$ such that the sequence $\{(\beta, y)\}$ converges to (α, x), then

$$\lim_{\{(\beta,y)\}} U(\beta, y) = U(\alpha, x).$$

Proof: We shall first show that $\lim_{\{(\beta,y)\}} \mu[S(\beta, y) \cap S(\alpha, x)^c] = 0.$

$S(\beta, y) \cap S(\alpha, x)^c = \{(u, v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u, v) \mid v \precsim_\alpha u \succ_\alpha x\} \cup$ $\{(u, v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u, v) \mid v \succ_\alpha u\}$. Hence $\mu[S(\beta, y) \cap S(\alpha, x)^c] =$ $\mu(\{(u, v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u, v) \mid v \precsim_\alpha u \succ_\alpha x\} +$ $\mu(\{(u, v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u, v) \mid v \succ_\alpha u\})$. We shall show that

$$\underline{\lim}_{\{(\beta,y)\}} \{(u, v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u, v) \mid v \precsim_\alpha u \succ_\alpha x\} =$$

$$\overline{\lim}_{\{(\beta,y)\}} \{(u, v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u, v) \mid v \precsim_\alpha u \succ_\alpha x\} = \phi.$$

Thus assume $(\bar{u},\bar{v}) \in \underline{X} \times \underline{X}$. In the topology of closed convergence the set $T = \{(\gamma, z) \mid \bar{u} \succ_\gamma z\}$ is open [4 , p. 18]. If $(\beta,y) \in T$, then $\bar{u} \succ_\beta y$. Hence $(\bar{u},\bar{v}) \notin \{(u,v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u,v) \mid v \precsim_\alpha u \succ_\alpha x\}$, and therefore $(\bar{u},\bar{v}) \notin \overline{\lim}_{\{(\beta,y)\}} \{(u,v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u,v) \mid v \precsim_\alpha u \succ_\alpha x\}$. It follows from Fatou's Lemma [3, p. 113], that

$\lim_{\{(\beta,y)\}} \mu[\{(u,v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u,v) \mid v \precsim_\alpha u \succ_\alpha x\}] = 0.$ Similarly

$$\lim_{\{(\beta,y)\}} \mu[\{(u,v) \mid v \precsim_\beta u \precsim_\beta y\} \cap \{(u,v) \mid v \succ_\alpha u\}] = 0.$$

We shall now show that $\lim_{\{(\beta,y)\}} \mu(S(\alpha,x) \cap S(\beta,y)^c \cap (\mathcal{P}(\alpha)^c \times \underline{X})) = 0.$

To show this we shall begin by demonstrating that

$$\overline{\lim_{\{(\beta,y)\}}} (S(\alpha,x) \cap S(\beta,y)^c) \subseteq [\overline{\lim_{\{(\beta,y)\}}} S(\alpha,x) \cap S(\beta,y)^c)] \cap [I(\alpha,x) \times \underline{\overline{X}}] \cup$$

$[\overline{\lim_{\{(\beta,y)\}}} S(\alpha,x) \cap S(\beta,y)^c] \cap I(\alpha)$. Indeed, if $(\bar{u},\bar{v}) \in \{(u,v) \mid v >_\beta u\} \cap S(\alpha,x)$ infinitely often, then $\bar{u} \sim_\alpha \bar{v}$ and if $(\bar{u},\bar{v}) \in \{(u,v)\, v \precsim_\beta u >_\beta y\} \cap S(\alpha,x)$ infinitely often, then $u \sim_\alpha x$. It follows that

$$\overline{\lim_{\{(\beta,y)\}}} S(\alpha,x) \cap S(\beta,y)^c \cap (\mathcal{J}(\alpha)^c \times \underline{\overline{X}}) \subseteq [\overline{\lim_{\{(\beta,y)\}}} S(\alpha,x) \cap S(\beta,y)^c] \cap (I(\alpha,x) \times \underline{\overline{X}}) \cap$$

$(\mathcal{J}(\alpha)^c \times \underline{\overline{X}}) \cup [\overline{\lim_{\{(\beta,y)\}}} S(\alpha,x) \cap S(\beta,y)^c] \cap I(\alpha) \cap [\mathcal{J}(\alpha)^c \times \underline{\overline{X}}]$. Hence, by Fatou's Lemma, $\overline{\lim_{\{(\beta,y)\}}} \mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathcal{J}(\alpha)^c \times \underline{\overline{X}})] \leq$

$$\mu[(\overline{\lim_{\{(\beta,y)\}}} S(\alpha,x) \cap S(\beta,y)^c) \cap (I(\alpha,x)^\circ \times \underline{\overline{X}}) \cap (\mathcal{J}(\alpha)^c \times \underline{\overline{X}})] +$$

$$\mu[(\overline{\lim_{\{(\beta,y)\}}} S(\alpha,x) \cap S(\beta,y)^c) \cap I(\alpha)^\circ \cap (\mathcal{J}(\alpha)^c \times \underline{\overline{X}})]$$, because

$\mu(I(\alpha)) = \mu(I(\alpha)^\circ)$, and $\nu(I(\alpha,x)) = \nu(I(\alpha,x)^\circ)$. However $(I(\alpha,x)^\circ \times \underline{\overline{X}}) \cap (\mathcal{J}(\alpha)^c \times \underline{\overline{X}}) = \phi$ and $I(\alpha)^\circ \cap [\mathcal{J}(\alpha)^c \times \underline{\overline{X}}] = \phi$. Thus

$$\lim_{\{(\beta,y)\}} \mu[S(\alpha,x) \cap S(\beta,y)^c \cap (\mathcal{J}(\alpha)^c \times \underline{\overline{X}})] = 0.$$

Now to show that $\lim_{\{(\beta,y)\}} U(\beta,y) = U(\alpha,x)$ it will suffice to show that $\lim\limits_{\{(\beta,y)\}} |U(\beta,y) - U(\alpha,x)| = 0$. But $|U(\beta,y) - U(\alpha,x)| = |\mu[S(\alpha,x)\cap(\mathscr{I}(\alpha)^c \times \bar{\underline{X}})] - \mu[S(\beta,y) \cap (\mathscr{I}(\beta)^c \times \bar{\underline{X}})]| = |\mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\alpha)^c \times \bar{\underline{X}})] + \mu[S(\alpha,x) \cap S(\beta,y)^c \cap (\mathscr{I}(\alpha)^c \times \bar{\underline{X}})] - \mu[S(\beta,y) \cap S(\alpha,x)^c \cap (\mathscr{I}(\beta)^c \times \bar{\underline{X}})] - \mu[S(\beta,y) \cap S(\alpha,x) \cap (\mathscr{I}(\beta)^c \times \bar{\underline{X}})]| \leq |\mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\alpha)^c \times \bar{\underline{X}})] - \mu[S(\beta,y) \cap S(\alpha,x) \cap (\mathscr{I}(\beta)^c \times \bar{\underline{X}})]| + |\mu[S(\alpha,x) \cap S(\beta,y)^c \cap (\mathscr{I}(\alpha)^c \times \bar{\underline{X}})]| + |\mu[S(\beta,y) \cap S(\alpha,x)^c \cap (\mathscr{I}(\beta)^c \times \bar{\underline{X}})]|$. Applying the two previously established results to the second and third terms of this expression, it follows that $\lim\limits_{\{(\beta,y)\}} |U(\beta,y) - U(\alpha,x)| \leq$

$$\lim_{\{(\beta,y)\}} |\mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\alpha)^c \times \bar{\underline{X}})] - \mu[S(\beta,y) \cap S(\alpha,x) \cap (\mathscr{I}(\beta)^c \times \bar{\underline{X}})]|.$$

But $|\mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\alpha)^c \times \bar{\underline{X}})] - \mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\beta)^c \times \bar{\underline{X}})]| =$

$|\mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\beta) \cap \mathscr{I}(\alpha)^c) \times \bar{\underline{X}})] + \mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\beta)^c \cap \mathscr{I}(\alpha)^c) \times \bar{\underline{X}})]$

$- \mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\alpha) \cap \mathscr{I}(\beta)^c) \times \bar{\underline{X}})]$

$- \mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\alpha)^c \cap \mathscr{I}(\beta)^c) \times \bar{\underline{X}})]|$

$\leq |\mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\beta) \cap \mathscr{I}(\alpha)^c) \times \bar{\underline{X}})]| + |\mu[S(\alpha,x) \cap S(\beta,y) \cap (\mathscr{I}(\alpha) \cap \mathscr{I}(\beta)^c) \times \bar{\underline{X}})]|$.

But $\mu[(\mathscr{I}(\beta) \cap \mathscr{I}(\alpha)^c) \times \bar{\underline{X}}] = \nu(\mathscr{I}(\beta) \cap \mathscr{I}(\alpha)^c) \cdot \nu(\bar{\underline{X}})$ and $\mu[(\mathscr{I}(\alpha) \cap \mathscr{I}(\beta)^c) \times \bar{\underline{X}}] = \nu((\mathscr{I}(\alpha) \cap \mathscr{I}(\beta)^c) \times \bar{\underline{X}})$. Thus, it will suffice to show that $\lim\limits_{\{(\beta,y)\}} \nu(\mathscr{I}(\alpha) \cap \mathscr{I}(\beta)^c) = 0 = \lim\limits_{\{(\beta,y)\}} \nu(\mathscr{I}(\alpha)^c \cap \mathscr{I}(\beta))$. However $\nu(\mathscr{I}(\alpha) \cap \mathscr{I}(\beta)^c) = \bar{\nu}[I(\alpha)^\circ \cap (I(\beta)^\circ)^c \cap \Delta]$ and $\nu(\mathscr{I}(\alpha)^c \cap \mathscr{I}(\beta)) = \bar{\nu}[(I(\alpha)^\circ)^c \cap I(\beta)^\circ \cap \Delta]$. Thus

$$\nu(\mathcal{J}(\alpha) \cap \mathcal{J}(\beta)^c) + \nu(\mathcal{J}(\alpha)^c \cap \mathcal{J}(\beta)) = \bar{\nu}[I(\alpha)^\circ \cap \Delta \oplus I(\beta)^\circ \cap \Delta] = F(I(\beta), I(\alpha)),$$

and $\lim_{\beta \to \alpha} F(I(\beta), I(\alpha)) = F(I(\alpha), I(\alpha)) = 0$, since $(\alpha, x) \in \mathcal{E}(B) \times \bar{\underline{X}}$ and $\{\beta, y\} \subset \mathcal{E}(B) \times \bar{\underline{X}}$. This completes the proof.

Theorem 1 characterizes a class of agents on which U is jointly continuous, in terms of the indifference correspondence. This condition naturally raises the question, For which agents is it satisfied? We study this question in the next section.

Section 3.

One would naturally be interested in identifying the largest subclass of agents on which the function U is a continuous utility indicator. Unfortunately when $\bar{X}$ is the closure of its interior in $\mathbb{R}^\ell$ and ν is the measure given by the Normal distribution on $\mathbb{R}^\ell$, there is no largest class under the partial ordering by inclusion. To see this we note first that the set of agents satisfying the assumptions in Neuefeind [11] is an admissible class in the sense of Definition 2. Briefly, the indifference classes have Lebesgue measure zero, and hence satisfy $\nu(I(\alpha,x) - \overset{\circ}{I(\alpha,x)}) = 0$. It follows from Fubini's Theorem that $\mu(I(\alpha)) = 0 = \mu(\overset{\circ}{I(\alpha)})$. The function F satisfies condition 3 of Definition 2, since $\overset{\circ}{I(\alpha)} = \phi$ for all α in this class. To verify the Cantor condition, if α is an agent with $\mathcal{J}(\alpha) = \phi$, it follows that $\mathcal{J}(\alpha)^c = \bar{X}$. Thus, for any open set $B \subset \bar{X}$, $B \times B \not\subset I(\alpha)$, and $\mu(\Gamma(\alpha)^c \cap (\mathcal{J}(\alpha)^c \times \bar{X}) \cap (B \times \bar{X})) = \mu[\Gamma(\alpha)^c \cap (B \times \bar{X})]$

$= \mu[\{u,v) \mid v >_\alpha u \text{ and } u \in B\}] \neq 0$, because $\{(u,v) \mid v >_\alpha u\}$ is open, and hence is a set of positive measure.[5/]

If there were a largest class of agents on which U is continuous, that class must include the agents satisfying Neuefeind's assumptions. It must also include the class consisting of the single agent who is indifferent between any two elements of $\bar{X}$. We call this agent $\bar{\alpha}$, the "flat" agent. Therefore the largest class must include the union of these two sets. But U is not continuous on that class, as the following example shows.

5/ The assumption that μ assigns positive measure to open sets is used here.

Example 1. Let $\overline{X} = [0,1]$.[6/] Define a sequence of preferences on $\overline{X}$ as follows. Subdivide $\overline{X}$ into an even number of subintervals of equal length, using the subdivisions

$$\delta_n = \{ \frac{m}{2^n} \text{ when } 0 \leq m \leq 2^n \}, \; n = 1,2,\ldots \quad .$$

Let the $n^{\underline{th}}$ preference α_n be monotone decreasing on intervals $[\frac{m-1}{2n}, \frac{m}{2n}]$ for m odd, and strictly increasing on those intervals with m even (see Figure 1.a.). Then, the point $\frac{1}{2}$ is a point of δ_n for every n, and is a maximal point of [0,1] for α_n, for all n. Furthermore, since $\mathcal{P}(\alpha_n)^{\circ} = \phi$ for all n,

$$U(\alpha_n,\tfrac{1}{2}) = \mu(S(\alpha_n,\tfrac{1}{2}) \cap ((\mathcal{P}(\alpha)^{\circ})^c \times \overline{X}) = \mu(S(\alpha_n,\tfrac{1}{2})) = \mu(\overline{X} \times \overline{X}),$$ a

constant not equal to zero.

However, α_n converges to $\overline{\alpha}$, the flat agent on [0,1]. Figure 1.b shows the graph of α_3. As $n \to \infty$ the number of shaded diamonds becomes large, while the size of each becomes small. The distance between these graphs and the square $\overline{X} \times \overline{X}$ goes to zero, while the measure of the shaded diamonds stays at $\frac{1}{2}$ the total measure of $\overline{X} \times \overline{X}$. Since $(\mathcal{P}(\overline{\alpha})^{\circ})^c = \phi$, it follows that

$$U(\overline{\alpha},\tfrac{1}{2}) = \mu(S(\alpha,\tfrac{1}{2}) \cap ((\mathcal{P}(\overline{\alpha})^{\circ})^c \times \overline{X})) = 0.$$

6/ We take $\overline{X}$ to be compact in this example just to simplify drawing pictures.

The question arises, whether there are agents whose indifference relations form an admissible class, and, if so, how can they be characterized?

Definition 4:[7/] An equivalence relation α is said to have an ε - threshold if there exists a real number $\varepsilon(\alpha) > 0$ such that if $(u,v) \in \overset{\circ}{I(\alpha)}$, then there exists a point $(u',v') \in \overset{\circ}{I(\alpha)}$ such that

(1) $d[(u,v), (u',v')] < \frac{\varepsilon(\alpha)}{2}$, and

(2) the open ball of radius $\frac{\varepsilon(\alpha)}{2}$ with center (u',v'), is contained in $\overset{\circ}{I(\alpha)}$:

The number $\varepsilon(\alpha)$ will be called a threshold for $I(\alpha)$. An agent $a \in \mathcal{A}$ has a threshold if $I(\alpha)$ has a threshold.

Lemma 5: Suppose α is an agent with a threshold $\varepsilon(\alpha)$. If $d(I(\alpha),I(\beta)) < \varepsilon/2$, where $0 < \varepsilon < \varepsilon(\alpha)$, then $d(\overset{\circ}{I(\alpha)},\overset{\circ}{I(\alpha)} \cap I(\beta)) < \varepsilon$.

Proof: Suppose $(u,v) \in \overset{\circ}{I(\alpha)}$. By assumption there exists a point $(u',v') \in \overset{\circ}{I(\alpha)}$ such that $d((u,v),(u',v')) < \varepsilon(\alpha)/2$ and such that the ball of radius $\varepsilon(\alpha)/2$ around (u',v') is contained in $\overset{\circ}{I(\alpha)}$. Because $d(I(\alpha),I(\beta)) < \varepsilon/2$, there is a point $(u'',v'') \in I(\beta)$ such that $d(u'',v''),(u',v')) < \varepsilon/2$. But then $(u'',v'') \in \overset{\circ}{I(\alpha)}$ and $d((u,v),(u'',v'')) < d(u,v),(u',v')) + d((u',v'),(u'',v'')) < \varepsilon$.

7/ We are indebted to R. Aumann and K. J. Arrow for comments which suggested this formulation of the concept of ε-threshold.

Definition 5: (i) An agent $\beta \in \mathcal{S}$ is said to satisfy Condition 3 (the outer condition) with respect to a class $\mathcal{A}$ of equivalence relations on $\bar{X}$ if for each equivalence relation $\alpha \in \mathcal{A}$ which has a threshold there is a constant $K_1(\alpha) > 0$, independent of β, such that

$$|\bar{\nu}(I(\alpha)^{\circ} \cap \Delta) - \bar{\nu}(I(\alpha)^{\circ} \cap I(\beta)^{\circ} \cap \Delta)| \leq K_1(\alpha)d(I(\alpha)^{\circ}, I(\alpha)^{\circ} \cap I(\beta)).$$

(ii) An agent $\beta \in \mathcal{S}$ is said to satisfy Condition 4 (the inner condition) with respect to a class $\mathcal{A}$ of equivalence relations on $\bar{X}$, if for each equivalence relation $\alpha \in \mathcal{A}$ which has a threshold there is a constant $K_2(\alpha) \geq \nu(\bar{X})$, independent of β, such that

$$|\bar{\nu}(I(\alpha)^{\circ} \cap \Delta) - \bar{\nu}(I(\alpha)^{\circ} \cup I(\beta)^{\circ} \cap \Delta)| \leqq K_2(\alpha) \cdot d(I(\alpha), I(\alpha) \cup I(\beta))$$

Definition 6: Let $\mathcal{A}^*$ denote the class of equivalence relations of agents in $\mathcal{S}$ which have thresholds. Let $\mathcal{S}^*$ denote the class of agents in $\mathcal{S}$ which (a) have thresholds (b) satisfy the inner and outer conditions with respect to the class $\mathcal{A}^*$ for some functions K_1 and K_2 on $\mathcal{A}^*$ (i.e. satisfy Conditions 3 and 4 with respect to $\mathcal{A}^*$ for $K_1(\cdot)$ and $K_2(\cdot)$, (c) are such that $\mu(I(\alpha)) = \mu(I(\alpha)^{\circ})$ and $\nu(I(\alpha,s)) - \nu(I(\alpha,x)^{\circ}) = 0$ (i.e., Conditions 1 and 2 of Definition 2)and (d) also satisfy the Cantor Condition.

Condition 4 requires that if β is an agent whose indifference correspondence approximates that of α in distance "from the inside", then the measure of the interior of $I(\beta)$ on the diagonal is at least as large as the interior of $I(\alpha)$ on the diagonal. This inner condition serves to rule out the convergence in distance to the flat agent of a sequence of agents whose indifferences have empty interiors (thin agents), while the measures do not converge. Example 1 shows such a case. Condition 3 requires that if β is an agent whose indifference correspondence approximates that of α "from the outside" in distance, then the measure on the diagonal of the interior of $I(\beta)$ is at most that of the interior of $I(\alpha)$. The outer condition serves to rule out convergence to a thin agent by agents with non-empty indifference interiors whose measure does not go to zero. Example 2 shows such a case.

Example 2. Let $\underline{X} = [0,1]$. Let the $n^{\underline{th}}$ subdivision of $[0,1]$ consist of the points $[0, \frac{1}{2n}, \ldots, \frac{m}{2n}, \ldots, \frac{2n}{2n}]$. Let α_n be monotone on the intervals $[\frac{m-1}{2n}, \frac{m}{2n}]$ for m odd and let α_n be flat on the intervals $[\frac{m-1}{2n}, \frac{m}{2n}]$ for m even, such that $x \in (\frac{m-1}{2n}, \frac{m}{2n})$, m even implies $x \underset{\alpha_n}{\sim} \frac{m-1}{2n}$. The graph of α_n is shown in Figure 2.a, for $n = 4$ and Figure 2.b shows a utility function for α_4. The (Lebesgue) measure λ of $(\overset{\circ}{I(\alpha_n)} \cap \Delta)$ is $n \cdot \frac{1}{2n} = \frac{1}{2}$ and hence the $\lim_{n\to\infty} \lambda(\overset{\circ}{I(\alpha_n)} \cap \Delta = \frac{1}{2}$. However, α_n converges to the monotone agent, the measure of whose indifference correspondence is 0. Condition 3 serves to exclude this type of case.

Theorem 2: The utility function U is continuous on $\mathcal{E}^* \times \overline{X}$.

Proof: We need only show that the class B^* of indifference correspondences of agents in $\mathcal{E}^*(K_1(\cdot),K_2(\cdot))$ is admissible. Thus suppose $\beta_i \in B^*$, $\alpha \in B^*$ such that $\lim_i \beta_i = \alpha$. The inner condition is satisfied for each $\beta_j \in B^*$ by definition of $\mathcal{E}^*$. Note that

$$|\bar{\nu}(I(\alpha)^\circ \cap \Delta \oplus I(\beta)^\circ \cap \Delta)| = |\bar{\nu}(I(\alpha)^\circ \cup I(\beta)^\circ)\cap \Delta) - \bar{\nu}(I(\alpha)^\circ \cap I(\beta)^\circ \cap \Delta)| =$$

$$|\bar{\nu}((I(\alpha)^\circ \cup I(\beta)^\circ) \cap \Delta) - \bar{\nu}(I(\alpha)^\circ \cap \Delta) + \bar{\nu}(I(\alpha)^\circ \cap \Delta) - \bar{\nu}(I(\alpha)^\circ \cap I(\beta)^\circ \cap \Delta)| \leqq$$

$$|\bar{\nu}(I(\alpha)^\circ \cup I(\beta)^\circ)\cap \Delta) - \bar{\nu}(I(\alpha)^\circ \cap \Delta)| + |\bar{\nu}(I(\alpha)^\circ \cap \Delta) - \bar{\nu}(I(\alpha)^\circ \cap I(\beta)^\circ \cap \Delta)|$$

$$\leq K_2(\alpha)\cdot d(I(\alpha)\ ,I(\alpha) \cup I(\beta)\ ) + K_1(\alpha)\cdot d(I(\alpha)^\circ, I(\alpha)^\circ \cap I(\beta).$$

By Lemma 5 , $\beta \to \alpha$ implies $d(I(\alpha)^\circ, I(\alpha)^\circ \cap I(\beta)) \to 0$. It follows from Lemma 3 that when $\beta \to \alpha$, $I(\beta) \to I(\alpha)$. Since $I(\alpha) \subset I(\alpha) \cup I(\beta)$, it suffices to show that given $\epsilon > 0$, if $\dot{z} \in I(\beta)$ then there is an element $w \in I(\alpha)$ within ϵ of z. But if the distance between β and α is ϵ, then there must be such an element w.

We have thus shown that $\lim_{\beta\to\alpha} F(I(\beta),I(\alpha)) = 0$, which completes the proof.

Lemma 7: The flat agent $\bar{\alpha}$, is a member of $\mathcal{E}^*$. I.e., $I(\bar{\alpha}) = \underline{\overline{X}} \times \underline{\overline{X}}$ implies $\bar{\alpha} \in \mathcal{E}^*$.

Proof: Clearly $\bar{\alpha}$ has a threshold. Thus, we must show that $\bar{\alpha}$ satisfies the inner and outer conditions. If $I(\alpha)$ has a threshold, then we must show there is a constant $K_1(\alpha)$ such that

$$|\bar{\nu}(\overset{\circ}{I(\alpha)} \cap \Delta) - \bar{\nu}(\overset{\circ}{I(\alpha)} \cap \bar{\underline{X}} \times \bar{\underline{X}} \cap \Delta)| \leq K_1(\alpha)\, d(\overset{\circ}{I(\alpha)}, \overset{\circ}{I(\alpha)} \cap I(\bar{\alpha})).$$

But this condition is trivially satisfied for any $K_1(\alpha) \geq 0$ because

$$(\overset{\circ}{I(\alpha)} \cap \bar{\underline{X}} \times \bar{\underline{X}} \cap \Delta) = (\overset{\circ}{I(\alpha)} \cap \Delta).$$

The inner condition is also satisfied by $\bar{\alpha}$, for $K_2(\alpha) = \dfrac{\nu(\bar{\underline{X}})}{d(I(\alpha), \bar{\underline{X}} \times \bar{\underline{X}})} > 0$,

since $|\bar{\nu}(\overset{\circ}{I(\alpha)} \cap \Delta) - \bar{\nu}((\overset{\circ}{I(\alpha)} \cup \bar{\underline{X}} \times \bar{\underline{X}}) \cap \Delta)| = |\bar{\nu}((\overset{\circ}{I(\alpha)})^c \cap \Delta)|$

$$\leq \nu(\bar{\underline{X}}) = \frac{\nu(\bar{\underline{X}})}{d(I(\alpha), \bar{\underline{X}} \times \bar{\underline{X}})} \cdot d(I(\alpha), \bar{\underline{X}} \times \bar{\underline{X}}) = K_2(\alpha) \cdot d(I(\alpha), I(\alpha) \cup \overset{\circ}{I(\bar{\alpha})}).$$

If $d(I(\alpha), \bar{\underline{X}} \times \bar{\underline{X}}) = 0$, then $\alpha = \bar{\alpha}$ and the inner condition is trivially satisfied. It is clear that Conditions 1, 2 and the Cantor condition are all satisfied by $\bar{\alpha}$.

Lemma 8: The (strictly) monotone agents are members of $\mathcal{E}^*$. I.e., we may choose functions $K_1(\cdot), K_2(\cdot)$ on the set of agents with thresholds such that the monotone agents are in $\mathcal{E}^*((K_1(\cdot), K_2(\cdot))$.

Proof: Let β be a monotone agent and let α be an agent with a threshold. The inner condition (Condition 4) is satisfied trivially, because $(\overset{\circ}{I(\alpha)} \cup \overset{\circ}{I(\beta)}) \cap \Delta = (\overset{\circ}{I(\alpha)} \cap \Delta)$.

To verify the outer condition, we consider two cases . If $\overset{\circ}{I(\alpha)} = \emptyset$, then $\bar{\nu}(\overset{\circ}{I(\alpha)} \cap \Delta) = 0 = \bar{\nu}(\overset{\circ}{I(\alpha)} \cap \overset{\circ}{I(\beta)} \cap \Delta)$, and hence the outer condition is satisfied in this case.

Suppose $\overset{\circ}{I(\alpha)} \neq \emptyset$. Then there exists a ball B of radius $\varepsilon > 0$ such that $\bar{B} \times \bar{B} \subset \overset{\circ}{I(\alpha)}$. Let D denote the ray from the origin through the point $(1,1,\ldots 1)$ in $\bar{\underline{X}} \times \bar{\underline{X}}$, and let p and p' denote the points of intersection of D with the boundary of $\bar{B} \times \bar{B}$. Then $p, p' \in \overset{\circ}{I(\alpha)}$, and the closest points to p,p' in $I(\beta) \cap \bar{B} \times \bar{B}$ must be q,q', where q and q' are elements of the set

$$\{\{v \in \bar{\underline{X}} \mid v = \frac{p+p'}{2} + z,\ z \gg 0\} \cup \{v \in \bar{\underline{X}} \mid v = \frac{p+p'}{2} - z,\ z \gg 0\}\}^c$$

It follows that the distance between (p,p') and (q,q') is bounded below by a positive function of ε. E.g., in $\mathbb{R}^2$ that distance would be at least $\frac{\varepsilon}{\sqrt{2}}$, using the Euclidean distance for $\mathbb{R}^2$. It is also clear that Conditions 1, 2 and the Cantor Condition are satisfied by a monotone agent.

Section 4.

The utility function given in Definition 1 is, in a sense, constructed on the basis of the structure of the commodity space and data from the individual preference relations. So are the utility functions of Kannai [6] Hildenbrand [4] and Neuefeind [11]. However, none of these functions is a continuous indicator for the full class of continuous preference relations on $\mathbb{R}_+^\ell$ (or $\mathbb{R}^\ell$). Mas-Colell's result [7] tells us that a continuous indicator exists for the class of continuous preferences on $\mathbb{R}^\ell$ but we do **not** as yet know how to construct one. Indeed, it is a question whether such a function can be "constructed". To give precise meaning to that statement we must first say what we mean by "constructed", and that seems to be difficult to do.

Informally we have in mind procedures with the following characteristics. First, the basic input to the procedure should be data from each individual preference relation. We regard such data as an idealized form of the results of a choice experiment. Secondly, a measurement is performed on those data, a measurement which is defined independently of preferences, using only properties of the commodity space. A procedure of this kind would involve only properties of the measurement of commodities and the individual preferences, and would not, for example, depend on any relationship among different preference relations. One class of such procedures is the following.

We suppose that data from a preference relation takes the form of a subset of a space related to the commodity space. E.g., the lower contour set in the consumption set of a point x can represent the information given by the preference relation at the point x. This set is then evaluated by a measurement which relies only on properties of the commodity space.

More formally, let $\mathbb{R}^k$ be k-dimensional euclidean space, $1 \leq k$; let $\mathcal{B}$ denote the family of Borel sets of $\mathbb{R}^k$. We say that a pair (s,u) is a <u>measurement process</u> (on a family of agents $\mathcal{G}$ with common consumption set $\underline{\bar{X}} \subset \mathbb{R}^\ell$), if

$$s: \mathcal{G} \times \underline{\bar{X}} \to \mathcal{B}$$

is a function [8/], and

$$\mu: \mathcal{B} \to \mathbb{R},$$

is a real valued function which is (i) countably additive and (ii) gives positive measure to open sets. The requirement that the function μ give positive measure to open sets reflects the quantification of commodities and expresses a requirement that this measurement of commodities carry over in a sense to $\mathbb{R}^k$. The requirements that μ be countably additive is an idealization to countable

8/ It would be natural to require of s in addition that if $\alpha \in \mathcal{G}$ and $x \in \underline{\bar{X}}$, $y \in \underline{\bar{X}}$ such that $x \underset{\alpha}{<} y$, then $s(\alpha,x) \subset s(\alpha,y)$.

collections of the property of finite additivity that processes of measurement commonly have.

We say that a utility function $u: \mathcal{E} \times \bar{\underline{X}} \to \mathbb{R}$, for the class $\mathcal{E}$, is <u>constructible by a measurement process</u> if there exists a measurement process (s,μ) on $\mathcal{E} \times \bar{\underline{X}}$, such that

$$u = \mu \circ s$$

A function constructed by measurement process has the property that if two different agents α and β should happen to have the same set for perhaps different commodity points, the function must give those points the same utility. Thus, if

$$s(\alpha,x) = S(\beta,y),$$

then

$$\mu\ (S(\alpha,x)) = \mu\ (S(\beta,y)).$$

This property, which can be stated more generally without reference to measurement processes, is reminiscent of "independence of irrelevant alternatives", and might be made a part of an axiomatic approach to constructibility. It is clear that functions based on the Michael's selection theorem need not have this property.

Several examples of utility functions constructible by a measurement process are available.

Example 1. Let the process be defined by taking $k = 1$, letting

$$\mathbb{R}^k_+ \equiv \Delta_+ = \{y \in \mathbb{R}^\ell \mid y = a(1,\ldots,1),\ a \text{ real}\},$$

and defining s by

$$s(\alpha,x) = \{y \in \mathbb{R}^\ell_+ \mid y \underset{\alpha}{\lesssim} x\} \cap \Delta_+ \quad \text{for} \quad (\alpha,x) \in \mathcal{S} \times \bar{X}.$$

Since α is a continuous agent, $s(\alpha,x)$ is closed in Δ_+ and hence is a Borel set. Taking μ to be (1-dimensional) Legesgue measure on $\mathbb{R}$, $\mu(s(\alpha,x))$ is just the length of the segment of the diagonal between the origin and the point α-indifferent to x. Let $\mathcal{S}_m \subset \mathcal{S}$ be the class of strictly monotone continuous agents on $\bar{X} = \mathbb{R}^\ell_+$. Then the function $u = \mu \circ s$ is well-defined when $\alpha \in \mathcal{S}_m$, and is the utility constructed by Hildenbrand for the class of monotone agents.

Example 2. Neuefeind's construction is obtained by taking $\mathbb{R}^k = \mathbb{R}^\ell$, and letting $s(\alpha,x)$ be the lower contour set (in the commodity space $\mathbb{R}^\ell$) of the point x preference relation α. The function μ is the probability of $s(\alpha,x)$ according to the Normal distribution on $\mathbb{R}^\ell$. Neuefeind's procedure constructs a utility indicator by a measurement process for the class of continuous preferences whose indifference sets have Lebesgue measure 0. (He also assumes that the consumption set is connected and is the closure of its interior in $\mathbb{R}^\ell$.)

Example 3. It is easy to verify that the utility function given in Definition 1, section 2 is constructed by a measurement process. Take $k = 2\ell$, and let $s: \mathcal{S} \times \bar{X} \to \mathcal{B}$ be given by $s(\alpha,x) \equiv S(\alpha,x) \cap (\mathcal{P}^c(\alpha) \times \bar{X})$, which is a Borel set of $\underline{\bar{X}} \times \underline{\bar{X}}$ when α is a continuous preference relation, since $S(\alpha,x)$ and $\mathcal{P}^c(\alpha) \times \underline{\bar{X}}$ are both closed subsets of $\underline{\bar{X}} \times \underline{\bar{X}}$. Finally, μ is a measure on $\mathbb{R}^{2\ell}$ which gives finite measure to $\underline{\bar{X}} \times \underline{\bar{X}}$, is countably additive and gives positive measure to open sets, since it is the product measure $\nu \times \nu$ and ν has these properties.

It is possible to think of other bases for construction. For example, one might require only that the function s be defined on closed sets and be subadditive. Then functions based on distance in the commodity space can be used to assign to each subset the distance from its closure to a chosen point. The method which Arrow-Hahn [1] use to construct an indicator for one given preference relation satisfying local non-satiation, and its extension in [10] to the class of continuous agents satisfying local non-satiation, is of this kind. However, the distance associated with a set seems too insensitive to the structure of the set far from the point at which the distance is assumed to yield a continuous indicator, except in restricted situations.

The utility construction based on distance, like the one based on measure, defines a real valued set function. However, the function based on distance is generally not additive. If such a function, denoted γ, is so defined as to be subadditive in the sense that, if A and B are sets such that $A \cap B = \emptyset$, then

$$\gamma(A \cup B) \leqq \gamma(A) + \gamma(B),$$

and if the empty set is assigned the value 0, then a definition of constructibility in terms of outer measure suggests itself. Whether this would lead to anything useful remains to be explored.

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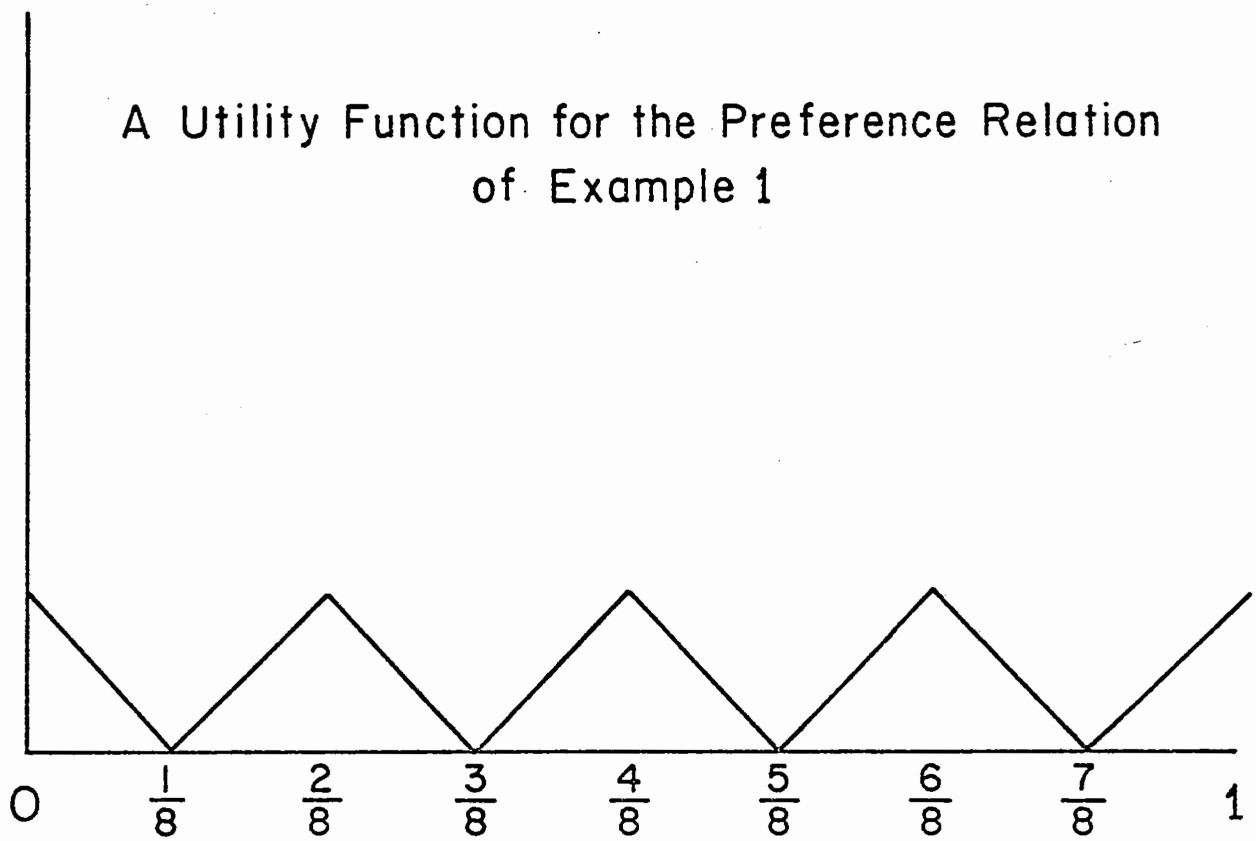
A Utility Function for the Preference Relation of Example 1
0
1/8
2/8
3/8
4/8
5/8
6/8
7/8
1

Figure 1a

Graph of the Preference Relation α_4.

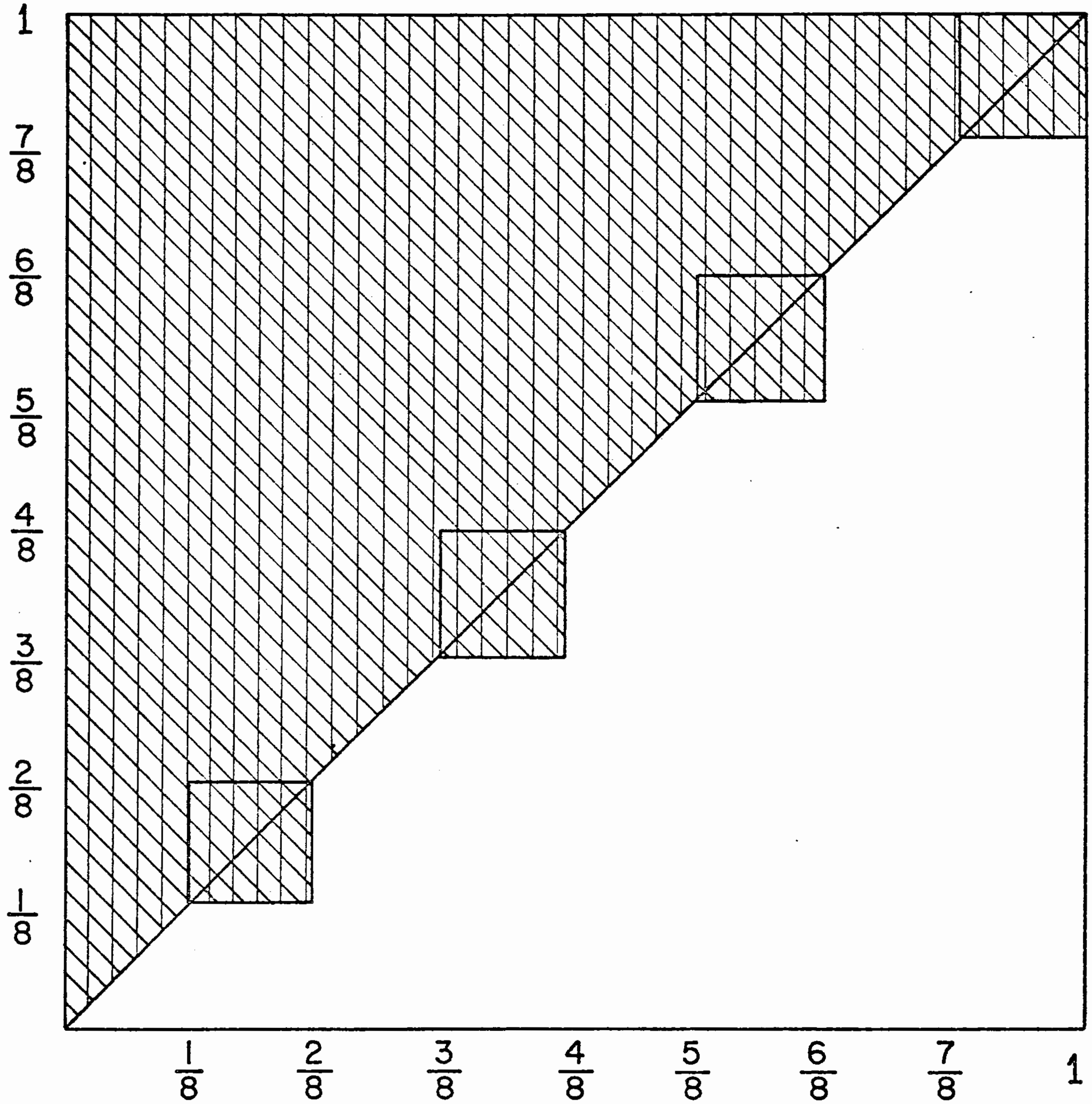

Figure 2a

Graph of the Preference Relation of Example 1, for n = 3.

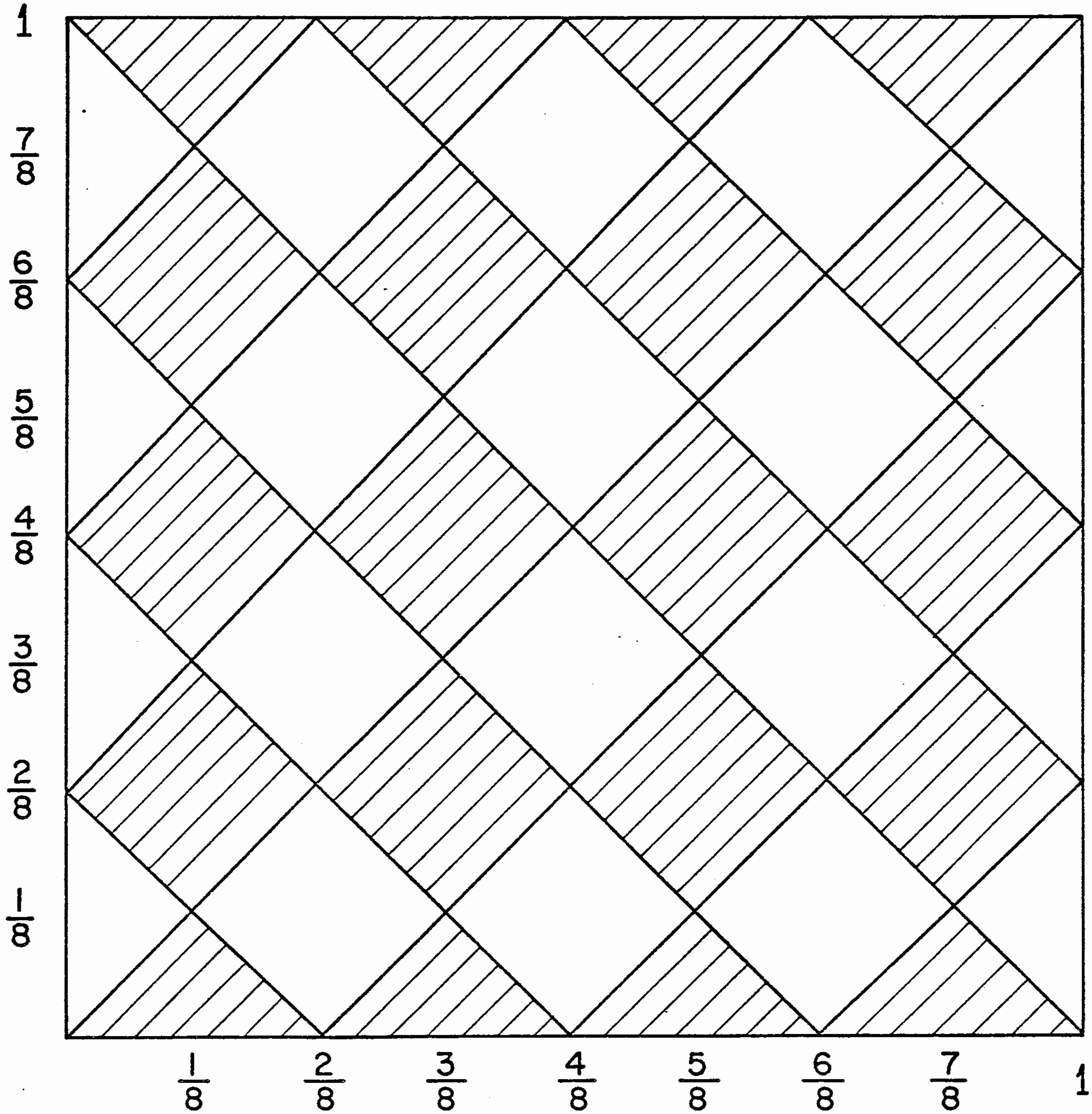

Figure 1 b

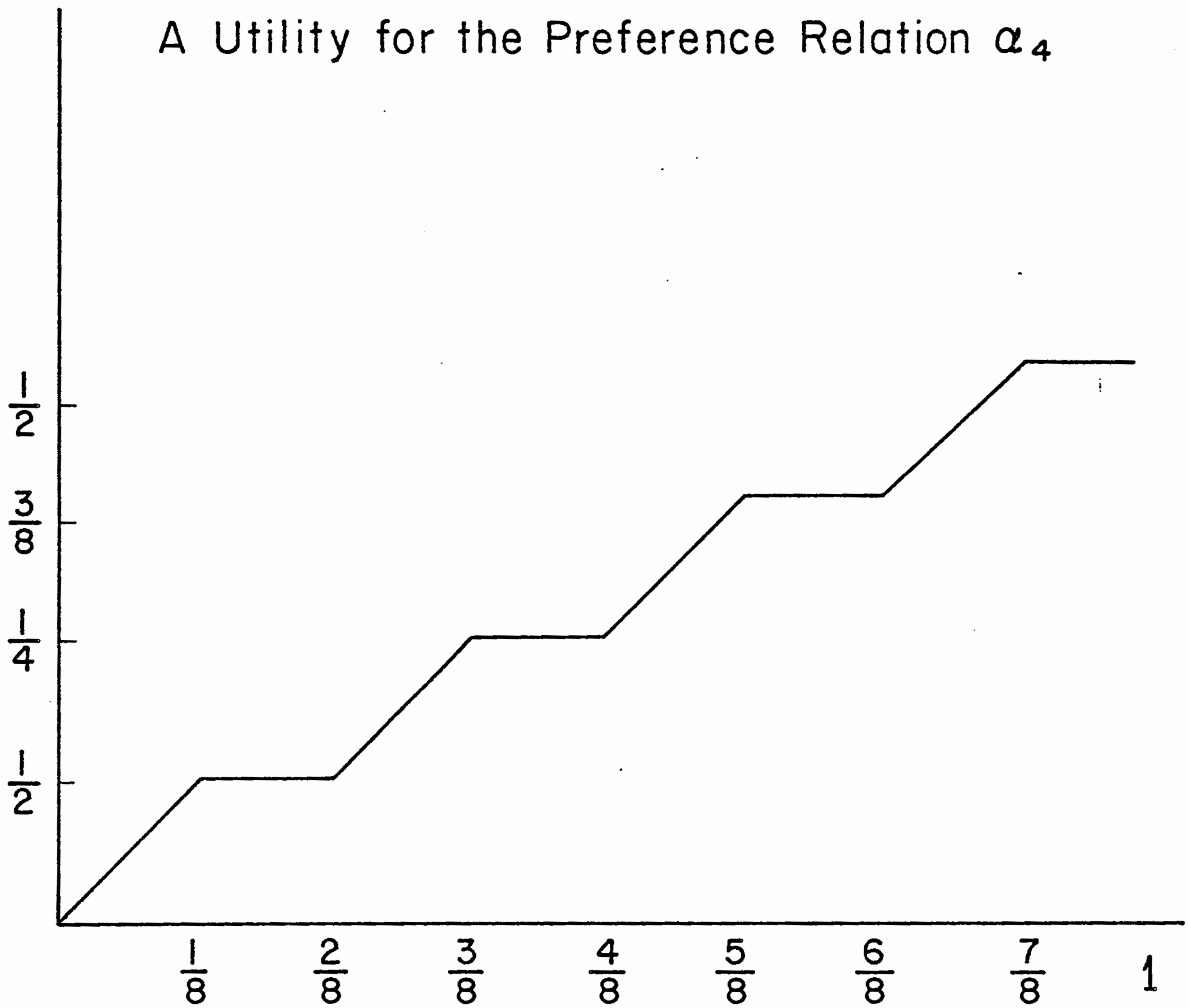
A Utility for the Preference Relation α_4
1/2
3/8
1/4
1/2
1/8
2/8
3/8
4/8
5/8
6/8
7/8
1

Figure 2b