

Discussion Paper No. 162

ON THE NUMBER OF SOLUTIONS TO THE
LINEAR COMPLEMENTARITY PROBLEM

by

Arie Tamir

July 1975
(Revised)

ABSTRACT

Given an $n \times n$ matrix A and an n -dimensional vector q let $N(A,q)$ be the cardinality of the set of solutions to the linear complementarity problem defined by A and q . It is shown that if A is nondegenerate then $N(A,q) + N(A,-q) \leq 2^n$, which in turn implies $N(A,q) \leq 2^{n-1}$ if A is also a Q -matrix.

It is then demonstrated that $\min_{q \neq 0} N(A,q) \leq 2^{n-1} - 1$, which concludes that the complementary cones cannot span \mathbb{R}^n more than $2^{n-1} - 1$ times around. For any n , an example of an $n \times n$ nondegenerate Q -matrix spanning all \mathbb{R}^n , but a subset of empty interior, $2^{\lfloor \frac{n}{3} \rfloor}$ times around is given.

Given a square real matrix A of order n and an n -dimensional vector q , the Linear Complementarity Problem, (LCP), denoted by (q,A) , is to find n -dimensional vectors x and y satisfying

$$y = Ax + q, \quad x \geq 0, \quad y \geq 0 \quad x'y = 0 \quad (1)$$

(See [3,5,6,7] for the bibliography and history of the LCP.)

Letting $N(A,q)$ be the cardinality of the set of solutions to (q,A) we study the quantities $N^*(A) = \max_{q \in \mathbb{R}^n} N(A,q)$ and $N_*(A) = \min_{0 \neq q \in \mathbb{R}^n} N(A,q)$ and establish bounds on both. The latter quantity has the following geometrical interpretation. If we consider the 2^n complementary cones generated by the columns of $[I,-A]$, (see [8]), then $N_*(A)$ is the number of times \mathbb{R}^n is spanned around by these cones.

This study shows that this (spanning) number is bounded above by $2^{n-1}-1$ if the matrix A is nondegenerate, i.e. all its principal minors are nonzero. For any n , an example of an $n \times n$ nondegenerate matrix spanning all \mathbb{R}^n , but a subset of empty interior, $2^{\lfloor \frac{n}{3} \rfloor}$ times around is given.

It is also demonstrated that the nondegeneracy of A implies $N(A,q) + N(A,-q) \leq 2^n$ for all q . Thus, if A is a Q -matrix (i.e. $N(A,q) \geq 1$ for all q) then $N^*(A) \leq 2^{n-1}$.

This work was motivated by a conjecture due to Ingleton [4] and Censor [1]. Studying nondegenerate matrices Ingleton showed that if for some q^* (q^*,A) has a unique solution (x^*,y^*) and $x^* + y^*$ is strictly positive, then A is a Q -matrix. (Extensions of this result appear in [5,8,9,10]). Ingleton conjectured that under the above conditions the maximal number of solutions to (q,A) is 2^{n-1} , provided that $N(A,\bar{q}) > 1$ for some \bar{q} .

The conjecture was studied and partially resolved by Censor, [1], who also raised the following general question: Given an $n \times n$ nondegenerate Q -matrix A and supposing that $N(A, \bar{q}) > 1$ for some \bar{q} , is the maximal number of solutions to (1) $2^n - 1$ for all q in R^n ?

The following result demonstrates that $2^n - 1$ is indeed a valid bound. (We later show that the bound is not necessarily attained.)

Lemma 1: Let A be an $n \times n$ matrix such that $N(A, q) \geq 1$ for all $q \leq 0$. Then for any q $N(A, q) < \infty$ implies $N(A, q) \leq 2^n - 1$.

Proof:

For any $I \subseteq \{1, 2, \dots, n\}$ consider the (complementary) cone generated by the n columns $-A \cdot i$, $i \in I$ and e_i , $i \in \bar{I}$, where $A \cdot i$ is the i^{th} column of A and e_i is the i^{th} unit vector. Note that the correspondence between subsets of $\{1, 2, \dots, n\}$ and the complementary cones is not necessarily one to one.

It is obvious that if there are two different solutions to (2)

$$y = Ax + q, x \geq 0, y \geq 0, x'y = 0, \quad \begin{array}{l} x_i > 0 \quad i \in I \\ \text{and } x_i = 0 \quad i \notin I \end{array} \quad (2)$$

for some $I \subseteq \{1, \dots, n\}$ then $N(A, q)$ is not finite. (Consider convex combinations of these two solutions.)

Hence, if $N(A, q)$ is finite each set $I \subseteq \{1, 2, \dots, n\}$ contributes at most one solution to (q, A) and $N(A, q) \leq 2^n$. If $q \not\leq 0$, then the cone defined by $I = \emptyset$ does not contribute a solution and $N(A, q) \leq 2^n - 1$. Note that $N(A, 0)$ is finite only if $N(A, 0) = 1$. Consider now a non-zero, nonnegative q . By the lemma's assumption $(-q, A)$ has a solution.

Defining a cone to be nondegenerate if it has a nonempty interior, we observe that the union of all the degenerate complementary cones and the (proper) faces of the nondegenerate cones is closed and nowhere dense in \mathbb{R}^n . Thus there exists a sequence $\{q^k\}$, converging to $-q$ where $\forall k \ q^k \leq 0$ and q^k belongs to a nondegenerate cone. Since there is a finite number of complementary cones we can assume without loss of generality (choose a subsequence if necessary) that $\forall k \ q^k$ belongs to the same nondegenerate cone. Hence $-q$ is in that cone. It is then clear that q is not contained in this nondegenerate cone, since otherwise we would have that the n generators are linearly dependent - a contradiction to the nondegeneracy assumption. Hence $N(A,q) \leq 2^n - 1$.

From Lemma 1 $N^*(A) \leq 2^n - 1$ for any nondegenerate Q-matrix A. Censor, [1], provided an example of an $n \times n$ nondegenerate Q-matrix, (which we denote by A_n), such that $N^*(A_n) = 2^n - 1$. Murty, [8], illustrated that the matrix

$$M = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad (3)$$

is a nondegenerate Q-matrix and $N^*(M) = 4$. Combining the above two illustrations and using the direct sum operation we show that for each $n \ 2^n - 1$ is not always a sharp bound.

Recalling that the direct sum of the square matrices A and B, $A \oplus B$, is given by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

we notice that

$$N(A \oplus B, q) = N(A, q^1) N(B, q^2) \quad (4)$$

where (q^1, q^2) is the corresponding partition of q .

Lemma 2: Given nonnegative integers n, r, t , such that $3t+r \leq n$ there exists an $n \times n$ nondegenerate Q -matrix A such that $N^*(A) = 4^t \max\{1, 2^r - 1\}$.

Proof: Define $A = \underbrace{M \oplus \dots \oplus M}_t \oplus A_r \oplus I_{n-3t-r}$, where M is given by (3),

A_r is the above mentioned matrix due to Censor [1] and I_{n-3t-r} is the identity of order $n-3t-r$.

Having established an upper bound on $N(A, q)$, we next turn to the lower bound question and prove that if A is an $n \times n$ nondegenerate Q -matrix then $\min_{q \neq 0} N(A, q) \leq 2^{n-1} - 1$, provided $n \geq 2$.

Lemma 3: Let A be an $n \times n$ matrix and $q \in \mathbb{R}^n$. If $N(A, q) + N(A, -q) < \infty$, then $N(A, q) + N(A, -q) \leq 2^n$.

Proof:

If $q = 0$, then $N(A, 0) < \infty$ implies $N(A, 0) = 1$. Thus consider $q \neq 0$. Let $I \subseteq \{1, 2, \dots, n\}$, then we show that the complementary cone corresponding to I does not contain both q and $-q$. This will prove the lemma. Observe first that if the cone corresponding to I is nondegenerate, i.e. has an interior, then $q \neq 0$ implies that the cone cannot contain both q and $-q$. Suppose that there exists a cone C containing q and $-q$. Let B be an $n \times n$ matrix corresponding to

the n generators of C , then $Bx^1 = q$ and $Bx^2 = -q$ for some $x^1 \geq 0$ and $x^2 \geq 0$. Hence $\forall t \geq 0$ $B[(1+t)x^1 + tx^2] = q$ and there exists a (complementary) ray of solutions - a contradiction to $N(A,q) < \infty$.

Note that Lemma 3, which has been proved independently of Lemma 1, also implies that $N(A,q) \leq 2^{n-1}$ for any nondegenerate $n \times n$ Q -matrix. Another consequence is that $\min_{q \neq 0} N(A,q) \leq 2^{n-1}$.

We now refine this bound to $2^{n-1}-1$, applying a result due to Murty [8].

Theorem 4: Let A be an $n \times n$ nondegenerate matrix.

Then $\min_{q \neq 0} N(A,q) \leq 2^{n-1}-1$ when $n \geq 2$.

Proof:

From Lemma 3 it follows that $\min_{q \neq 0} N(A,q) \leq 2^{n-1}$. Suppose that for all $q \neq 0$ $N(A,q) = 2^{n-1}$. We then apply Theorem (7.2) of [8] to have $N(A,q) = 1$ for all q . Hence the proof is complete.

Theorem 4 concludes that the complementary cones cannot span the space R^n more than $2^{n-1}-1$ times around. This bound is trivially attained for 2×2 Q -matrices.

Finally we show that for any $n \geq 3$ there exists an $n \times n$ nondegenerate Q -matrix A such that the complementary cones defined by the columns of $[I, -A]$ span $R^n - R(A)$ $2^{\lfloor \frac{n}{3} \rfloor}$ times around, where $R(A)$ is a subset of R^n with empty interior.

To define $R(A)$ precisely, we recall that q in R^n is said to be nondegenerate with respect to the $n \times n$ matrix A if for any (x,y)

solving $(q,A) x + y$ is a strictly positive vector (i.e. q is neither contained in a complementary cone with no interior nor in a proper face of a nondegenerate cone). q is degenerate if it is not nondegenerate. $R(A)$ is defined as the set of vectors in R^n that are degenerate with respect to A .

We need the following lemma for our discussion.

Lemma 5: Let A be an $n \times n$ nondegenerate matrix and consider q in R^n which is not contained in any face of dimension less than $n - 1$ of a complementary cone defined by the columns of $[I, -A]$. Then, $\forall \epsilon > 0$ there exist nondegenerate vectors P^1 and P^2 such that

$$\|P^1 - q\| < \epsilon, \|P^2 - q\| < \epsilon \text{ and } N(A, q) = \frac{1}{2}[N(A, P^1) + N(A, P^2)].$$

Proof:

If q itself is nondegenerate choose $P^1 = P^2 = q$.

Suppose that q is contained in the intersection of k different $(n-1)$ dimensional faces C_1, \dots, C_k of complementary cones but not in any face of a lower dimension. Choose $\epsilon > 0$ such that $S(q, \epsilon) = \{x \mid \|x - q\| \leq \epsilon\}$ is contained in any complementary cone having q in its interior, and such that each complementary cone having one of C_1, \dots, C_k as its $(n-1)$ dimensional face contains the corresponding "half" ball chopped off by that face. ϵ is well defined since each complementary cone is nondegenerate and q is not in any face of dimension less than $n-1$.

Let $u \in \mathbb{R}^n$, $\|u\| < \epsilon$, such that $q + u$ and $q - u$ are nondegenerate vectors. We then have that $P^1 = q + u$ and $P^2 = q - u$ are on opposite sides of each C_i , $i=1, \dots, k$. Also, P^1 and P^2 are contained in any complementary cone having q in its interior.

Consider the i^{th} face C_i , and let t_i be the number of different complementary cones having C_i as their face. Using the assumption that q is not on a face of dimension less than $n - 1$ we may suppose, without loss of generality, that \bar{t}_i cones, $0 \leq \bar{t}_i \leq t_i$ contain P^1 and $t_i - \bar{t}_i$ contain P^2 .

Let \bar{T}_i ($T_i - \bar{T}_i$) be the cardinality of the set of solutions to (P^1, A) ((P^2, A)) corresponding to the above \bar{t}_i ($t_i - \bar{t}_i$) cones. (Note that we introduce T_i and \bar{T}_i since the correspondence of subsets of $\{1, 2, \dots, n\}$ to the different complementary cones is not injective.) Note that T_i is even by the nondegeneracy assumption. Thus the t_i cones contribute $\frac{T_i}{2}$ solutions to (q, A) , \bar{T}_i solutions to (P^1, A) and $T_i - \bar{T}_i$ solutions to (P^2, A) . Noting that P^1 and P^2 are on opposite sides of each C_i , $i=1, \dots, k$ and denoting by r the number of nondegenerate solutions to (q, A) we have

$$N(A, P^1) = r + \sum_{i=1}^k \bar{T}_i, \quad N(A, P^2) = r + \sum_{i=1}^k (T_i - \bar{T}_i) \quad \text{and} \quad N(A, q) = r + \sum_{i=1}^k \frac{T_i}{2}$$

Hence $N(A, \bar{q}) = \frac{1}{2}[N(A, P^1) + N(A, P^2)]$ and the proof is complete.

Dealing with the 3×3 matrix M defined in (3), Murty [8] claims that $N(M, q)$ is positive and even for all nondegenerate q . Lemma 5 and a simple inspection of the six generators of $[I, -M]$ show that $N(M, q) \geq 2$ for all $q \neq 0$. Clearly $N(M, 0) = 1$.

Let M be given by (3) and define $M^{(j)} = \underbrace{M \oplus \dots \oplus M}_j$. If q is in R^{3j} then we have $q = (q^1, q^2, \dots, q^j)$, where q^t , $1 \leq t \leq j$, is in R^3 . We then observe that for $q \in R^{3j}$ such that $q^t \neq 0$, $t=1, \dots, j$ $N(M^{(j)}, q) \geq 2^j$, and that $\{q \mid q^t = 0 \text{ for some } 1 \leq t \leq j\}$ is (strictly) contained in $R(M^{(j)})$, the set of degenerate vectors corresponding to $M^{(j)}$. Thus the following theorem is implied:

Theorem 6:

For any $n \geq 3$ there exists an $n \times n$ nondegenerate Q -matrix A such that $N(A, q) \geq 2^{\lfloor \frac{n}{3} \rfloor}$ for any q in $R^n - B$, where B is a subset of $R(A)$, the set of degenerate vectors corresponding to A .

Finally, while noting that for $j \geq 2$ $N(M^{(j)}, e_1) = 2$ where $e_1 = (1, 0, \dots, 0)^1$ we conclude that Lemma 5 does not hold for all $q \neq 0$. In fact, the above illustrates that if q is contained in a face of dimension less than $n-1$, where n is the order of the matrix A , the existence of nondegenerate vectors P^1 and P^2 such that $N(A, P^1) \leq N(A, q) \leq N(A, P^2)$ is not guaranteed.

References

- [1] Y. Censor, "On the maximal number of solutions of a problem in linear inequalities," Israel J. Math. 9, (1971).
- [2] R. W. Cottle, "Solution rays for a class of complementarity problems," to appear in Mathematical Programming Studies.
- [3] R. W. Cottle and G. B. Dantzig, "Complementary pivot theory of mathematical programming," Linear Algebra and its Appl., 1 (1968), 103-125.
- [4] A. W. Ingleton, "A problem in linear inequalities," Proc. London Math. Soc., Third Series, 16, (1966), 519-536.
- [5] S. Karamardian, "The Complementarity Problem," Mathematical Programming, 2 (1972), 107-129.
- [6] C. E. Lemke, "Bimatrix equilibrium and mathematical programming," Management Science, 11 (1965), 681-689.
- [7] _____, "Recent results on complementarity problems," in J. B. Rosen, O. L. Mangasarian and K. Ritter, eds., Nonlinear Programming, Academic Press, New York, 1970, pp.349-384.
- [8] K. G. Murty, "On the number of solutions to the complementarity problem and spanning properties of complementary cones," Linear Algebra and its Appl., 5, (1972), 65-108.
- [9] R. Saigal, "A characterization of the constant parity property of the number of solutions to the linear complementarity problem," SIAM J. Appl. Math. 23, (1972), 40-45.
- [10] A. Tamir, "On the complementarity problem of mathematical programming," Doctoral dissertation, Case Western Reserve University, 1973.