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# ON THE FOUNDATIONS OF THE THEORY OF MONOPOLISTIC COMPETITION *

by

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## INTRODUCTION

This paper is concerned with the foundations of the theory of monopolistic competition. Since the pioneering work of Negishi [8], a number of studies have been directed towards incorporating firms which recognize their ability to influence prices (but behave non-cooperatively towards one another) into the Arrow-Debreu model of general equilibrium. Of particular interest are the studies by Arrow and Hahn [1], Fitzroy [3], Gabszewicz and Vial [4], Laffont and Laroque [5] and Marschak and Selten [7]. Each of these presents a theorem establishing existence of market-clearing prices when some agents behave competitively while others maximize profits given the demand relations in the competitive sector and the choices made by the other imperfect compeitors.

Despite these important contributions, the problem of existence of such mixed, Cournot-Chamberlin-Walras equilibrium is not yet adequately resolved, since each of the above mentioned theorems employs assumptions made directly on the constructs to be used in the proofs, and the properties thus assumed are not derived from hypotheses on the fundamental data of preferences, endowments and technology. This is, of course, in sharp contrast with the theorems for the purely competitive case, in which, for example, all the properties of the excess-demand correspondence used in the proofs are derived from conditions on the individual agents' characteristics.

One such ad hoc assumption plays a crucial role in all these studies: that the optimal choices by each firm should define a convex-valued correspondence. The particular form taken by this assumption differs between the studies cited, but in each it is used to permit immediate application of the Kakutani fixed-point theorem. In Arrow and Hahn the assumption appears directly as their A.6.16 [1,p.158], which requires that the quantities chosen by each monopolistic firm should define a continuous function. In Fitzroy's study the crucial assumption is his A.8 [3,p.12], which specifies that the firms can be ordered so that, given the optimal choices of the firms preceding it in the ordering, each firm has a unique optimal output. Gabszewicz and Vial, in their conditions A.1 and A.2 [4,pp.384 and 388], assume that there is a unique market-clearing price vector for any choice of outputs by the monopolistic firms, and that these prices depend on each firm's outputs in such a way that the individual profit functions are strictly quasi-concave. The corresponding assumptions in the Laffont and Laroque study are their Hypotheses 10 and 11 [5,p. ]. These require the existence of inverse demand functions which yield strictly concave profit functions for each imperfectly competitive firm. In the Marschak and Selten study, in which prices are the choice variables, their assumption III-3 [7,p.49] requires that each firm's profit function should be single-peaked in the prices it controls. In Negishi's work [8],[9], the firms' perceived inverse demand functions are taken as linear, so that profit is a quadratic function of output.

Arrow and Hahn recognized that employing such assumptions is less than completely satisfactory.[1] They suggest that an "open and potentially important research area is the specification of conditions under which monopolistic behavior ... is, in fact, continuous" [1,p.166]. This paper is addressed to opening this investigation. Our results are very negative in tone and serve to indicate a fundamental weakness in the foundations of the theory of monopolistic competitions.

Specifically, we present two examples, neither of which can be considered pathological, of economies in which the convex-valuedness assumption made in the literature fails and, as a result, no equilibrium exists. In both examples there are three commodities and two firms, each of which costlessly produces one commodity. In the first example there is only one consumer, who behaves competitively, while the firms act as non-cooperative quantity-setters, with each taking the other's output as given in determining its own output level. The second example involves four consumers with homogeneous utility functions. Here the firms act non-cooperatively as price-setters, each taking the other's price as given in determining the price it will charge. In each example the reaction curve of one firm, which gives the profit-maximizing values of its decision variable (quantity or price) as a function of the other's decision, is not convex-valued and no equilibrium exists. This happens even though the preferences of consumers and the technologies of firms satisfy every one of the standard assumptions of general equilibrium analysis. Moreoever, the lack of equilibrium which the examples illustrate is essential in the sense that equilibrium cannot be restored by making small perturbations in the economies.

Before presenting the examples, it may be useful to examine the reaction curves that can arise in the context of the familiar, partial equilibrium model of Cournot duopoly. Suppose then that there are two firms costlessly producing a single homogeneous good. The inverse demand for this good is given by $p = f(x_1+x_2)$, where $x_i$ is the output of firm i. Taking the other firm's output $x_j$ as given, firm i seeks to maximize its profit $px_i = f(x_i+x_j)x_i$ by its choice of $x_i$. This procedure is illustrated in Figure 1, where the rectangular hyperbola is an iso-profit curve and the other line is the residual demand curve facing firm i when it takes $x_j$ as given. As indicated, the maximizing values of $x_i$ corresponding to a given level of $x_j$ may well form a non-convex set if the inverse demand function is sufficiently non-concave. Thus, even in this simple case, the assumption of a convex-valued reaction curve may fail. Note that by reinterpreting the $x_i$ as prices and p as quantities, so that f is the (common) demand function for a pair of commodities consumed in fixed proportions, Figure 1 also applies to models in which prices are the choice variables of the firms.[2]

In an earlier paper [10], we showed that the special structure of this version of Cournot's model, including especially its symmetry, permits the demonstration of the existence of equilibrium even though the reaction curves are not convex-valued.[3] In the examples we will present here, however, no such demonstration is possible, since equilibrium does not exist.

Before beginning the formal analysis it may be useful to say something more about the approach we have adopted. In a fundamental and very real sense, the study of competitive equilibrium is the study of (systems of) demand functions: the theorems on competitive markets (existence of equilibrium, comparative statics, etc.) rest on the properties of demand and supply functions. In a parallel fashion, the theory of monopolistic or imperfect competition has at its foundation the theory of the reaction function, and theorems on monopolistically competitive equilibrium should rest on the properties of reaction functions. In the competitive case, the theory of the demand function and its properties is well-established. Such is not yet the case with the reaction function: the properties of reaction curves used in the existing theories of imperfectly competitive equilibrium have not been derived from the technological conditions and the behavior these theories claim to address. Thus, to provide a proper foundation for the theory of imperfect competition one must answer the question of "what functions can be reaction functions." In this context, what we will be presenting in the following pages is a contribution to the as-yet-incomplete theory of the reaction function, which shows that the assumptions that have been made on these functions are unwarranted. More specifically, we show that there are no nice general conditions on preferences and technologies which will generate a class of reaction functions for which non-cooperative, Cournot-Chamberlin-Walras equilibrium will always exist.

## The Model with Quantity-Setting Firms

The model we present here is the simplest possible extension of the Cournot model within a general equilibrium framework. It consists of two monopolists, each producing a single good costlessly, and a single consumer, described in the usual way by his preferences, endowment and ownership of the firms. The consumer's preferences are strictly increasing in each commodity and are described by smooth indifference surfaces of the usual shape. Thus, his marginal rates of substitution define a unique vector $(p_1,p_2,1)$ of normalized prices associated with each point $(x_1,x_2,x_3)$ in the non-empty interior of his consumption set. Firm i, i=1,2, holds one unit of good i (or can produce up to one unit at no cost), while the consumer is endowed with one unit of the third good. The consumer is assumed to act competitively, maximizing his utility subject to the budget constraint he faces, taking prices and profits as given. He uses his initial holding of good three to purchase goods one and two, but since he "owns" both firms and receives all the profits they generate, his expenditures on the first two goods are always returned to him as profit. Thus, given prices $(p_1,p_2,1)$, his budget constraint is $p_1x_1 + p_2x_2 + x_3 = p_1\bar{x}_1 + p_2\bar{x}_2 + 1$, where $\bar{x}_i$ is the production of firm i. Further, if the prices $p_i$ are those corresponding to the marginal rates of substitution at $(\bar{x}_1,\bar{x}_2,1)$, then the consumer's unique utility maximizing choice is $x_1 = \bar{x}_1$, $x_2 = \bar{x}_2$, $x_3 = 1$.

In this model, with a single consumer whose expenditures are returned to him as profits, the utility function directly serves the role of an inverse demand function: the marginal rates of substitution at any point $(x_1,x_2,1)$ in the given range give the normalize prices $(p_1(x_1,x_2), p_2(x_1,x_2),1)$ at which these quantities will be purchased. Thus we calculate the inverse demand at $(\bar{x}_1,\bar{x}_2,1)$ by $p_1(\bar{x}_1,\bar{x}_2) = U_1(\bar{x}_1,\bar{x}_2,1)/U_3(\bar{x}_1,\bar{x}_2,1)$ and $p_2(\bar{x}_1,\bar{x}_2) = U_2(\bar{x}_1,\bar{x}_2,1)/U_3(\bar{x}_1,\bar{x}_2,1)$.

The firms behave as non-cooperative, quantity-setting profit maximizers: given the quantity $\bar{x}_j$ selected by the other firm, the prices available to firm i depend only on his choice of $x_i$. If he selects $\bar{x}_i$, the price he receives is $p_i(\bar{x}_1,\bar{x}_2)$, and his profits are $p_i(\bar{x}_1,\bar{x}_2)\bar{x}_i$. Thus, firm 1 seeks to maximize $p_1(x_1,x_2)x_1$, given $x_2$, and correspondingly for firm 2. Equilibrium then consists of quantities $\bar{x}_1$ and $\bar{x}_2$ and prices $(\bar{p}_1,\bar{p}_2,1)$, $\bar{p}_i = p_i(\bar{x}_1,\bar{x}_2)$, such that $\bar{x}_1$ maximizes

$$p_1x_1 = [U_1(x_1,\bar{x}_2,1)/U_3(x_1,\bar{x}_2,1)]x_1 = x_1 MRS^1(x_1,\bar{x}_2)$$

over $[0,1]$ and $\bar{x}_2$ maximizes

$$p_2x_2 = [U_2(\bar{x}_1,x_2,1)/U_3(\bar{x}_1,x_2,1)]x_2 = x_2 MRS^2(\bar{x}_1,x_2)$$

over $[0,1]$. (These maximization problems define the two firms' reaction curves, $x_1 = R^1(x_2)$ and $x_2 = R^2(x_1)$, respectively.) At such an equilibrium, the consumer is maximizing his utility by selecting as his consumption the amounts produced of the two goods.

A simple example may help clarify the situation. Suppose the consumer's preferences are given by the utility function

$U(x_1,x_2,x_3) = 2x_1 + 2x_2 - \frac{2}{3}x_1^2 - \frac{2}{3}x_2^2 - \frac{1}{2}x_1x_2 + x_3$. Then, for $x_3 = 1$, the functions $x_i MRS^i$ are given by

$$x_1 MRS^1(x_1,x_2) = 2x_1 - \frac{4}{3}x_1^2 - \frac{1}{2}x_1x_2,$$

$$x_2 MRS^2(x_1,x_2) = 2x_2 - \frac{4}{3}x_2^2 - \frac{1}{2}x_1x_2.$$

Maximization by each firm with respect to its output then yields the reaction curves

$$R^1(x_2) = \frac{3}{16}(4-x_2),$$

$$R^2(x_1) = \frac{3}{16}(4-x_1).$$

In this example, the reaction curves are continuous functions and, of course, equilibrium exists (at $x_1 = x_2 = \frac{12}{19}$, yielding prices $p_1 = p_2 = \frac{16}{19}$). However, it is easily seen that in general one should not expect that the reaction curves will be convex-valued. Indeed, unless the function $x_i MRS^i(x_1,x_2)$ is concave in $x_i$, one must rather expect that the set of profit-maximizing levels of $x_i$ will be disconnected for some values of $x_j$. Further, the concavity of this function is a very special and restrictive requirement. Short of separability or restrictions on the third order partial derivatives of the utility function, it is not at all clear it can be assured even in a one consumer world. Thus, in general we must be prepared to encounter the situation depicted in Figure 2, where the profit of firm 1 is plotted as a function of $x_1$ for various values of $x_2$. When $x_2 = \bar{x}_2$, the values of $x_1 = \hat{x}_1$ and $x_1 = \check{x}_1$

both maximize $x_1 MRS^1(x_1,\bar{x}_2)$, while intermediate values are not maximizers. Note too that the reaction curve, $R^1$, graphed in Figure 3, does not admit a continuous selection.

These remarks suggest that the assumption of convex-valued reaction curves made in the literature is not likely to be met without placing restrictions on consumers' preferences that go far beyond the usual. The difficulty in constructing an explicit example in which the reaction curves are not convex-valued and, as a result, equilibrium fails to exist is that the relationship between the utility function and the corresponding reaction curves is a highly complex one and not easily visualized. As well, it is clear that we must work with relatively "complicated" preferences since, for example, preferences representable by a separable function would give constant-valued reaction curves. In fact, we will not present an example in which an algebraic specification of preference is given. Rather, we adopt a geometric approach in which preferences with the desired properties are shown to exist. The following technical result is designed to facilitate our study.

Lemma. Let a and b be strictly positive, twice continuously differentiable, real-valued functions defined on an open region S in $\mathbb{R}^2$ containing $B \equiv [0,1] \times [0,1]$, and let U be a strictly monotonic, strictly quasi-concave utility function on $S \times \{0\}$. Assume that for every $(x_1,x_2) \in B$, the ratio $-a(x_1,x_2)/b(x_1,x_2)$

is identically equal to the slope of the indifference curve through $(x_1,x_2)$. Assume that the derivative of this slope with respect to $x_1$ is continuous and uniformly bounded away from zero on B, i.e., there is an $\varepsilon$ such that

$$\frac{-[a_1b^2-aa_2b-abb_1+a^2b_2]}{b^3} \geqq \varepsilon > 0$$

for all $(x_1,x_2) \in B$. Then there exist a $\delta > 0$ and a monotonic, quasi-concave utility function V which extends U on $B \times \{0\}$ to $B^\delta \equiv [0,1] \times [0,1] \times [-\delta,\delta]$ and which has the property that for each $(\bar{x}_1,\bar{x}_2) \in B$

$$H(\bar{x}_1,\bar{x}_2) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid a(\bar{x}_1,\bar{x}_2)(x_1-\bar{x}_1) + b(\bar{x}_1,\bar{x}_2)(x_2-\bar{x}_2) + x_3 = 0\}$$

is the unique supporting plane at $(\bar{x}_1,\bar{x}_2,0)$ to $\{(x_1,x_2,x_3) \in B^\delta \mid V(x_1,x_2,x_3) \geqq V(\bar{x}_1,\bar{x}_2,0)\}$.

This result, a proof of which is given in the Appendix, means that any "nice" utility function on the unit square in the plane can be extended to a utility function defined on a cube in three-space in such a way that the relative steepness of the indifference surfaces where they cut the plane can be specified in an essentially arbitrary fashion. This is the key to constructing the example.

## The First Example

The reaction curves we wish to generate for the two firms are diagrammed in Figure 4. The first firm's curve, $R^1$, is given by

$$R^1(x_2) = \begin{cases} .1 & x_2 < .75 \\ \{.1,.9\} & x_2 = .75 \\ .9 & x_2 > .75 \end{cases},$$

while $R^2$ is given by

$$R^2(x_1) = 1 - .5x_1, \quad 0 \leqq x_1 \leqq 1.$$

Since $R^1$ and $R^2$ have no point in common, no equilibrium exists.

We begin by giving the marginal rates of substitution between $x_1$ and $x_2$ along the curves we want to be the reaction curves. It is clearly possible to specify a utility function U on $[0,1] \times [0,1] \times \{1\}$ consistent with these marginal rates of substitution and (after translating to $[0,1] \times [0,1] \times \{0\}$) satisfying the conditions assumed in the Lemma. The following values are assigned for the marginal rate of substitution of good one for good two:

$$x_1 = .1, \quad x_2 \leqq .75, \quad \text{MRS} = 3/2$$

$$x_1 = .9, \quad x_2 \geqq .75, \quad \text{MRS} = 1/3$$

$$x_1 \in [0,1), \quad x_2 = 1-.5x_1, \quad \text{MRS} = -1.14 + .15x_1 + \exp[(1-x_1)\log_e 1.49].$$

The idea now is to assign to each point $(x_1,x_2,1)$ a vector $(a(x_1,x_2),b(x_1,x_2),1)$ such that:

(1) $a(x_1,x_2)/b(x_1,x_2)$ equals the MRS of good one for good two under the utility function U at $(x_1,x_2,1)$;

(2) $x_1 a(x_1,\bar{x}_2)$ is maximized at $R^1(\bar{x}_2)$ for each $\bar{x}_2$ in $[0,1]$; and

(3) $x_2 b(\bar{x}_1,x_2)$ is maximized at $R^2(\bar{x}_1)$ for each $\bar{x}_1$ in $[0,1]$.

Along the reaction curves we assign the following values:

for $x_1 = .1$, $\quad x_2 \leqq .75$, $\quad a(x_1,x_2) = 3$,

$$b(x_1,x_2) = 2;$$

for $x_1 = .9$ $\quad x_2 \geqq .75$, $\quad a(x_1,x_2) = 1/3$,

$$b(x_1,x_2) = 1;$$

for $x_1 \in [0,1]$, $x_2 = 1-.5x_1$, $a(x_1,x_2) = -3.42 + .45x_1$

$$+ 3\exp[(1-x_1)\log_e 1.49]$$

$$b(x_1,x_2) = 3.$$

With this assignment, the profit $\pi^1$ of firm one along $R^1$ (when evaluated in terms of the numeraire good three) is .3, while one can check that $\pi^1 < .3$ along $R^2$. The profit $\pi^2$ of firm two along $R^2$ is $3x_2$. On the portion of $R^1$ with $x_1 = .1$, we have $\pi^2 = 2x_2$, and along the portion where $x_1 = .9$, we have $\pi^2 = x_2$. Thus, $\pi^1$ is higher on $R^1$ than $R^2$ and $\pi^2$ is higher on $R^2$ that on $R^1$.

If the values of a and b are selected to be sufficiently low off the two reaction curves, then, for any $x_j$, firm i will find its profits maximized at $R^i(x_j)$. This assignment can be made to satisfy the conditions of the Lemma. We thus are guaranteed the existence of a $\delta > 0$ and a utility function V which extends U to $[0,1] \times [0,1] \times [1-\delta,1+\delta]$ and satisfies the condition that $(a(x_1,x_2),b(x_1,x_2),1)$ defines the unique supporting plane to the V upper contour set at $(x_1,x_2,1)$. This utility function V generates the reaction curves $R^1$ and $R^2$. This completes the first example.

## The Second Example

The preferences of the single consumer in the previous example are quite standard, fulfilling even stronger conditions than are normally used in the literature on existence of competitive equilibrium. However, they would give rise to marked, non-linear income effects, and this might lead one to hope that if individual preferences were even better-behaved (so that, say, no individual demand curves showed negative income effects), one would obtain convex-valued reaction curves. Our second example shows that if we allow for multiple consumers, then even requiring all preferences to be homothetic (so that the individual Engel curves are linear) is not sufficient to obtain an existence theorem. This example also shows that the assumption that the firms select prices rather than quantities does not obviate the difficulties with existence.

As in the previous example, there are three commodities and two firms. Each firm costlessly produces one commodity in amounts not exceeding one unit and uses price as its strategic variable. Let commodity three be the numeraire ($p_3$=1) and let $f_i$, i=1,2, which associates to each pair of prices $p_1$ and $p_2$ the demand for the commodity i, satisfy the condition that for $p_j \in [.1,1.1], j \neq i, j \in \{1,2\}$, $p_i f_i(p_1,p_2)$ is maximized with respect to $p_i$ at $Q^i(p_j)$, where the $Q^i$ functions are defined by

$$Q^1(p_2) = \begin{cases} .43 & .1 \leqq p_2 < 6 \\ \{.43,.77\} & p_2 = .6 \\ .77 & .6 \leqq p_2 \leqq 1.1 \end{cases}$$

$$Q^2(p_1) = \min[1,(6-5p_1)/5] \quad .1 \leqq p_1 \leqq 1.1$$

Assume further that the $f_i$ are such that prices outside the unit interval [.1,1.1] are never chosen. (For $p_i$ small the profit of the $i^{th}$ producer is bounded by the assumption that he can place only up to one unit on the market, and for $p_i$ greater than 1.1 the excess demand for the $i^{th}$ commodity can be chosen to be negative.) To complete the example it remains to show that the functions $f_1$ and $f_2$ could arise as the excess demands of competitive consumers. By a result of Rolf Mantel [6], (see also Sonnenschein [11] and Debreu [2]), there exist three consumers, whose utility functions that are homogeneous of degree one and whose incomes are derived solely from their initial endowments of the three commodities, such that the aggregate excess demands for the first two commodities from these consumers satisfy the properties attributed to $f_1$ and $f_2$. Now, let all ownership claims in profits in the two firms be vested in a fourth consumer who is interested only in the third good and holds no initial endowments of the first two commodities. Since the fourth consumer will neither demand nor supply any of the first two commodities, the demands facing the firms will be $f_1$ and $f_2$ respectively, the reaction curves will be $Q^1$ and $Q^2$, and no equilibrium will exist.

In fact, a more general result has been established. Since any upper hemi-continuous correspondence from the unit interval into itself can be represented in the form $R(x) = \{y \in [0,1] | y$ maximizes $F(x,y)\}$, where $F: [0,1]^2 \to R$ is a $C^\infty$ function, the argument given above yields the following striking result: the class of pairs of reaction curves that can arise in economies with two price-setting monopolisticly competitive firms includes all pairs of upper hemi-continuous correspondences on the unit interval. This is true even when all consumers have homothetic preferences.

Implications for Future Work

We have shown that the simple grafting of the Cournot-Chamberlin model of non-cooperative firms with market power onto the usual Arrow-Debreu model of general economic equilibrium fails to provide an integration of the two theories, since it does not provide an explanation of price and quantity determination in even the most simple and non-pathological cases. The question is then that of what one's response to these examples should be: what are the implications for the directions of future work?

If one insists on holding to the standards usually adopted in treating questions of existence of general equilibrium, which require that the conditions for existence should be stated in terms of the primitive data of tastes, endowments and technology, then one might be led to attempt to establish restrictions on this data which would ensure that the demand arising from the competitive sector is sufficiently well-behaved that equilibrium could be shown to exist. This would not appear to be a very promising line of research. Even in the context of the models used here, any conditions sufficient to guarantee the convex-valued reaction curves needed to apply Kakutani's theorem would appear to be very restrictive. If we attempt to enrich the model by introducing costly production, multi-product firms, several firms producing a given commodity, etc., one must suspect that any conditions sufficient for existence which could be obtained would be so restrictive as to leave the theorems essentially without interest.

If this pessimism is justified, one might adopt the view that the import of the example is that we cannot really hope to do better than to assume directly that the reaction curves are convex-valued. Thus the example would serve to justify in some sense the _ad hoc_ assumptions made in the literature. In adopting such a position, one might attempt to draw a parallel with the use of the gross substitutes assumption in the literature on stability of the tatonnement process. Still, one must fact the unpleasant fact that the theory may have little or nothing to say about the formation of prices in a class of situations which may be quite large.

A third alternative would be to accept the failure of existence in the model as indicating that equilibrium might well not be achieved in an actual economy in which firms with very complete knowledge of demand conditions behave in the extreme non-cooperative fashion assumed here. In this case, three different approaches suggest themselves. First, one might attempt to relax the assumption of full knowledge of the demand conditions. In fact, Negishi's papers do this in assuming that the firms base their decisions on linear perceived inverse demand curves, and Arrow and Hahn have used the notion of perceived demands to attempt to motivate their continuity assumption. Of course, these perceived curves are completely ad hoc, although one might attempt to construct a theory to explain them. Within such a theory, however, one would presumably want to allow for learning leading to the perceived curves more and more closely approximating the true relationships. In this case, one is confronted again with the original problem of non-existence. A second approach would be to replace the assumption of non-cooperative behavior. One might then look to the sort of limited, implicit cooperation observed in repeated play of prisoners' dilemma games, or to more explicit cooperative behavior. In doing so, however, one is not only abandoning the attempted integration of our partial and general equilibrium models, one is also immediately faced with all the unresolved partial equilibrium problems associated with oligopoly models other than that of Cournot. Finally, one might attempt to build a model based on non-cooperative behavior which would give some explanation of price formation while allowing for situations of persistent disequilibrium. This would not seem to be an easy task, but the rewards to its successful completion should be high.

In closing, we should note that the examples here demonstrate the problems with finding an equilibrium in pure strategies. Such problems are well known in game theory, and there the standard response is to look for an equilibrium in mixed strategies. Such expected profit maximizing probability distributions over output levels or prices would exist in the situations considered here, and in much more general models. The problem is to reconcile such systematic randomization with standard economic theory and observed behavior.

## APPENDIX

We offer here a proof of the Lemma stated in the text.

Let s be a Lipschitz function from $[0,1] \times [0,1]$ into $\mathbb{R}^2$, and let $\ell(x)$ denote, for each $x \in [0,1] \times [0,1] \times \{0\}$, the line in $\mathbb{R}^3$ given parametrically by $y = x + \lambda(s(x),1)$, $\lambda \in \mathbb{R}$. Then there exists $\delta > 0$ such that for no two distinct $x'$, $x''$ in $[0,1] \times [0,1]$ does $\ell(x') \cap \ell(x'')$ lie in $B^\delta = [0,1] \times [0,1] \times [-\delta,\delta]$. To see this, note that if $\ell(x')$ and $\ell(x'')$ intersect at y, then $|y_3| = \|x'-x''\| / \|s(x')-s(x'')\| \geq 1/K$, where K is the Lipschitz constant for s.

Now, for each $\bar{x} \in S \times \{0\} \supset B \times \{0\}$, let $H(\bar{x})$ be the plane given by $a(\bar{x})(x_1-\bar{x}_1) + b(\bar{x})(x_2-\bar{x}_2) + x_3 = 0$, and let $D(\bar{x})$ be the convex set formed as the intersection over all x for which $U(x) = U(\bar{x})$ of the half-spaces above the planes $H(x)$. Let $\ell(\bar{x})$ be the line defined as the limit of $(H(y^n) \cap H(z^n))$, where $y^n, z^n \in S$, $U(y^n) = U(z^n) = U(\bar{x})$, $y_1^n = \bar{x}_1 + \frac{1}{n}$, $z_1^n = \bar{x}_1 - \frac{1}{n}$. Then $\ell(\bar{x})$ may be computed to be of the form $x = \bar{x} + \lambda(s(\bar{x}),1)$, where

$$s(\bar{x}) = (s^1(\bar{x}),s^2(\bar{x}))$$

$$\equiv \left[\frac{b(\bar{x})b_1(\bar{x})-a(\bar{x})b_2(\bar{x})}{d(\bar{x})}, \frac{a(\bar{x})a_2(\bar{x})-a_1(\bar{x})b(\bar{x})}{d(\bar{x})}\right]$$

and

$$d(\bar{x}) \equiv a_1(\bar{x})b(\bar{x})^2 - a(\bar{x})a_2(\bar{x})b(\bar{x}) - a(\bar{x})b(\bar{x})b_1(\bar{x}) + a(\bar{x})^2 b_2(\bar{x}).$$

By assumption, d is bounded from zero and a and b are $C^2$. Thus, s is Lipschitz and so there exists a $\delta > 0$ such that no two of these lines intersect in $B^\delta$. Let $\ell^\delta(\bar{x}) = \ell(\bar{x}) \cap B^\delta$.

Let $I(\bar{x})$ be the boundary of $D(\bar{x})$, let $I^\delta(\bar{x}) = I(\bar{x}) \cap B^\delta$ and note that

$$I^\delta(x) = \bigcup_{\{x:U(x)=U(\bar{x})\}} [B^\delta \cap H(x) \cap D(\bar{x})].$$

But for any $x'$ with $U(x') = U(\bar{x})$, if $y \in B^\delta \cap H(x') \cap D(\bar{x})$ then $y \in \ell^\delta(x')$. To see this, let $y \in B^\delta \cap H(x') \cap D(\bar{x})$ but suppose, without loss of generality, that $y_1 > x_1' + y_3 s^1(x')$, $y_2 < x_2 + y_3 s^2(x')$, where $|y_3| \leqq \delta$. Let $x_2 = f(x_1)$ be the indifference curve through $(x_1', x_2', 0)$, and consider

$$\Delta(x_1) = a(x_1, f(x_1))(y_1 - x_1) + b(x_1, f(x_1))(y_2 - f(x_1)) + y_3.$$

Note that $\Delta(x_1') = 0$, since $y \in H(x')$, while sign $[d\Delta/dt]$ evaluated at $x_1'$ is negative, since it is the same as that of $d(x')$. Thus, for values of x on the indifference curve sufficiently close to $x'$ but with $x_1 > x_1'$, y lies below $H(x)$. Thus $y \in D(\bar{x})$, and we have a contradition. Thus, $I^\delta(\bar{x})$ is the union of the $\ell^\delta(x)$ for x indifferent to $\bar{x}$. Further, any point y on $I^\delta(x)$ can lie on at most one of the $\ell^\delta$ lines, since $|y_3| \leqq \delta$. This establishes that each $I^\delta(\bar{x})$ is the disjoint union of the $\ell^\delta(x)$ for $U(x) = U(\bar{x})$, and, in turn, that the $I^\delta$ surfaces do not intersect one another. They then define the level surfaces of the utility function V.

FOOTNOTES

1. A later but even more explicit recognition of the desirability of avoiding such assumptions can be found in Laffont and Laroque [5].

2. For an elaboration of the point see the ninth chapter of A. Cournot's Researchs into the Mathematical Principles of the Theory of Wealth.

3. The argument in [10] also applies when production takes place under identical constant returns to scale technologies for the two firms, with input prices either given competitively or depending monopsonistically on total industry demand for the factors.

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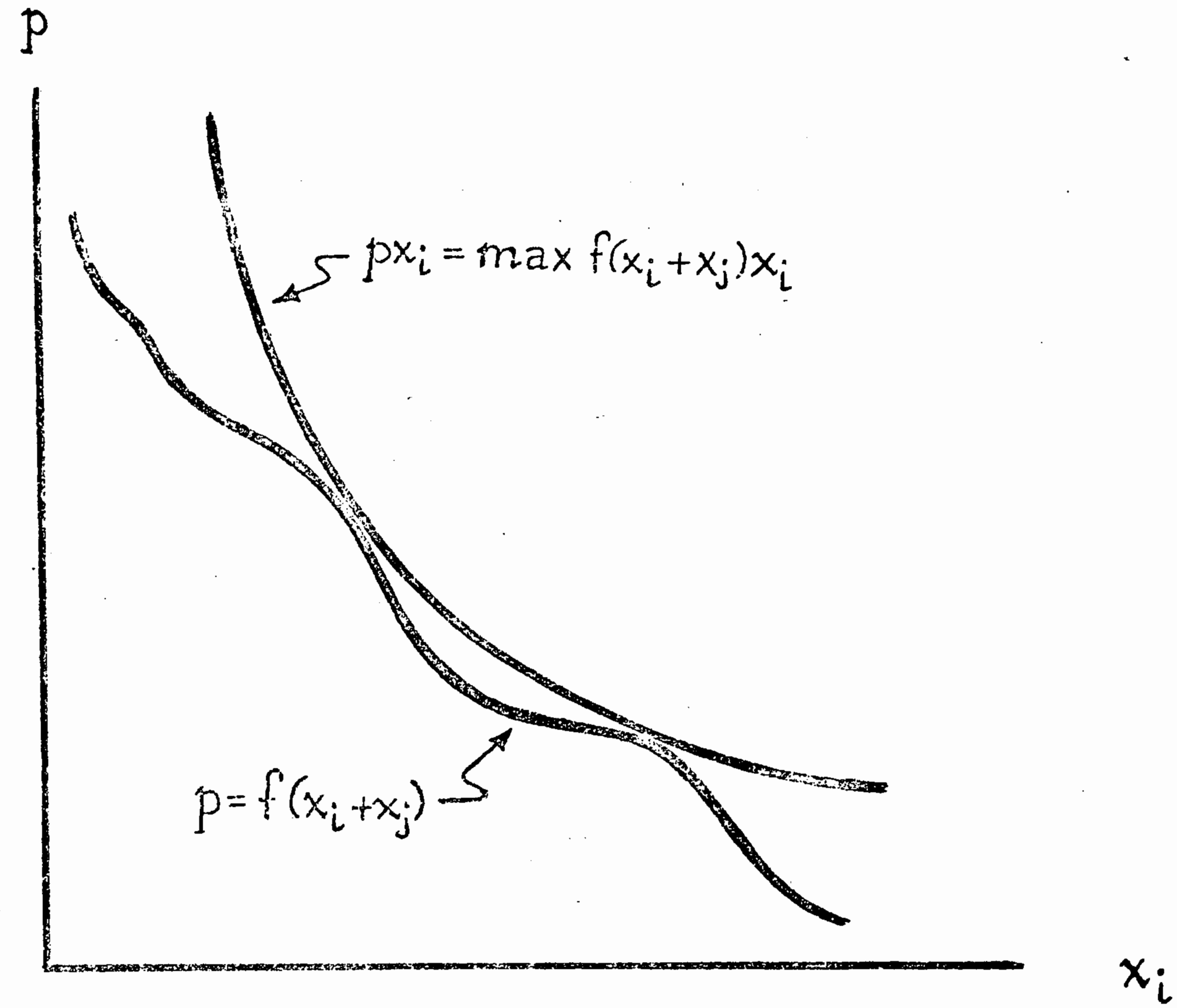
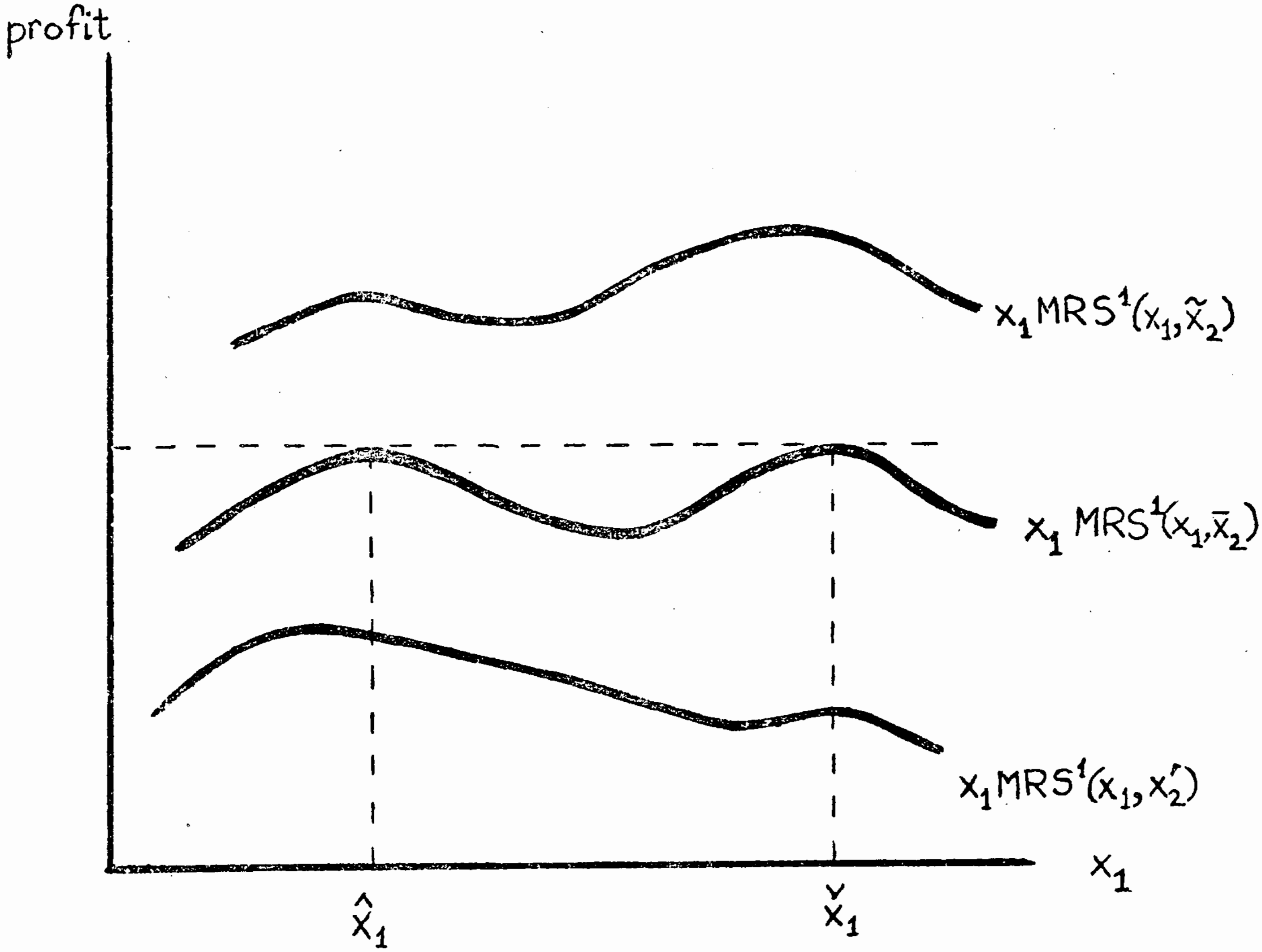
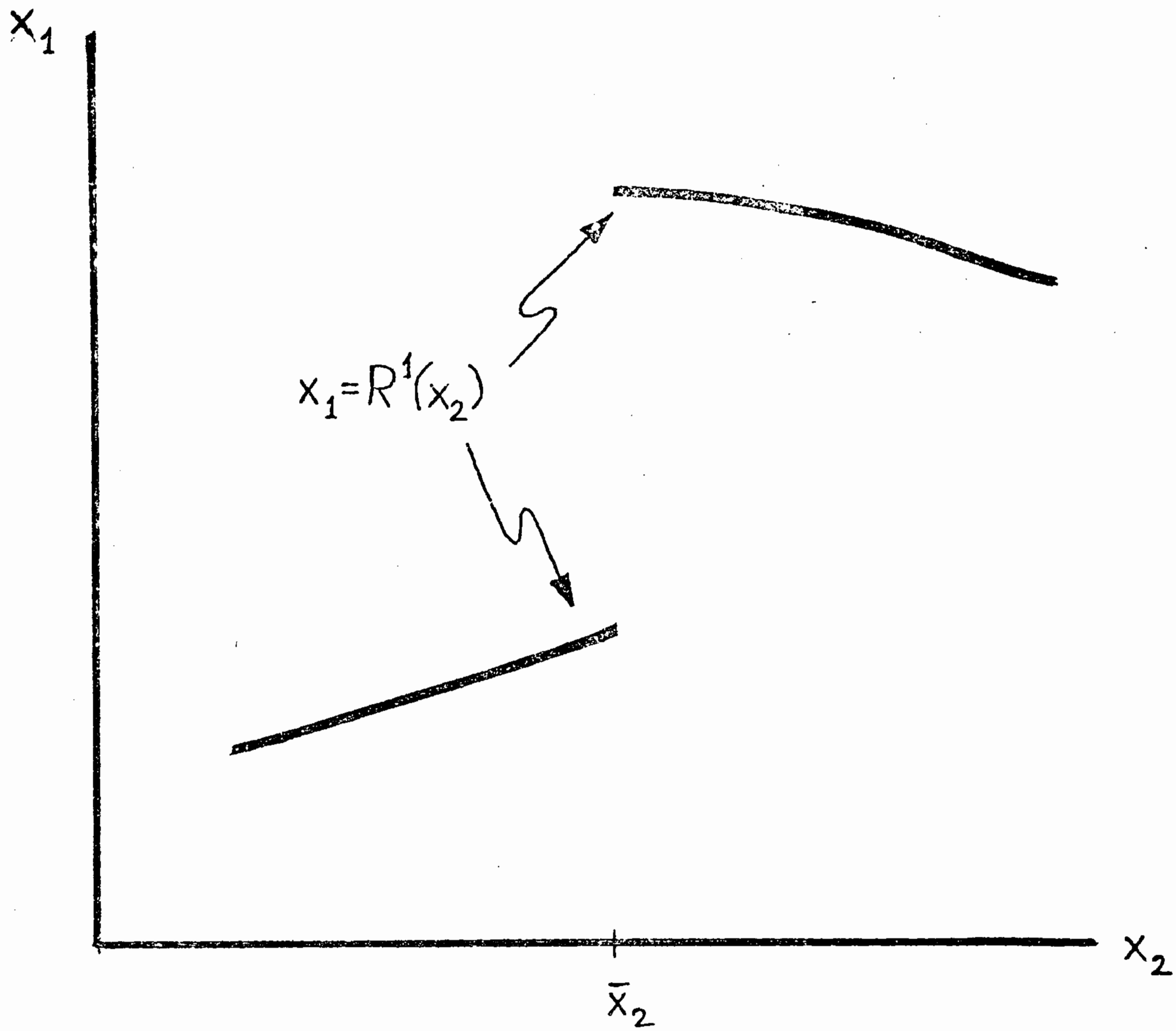



ROBERTS – SONNENSCHEIN – FIGURE 1




ROBERTS - SONNENSCHEIN : FIGURE 2




ROBERTS-SONNENSCHEIN- FIGURE 3