DISCUSSION PAPER NO. 167

SOME RECENT RESULTS ON THE EXISTENCE OF EQUILIBRIUM IN FINITE PURELY COMPETITIVE ECONOMIES

by

Hugo Sonnenschein

August 1975

SOME RECENT RESULTS ON THE EXISTENCE OF EQUILIBRIUM IN FINITE PURELY COMPETITIVE ECONOMIES */

Hugo Sonnenschein, Northwestern University **/

The subject of the existence of equilibrium in finite purely competitive economies has been well worked during the past twenty-five years. Thus, it may seem surprising that a session devoted to "new developments" should include a paper in this area. To make it clear at the outset that this inclusion is in fact appropriate, I shall begin by describing a theorem due to A. Mas-Colell [8]. 1/ His result is central to my presentation.

A consumer with nonordered preferences is a triple (X_i, ω_i, P_i), where the consumption set X_i is a subset of the positive orthant $\Omega \subset R^{\ell}$, the initial endowment ω_i is a point in R^{ℓ}, and P_i is a binary (preference) relation on X_i. A (pure exchange) economy with nonordered preferences $(X_i, \omega_i, P_i)_{i=1}^{i=n}$ is a collection of n consumers with nonordered preferences. An equilibrium for such an economy is a point $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) \in \Pi X_j = X$ and a price vector $\overline{p} \in \Delta = \{p \in \Omega : \Sigma p_i = 1\}$ satisfying

e1. $\Sigma \overline{x}_i \leqq \Sigma \omega_i$,

e2. for each i, $\overline{p} \cdot \overline{x}_i = \overline{p} \cdot \omega_i$, and

e3. for each i, $x_i' \in P_i(\overline{x}_i) = \{x_i : x_i P_i \overline{x}_i\}$ implies $\overline{p} \cdot x_i' > \overline{p} \cdot \omega_i$.

<u>Theorem 1</u> (A. Mas-Colell [8]): If $\mathcal{E}$ is an economy with nonordered preferences satisfying for each i

*/ For presentation at the Third World Congress of the Econometric Society, Toronto, August 1975.

**/ My ideas on this subject have been developed in such close association with Wayne Shafer that he should be regarded as coauthor of whatever is useful in this exposition. I am also very indebted to Andreu Mas-Colell. Not only is his work central to the substance of this survey, but in addition he has been most generous with his comments and advice.

X_i is convex and closed,

$\omega_i \in$ interior X_i,

$P_i(x_i)$ is convex and nonempty for each $x_i \in X_i$,

P_i is open (in $X_i \times X_i$), and

$x_i \notin P_i(x_i)$ for each x_i (irreflexivity),

then $\mathcal{E}$ has an equilibrium.

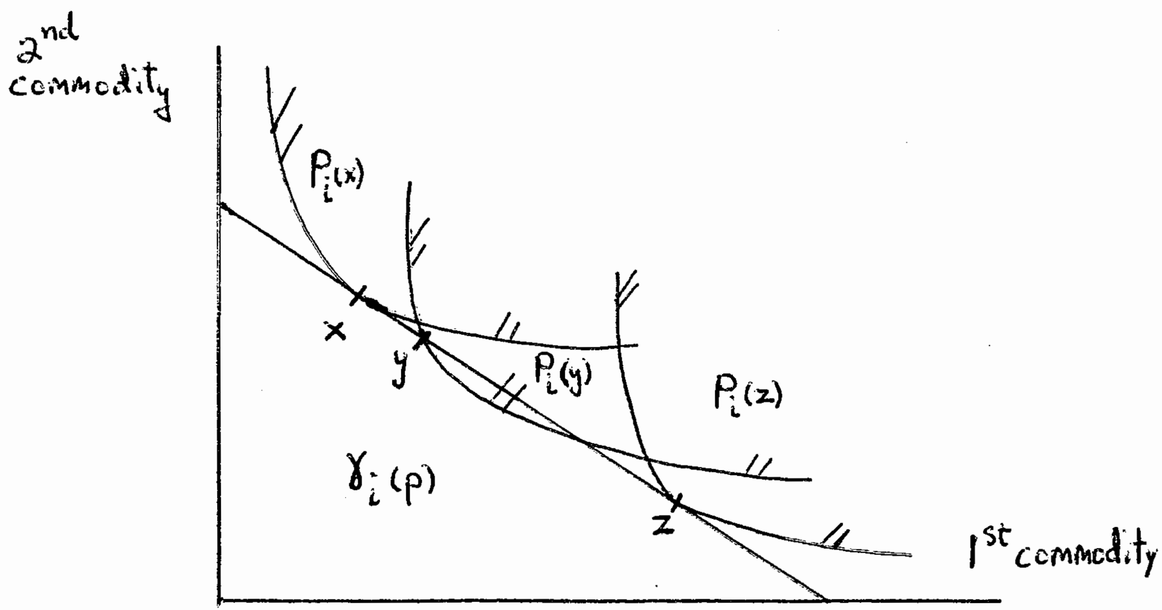

FIGURE 1

Note that the hypotheses of Theorem 1 do not imply that the P_i can be derived from "as good as" relations which are either transitive or complete. Thus, for the three points x, y, and z on the boundary of the budget $\gamma_i(p)$, it can be the case that the "preferred" sets $P_i(x)$, $P_i(y)$, and $P_i(z)$ are as drawn.

The theorem of Mas-Colell is striking for two reasons. First, observe that without a utility function (in particular without transitivity) it is not at all clear that there is a P_i maximal element in each budget constraint ([15]) and [10] are relevant here), and second, note that the set of P_i maximal elements in a budget need not be connected. (This is the case in Figure 1.) As a result the aggregate excess demand function may have values which are not connected sets (and in fact it may not admit a selection which is both upper hemicontinuous and convex valued). Thus proofs of the existence of equilibrium which apply the Kakutani fixed point theorem to a modification of the aggregate excess demand function are likely to fail. I also hasten to point out that this result may be even more of a surprise from an intuitive standpoint than a technical one, for it demonstrates that prices and individual preference maximization can lead to a coherent allocation of resources even when individuals' preferences are not consistent! (When your psychologist friends tells you that all of economic theory is based on the assumption that individuals' preferences are consistent, and pulls out ten papers which refute that hypothesis, recite Mas-Colell's theorem to him.)

Well, Mas-Colell's theorem is quite correct, and the result is sufficiently fine so that one might naturally ask whether there were any hints around which might have enabled us to guess at the result ourselves and so aid **our** understanding of the theorem. In fact, there is a theorem in Arrow and Hahn's General Competitive Analysis which, when properly viewed, is of some help. [2/] The Arrow-Hahn-Starrett theorem is a descendent of McKenzie's 1955 theorem [9] on the existence of equilibrium with externalities. For the case of pure exchange the result is the following. A consumer with externalities in consumption is a triple $(X_i, \omega_i, {}^{ext}u_i)$, where the first two coordinates are as before and

${}^{ext}u_i : X \times X_i \to R$.[3/] A (pure exchange) economy with externalities in consumption is a collection of n consumers with externalities in consumption. An equilibrium is a $(\bar{x},\bar{p}) \in X \times \Delta$ which satisfies e1,e2, and e3': for each i, ${}^{ext}u_i(\bar{x},y_i) > {}^{ext}u_i(\bar{x},\bar{x}_i)$ implies $\bar{p} \cdot y_i > \bar{p} \cdot \omega_i$.

Theorem 2 (Arrow-Hahn-Starrett): If $\mathcal{E}$ is an economy with externalities in consumption satisfying for each i

X_i is convex and closed,

$\omega_i \in \text{int } X_i$,

$u_i(x, \cdot)$ is quasi concave and monotonic with respect to $\geq$ ($x \geq y$ if $x \neq y$ and $x_i \geqq y_i$ for all i),[4/] and

u_i is continuous,

then $\mathcal{E}$ has an equilibrium.

This is how a version of Theorem 1 can be deduced from Theorem 2. (These remarks are amplified in [12].) Consider an economy $\mathcal{E} = (X_i,\omega_i,P_i)_{i=1}^{i=n}$ which satisfies the conditions of Theorem 1 and assume for the moment that each X_i is Ω and $P_i(x_i) + \Omega \subset P_i(x_i)$ for all i and all $x_i \in X_i$. For each consumer i and each point $(x,y_i) \in X \times X_i$, define ${}^{ext}u_i(x,y_i) = u_i^x(y_i)$, where u_i^x is the unique linear homogeneous utility function satisfying $u_i^x(1,1,\ldots,1) = 1$ and u_i^x is constant on the boundary of $P_i(x_i)$. (See Figure 2.) Then, $(X_i,\omega_i,{}^{ext}u_i)_{i=1}^{i=n}$ is an economy with externalities and its equilibria are also equilibria of $(X_i,\omega_i,P_i)_{i=1}^{i=n}$.[5/] Loosely speaking, $(X_i,\omega_i,{}^{ext}u_i)_{i=1}^{i=n}$ will satisfy the conditions of Theorem 2 (and thus an equilibrium will exist) provided that the preference relations are strongly

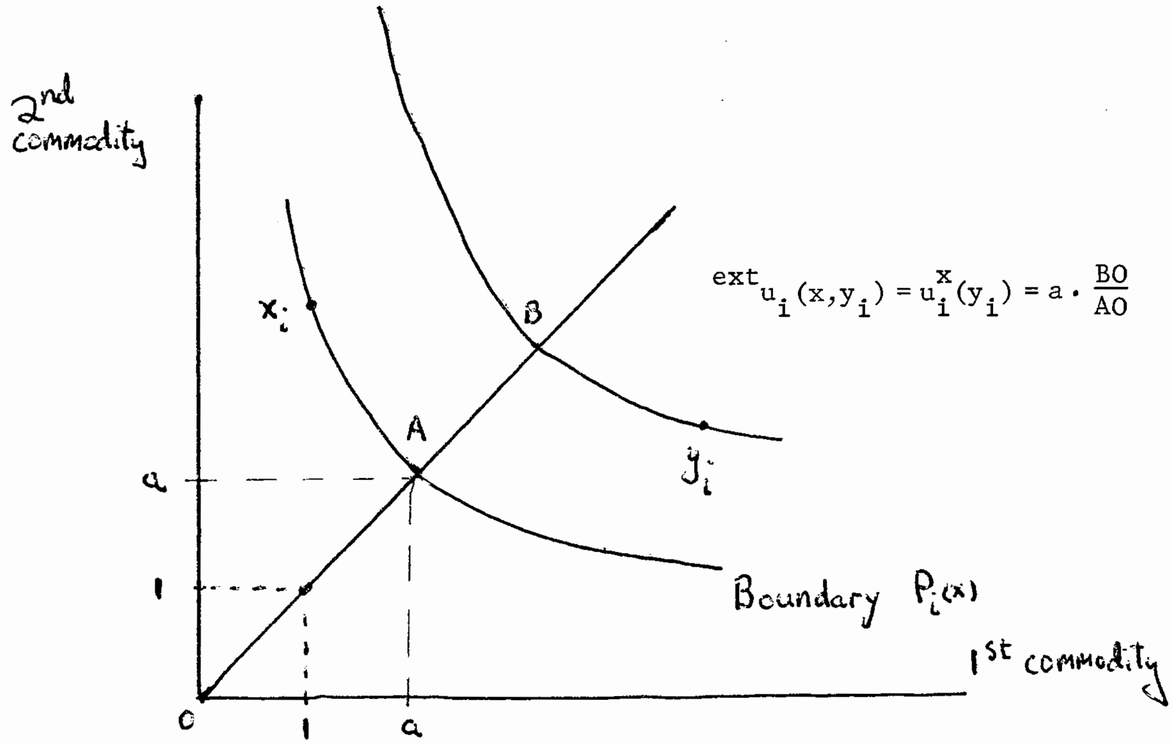

FIGURE 2

monotone $(P_i(x_i) + \Omega \subset P_i(x_i))$ and P_i, rather than being open, has a continuous complement.

The foregoing suggests that the problems of existence with externalities and existence with nonordered preferences may have quite a lot in common. This is accentuated by observing that the proofs of the theorems we have just discussed rely on mappings from a domain which is at least the dimension of the set of allocations (as opposed to once or twice the dimension of the commodity space). It is a natural next step to join the literatures together by proving the existence of equilibrium in a situation where preferences are not only nonordered, but in which externalities (both in the form of price dependent and allocation dependent preferences) are present as well. The perfect tool to accomplish this union is the original Debreu lemma on the existence of equilibrium in abstract economies [4], modified so as to allow for preference correspondences [13].

Lemma (Debreu, modified by Shafer and Sonnenschein): An abstract economy is defined by n ordered triples $(X_i, \mathcal{A}_i, P_i)$, where $\mathcal{A}_i : \Pi X_j \twoheadrightarrow X_i$ and $P_i : \Pi X_j \twoheadrightarrow X_i$ are correspondences. Let $\Gamma = (X_i, \mathcal{A}_i, P_i)_{i=1}^{i=n}$ be an abstract economy satisfying for each i

> X_i is a nonempty, compact and convex subset of R^{ℓ}, $X = \Pi X_j$,
>
> $\mathcal{A}_i$ (the constraint correspondence) is continuous,
>
> for each $x \in X, \mathcal{A}_i(x)$ is nonempty and convex,
>
> P_i (the preference correspondence) has an open graph in $X \times X_i$ and in convex valued,
>
> for each $x \in X$, $x_i \notin P_i(x)$.

Then Γ has an equilibrium; i.e., there exists $\bar{x} \in X$ satisfying for each i, $\bar{x}_i \in \mathcal{A}_i(\bar{x})$ and $P_i(\bar{x}) \cap \mathcal{A}_i(\bar{x}) = \emptyset$.

With this lemma one can prove the existence of equilibrium in a model where consumers' preferences are represented by correspondences $P_i : X \times \Delta \twoheadrightarrow X_i$ which are convex and nonempty valued, have open graphs, and satisfy $x_i \notin P_i(x,p)$ for each $(x,p) \in X \times \Delta$. (One interprets $P_i(x_1, x_2, \ldots, x_n, p)$ as the set of consumption plans which the ith consumer preferes to x_i given that other agents choices are set at $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and prices are p.) The proof is obtained by following Arrow and Debreu [1]. A market player with choice set Δ

is adjoined to the economy. For any state (x,p) which includes the actions of the consumer agents, the market player prefers prices which yield a higher value of excess demand than does p. An equilibrium of the $n+1$ person abstract economy so defined (which exists by the Lemma) is then an equilibrium for the economy with preference correspondences which depend on allocation and price variables. The existence theorems of Mas-Colell (nonordered preferences) and Arrow-Hahn-Starrett (preferences with externalities) are special cases. (In particular we note that the Arrow-Hahn-Starrett type theorem which follows from this result applies to some situations in which individual utility functions are not continuous; (see [12])). Also, if one selects $B^{\ell} = \{x \in R^{\ell} : \|x\| \leqq 1\}$ as the choice set of the market player and defines the constraint correspondences of the consumers by $\mathcal{O}_i(x,p) = \{x_i' : p \cdot x_i' \leqq p \cdot \omega_i + 1 - \|p\|\}$, it can be verified that an equilibrium of the abstract economy so defined is a competitive equilibrium without free disposal. 6/ The above modification of the constraint set is due to T. Berstrom [3] and is carried out explicitly in the present context by Wayne Shafer [11]. I should say that it works extremely well and ask the listener to compare it to the pioneering techniques of McKenzie [9] and Debreu [5]. 7/

Finally, let me use a device of Gale and Mas-Colell [6] to illustrate how the results we have considered can serve to simplify some of the work in equilibrium analysis. For well known reasons proofs of the existence of competitive equilibrium are more cumbersome without the assumption of local nonsatiation. In its absence a price vector $\bar{p}$ may make the excess demand function nonpositive in every coordinate and yet expenditure may fall short of income for some consumer. As a result, it may not be possible to show that a commodity must be free if its supply at price $\bar{p}$ exceeds its demand. But with the above theorem local satiation can be handled simply as follows. Assume that u is a continuous and quasiconcave utility function on Ω which exhibits local satiation but does not have a global maximum. Replace it by the nonordered preference relation P defined by "$P(x)$ is the interior of the convex hull of $\{x\} \cup \{x' : u(x') > u(x)\}$". The relation so defined cannot be represented by a utility function; however, it satisfies the conditions on preferences of the Mas-Colell theorem. If for each individual this replacement is made in a given economy $\mathcal{E}$ with local satiated preferences, one obtains an economy $\mathcal{E}'$ with nonordered preferences, and the Mas-Colell theorem can be applied to establish the existence of equilibrium in $\mathcal{E}'$. Furthermore, since for each x_i, $\{x_i' : u_i(x_i') > u_i(x_i)\} \subset P_i(x_i)$ and $x_i \in$ Boundary $P_i(x_i)$, an equilibrium of the economy $\mathcal{E}'$ with nonordered preferences is an equilibrium of the original economy $\mathcal{E}$ in which individuals spend all of their income. Thus, supply can only exceed demand for free goods.

This concludes my presentation. The material I have communicated is important for two reasons. First, it directly extends the class of economic environments for which prices and individual preference maximization lead to a coherent allocation of resources. Second, the techniques which have been employed appear to simplify the analysis of competitive markets.[8/] It is my hope that you will find some of these new theorems and techniques useful in your own work.

FOOTNOTES

1/ Mas-Colell's original proof was quickly followed by an improved version due to David Gale and Mas-Colell [6].

2/ The section in which it appears is acknowledged as having been developed jointly with David Starrett.

3/ Note that if (x,y_i) is a generic element of $X \times X_i$, then y_i is not necessarily the same as the ith co-ordinate of x.

4/ This is much stronger than is needed, but is useful in this exposition.

5/ To verify this claim observe that at an equilibrium pair $(\bar{x},\bar{p})$ the set $P_i(\bar{x}_i)$ is contained in $\{y_i : {}^{ext}u_i(\bar{x},y_i) > {}^{ext}u_i(\bar{x},\bar{x}_i)\}$. In fact, if $\bar{x}_i \in$ Boundary $P_i(\bar{x}_i)$, then these sets coincide.

6/ This requires that aggregate supply and demand exactly balance; however, negative prices are admitted.

7/ A recent interesting proof of existence without free disposal is due to O. Hart and H. Kuhn [7].

8/ Let me note that Wayne Shafer and I have recently used the modified Debreu Lemma (above) to provide a proof of the existence of equilibrium in economies with very general forms of commodity taxation, externalities, and public goods, [14].