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A THEOREM ON CORRESPONDENCES AND SOME APPLICATIONS

by

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ABSTRACT

A theorem on correspondences is proved, and is then used to verify the upper hemi-continuity of several correspondences which are of interest in studies of social phenomena: Pareto correspondences, the outcome correspondence for voting procedures, and the equilibrium correspondence, under a general definition of social equilibrium. Included as a special case is the upper hemi-continuity of the core correspondence for a fairly general exchange economy (of fixed finite size).

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As Gerard Debreu observed in his Presidential Address to the Econometric Society, [1] there are several conditions of "adequacy" that we ought to impose upon any concept of equilibrium or outcome which is introduced into the study of social systems. In particular, according to Debreu, we ought to require that, as the data defining the social system undergo change, (a) an equilibrium will always exist, (b) the equilibrium will always be, in some sense, unique, and (c) the equilibrium will change continuously. It is the purpose of this essay to provide a tool which can often be applied quite easily to verify that an equilibrium or outcome satisfies the third of these criteria.

The tool -- a theorem on correspondences -- is provided in the first section of the paper; it is a generalization of a well-known theorem of Claude Berge. For many correspondences which are of interest in studies of social phenomena, the theorem can be applied, in very direct fashion, to establish that the graph of the correspondence is closed, and hence, ultimately, that the correspondence is upper hemicontinuous. Three applications are provided here: to Pareto correspondences, to outcome correspondences for voting procedures, and to a "social equilibrium" correspondence, of which the core of a fairly general exchange economy is a special case.

I. A THEOREM ON CORRESPONDENCES

A theorem of Claude Berge, concerning the continuity of "maximizing" correspondences, has become an essential tool for, among other things, establishing the existence of a market equilibrium -- that is, a price list and an allocation of production and consumption which together form an equilibrium of the "price mechanism". Berge's theorem is stated here.

Theorem 1 (Berge [2, p. 116]): Let E and X be topological spaces; let $u: E \times X \to \mathbb{R}$ be a continuous real-valued function; let $F: E \twoheadrightarrow X$ be a continuous correspondence;[2] and, for each $e \in E$, let $M(e) = \{x \in F(x) \mid \forall y \in F(x): u(e,x) \geq u(e,y)\}$. Then the correspondence $M: E \twoheadrightarrow X$ is closed, and if F is compact-valued, then M is upper hemi-continuous.

Berge's theorem is typically used to verify that agents' "choice (demand and supply) correspondences" are upper hemi-continuous, which guarantees the same property for the aggregate net demand correspondence; this fact (along with several others) allows one to invoke a topological fixed-point theorem, and thereby to insure that a market equilibrium exists.[3]

In order to interpret Berge's theorem as a statement about demand and supply correspondences, one makes the following interpretation of its abstract entities (where $\mathbb{R}^{\ell}$ is interpreted as the "commodity space"): $E \subseteq \mathbb{R}^{\ell}_{+}$ is a set of price ℓ-tuples;[4] $X \subseteq \mathbb{R}^{\ell}$ is a set of conceivable consumption or production plans for an agent; u is the agent's utility function; $F(e)$ is the budget set (of a "consuming agent") or the technologically feasible set (of a "producing agent"); and M is the agent's demand or supply correspondence.

In order to motivate the main theorem, which we are about to state and prove, an analogous description of the interpretation we will utlimately make is hereby given: E is a set of environments or data of a social system; Y is a set of conceivable "proposals" for social action; X is a set of conceivable "counterproposals;" $H(e,y)$ is the set of counterproposals which are actually available in the instance (e,y); $D(e)$ is a preference or dominance relation; and $M(e)$ is the set of proposals which will not be upset or "blocked."

Theorem 2: Let E, X, and Y be topological spaces, and let $H: E \times Y \twoheadrightarrow X$ be a lower hemi-continuous correspondence. For each $e \in E$, let $D(e)$ be a subset of $X \times Y$, and let the set $\{(e,x,y) \in E \times X \times Y \mid (x,y) \in D(e)\}$ be open. Then the correspondence $M: E \twoheadrightarrow Y$ is closed, where M is defined by

$$M(e) = \{y \in Y \mid x \in H(e,y) \Rightarrow (x,y) \notin D(e)\} .$$

Proof:[5]

Let $e_0 \in E$ and $y_0 \in Y$ be such that $y_0 \notin M(e_0)$. We must show that there are neighborhoods U of e_0 and W of y_0 for which $e \in U \Rightarrow M(e) \cap W = \emptyset$. Since $y_0 \notin M(e_0)$, there is an $x_0 \in H(e_0, y_0)$ for which $(x_0, y_0) \in D(e_0)$. Since $\{(e,x,y) \mid (x,y) \in D(e)\}$ is open, there are neighborhoods U_1 of e_0, V of x_0, and W_1 of y_0 such that $(e,x,y) \in U_1 \times V \times W_1 \Rightarrow (x,y) \in D(e)$. Because H is lower hemi-continuous, the set $H^{-1}(V) = \{(e,y) \in E \times Y \mid H(e,y) \cap V \neq \emptyset\}$ is open; because $H(e_0, y_0) \cap V \neq \emptyset$ (the set contains x_0), $H^{-1}(V)$ is a neighborhood of (e_0, y_0). Let U_2 and W_2 be neighborhoods of e_0 and y_0 such that $U_2 \times W_2 \subseteq H^{-1}(V)$. Finally, let $U = U_1 \cap U_2$, and let $W = W_1 \cap W_2$.

We now show that U and W are the desired neighborhoods. Let $e \epsilon U$ and $y \epsilon W$; we must show that $y \notin M(e)$. Because $(e,y) \epsilon U_2 \times W_2$, we have $H(e,y) \cap V \neq \emptyset$; let $x \epsilon H(e,y) \cap V$. Since $e \epsilon U_1$, and $x \epsilon V$, and $y \epsilon W_1$, then we have $(x,y) \epsilon D(e)$, which together with $x \epsilon H(e,y)$, implies that $y \notin M(e)$. ||

Corollary 2.1: If H is closed as well, then the correspondence $\mathring{M} = M \cap H$ is closed.

Corollary 2.2: If H is compact-valued, then M is upper hemi-continuous; if H is also closed, then $\mathring{M} = M \cap H$ is u.h.c.

Since any upper hemi-continuous correspondence which maps into a Hausdorff space is closed, we have the following additional corollary.

Corollary 2.3: If H is continuous and compact-valued, and if X is a Hausdorff space, then $\mathring{M} = M \cap H$ is upper hemi-continous.

We are generally interested in knowing whether, for each $e \epsilon E$, the set $\mathring{M}(e)$ will be non-empty. The following theorem gives conditions under which the answer is affirmative.

Theorem 3:[6] Let X be a non-empty compact space, and let P be an acyclic relation on X such that, for each $x \epsilon X$, the lower contour set $x\,P = \{y \epsilon X \mid (x,y) \epsilon P\}$ is open. Then the set of P-maximal members of X,

$$\mathring{M} = \{y \epsilon X \mid x \epsilon X \Rightarrow (x,y) \notin P\} = \bigcap_{x \epsilon X} \sim xP\,,$$

is non-empty

Corollary 2.4: If, in Corollary 2.3, $X = Y$, $D(e)$ is acyclic for each $e \varepsilon E$, and H is both non-empty-valued and compact-valued, then $\mathring{M}$ is non-empty-valued, as well as upper hemi-continous.

Previous generalizations of Theorem 1, all of which are included in Theorem 2 and its corollaries, have been given by Debreu [5], Sonnenschein [14], and Hildenbrand [8].

Before turning to some applications, in which the results above are used to show that particular correspondences are upper hemi-continous, it might be well to say a few words here about the usefulness of such results. Since upper hemi-continous correspondences may be "very discontinuous", they don't seem to really satisfy Debreu's third desideratum. However, if a correspondence is upper hemi-continuous on a space E, and is single-valued on an open subspace E', then it is a continuous function on E'. If E' is dense in E as well, then the desiderata are satisfied on all but a "very small" set in E. In the economic analysis of competitive markets, for example, the equilibrium correspondence has been found to have such properties.

II. NOTATION

It will be helpful to set out in advance some notation that will be used throughout the sequel.

E is a topological space; its members $e \in E$ are called <u>environments</u>.

A is a set; its members $a \in A$ are called <u>agents.</u>

For each agent $a \in A$: X_a is a topological space; its members $x_a \in X_a$ are called (individual) <u>actions</u> (the actions that it is a priori possible for a to undertake).

For each $a \in A$ and each $e \in E$: $P_a(e)$ is a binary relation on X (i.e., a subset of $X \times X$), called agent a's <u>preference</u> at e.

X will denote the product [9] $\prod_{a \in A} X_a$, and x or (x_a) will denote a member of X; the members will be called (joint) <u>actions</u>.

For each $e \in E$: $F(e)$ is a non-empty subset of X, called the <u>feasible set</u> in environment e.

III. PARETO CORRESPONDENCES

In any multi-person decision situation, the set of Pareto-optimal decisions holds some claim on our interest by virtue of its strong normative flavor: Pareto-optimality seems to be a minimum condition of "justice" for any decision which affects more than one- person -- a sort of "ethical efficiency." We say that a decision x <u>Pareto-dominates</u> a decision y if there is some agent who prefers x and none who prefer y; the <u>Pareto-</u>

optimal decisions are the undominated ones. We will use the strong Pareto-relation here (for conditions under which the two Pareto relations coincide, see [11]).

Definition 1: For each $e \in E$, the (strong) Pareto relation on X is

$$D(e) = \bigcap_{a \in A} P_a(e), \text{ and the (weak) Pareto-optimal set is}$$

$$P(e) = \{y \in F(e) \mid x \in F(e) \Rightarrow (x,y) \notin D(e)\}.$$

We refer to $P: E \twoheadrightarrow X$ as the (weak) Pareto correspondence.

The following theorem is a direct application of the results of Section I.

Theorem 4: Let A be finite; if the correspondence F is l.h.c., and if, for each $a \in A$, the correspondence P_a is open, then the weak Pareto correspondence P is closed. If F is compact-valued, as well, then P is u.h.c., and if, in addition, each P_a is acyclic-valued, then P is non-empty-valued.

Often, when preferences are representable by utility functions, we are interested in the set of utility-values which are not Pareto-dominated; we call this set the Pareto-Frontier.

Definition 2: A correspondence $P_a : E \twoheadrightarrow X_a \times X_a$ is [continuously] representable if there is a [continuous] function $u_a : E \times X_a \to \mathbb{R}$ such that, for each $e \in E$, $u(e,x) > u(e,y)$ if and only if $(x,y) \in P_a(e)$. Such a function u_a is called a utility representation of P_a.

<u>Definition 3:</u> For each $a \in A$, let P_a be representable by the utility representation u_a; let $u = (u_a): E \times X \to \mathbb{R}^A$. For each $e \in E$, the feasible-utility set at e is the set $U(e) = \{r \in \mathbb{R}^A \mid \exists\, x \in X : u(x) = r\}$, and the (strong) Pareto-frontier is the set $\mathring{U}(e) = \{r \in U(e) \mid \nexists\, r' \in U(e): r' \gg r\}$

<u>Theorem 5:</u> Let A be finite, and for each $a \in A$, let $u_a: E \times X_a \to \mathbb{R}$ be a continuous representation of P_a. If the feasible-set correspondence $F: E \twoheadrightarrow X$ is continuous, then the Pareto-frontier correspondence $\mathring{U}: E \twoheadrightarrow \mathbb{R}^A$ is closed; if F is compact-valued as well, then $\mathring{U}$ is non-empty-valued and upper hemi-continuous.

<u>Proof</u>:[8]

$U: E \twoheadrightarrow \mathbb{R}^A$ is the composition of the lower hemicontinuous correspondences $(id_E \times F): E \twoheadrightarrow E \times X$ and $u: E \times X \to \mathbb{R}^A$, so is itself lower hemicontinuous (Theorems 1 and 4, pages 113, 114 of [2]). Let $D: E \twoheadrightarrow \mathbb{R}^A \times \mathbb{R}^A$ be the constant correspondence defined by $D(e) = \{(r,s) \mid r \gg s\}$; then $\{(e,r,s) \mid (r,s) \in D(e)\} = \{(e,r,s) \mid r \gg s\} = E \times \{(r,s) \mid r \gg s\}$, which is clearly open in $E \times \mathbb{R}^A \times \mathbb{R}^A$. Since F is upper hemi-continuous and $\mathbb{R}^A$ is Hausdorff, F is closed; hence $\mathring{U}$ is closed, according to Corollary 2.1. If F is compact valued, then, since $D(e)$ is always acyclic ($D(e) = \bigcap_{a \in A} P_a(e)$, and $P_a(e)$ is acyclic if it is representable), it follows from Corollary 2.4 that $\mathring{U}$ is non-empty-valued and upper hemicontinuous.

IV. VOTING PROCEDURES

Let the set A of agents be finite, and let α be a real number, with $0 \leq \alpha \leq 1$. The <u>α-majority voting procedure</u> aggregates the preferences or "votes" of the agents into a single (collective) dominance relation on the set X of actions;[10] specifically, an action x is said to dominate an action y if the proportion of all agents who prefer x to y is larger than α. An outcome, or equilibirum, of the procedure in a particular instance is a feasible action which is not dominated by any other feasible action.

<u>Definition 3</u>: For each $\alpha (\alpha \leqq 1)$, the set β_α of <u>α-majority coalitions</u> is the set of all subsets of A which have more than $\alpha|A|$ members.

<u>Definition 4</u>: For each $e \in E$, the <u>α-majority relation</u> D_α is defined by

$$D_\alpha = \bigcup_{S \in \beta_\alpha} \bigcap_{\alpha \in S} P_a,$$

and the set $\mathring{M}_\alpha(e) = F(e) \cap M_\alpha(e)$ is called the set of <u>α-majority outcomes</u>, where

$$M_\alpha(e) = \{y \in X | x \in F(e) \Rightarrow (x,y) \notin D_\alpha\}.$$

The correspondence $\mathring{M}: E \twoheadrightarrow X$ is called the <u>outcome correspondence</u> of the α-majority voting procedure.

The following theorem is simply a direct application of Theorem 2 and its corollaries.

Theorem 6:[9] Let A be finite; if the correspondence F is closed and l.h.c. and if each of the correspondences P_a is open, then the outcome correspondence $\overset{\circ}{M}: E \twoheadrightarrow X$ of the α-majority voting procedure is closed. If F is compact-valued as well, then $\overset{\circ}{M}$ is u.h.c.

It is, of course, well-known that in the absence of some restrictions, the correspondences D_α are generally not acyclic-valued, and the outcome correspondence is not non-empty-valued. Several restrictions are known which yield acyclic-valued correspondences D_α (for $\alpha \geq \frac{1}{2}$), and in these cases (if F is compact-valued), Corollary 2.3 guarantees that the outcome correspondence will be non-empty-valued.

The ideas in this section can apparently be generalized in several directions -- for example, to representative or hierarchical voting procedures, and to "simple games."

V. SOCIAL EQUILIBRIUM

In this section, we will want to consider the possibility that some agents may act in concert, coordinating their individual actions. In these cases, we will use the following notation.

$\mathcal{A}$ is a collection of subsets of A, its members $S \in \mathcal{A}$ are called coalitions.

For each $S \subseteq A$: X_S will denote the product $\prod_{a \in S} X_a$. If $x \in X$ and $S \in \mathcal{A}$, the projection of x into X_S will be denoted by x_S.

For each $S \in \mathcal{A}$, $e \in E$, and $\underset{\sim}{x}_s \in \underset{\sim}{X}_S$: $F_S(e,\underset{\sim}{x}_s)$ is a subset of X_S, called the feasible set for coalition S.

It remains, now, to specify a process by which the system arrives at an action $x \in X$, or at least to specify conditions which characterize equilibrium actions, leaving in the latter case no more than a very loose specification (possibly only implicit) of the entire social choice process.

Let us begin with a very general notion of equilibrium for a social system. Of course, at the most general level, an action is an equilibrium if it will not be upset by whatever social forces exist in the system; to attain this ultimate generality, though, we pay the ultimate price: this notion of equilibrium is virtually useless without further qualification. Let us say then, as in the definition of the core, that an action is an equilibrium if it will not be upset, or "blocked," by some coalition. A coalition S is said to upset an action (or a proposed action) $y \in X$ if there is an action $x_S \in X_S$ for which $x_S \neq y_S$, and such that, when S is faced with y, it is both willing and able to undertake x_X.

Formally, for each $S \in \mathcal{A}$:

$H_S : E \times X \twoheadrightarrow X_S$ is a correspondence; $H_S(e,y)$ is to be interpreted as the set of actions $x_S \in X_S$ by which S is able to upset y;

$D_S : E \twoheadrightarrow X_S \times X$ is a correspondence, and satisfies $(x_S,y) \in D_x(e) \Rightarrow x_S \neq y_S$; "$(x_S,y) \in D_S(e)$" is interpreted to mean that, for coalition S, x_S dominates y, if x_S is available (or "x_S is preferred to y").

$M_S : E \twoheadrightarrow X$ is the correspondence defined by $M_S(e) = \{y \in X | x_S \in H_S(e,y) \Rightarrow (x_S,y) \notin D_S(e)\}$; $M_S(e)$ is thus the set of all actions $y \in X$ which will not be upset by S.

The equilibrium actions, we have said, are just the (feasible) ones which cannot be upset by any coalition.

Definition 4: For each $e \in E$, the set of undominated actions is $M(e) = \bigcap_{S \in \mathcal{A}} M_S(e)$, and the set of equilibrium actions is $\mathring{M}(e) = F(e) \cap M(e)$.

The correspondence $\mathring{M}$: $E \twoheadrightarrow X$ is called the equilibrium correspondence.

The definition of equilibrium developed above is clearly in the spirit of Nash-equilibrium and the core. It does not, however, specify in what way H_S and D_S are to depend upon preferences and coalitions' feasible sets. Indeed, the definition of a social system need only specify a coalition structure $\mathcal{A}$, a feasible set correspondence F for the whole system, and correspondences H_S and D_S, in order that equilibrium be well-defined (i.e., it need not specify directly either the correspondences F_S or P_S). The definition given here can be applied, for example, the Rosenthal's "effectiveness" notion of equilibrium [12].

Of course, Theorem 2 and its corollaries can now be interpreted directly as statements about the continuity of the social equilibrium correspondence (note that $\mathcal{A}$ need not be finite here).

Theorem 7: If

(i) F is lower hemi-continuous;

(ii) for each $S \in \mathcal{A}$, H_S is closed and lower hemi-continuous;

(iii) for each $S \in \mathcal{A}$, D_S is open:

then the social equilibrium correspondence M is closed. If F (or, alternatively, each H_S) is compact-valued as well, then $\mathring{M}$ is upper hemi-continuous.[11]

It remains to show that there are interesting social systems in which natural specifications of the correspondences H_S and D_S satisfy the conditions of Theorem 7. An example -- an exchange economy -- is given in the following Section.

VI. THE CORE OF AN ECONOMY

A finite exchange economy is a quadruple $(A, X, \mathbb{P}, \omega)$, where

A is a finite set;

$X = \prod_{a \in A} X_a$, and each X_a is a subset of $\mathbb{R}^{\ell}$ (with its usual topology), for some positive integer ℓ, and each X_a is bounded below;

$\mathbb{P}$ is a member of $\mathcal{P}^A$, where $\mathcal{P}$ is a topological space whose members P are subsets of $X \times X$;

ω is a member of $\mathbb{R}^{\ell}_{+}$

We fix both A and X, and allow $\mathbb{P}$ and ω to vary; we refer to each member $(\mathbb{P}, \omega) \in \mathcal{P}^A \times \mathbb{R}^{\ell A}_{+}$ as an environment, and we write $E = \mathcal{P}^A \times \mathbb{R}^{\ell A}_{+}$, and $e = (\mathbb{P}, \omega)$. In each economic environment $e \in E$, the actions which are feasible for the whole economy are those in the set

$$F(e) = \{x \in X \mid \sum_{a \in A} x_a \leq \sum_{a \in A} \omega_a\}.$$

Finally, we fix a coalition structure $\mathcal{C}$.

We have specified A, E, X, F, and $\mathcal{C}$; as described in Section V, we must specify the correspondences H_S and D_S in order to define the equilibrium actions in this system. Let us consider first the correspondences H_S. For any action $y \in X$, the actions available to the coalition S

are of two kinds:

(i) "cooperative" actions, those $x_S \epsilon X_S$ for which $\sum_{a\epsilon S} x_a = \sum_{a\epsilon S} y_a$;

and (ii) "non-cooperative" actions, those $x_S \epsilon X_S$ for which

$$\sum_{a\epsilon S} x_a \leq \sum_{a\epsilon S} \omega_a .$$

Hence, for each $S \epsilon \mathcal{Q}$, $e \epsilon E$, and $y \epsilon X$, let us write

$$G_S^1(e,y) = \{x_S \epsilon X_S \mid \sum_{a\epsilon S} x_a = \sum_{a\epsilon S} y_a\} \tag{1}$$

and $$G_S^2(e,y) = \{x_S \epsilon X_S \mid \sum_{a\epsilon S} x_a \leq \sum_{a\epsilon S} \omega_a\} . \tag{2}$$

Then the correspondence $G_S^1 \cup G_S^2 \colon E \times X \twoheadrightarrow X_S$ is clearly the one in which we are interested, and we summarize in the following assumption.

Assumption 1: For each $S \epsilon \mathcal{Q}$, the correspondence H_S is given by $H_S = G_S^1 \cup G_S^2$, where G_S^1 and G_S^2 are as given in (1) and (2) above.

Turning now to the dominance relations D_S for the coalitions, we want to say, for a given action (or "proposal") $y \epsilon X$, which actions (or "counterproposals") $x_S \epsilon X_S$ are "better," from the point of view of S, than the original proposal y. The natural first requirement is that each member of the coalition must agree that x_S is better than y; this leads us to define for each agent $a \epsilon S$ a dominance relation D_a, and to say that $(x_S, y) \epsilon D_S$ if and only if $(x_S, y) \epsilon D_a$ for all $a \epsilon S$ -- i.e., $D_S = \bigcap_{a\epsilon S} D_a$.

Now we are left with the question which actions $x_S \epsilon X_S$ will be "preferred" by a member $a \epsilon S$ to the action $y \epsilon X$. In other words, if an action $y \epsilon X$ were proposed, when would an agent choose an action $x_S \epsilon X_S$ (if he could) to "block" proposal y? The following assumptiom provides a straightforward answer to the question -- it says that an agent will prefer x_S to y if, when the members of the set $\sim_S$ adhere to the proposal y, the agent prefers the outcome resulting from x_S to that resulting from y_S. This is, of course, just R. Aumann's "β-core" assumption [1].

Assumption 2: For each $S \epsilon \mathcal{A}$, each $a \epsilon S$, and each $e \epsilon E$, the set $D_a(e)$ is the subset of $X_S \times X$ defined as follows: $(x_S, y) \epsilon D_a(e)$ if and only if [12] $([x_S, y_{\sim S}], y) \epsilon P_a(e)$. For each $S \epsilon \mathcal{A}$, the correspondence D_S: $E \twoheadrightarrow X_S \times X$ is defined by $D_S = \bigcap_{a \epsilon S} D_a$.

This is clearly not the only reasonable behavior that we might ascribe to agents. Their behavior ought to be predicated on their expectation of response by other agents, and there are at least several reasonable kinds of such expectation. In the case of Assumptions 1 and 2, however, Theorem 8 shows that the usual restriction on the space $\mathcal{P}$ is sufficient to guarantee that the equilibrium correspondence is upper hemi-continuous.

Theorem 8: Let A be finite, and let $E = \mathcal{P}^A \times \mathbb{R}_+^{\ell A}$, a space of finite economic environments. For each $S \epsilon \mathcal{A}$, let H_S and D_S be correspondences which satisfy Assumptions 1 and 2. If the set $\{(P,x,y) \mid (x,y) \epsilon P\}$ is open in $\mathcal{P} \times \mathbb{R}^\ell \times \mathbb{R}^\ell$, then the equilibrium correspondence $\mathring{M}$: $E \twoheadrightarrow X$ is upper hemi-continuous (where $\mathring{M}$ is given by Definition 4).

Proof:

F is clearly closed, l.h.c. (in fact, continuous), and compact-valued; it is easy to show that each D_a is open, and, since each D_S is a finite intersection of correspondences D_a, this yields each D_S open; each G_S^1 and G_S^2 is clearly closed and l.h.c. (in fact, continuous), and since each H_S is a union of G_S^1 and G_S^2, this yields each H_S closed and l.h.c. Hence, Theorem 6 guarantees that $\mathring{M}$ is u.h.c. ||

It should be noted that when only "selfish" preferences are considered, there is no ambiguity in specifying the correspondences D_a, and the set of equilibria in that case is precisely the core. Specifically, a subset P of $X \times X$ is said to be an <u>a-selfish preference</u> if it satisfies the following condition:

$$[x_a = x'_a \text{ and } y_a = y'_a] \Rightarrow [(x,y) \in P \Leftrightarrow (x',y') \in P] .$$

Let $\mathcal{P}_a$ denote a space of a-selfish preferences, and let $\mathcal{P}_A = \prod_{a \in A} \mathcal{P}_a$. Assumption 2 is now the only natural one to make on the D-correspondences, and then $\mathring{M}(e)$ is the core of the economy defined by e. Theorem 8 is, of course, true in this case, if we substitute $E = \mathcal{P}_A \times \mathbb{R}_+^{\ell A}$. If the sets X_a are allowed to vary appropriately with e, the result will not be disturbed.

It should be emphasized that the economies treated here are of fixed finite size, and that the question of existence (i.e., whether $\mathring{M}$ is non-empty-valued) is not dealt with. Hence, the results of Kannai [10] and Hildenbrand and Mertens [9], for example, are much deeper than Theorem 8.

FOOTNOTES

1. [6], pages 603-605.

2. A correspondence C from a set X to a set Y (denoted $C:X \twoheadrightarrow Y$) is a subset of $X \times Y$ (equivalently, we can think of C as an assignment of a subset $C(x)$ of Y to each member $x \in X$, and of the corresponding subset of $X \times Y$ as the graph of C). Let X and Y be topological spaces, and $X \times Y$ their (topological) product, and let $C: X \twoheadrightarrow Y$ be a correspondence from X to Y.

   i) C is open if it is an open subset of $X \times Y$.

   ii) C is closed if it is a closed subset of $X \times Y$.

   iii) C is lower hemi-continuous (l.h.c.) if, for each $x_0 \in X$ and each open set $V \subseteq Y$ which intersects $C(x_0)$, there is a neighborhood U of x_0 for which $x \in U \Rightarrow C(x) \cap V \neq 0$.

   iv) C is upper hemi-continuous (u.h.c.) if, for each $x_0 \in X$ and each open set $V \subseteq Y$ which contains $C(x_0)$, there is a neighborhood U of x_0 for which $x \in U \Rightarrow C(x) \subseteq V$.

   v) C is continuous if it is both u.h.c. and l.h.c.

See Berge [2, Chapter 6] for a detailed exposition of correspondences and their topological properties.

3. See, for example, [4, Chapter 5, especially pages 82 and 86].

4. $\mathbb{R}^{\ell} = \{x \in \mathbb{R}^{\ell} \,|x_i \geq 0,\ i = 1, \ldots, \ell\}$.

5. This proof is similar to that in [5, p. 89]; the result there is more general than Theorem 1 here, but less general than Theorem 2.

6. This theorem has been discovered independently by several people; [15]

contains references, as well as a generalization of, and comment on, this and several related results. (The striking results of H. Sonnenschein [14, Theorem 4] and Shafer and Sonnenschein [13] which are much deeper than Theorem 3, should be pointed out as alternatives when $X = Y \subseteq \mathbb{R}^{\ell}$ and convexity assumptions are appropriate). The most straightforward proof of the theorem rests on the observation that a subset A of X has a P-maximal member if and only if the collection $\{xP \mid x \in A\}$ does not cover A. Suppose now that $\mathring{M} = \emptyset$; then $\{xP \mid x \in X\}$ is an open cover of the compact set X, and there is consequently a finite subset $A \subseteqq X$ such that $\{xP \mid x \in A\}$ covers X, and a fortiori covers A. Hence, A has no P-maximal member; however, it is easy to verify that a finite set must have a P-maximal member if P is acyclic, and we thus have $\mathring{M} \neq \emptyset$.

7. The product of topological spaces will always be understood to be the topological product.

8. Mount and Reiter [11] have given a proof of this theorem in the case in which $X_a = \mathbb{R}_+^{\ell}$ for each a, and E is a class of exchange economies over $\mathbb{R}_+^{\ell}$.

9. The idea for Theorem 6 is due to Don Campbell.

10. It is unnecessary here, and even a bit misleading, to think of X as a product space: we may take the set X to be fundamental ($x \in X$ is an "alternative social action," for example).

11. Notice that "each D_S acyclic-valued" is not sufficient for "M is non-empty-valued."

12. $[x_S, y_{\sim S}]$ is the member $z \in X$ for which $z_S = x_S$ and $z_{\sim S} = y_{\sim S}$.

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