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The Complementary Unboundedness of Dual Feasible Solution Sets in Convex Programming

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Abstract. F.E. Clark has shown that if at least one of the feasible solution sets for a pair of dual linear programming problems is nonempty then at least one of them is both nonempty and unbounded. Subsequently, M. Avriel and A.C. Williams have obtained the same result in the more general context of (prototype posynomial) geometric programming. In this paper we show that the same result is actually false in the even more general context of convex programming -- unless a certain regularity condition is satisfied.

We also show that the regularity condition is so weak that it is automatically satisfied in linear programming, (prototype posynomial) geometric programming, quadratic programming (with either linear or quadratic constraints), ℓ_p-regression analysis, optimal location, roadway network analysis, and chemical equilibrium analysis. Moreover, we develop an equivalent regularity condition for each of the usual formulations of duality.

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1. Introduction. In convex programming there are at least five different formulations of duality -- the original Fenchel formulation [5,12], the (generalized) geometric programming formulation [3,6,8], the Fenchel-Rockafellar formulation [10,12], the ordinary Lagrangian formulation [15,4,12], and the Rockafellar formulation [11,12]. Although each formulation has its own advantages and disadvantages, each can also be viewed as a special case of each of the other four (by virtue of the specializations given in [6,12]). In particular, a given result in one formulation has its counterparts in each of the other four formulations, all of which can be used to supply important information about various special programming types.

For simplicity, we begin by establishing our main result in the context of the most recent unconstrained geometric programming formulation [8]. This also provides a convenient mechanism for translating it into the context of the most recent constrained geometric programming formulation [8], as well as both the Fenchel-Rockafellar formulation [12] (including the original Fenchel formulation) and the more recent Rockafellar formulation [12]. In turn, the latter provides a convenient mechanism for translating it into the context of the ordinary Lagrangian formulation [12]. When appropriate, we also consider applications to special programming types.

Some familiarity with the usual concepts, terminology, notation, and facts from "convex analysis" [12] is assumed. In particular, given a nonempty convex set S in E_N (N-dimensional Euclidean space), its "recession cone"

$$(0^+S) \stackrel{\Delta}{=} \{\gamma \in E_N \mid s + \alpha\gamma \in S \text{ for each } s \in S \text{ and each } \alpha \geq 0\},$$

and its "barrier cone"

$$(\Delta S) \stackrel{\Delta}{=} \{\delta \in E_N \mid \text{there is a scalar } \beta \text{ for which } \langle s,\delta\rangle \le \beta \text{ for each } s \in S\}.$$

Moreover, given a nonempty convex cone K in E_N, its "polar cone"

$$K^\circ \stackrel{\Delta}{=} \{\delta \in E_N \mid \langle \gamma,\delta\rangle \le 0 \text{ for each } \gamma \in K\}.$$

2. <u>The Main Result</u>. Throughout this paper we assume that the following hypotheses are satisfied:

> $\mathcal{X}$ is a nonempty closed convex cone in E_n,
>
> g is a closed convex function with a nonempty (effective) domain $\mathcal{C} \subseteq E_n$.

Now, given $\mathcal{X}$ and g, consider the resulting "geometric programming problem" $\mathcal{A}$.

PROBLEM $\mathcal{A}$. <u>Using the feasible solution set</u>

$$\mathcal{S} \stackrel{\Delta}{=} \mathcal{X} \cap \mathcal{C},$$

<u>calculate both the problem infimum</u>

$$\varphi \stackrel{\Delta}{=} \inf_{x \in \mathcal{S}} g(x)$$

<u>and the optimal solution set</u>

$$\mathcal{S}^* \stackrel{\Delta}{=} \{x \in \mathcal{S} \mid g(x) = \varphi\}.$$

Geometric duality is defined in terms of both the "dual cone"

$$\mathcal{Y} \stackrel{\Delta}{=} \{y \in E_n \mid 0 \le \langle x,y\rangle \text{ for each } x \in \mathcal{X}\} = -\mathcal{X}^\circ$$

and the "conjugate transform function" h whose (effective) domain

$$\mathcal{D} \stackrel{\Delta}{=} \{y \in E_n \mid \sup_{x \in \mathcal{C}} [\langle y,x\rangle - g(x)] \text{ is finite}\}$$

and whose functional value

$$h(y) \stackrel{\Delta}{=} \sup_{x \in \mathcal{C}} [\langle y,x\rangle - g(x)].$$

In particular, given the geometric programming problem $\mathcal{A}$, consider the resulting "geometric dual problem" $\mathcal{B}$.

PROBLEM $\mathcal{B}$. <u>Using the feasible solution set</u>

$$\mathcal{T} \stackrel{\Delta}{=} \mathcal{Y} \cap \mathcal{D},$$

<u>calculate both the problem infimum</u>

$$\psi \stackrel{\Delta}{=} \inf_{y \in \mathcal{T}} h(y)$$

<u>and the optimal solution set</u>

$$\mathcal{T}^* \stackrel{\Delta}{=} \{y \in \mathcal{T} \mid h(y) = \psi\}.$$

It is important to note that the symmetry of conical duality (or, equivalently, the symmetry of conical polarity asserted by Theorem 14.1 on page 121 of [12]) together with the symmetry of functional conjugacy (asserted by Corollary 12.2.1 on page 104 of [12]) induces a symmetry on the theory that relates problem $\mathcal{A}$ to problem $\mathcal{B}$. In particular, under our given hypotheses, each statement about $\mathcal{A}$ and $\mathcal{B}$ automatically produces an equally valid "dual statement" about $\mathcal{B}$ and $\mathcal{A}$ (obtained by interchanging the symbols $\mathcal{X}$ and $\mathcal{Y}$, the symbols $\mathcal{C}$ and $\mathcal{D}$, the symbols g and h, the symbols $\mathscr{S}$ and $\mathscr{T}$, the symbols φ and ψ, the symbols $\mathscr{S}^*$ and $\mathscr{T}^*$, and the symbols x and y). However, to be concise, each dual statement is actually left to the reader.

Although there are dual problems $\mathscr{A}$ and $\mathscr{B}$ with nonempty feasible solution sets $\mathscr{S}$ and $\mathscr{T}$ for which both $\mathscr{S}$ and $\mathscr{T}$ are bounded, the main result of this paper shows that such problems $\mathscr{A}$ and $\mathscr{B}$ do not possess the following regularity.

DEFINITION. Given that

$$\text{there is a nonzero (direction) vector } \gamma \in \mathscr{X} \cap (0^+\mathscr{C}),$$

problem $\mathscr{A}$ is said to be <u>primal regular</u>. On the other hand, given that

$$\text{either } \mathscr{X} \text{ or } (0^+\mathscr{C}) \text{ is not a vector space}$$

and given that

$$(0^+\mathscr{C})^\circ \subseteq (\Delta\mathscr{C}),$$

problem $\mathscr{A}$ is said to be <u>dual regular</u>. A problem $\mathscr{A}$ that is either primal regular or dual regular is said to be <u>regular</u>.

It is important to note that neither the definition of primal regularity nor the definition of dual regularity explicitly involve (dual) problem $\mathscr{B}$. On the other hand, it is also important to note that the (unstated) duals of these definitions do explicitly involve (dual) problem $\mathscr{B}$, but not (primal) problem $\mathscr{A}$.

<u>Proposition</u>. <u>Given that at least one of the dual problems $\mathscr{A}$ and $\mathscr{B}$ is regular</u>, <u>if at least one of the corresponding feasible solution sets $\mathscr{S}$ and $\mathscr{T}$ is nonempty</u>, <u>then at least one of these sets is both nonempty and</u>

<u>unbounded</u>.

A proof of this proposition is given in Appendix A.

Counterexample (1) in Appendix B shows that this proposition loses its validity when our regularity assumption is deleted -- even when it is replaced with the assumption that both feasible solution sets $\mathcal{S}$ and $\mathcal{T}$ are nonempty and there is no "duality gap". On the other hand, we shall eventually see that our regularity assumption is actually so weak that very broad classes of dual convex programming problems satisfy it. In fact, the much stronger regularity assumption that requires both problem $\mathcal{A}$ and problem $\mathcal{B}$ to be both primal regular and dual regular is still not even strong enough to guarantee the absence of a duality gap -- a fact that can easily be demonstrated with the aid of the second example in Appendix C of [6].

The second example in Appendix C of [6] also shows that the preceding (strongest possible) regularity assumption is not strong enough to guarantee that <u>at most</u> one of the feasible solution sets $\mathcal{S}$ and $\mathcal{T}$ is unbounded. In fact, Theorem 3 in Appendix A along with its (unstated) dual shows that the existence of a nonempty bounded feasible solution set (either $\mathcal{S}$ or $\mathcal{T}$) is incompatible with the existence of a duality gap. On the other hand, counterexample (2) in Appendix B shows that the absence of a duality gap along with primal regularity (and hence regularity) for both problem $\mathcal{A}$ and problem $\mathcal{B}$ is also not strong enough to guarantee that <u>at most</u> one of the feasible solution sets $\mathcal{S}$ and $\mathcal{T}$ is unbounded.

The most effective applications of this proposition require several elementary (seemingly unrecorded) facts about convex sets and cones to establish our regularity assumption. In particular, the following elementary facts are crucial to many applications:

(A) Each Cartesian product $\mathcal{C} = \overset{r}{\underset{1}{\times}} \mathcal{C}_k$ has the property that $(0^+\mathcal{C}) = \overset{r}{\underset{1}{\times}} (0^+\mathcal{C}_k)$,

(B) A Cartesian product $\mathcal{V} = \overset{s}{\underset{1}{\times}} \mathcal{V}_k$ is not a vector space if and only if at least one of the factors $\mathcal{V}_k$ is not a vector space,

(C) A Cartesian product $\mathcal{C} = \overset{r}{\underset{1}{\times}} \mathcal{C}_k$ satisfies the condition $(0^+\mathcal{C})^\circ \subseteq (\Delta\mathcal{C})$ if and only if each factor $\mathcal{C}_k$ satisfies the condition $(0^+\mathcal{C}_k)^\circ \subseteq (\Delta\mathcal{C}_k)$.

Formal proofs of these facts are left as exercises for the reader.

The preceding theory can be applied directly to the unconstrained cases of the following special programming types: posynomial minimization, (convex) quadratic minimization and ℓ_p-regression analysis, (convex) optimal location, (convex) discrete optimal control with linear dynamics, and monotone network analysis -- simply by consulting examples 1 through 5 respectively in section 2 of [8]. In particular, it is easy to see that discrete optimal control and monotone network analysis are the only such examples for which nontrivial problems $\mathcal{A}$ are not necessarily primal regular. Moreover, it is also easy to see that the kind of monotone network analysis that arises in the context of roadway networks always involves a problem $\mathcal{A}$ that is dual regular. However, the possible states of regularity for discrete optimal control and the more classical kinds of monotone network analyses that arise in the context of electric and hydraulic networks have not yet been investigated in sufficient detail to report on here.

3. Other Formulations. For each of the other formulations, we specify problem $\mathcal{A}$ and then give the resulting problem $\mathcal{B}$ along with appropriate characterizations of the different kinds of regularity.

The following three subsections are pedagogically independent of one another, but the fourth subsection utilizes the third subsection.

3.1. <u>The Constrained Geometric Programming Formulation</u>. First, suppose that:

I and J are two nonintersecting (possibly empty) positive-integer index sets with finite cardinality $o(I)$ and $o(J)$ respectively;

x^k and y^k are independent vector variables in E_{n_k} for $k \in \{0\} \cup I \cup J$, and x^I and y^I denote the respective Cartesian products of the vector variables x^i, $i \in I$, and y^i, $i \in I$ while x^J and y^J denote the respective Cartesian products of the vector variables x^j, $j \in J$, and y^j, $j \in J$; so the Cartesian products $(x^0, x^I, x^J) \stackrel{\Delta}{=} x$ and $(y^0, y^I, y^J) \stackrel{\Delta}{=} y$ are independent vector variables in E_n, where

$$n \stackrel{\Delta}{=} n_0 + \sum_I n_i + \sum_J n_j;$$

α and λ are independent vector variables with respective components α_i and λ_i for $i \in I$, and β and κ are independent vector variables with respective components β_j and κ_j for $j \in J$;

X and Y are closed convex dual cones in E_n, and g_k and h_k are closed convex conjugate functions with respective domains $C_k \subseteq E_{n_k}$ and $D_k \subseteq E_{n_k}$ for $k \in \{0\} \cup I \cup J$.

Now, let

$$\mathcal{X} \stackrel{\Delta}{=} \{(x^0, x^I, \alpha, x^J, \kappa) \in E_{\eta} \mid (x^0, x^I, x^J) \in X;\ \alpha = 0;\ \kappa \in E_{o(J)}\},$$

where $n + o(I) + o(J) = \mathcal{n}$. In addition, let

$$\mathcal{C} \overset{\Delta}{=} \{(x^0, x^I, \alpha, x^J, \kappa) \in E_{\mathcal{n}} \mid x^0 \in C_0; x^i \in C_i,\ \alpha_i \in E_1, \text{ and } g_i(x^i) + \alpha_i \le 0,\ i \in I;\ (x^j, \kappa_j) \in C_j^+,\ j \in J\},$$

and let

$$\mathcal{g}(x^0, x^I, \alpha, x^I, \kappa) \overset{\Delta}{=} g_0(x^0) + \sum_J g_j^+(x^j, \kappa_j) \overset{\Delta}{=} G(x, \kappa),$$

where the (closed convex) function g_j^+ has a domain

$$C_j^+ \overset{\Delta}{=} \{(x^j, \kappa_j) \mid \text{either } \kappa_j = 0 \text{ and } \sup_{d^j \in D_j} \langle x^j, d^j \rangle < +\infty, \text{ or } \kappa_j > 0 \text{ and } x^j \in \kappa_j C_j\}$$

and functional values

$$g_j^+(x^j, \kappa_j) \overset{\Delta}{=} \begin{cases} \sup_{d^j \in D_j} \langle x^j, d^j \rangle & \text{if } \kappa_j = 0 \text{ and } \sup_{d^j \in D_j} \langle x^j, d^j \rangle < +\infty \\ \kappa_j g_j(x^j/\kappa_j) & \text{if } \kappa_j > 0 \text{ and } x^j \in \kappa_j C_j \end{cases}$$

Then, section 6 of [7] shows that

$$\mathcal{Y} = \{(y^0, y^I, \lambda, y^J, \beta) \in E_{\mathcal{n}} \mid (y^0, y^I, y^J) \in E_{o(I)};\ \beta = 0;\ \lambda \in E_{o(I)}\}.$$

Section 6 of [7] also shows that

$$\mathcal{D} = \{(y^0, y^I, \lambda, y^J, \beta) \in E_{\mathcal{n}} \mid y^0 \in D_0;\ (y^i, \lambda_i) \in D_i^+,\ i \in I;\ y^j \in D_j, \beta_j \in E_1, \text{ and } h_j(y^j) + \beta_j \le 0,\ j \in J\},$$

and that

$$\mathcal{h}(y^0, y^I, \lambda, y^J, \beta) = h_0(y^0) + \sum_I h_i^+(y^i, \lambda_i) \overset{\Delta}{=} H(y, \lambda),$$

where the (closed convex) function h_i^+ has a domain

$$D_i^+ \stackrel{\Delta}{=} \{(y^i,\lambda_i) \mid \text{either } \lambda_i = 0 \text{ and } \sup_{c^i \in C_i} \langle y^i, c^i\rangle < +\infty, \text{ or } \lambda_i > 0 \text{ and } y^i \in \lambda_i D_i\}$$

and functional values

$$h_i^+(y^i,\lambda_i) \stackrel{\Delta}{=} \begin{cases} \sup_{c^i \in C_i} \langle y^i, c^i\rangle & \text{if } \lambda_i = 0 \text{ and } \sup_{c^i \in C_i} \langle y^i,c^i\rangle < +\infty \\ \lambda_i h_i(y^i/\lambda_i) & \text{if } \lambda_i > 0 \text{ and } y^i \in \lambda_i D_i. \end{cases}$$

The symmetry of this formulation becomes even more apparent with the observation that the specified problem $\mathcal{A}$ consists essentially of minimizing $G(x,\kappa)$ for $x \in X$ and $g_i(x^i) \le 0$, $i \in I$ while the resulting problem $\mathcal{B}$ consists essentially of minimizing $H(y,\lambda)$ for $y \in Y$ and $h_j(y^j) \le 0$, $j \in J$. (In fact, this is the only known completely symmetric formulation of duality for general convex programming with explicit constraints.)

Now, fact (A) and the displayed formula for $\mathcal{C}$ imply that

$$(0^+\mathcal{C}) = \{(x^0,x^I,\alpha,x^J,\kappa) \in E_n \mid x^0 \in (0^+C_0);\ x^i \in E_{n_i},\ \alpha_i \in E_1, \text{ and}$$
$$\sup_{d^i \in D_i} \langle x^i, d^i\rangle + \alpha_i \le 0,\ i \in I;$$
$$(x^j,\kappa_j) \in C_j^+,\ j \in J\},$$

by virtue of both the equation

$$(0^+\{(x^i,\alpha_i) \mid x^i \in C_i, \alpha_i \in E_1, \text{ and } g_i(x^i) + \alpha_i \le 0\}) =$$
$$\{(x^i,\alpha_i) \mid x^i \in E_{n_i},\ \alpha_i \in E_1, \text{ and } \sup_{d^i \in D_i} \langle x^i,d^i\rangle + \alpha_i \le 0\}$$

and the equation

$$(0^+C_j^+) = C_j^+.$$

To derive the latter equation, simply observe that C_j^+ is a convex cone and then use Theorem 8.1 on page 61 of [12]. To derive the former equation, first note that

$$\{(x^i,\alpha_i) \mid x^i \in C_i, \alpha_i \in E_1, \text{ and } g_i(x^i)+\alpha_i \le 0\} =$$

$$\{(x^i,\alpha_i) \mid (x^i, -\alpha_i) \in (\text{epi } g_i)\};$$

and then infer from Corollary 8.3.4 on page 64 of [12] that

$$(0^+\{(x^i,\alpha_i) \mid (x^i, -\alpha_i) \in (\text{epi } g_i)\}) = \{(x^i,\alpha_i) \mid (x^i,-\alpha_i) \in (0^+ (\text{epi } g_i))\}.$$

Now, use the definition of $g_i 0^+$ at the top of page 66 in [12]; and then invoke Theorem 13.3 on page 116 of [12].

To determine whether the specified problem $\mathcal{A}$ is primal regular, simply note from the displayed formulas for $\mathcal{X}$ and $(0^+\mathcal{C})$ that

problem $\mathcal{A}$ is primal regular if and only if there is a nonzero (direction) vector $(\gamma^0,\gamma^I,\gamma^J,\gamma_J)$ such that: $(\gamma^0,\gamma^I,\gamma^J) \in X$; $\gamma^0 \in (0^+C_0)$; $\gamma^i \in E_{n_i}$ and $\sup_{d^i \in D_i} \langle \gamma^i, d^i\rangle \le 0$, $i \in I$; $(\gamma^j,\gamma_j) \in C_j^+$, $j \in J$ (where γ_J has components γ_j for $j \in J$).

To determine whether the specified problem $\mathcal{A}$ is dual regular, note from facts (B) and (C) and the displayed formulas for $\mathcal{X}$, $\mathcal{C}$ and $(0^+\mathcal{C})$ that

problem $\mathcal{A}$ is dual regular if and only if: either X is not a vector space, or (0^+C_0) is not a vector space, or $\{(x^i,\alpha_i) \mid x^i \in E_{n_i}, \alpha_i \in E_1, \text{ and } \sup_{d^i \in D_i} \langle x^i,d^i\rangle + \alpha_i \le 0\}$ is not a vector space for some $i \in I$, or J is not empty; and both

$(0^+C_0)^\circ \subseteq (\Delta C_0)$ and $\{(x^i,\alpha_i) \mid x^i \in E_{n_i},\ \alpha_i \in E_1$, and

$\sup_{d^i \in D_i} \langle x^i,d^i\rangle + \alpha_i \le 0\}^\circ \subseteq (\Delta\{(x^i,\alpha_i) \mid x^i \in C_i,\ \alpha_i \in E_1,$

and $g_i(x^i) + \alpha_i \le 0\})$ for each $i \in I$.

Needless to say, the state of regularity for the resulting problem $\mathcal{B}$ can be determined with the aid of the dual of each of the preceding displayed statements (obtained by interchanging the symbols $\mathcal{A}$ and $\mathcal{B}$, the symbols γ and δ, the symbols X and Y, the symbols C and D, the symbols g and h, the symbols I and J, the symbols x and y, the symbols α and β, the symbols i and j, and the symbols c and d).

The theory established in section 2 can now be applied to the constrained cases of the following special programming types: (prototype) posynomial programming, linear programming, (convex) ordinary programming, and chemical equilibrium analysis -- simply by consulting examples 6 through 9 respectively in section 2 of [8]. Moreover, the interested reader should also have no trouble making applications to (convex) quadratic programming (with either linear or quadratic constraints), constrained ℓ_p-regression analysis, and constrained (convex) optimal location -- simply by consulting the references alluded to in section 2 of [8]. In particular, it is not difficult to see that ordinary programming and chemical equilibrium analysis are the only such examples for which the resulting problem $\mathcal{B}$ is not necessarily dual regular. In fact, for all other such examples, $(0^+\mathcal{D})$ is not a vector space because I is not empty; and $(0^+\mathcal{D})^\circ \subseteq (\Delta\mathcal{D})$ because J is empty and because rather elementary computations show that $(0^+D_0)^\circ = (\Delta D_0)$. Although the state of regularity for ordinary programming depends on the particular nature of the functions g_0 and g_i, $i \in I$, the specified problem $\mathcal{A}$ is always dual regular in the context of chemical equilibrium analysis. In fact, for chemical equilibrium analysis, $(0^+\mathcal{C})$ is not a vector space because J is not empty; and $(0^+\mathcal{C})^\circ \subseteq (\Delta\mathcal{C})$ because I is empty and because rather elementary computations show that $(0^+ C_0)^\circ = (\Delta C_0)$. Moreover, in chemical equilibrium analysis, the signs of the components of the specified generators for $\mathcal{X}^\perp$ along with the specific nature of C_j^+, $j \in J$ readily imply that the resulting problem $\mathcal{A}$ is never primal regular; so Theorem 2 and the (unstated) dual of Theorem 1 in Appendix A clearly imply that $\mathcal{T}$, but not $\mathcal{S}$, is unbounded in chemical equilibrium analysis.

Although the preceding observation about linear programming produces the complementarity theorem of Clark [2], stronger complementarity theorems about linear inequalities have subsequently been given by Williams [13,14]. Moreover, stronger complementarity theorems about posynomial programming have subsequently been given by Avriel and Williams [1]; and stronger complementarity theorems about some of the other special programming types are, no doubt, possible. However, the generalizability of such theorems to much broader classes of convex programming problems has, to the best of the author's knowledge, not yet been investigated.

3.2. <u>The Fenchel-Rockafellar Formulation</u>. Let

$$\mathcal{X} \overset{\Delta}{=} \text{the column space of } \begin{bmatrix} I_r \\ M \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix, M is an arbitrary $s \times r$ matrix, and $r + s = n$. In addition, let

$$\mathcal{C} \overset{\Delta}{=} C^1 \times C^2 \text{ and } g(x) \overset{\Delta}{=} g^1(x^1) - g^2(x^2),$$

where the closed convex function g^1 has domain $C^1 \subseteq E_r$, and the closed <u>con-</u>

<u>cave</u> function g^2 has domain $C^2 \subseteq E_s$.

Then, section 5 of [6] shows that

$$\mathcal{Y} = \text{the column space of } \begin{bmatrix} M^t \\ -I_s \end{bmatrix},$$

where M^t is the transpose of M, and I_s is the $s \times s$ identity matrix. Section 5 of [6] also shows that

$$\mathcal{D} = D^1 \times \left[-D^2\right] \text{ and } h(y) = h^1(y^1) - h^2(-y^2),$$

where $h^1:D^1$ is the (convex) conjugate transform of $g^1:C^1$, and $h^2:D^2$ is the <u>concave</u> conjugate transform of $g^2:C^2$ -- obtained by replacing sup with inf in the definition of the conjugate transform.

The symmetry of this formulation becomes apparent with the observation that the specified problem $\mathcal{A}$ obviously consists of minimizing $g^1(z^r) - g^2(Mz^r)$ for $z^r \in C^1$ and $Mz^r \in C^2$ while the resulting problem $\mathcal{B}$ obviously consists of minimizing $h^1(M^t z^s) - h^2(z^s)$ for $M^t z^s \in D^1$ and $z^s \in D^2$. (Actually, Fenchel and Rockafellar prefer to rephrase the resulting problem $\mathcal{B}$ in the (still symmetric) form of maximizing $h^2(z^s) - h^1(M^t z^s)$ for $z^s \in D^2$ and $M^t z^s \in D^1$.)

Now, fact (A) and the displayed formula for $\mathcal{C}$ imply that

$$(0^+\mathcal{C}) = (0^+ C^1) \times (0^+ C^2).$$

To determine whether the specified problem $\mathcal{A}$ is primal regular, simply note from the displayed formulas for $\mathcal{X}$ and $(0^+\mathcal{C})$ that

problem $\mathcal{A}$ is primal regular if and only if there is a nonzero vector γ^r such that $\gamma^r \in (0^+ C^1)$ and $M\gamma^r \in (0^+ C^2)$.

To determine whether the specified problem $\mathcal{A}$ is dual regular, simply note from facts (B) and (C) and the displayed formulas for $\mathcal{X}$ and $(0^+\mathcal{C})$ that

> problem $\mathcal{A}$ is dual regular if and only if either $(0^+ C^1)$ or $(0^+ C^2)$ is not a vector space and both $(0^+ C^1)^\circ \subseteq (\Delta C^1)$ and $(0^+ C^2)^\circ \subseteq (\Delta C^2)$.

Needless to say, the state of regularity for the resulting problem $\mathcal{B}$ can be determined with the aid of the dual of each of the preceding dis-payed statements (obtained by interchanging the symbols $\mathcal{A}$ and $\mathcal{B}$, the symbols γ^r and δ^s, the symbols C^1 and D^2, the symbols C^2 and D^1, and the symbols M and M^t).

It should be mentioned that the original Fenchel formulation can be obtained by letting $M \stackrel{\Delta}{=} I_r$, in which case $s = r$. Moreover, it is worth noting that the (reverse) specialization described by the "note added in proof" at the end of [6] shows that the counter examples in Appendix B serve equally well as counterexamples in the context of the original Fenchel formulation (and hence the preceding Fenchel-Rockafellar Formulation).

3.3. <u>The Rockafellar Formulation</u>. Let

$$\mathcal{X} \stackrel{\Delta}{=} \text{the column space of } \begin{bmatrix} I_p \\ 0_q \end{bmatrix},$$

where I_p is the $p \times p$ identity matrix, 0_q is the $q \times p$ zero matrix, and $p + q = n$.

Then, section 5 of [6] shows that

$$\mathcal{Y} = \text{the column space of } \begin{bmatrix} 0_p \\ I_q \end{bmatrix},$$

where I_q is the $q \times q$ identity matrix and 0_p is the $p \times q$ zero matrix.

The symmetry of this formulation becomes even more apparent with the observation that the specified problem $\mathcal{A}$ obviously consists of minimizing $g(z^p,0)$ for $(z^p,0) \in \mathcal{C}$ while the resulting problem $\mathcal{B}$ obviously consists of minimizing $h(0,z^q)$ for $(0,z^q) \in \mathcal{D}$. (Actually, Rockafellar prefers to rephrase the resulting problem $\mathcal{B}$ in the (still symmetric) form of a maximiation problem by placing minus signs at crucial (and difficult to remember) places, but those details need not be of any direct concern here.)

To determine whether the specified problem $\mathcal{A}$ is primal regular, simply note from the displayed formula for $\mathcal{X}$ that

> problem $\mathcal{A}$ is primal regular if and only if there is a nonzero vector γ^p such that $(\gamma^p,0) \in (0^+\mathcal{C})$.

To determine whether the specified problem $\mathcal{A}$ is dual regular, simply note from the displayed formula for $\mathcal{X}$ that

> problem $\mathcal{A}$ is dual regular if and only if $(0^+\mathcal{C})$ is not a vector space and $(0^+\mathcal{C})^\circ \subseteq (\Delta\mathcal{C})$.

Needless to say, the state of regularity for the resulting problem $\mathcal{B}$ can be determined with the aid of the dual of each of the preceding displayed statements (obtained by interchanging the symbols $\mathcal{A}$ and $\mathcal{B}$, the symbols γ^p and δ^q, the symbols $(\gamma^p,0)$ and $(0,\delta^q)$, and the symbols $\mathcal{C}$ and $\mathcal{D}$).

It is worth noting that the (reverse) specialization described in Appendix A of [6] shows that the counterexamples in Appendix B (of this paper) serve equally well as counterexamples in the context of the preceding Rockafellar formulation (and hence the original Rockafellar formulation).

3.4. <u>The Ordinary Lagrangian Formulation</u>. In the context of the preceding Rockafellar formulation, suppose that

$$x \stackrel{\Delta}{=} (z,u) \text{ where } z \in E_p \text{ and } u \in E_q .$$

Then, let

$$\mathcal{C} \stackrel{\Delta}{=} \{(z,u) \mid z \in C \text{ and } G_i(z) + u_i \le 0,\ i \in I\} \text{ and } g(z,u) \stackrel{\Delta}{=} G_0(z),$$

where I is a (possibly empty) positive - integer index set with finite cardinality o(I), and the G_k, $k \in \{0\} \cup I$, are closed convex functions with a common domain $C \subseteq E_p$.

Now, suppose that

$$y \stackrel{\Delta}{=} (\mu,\lambda) \text{ where } \mu \in E_p \text{ and } \lambda \in E_q .$$

Then, an elementary computation (given essentially in section 30 of [12]) shows that

$$\mathcal{D} = \{(\mu,\lambda) \mid \lambda \ge 0 \text{ and } \sup_{z \in C} [\langle \mu, z\rangle - G_0(z) - \sum_I \lambda_i G_i(z)] < +\infty\}$$

and

$$h(\mu,\lambda) = \sup_{z \in C} [\langle \mu, z\rangle - G_0(z) - \sum_I \lambda_i G_i(z)].$$

The lack of any transparent symmetry in this formulation becomes apparent with the observation that the specified problem $\mathcal{A}$ obviously consists of minimizing $G_0(z)$ for $z \in C$ and $G_i(z) \le 0$, $i \in I$ while the resulting problem $\mathcal{B}$ obviously consists of minimizing over $\lambda \ge 0$ the supremum over $z \in C$

of the negative of the ordinary Lagrangian $G_0(z) + \sum_I \lambda_i G_i(z)$. (Actually, standard treatises prefer to rephrase the resulting problem $\mathcal{B}$ in the (still unsymmetric) form of maximizing over $\lambda \geq 0$ the infimum over $z \in C$ of the ordinary Lagrangian $G_0(z) + \sum_I \lambda_i G_i(z)$.)

Now, the displayed formula for $\mathcal{C}$ implies that

$$(0^+\mathcal{C}) = \bigcap_I (0^+\{(z,u) \mid z \in C \text{ and } G_i(z) + u_i \leq 0\}),$$

by virtue of Corollary 8.3.3 on page 64 of [12].

To determine whether the specified problem $\mathcal{A}$ is primal regular, note from the corresponding displayed statement in the preceding subsection 3.3 and the displayed formula for $(0^+\mathcal{C})$ along with Theorem 8.6 on page 68 of [12] that

> problem $\mathcal{A}$ is primal regular if and only if there is a nonzero vector γ^p such that γ^p is in the "recession cone" of G_i for each $i \in I$.

To determine whether the specified problem $\mathcal{A}$ is dual regular, note from the corresponding displayed statement in the preceding subsection 3.3 and the displayed formulas for $\mathcal{C}$ and $(0^+\mathcal{C})$ that

> problem $\mathcal{A}$ is dual regular if and only if $\bigcap_I (0^+\{(z,u) \mid z \in C$ and $G_i(z) + u_i \leq 0\})$ is not a vector space and $[\bigcap_I (0^+\{(z,u) \mid z \in C$ and $G_i(z) + u_i \leq 0\})]^\circ \subseteq (\Delta \bigcap_I \{(z,u) \mid z \in C$ and $G_i(z) + u_i \leq 0\})$.

Needless to say, the lack of any transparent symmetry in this formulation necessitates another return to the preceding subsection 3.3 to characterize the state of regularity for the resulting problem $\mathcal{B}$.

The characterization that comes from also using Theorem 8.6 on page 68 of [12] is that

problem $\mathcal{B}$ is primal regular if and only if there is a nonzero vector δ^q such that δ^q is in the recession cone of $\{\lambda \geq 0 \mid \sup_{z \in C} [\langle \mu, z \rangle - G_0(z) - \sum_I \lambda_i G_i(z)] < +\infty\}$ for at least one vector $\mu \in E_p$ for which $\{\lambda \geq 0 \mid \sup_{z \in C} [\langle \mu, z \rangle - G_0(z) - \sum_I \lambda_i G_i(z)] < +\infty\}$ is nonempty.

and

problem $\mathcal{B}$ is dual regular if and only if $(0^+\{(\mu,\lambda) \mid \lambda \geq 0$ and $\sup_{z \in C} [\langle \mu, z \rangle - G_0(z) - \sum_I \lambda_i G_i(z)] < +\infty\})$ is not a vector space and $(0^+\{(\mu,\lambda) \mid \lambda \geq 0$ and $\sup_{z \in C} [\langle \mu, z \rangle - G_0(z) - \sum_I \lambda_i G_i(z)] < +\infty\})^\circ \subseteq (\Delta\{(\mu,\lambda) \mid \lambda \geq 0$ and $\sup_{z \in C} [\langle \mu, z \rangle - G_0(z) - \sum_I \lambda_i G_i(z)] < +\infty\})$.

It is worth noting that the (reverse) specialization indicated at the top of page 298 in [12] shows that the counterexamples in Appendix B serve equally well as counterexamples in the context of the preceding ordinary Lagrangian formulation.

Appendix A. The following sequence of theorems culminates in a proof for the proposition stated in section 2. (Actually, it may be worth noting that the closedness of $\mathcal{X}$ and $\mathcal{G}$ hypothesized in section 2 is needed in neither the proof of Theorem 1 nor the proof of Theorem 2 -- though such closedness is needed when utilizing the duals of Theorem 1 and

Theorem 2 to prove the proposition.)

The following theorem proves the proposition under the much stronger assumptions of primal regularity and consistency for problem $\mathcal{A}$

Theorem 1. Given that

$$\text{there is a nonzero (direction) vector } \gamma \in \mathcal{X} \cap (0^{+}\mathcal{C}),$$

if $\mathcal{S}$ is nonempty, then $\mathcal{S}$ is unbounded.

Proof. Given an $x \in \mathcal{X} \cap \mathcal{C}$, the convex conicality of $\mathcal{X}$ and the definition of $(0^{+}\mathcal{C})$ imply that $x+s\gamma \in \mathcal{X} \cap \mathcal{C}$ for every $s \geq 0$; so $\mathcal{X} \cap \mathcal{C}$ is unbounded (in the direction γ). q.e.d.

The following theorem shows that dual regularity for problem $\mathcal{A}$ along with a lack of primal regularity for problem $\mathcal{A}$ implies primal regularity for problem $\mathcal{B}$.

Theorem 2. Given that

either $\mathcal{X}$ or $(0^{+}\mathcal{C})$ is not a vector space

and given that

$$(0^{+}\mathcal{C})^{\circ} \subseteq (\Delta\mathcal{C}),$$

if

$$\text{there is no nonzero (direction) vector } \gamma \in \mathcal{X} \cap (0^{+}\mathcal{C}),$$

then

$$\text{there is a nonzero (direction) vector } \delta \in \mathcal{Y} \cap (0^{+}\mathcal{D}).$$

Proof. The following lemmas are needed and may be of some interest in their own right.

Lemma 1. Each convex cone K in E_n for which $0 \in (\mathrm{ri}\, K)$ is actually a vector space (where the "relative interior" (ri K) of K is defined to be the "interior" of K "relative to" the "affine hull" of K).

Proof. Such a K is obviously nonempty, and hence Theorem 6.4 on page 47 of [12] readily implies that such a $K = -K$; from which we infer via Theorem 2.7 on page 15 of [12] that such a K is actually a vector space. q.e.d.

Lemma 2. Given that either $\mathcal{X}$ or another fixed nonempty convex cone $\mathcal{K}$ in E_n is not a vector space, if there is no nonzero (direction) vector $\gamma \in \mathcal{X} \cap \mathcal{K}$, then there is a nonzero (direction) vector $\delta \in \mathcal{Y} \cap \mathcal{K}^{\circ}$.

Proof. Our assumptions clearly imply that $(\mathrm{ri}\,\mathcal{X}) \cap (\mathrm{ri}\,\mathcal{K})$ is empty, by virtue of Lemma 1 and the obvious fact that $(\mathrm{ri}\,\mathcal{X}) \cap (\mathrm{ri}\,\mathcal{K}) \subseteq \mathcal{X} \cap \mathcal{K}$. Taking account of Theorem 6.2 on page 45 of [12], we now infer from Theorem 11.3 on page 97 of [12] that there is a hyperplane "separating $\mathcal{X}$ and $\mathcal{K}$ properly". According to Theorem 11.1 on page 95 of [12], this means that there is a vector $\delta \in E_n$ such that

$$\sup_{\gamma \in \mathcal{X}} \langle \gamma, \delta \rangle > \inf_{\gamma \in \mathcal{K}} \langle \gamma, \delta \rangle$$

and such that

$$\inf_{\gamma \in \mathcal{X}} \langle \gamma, \delta \rangle \geq \sup_{\gamma \in \mathcal{K}} \langle \gamma, \delta \rangle .$$

The (strictness of the) former inequality obviously implies that δ is not the zero vector; and the latter inequality along with the conicality of $\mathcal{X}$ and $\mathcal{K}$ clearly implies that

$$\inf_{\gamma \in \mathcal{X}} \langle \gamma, \delta \rangle = 0 = \sup_{\gamma \in \mathcal{K}} \langle \gamma, \delta \rangle .$$

Finally, the first and second equations in the preceding relation simply assert that $\delta \in \mathcal{Y}$ and $\delta \in \mathcal{K}^{o}$ respectively. q.e.d.

Lemma 3. The barrier cone $(\Delta\mathcal{C})$ is a subcone of the recession cone $(0^{+}\mathcal{D})$.

Proof. Obviously, each δ in $(\Delta\mathcal{C})$ has the property that $\sup_{c \in \mathcal{C}} \langle c, \delta \rangle < +\infty$ and according to Theorem 13.3 on page 116 of [12] such a "support" δ for $\mathcal{C}$ must be in the domain for the "recession function" of h. Finally, Theorem 12.2 on page 104 of [12] and (the final assertion of) Theorem 8.5 on page 66 of [12] clearly imply that such a δ must be in $(0^{+}\mathcal{D})$. q.e.d.

Now, Theorem 8.1 on page 61 of [12] asserts that $(0^{+}\mathcal{C})$ is a nonempty convex cone. Hence, letting $\mathcal{K} \stackrel{\Delta}{=} (0^{+}\mathcal{C})$, we immediately infer from Lemma 2 and Lemma 3 the desired conclusion of Theorem 2. q.e.d.

The following theorem shows that consistency for problem $\mathcal{A}$ implies that either $\mathcal{S}$ is unbounded or problem $\mathcal{B}$ is consistent.

Theorem 3. If $\mathcal{S}$ is nonempty but bounded, then $\mathcal{T}$ is nonempty (in fact, $\mathcal{T}$ contains a vector $y^{o} \in (\text{ri}\,\mathcal{Y}) \cap (\text{ri}\,\mathcal{D})$, and hence problems $\mathcal{A}$ and $\mathcal{B}$ have no duality gap).

Proof. The assumptions obviously imply that each objective function "level set" $\mathcal{L}_r \overset{\Delta}{=} \{x \in \mathcal{S} \mid g(x) \leq r\}$ is bounded; so Theorem 3 in [9] asserts the existence of a vector $y^o \in (\text{ri}\,\mathcal{Y}) \cap (\text{ri}\,\mathcal{D})$. The absence of a duality gap is then asserted by the (geometric programming) version of "Fenchel's theorem" given as Theorem 31.4 on page 335 of [12]. q.e.d.

To prove the proposition, we first consider the case in which problem $\mathcal{A}$ is regular. In doing so, we also consider two mutually exhaustive subcases.

If $\mathcal{T}$ is nonempty, then either $\mathcal{T}$ is unbounded (in which event our proof is complete) or $\mathcal{T}$ is bounded. In the latter event, the (unstated) dual of Theorem 3 asserts that $\mathcal{S}$ is nonempty -- the defining condition for the second subcase.

If $\mathcal{S}$ is nonempty, then either $\mathcal{S}$ is unbounded (in which event our proof is complete) or $\mathcal{S}$ is bounded. In the latter event, Theorem 1 clearly implies that problem $\mathcal{A}$ is not primal regular; from which we infer via our regularity assumption and Theorem 2 that problem $\mathcal{B}$ must be primal regular. Since Theorem 3 asserts that $\mathcal{T}$ must be nonempty, we now conclude from the (unstated) dual of Theorem 1 that $\mathcal{T}$ must be unbounded.

To treat the only other possible case -- the case in which problem $\mathcal{B}$ is regular -- simply utilize the dual of the preceding argument. q.e.d.

Appendix B. The following counterexamples place limitations on any strengthening of the proposition stated in section 2.

Counterexample 1. Let $\mathcal{X}$ be an arbitrary vector space in E_n, and let g be the identically zero function on $\mathcal{C} \overset{\Delta}{=} \mathcal{X}^{\perp}$.

Then, $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal complementary vector spaces in E_n, and h is the identically zero function on $\mathcal{D}=\mathcal{Y}^{\perp}$.

Obviously, neither this problem $\mathcal{A}$ nor this problem $\mathcal{B}$ is regular. Moreover, note that $\mathcal{S}=\mathcal{X}\cap\mathcal{X}^{\perp}=\{0\}$ and $\mathcal{T}=\mathcal{Y}\cap\mathcal{Y}^{\perp}=\{0\}$. Consequently, both $\mathcal{S}$ and $\mathcal{T}$ are nonempty, but neither $\mathcal{S}$ nor $\mathcal{T}$ is unbounded.

Counterexample 2. Let $\mathcal{X}$ be an arbitrary nontrivial vector space in E_n, and let g be the identically zero function on $\mathcal{C}\overset{\Delta}{=}\mathcal{X}$.

Then, $\mathcal{X}$ and $\mathcal{Y}$ are nontrivial orthogonal complementary vector spaces in E_n, and h is the identically zero function on $\mathcal{D}=\mathcal{Y}$.

Obviously, both this problem $\mathcal{A}$ and this problem $\mathcal{B}$ are primal regular (and hence regular). Moreover, note that $\mathcal{S}=\mathcal{X}\cap\mathcal{X}=\mathcal{X}$ and $\mathcal{T}=\mathcal{Y}\cap\mathcal{Y}=\mathcal{Y}$. Consequently, both $\mathcal{S}$ and $\mathcal{T}$ are (nonempty) nontrivial vector spaces, and (hence) both $\mathcal{S}$ and $\mathcal{T}$ are unbounded.

Note that in both of these counterexamples the infima φ and ψ are both zero; so in both of these counterexamples $0=\varphi+\psi$, and hence both of these pairs of dual problems $\mathcal{A}$ and $\mathcal{B}$ do not have a duality gap.

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