“A universal bound on probabilistic influence”

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Abstract
This paper establishes a new probability result: Fix η ∈ (0, 1/2] and let \{X_i\}_{i \in \mathbb{N}} denote a family of independent random variables such that \( Pr(X_i = 1) = 1 - Pr(X_i = 0) = p_i \in [\eta, 1-\eta] \). For any \( K > 0, \varepsilon > 0 \), and function \( f : \{0, 1\}^n \to [0, K] \),

\[ \left| \left\{ i : E[f(X_1, \ldots, X_n)|X_i = 1] - E[f(X_1, \ldots, X_n)|X_i = 0] > \varepsilon \right\} \right| \leq N(\varepsilon, K, \eta) + 1 \]  \hspace{1cm} (1)

where \( N(\varepsilon, K, \eta) = \frac{K^2}{2\pi^2 \eta^2} \).

In many environments, a dependent variable potentially depends on a large number of independent binary variables. In economics, an agent’s utility may depend on a large number of stochastic consumption events, a market price can depend on a large number of individual decisions, a production function may depend on many stochastic factors, etc. In econometrics, the dependent variable in a nonlinear regression can potentially depend on many independent variables. In genomics, the occurrence of disease can potentially depend on a large number of genes. In quantum mechanics, the realization of a phenomenon can depend on the spin positions of large number of particles, etc. This paper shows, however, that if the variable of interest is bounded, it can only depend significantly on at most a fixed number of independent variables. This number depends only on the significance level of interest and on the probability range taken by the independent variables.

Lemma 1 Let \( \{X_i\}_{i \in \mathbb{N}} \) denote a family of independent random variables such that \( Pr(X_i = 1) = Pr(X_i = 0) = 1/2 \). For any \( K > 0, \varepsilon > 0 \), and function \( f : \{0, 1\}^n \to [0, K] \),

\[ \left| \left\{ i \in \{1, \ldots, n\} : E[f(X_1, \ldots, X_n)|X_i = 1] - E[f(X_1, \ldots, X_n)|X_i = 0] > \varepsilon \right\} \right| \leq N(\varepsilon, K) + 1 \]

where \( N(\varepsilon, K) = \frac{K^2}{2\pi^2} \).

This implies in particular that the fraction of independent random variables whose positive influence exceeds \( \varepsilon \) goes to zero as \( n \) gets large, at rate \( 1/n \). Obviously, a symmetric statement with the same bound holds for negative influence.

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Proof. Fix some $K, \varepsilon, n$, and $f$, and let $\mathcal{N}_f = \{i : E[f(X_1, \ldots, X_n)|X_i = 1] - E[f(X_1, \ldots, X_n)|X_i = 0] > \varepsilon\}$ and $n_f = |\mathcal{N}_f|$. We will derive an upper bound $N$ on $n_f$. Any bound $N$ obtained for the parameters $(\varepsilon, K)$ is also a bound for the parameters $(\alpha \varepsilon, \alpha K)$, for any $\alpha > 0$. Therefore, we focus on $K = 1$.

By relabeling coordinates, we can assume without loss of generality that $N_f = \{1, \ldots, n_f\}$. By letting $g(X_1, \ldots, X_{n_f}) = E[f(X_1, \ldots, X_n)|X_i = 1] - E[f(X_1, \ldots, X_n)|X_i = 0] = E[f(X_1, \ldots, X_n)|X_i = 1] - E[f(X_1, \ldots, X_n)|X_i = 0] > \varepsilon$ and hence $n_g = n_f$. Therefore we may focus without loss on functions like $g$ for which the influence of each variable exceeds $\varepsilon$.

Thus suppose that $n_f = n$ and let $\mathcal{X} = \{0,1\}^n$. Note that any bound $N$ obtained when $n$ is even implies a bound $N + 1$ for $n$ odd. To economize on notation, we focus on $n$ even. By assumption, we have for each $i \in \{1, \ldots, n\}$

$$\sum_{x \in \mathcal{X}: x_i = 1} f(x) - \sum_{x \in \mathcal{X}: x_i = 0} f(x) > \varepsilon 2^{n-1}$$

Notice that if $x$ has $j$ zeros and $n - j$ ones, the term $f(x)$ is counted positively in $n - j$ of the previous inequalities and negatively in the $j$ remaining ones. Summing up the $n$ inequalities and rearranging the terms therefore yields

$$\sum_{j=0}^{n} (n - 2j) \sum_{x \in \mathcal{X}^j} f(x) > \varepsilon n2^{n-1},$$

where $\mathcal{X}^j$ denotes the set of all elements of $\{0,1\}^n$ with exactly $j$ zeros. Since $f(x)$ is nonnegative and bounded above by 1, this implies that

$$\sum_{i=0}^{n/2} (n - 2i) \binom{n}{i} > \varepsilon n2^{n-1}. \tag{2}$$

Because $\binom{n}{i} = \binom{n}{n-i}$ and $\sum_{i=0}^{n} \binom{n}{i} = 2^n$, the left-hand side of (2) is equal

$$n \sum_{i=0}^{n/2} \binom{n}{i} - 2 \sum_{i=1}^{n/2} i \binom{n}{i} = \frac{n}{2} \left(2^n + \binom{n}{n/2}\right) - 2 \sum_{i=1}^{n/2} i \binom{n}{i}.$$ 

Since $i \binom{n}{i} = \binom{n-1}{i-1}$ and $\binom{n}{i} = \binom{n-1}{i}$, the last term of the previous expression equals

$$2n \sum_{j=0}^{n/2-1} \binom{n-1}{j} = n \sum_{j=0}^{n-1} \binom{n-1}{j} = n2^{n-1}$$

Therefore, (2) implies that

$$\frac{n}{2} \binom{n}{n/2} > \varepsilon n2^{n-1},$$

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or
\[
\frac{n!}{(n/2)!^2} > \varepsilon 2^n.
\]
Using Stirling’s formula,\(^1\) this implies that
\[
\sqrt{\frac{2}{\pi n}} > \varepsilon
\]
or
\[
n < \frac{2}{\pi \varepsilon^2}.
\]
This shows the lemma for
\[
N(\varepsilon, K) = \frac{2 K^2}{\pi \varepsilon^2}.
\]

**Lemma 2** Fix \(q \in \mathbb{N}_{++}\) and \(\eta \in (0, 1/2]\), and let \(\{X_i\}_{i \in \mathbb{N}}\) denote a family of independent random variables such that \(Pr(X_i = 1) = 1 - Pr(X_i = 0) = a_i 2^{-q}\) for some integer \(a_i\) such that \(a_i 2^{-q} \in [\eta, 1 - \eta]\). For any \(K > 0\), \(\varepsilon > 0\), and function \(f : \{0, 1\}^n \to [0, K]\),
\[
|\{i : E[f(X_1, \ldots, X_n) | X_i = 1] - E[f(X_1, \ldots, X_n) | X_i = 0] > \varepsilon\}| \leq N(\varepsilon, K, \eta) + 1
\]
where \(N(\varepsilon, K, \eta) = \frac{K^2}{2 \pi \varepsilon^2 n^q}\).

**Proof.** As with Lemma 1 we can focus on \(K = 1\) and \(n\) even and obtain the result for general \(K > 0\) by jointly rescaling \(K\) and \(\varepsilon\) and adding 1 to the upper bound to allow for \(n\) odd. Fix some integer \(n\) and function \(f : \mathcal{X} = \{0, 1\}^n \to [0, 1]\). For each \(i \in \{1, \ldots, n\}\) we will decompose \(X_i\) into \(q\) binary variables \(\{\tilde{X}_i^k\}_{k=1,\ldots,q}\) which are i.i.d. with \(Pr(\tilde{X}_i^k = 1) = Pr(\tilde{X}_i^k = 0) = 1/2\). For each \(i \leq n\), let \(\mathcal{A}_i\) denote a subset of \(\{0, 1\}^q\) with \(a_i\) elements, containing all elements such that \(X_i^1 = 1\) if \(a_i \geq 2^{q-1}\), and covered by these elements otherwise. Letting \(\tilde{X}_i = (\tilde{X}_i^1, \ldots, \tilde{X}_i^q)\), notice that \(Pr(\tilde{X}_i \in \mathcal{A}_i) = a_i/2^q\). We now show that
\[
Pr(\tilde{X}_i \in \mathcal{A}_i | \tilde{X}_i^1 = 1) = Pr(\tilde{X}_i \in \mathcal{A}_i | \tilde{X}_i^1 = 0) \geq 2\eta.
\]
Suppose first that \(a_i \geq 2^q - 1\). In this case, by construction \(\tilde{X}_i\) surely belongs to \(\mathcal{A}_i\) if \(\tilde{X}_i^1 = 1\). Let \(\delta = Pr(\tilde{X}_i \in \mathcal{A}_i | \tilde{X}_i^1 = 0)\). We have \(1/2 + \delta/2 = a_i/2^q\). By assumption, \(a_i/2^q \leq 1 - \eta\). Combining this yields \(1 - \delta \geq 2\eta\), as desired. If instead \(a_i < 2^{q-1}\), \(\tilde{X}_i\) cannot belong to \(\mathcal{A}_i\) if \(\tilde{X}_i^1 = 0\), and belongs to it with probability \(a_i/2^{q-1} \geq 2\eta\) if \(\tilde{X}_i^1 = 1\), which yields again (3).

Let \(\tilde{f} : \{0, 1\}^{np} \to [0, 1]\) be defined by
\[
\tilde{f}(\tilde{x}_1, \ldots, \tilde{x}_n) = f(1_{\tilde{x}_1 \in \mathcal{A}_1}, \ldots, 1_{\tilde{x}_n \in \mathcal{A}_n}).
\]

\(^1\)While Stirling’s formula is an approximation, the bound obtained here is exact: from Robbins (1955), we have
\[
\sqrt{2\pi n^{n+1/2} e^{-n} e^{1/(12n+1)}} < n! < \sqrt{2\pi n^{n+1/2} e^{-n} e^{1/12n}}.
\]
This implies that \(n!/(n/2)!^2\) is bounded above by \((2\pi n)^{-1/2} 2^{n+1} e^{1/(12n) - 2/(6n+1)}\). The last factor is less than one for all \(n \geq 1\) and can thus be dropped from the upper bound.
By construction, the random variable $\tilde{f}(\tilde{X})$ has the same distribution as $f(X)$. Moreover,

$$E[f(X)|X_i = 1] = E[f(\tilde{X})|\tilde{X}_i \in A_i]$$

and

$$E[f(X)|X_i = 0] = E[f(\tilde{X})|\tilde{X}_i \notin A_i].$$

The function $\tilde{f}$ depends on $nq$ iid symmetric Bernoulli variables. Lemma 1 can thus be applied to $\tilde{f}$, and shows that

$$E[\tilde{f}(\tilde{X})|\tilde{X}_i = 1] - E[\tilde{f}(\tilde{X})|\tilde{X}_i = 0] \leq \varepsilon$$

except for at most $N = \lceil 2/\pi \varepsilon^2 \rceil$ of these $np$ variables. If $n > N$, this implies that there is a subset $\mathcal{N}$ of $\{1, \ldots, n\}$ with at least $n - N$ elements such that for all $i \in \mathcal{N}$

$$\Delta_i = E[\tilde{f}(\tilde{X})|\tilde{X}_i = 1] - E[\tilde{f}(\tilde{X})|\tilde{X}_i = 0] \leq \varepsilon$$

for all $k \in \{1, \ldots, q\}$, and in particular for $k = 1$. Notice that for $r \in \{0, 1\}$,

$$E[\tilde{f}(\tilde{X})|\tilde{X}_i^k = r] = Pr(\tilde{X}_i \in A_i|\tilde{X}^k = r)E[f(X)|X_i = 1] + (1 - Pr(\tilde{X}_i \in A_i|\tilde{X}^k = r))E[f(X)|X_i = 0].$$

This implies that

$$\Delta_i = (E[f(X)|X_i = 1] - E[f(X)|X_i = 0])(Pr(\tilde{X}_i \in A_i|X_i^1 = 1) - Pr(\tilde{X}_i \in A_i|X_i^1 = 0))$$

Since the left-hand side is less than $\varepsilon$ for $i \in \mathcal{N}$ and the probability difference on the right-hand side exceeds $\eta$, we conclude that

$$E[f(X)|X_i = 1] - E[f(X)|X_i = 0] \leq \varepsilon/2\eta.$$ 

Setting $\varepsilon' = \varepsilon/2\eta$, this proves the lemma with $N(\varepsilon', 1, \eta) = 1/2\pi \varepsilon'^2 \eta^2$ and, by rescaling, $N(\varepsilon, K, \eta) = K^2/2\pi \varepsilon^2 \eta^2$.

**Theorem 1** Fix $\eta \in (0, 1/2]$ and let $\{X_i\}_{i \in \mathbb{N}}$ denote a family of independent random variables such that $Pr(X_i = 1) = 1 - Pr(X_i = 0) = p_i \in [\eta, 1 - \eta]$. For any $K > 0$, $\varepsilon > 0$, and function $f : \{0, 1\}^n \rightarrow [0, K]$,

$$|\{i : E[f(X_1, \ldots, X_n)|X_i = 1] - E[f(X_1, \ldots, X_n)|X_i = 0] > \varepsilon\}| \leq N(\varepsilon, K, \eta) + 1 \quad (4)$$

where $N(\varepsilon, K, \eta) = \frac{K^2}{2\pi \varepsilon^2 \eta^2}$.

**Proof.** Fix some $n$ even, function $f : \{0, 1\} \rightarrow [0, K]$ and $n$-dimensional vector $p$ such that $p_i \in [\eta, 1 - \eta]$ for all $i$. For each $q \in \mathbb{N}_{+}$, let $p_q^i$ denote an dimensional vector taking values in the set $\{1/2^p, 2/2^p, \ldots, 1 - 1/2^p\}$ such that $p_q^i \in [\eta, 1 - \eta]$ and $|p_q^i - p_i| \leq 1/2^q$ for all $i$. Let $E_q$ denote
the expectation taken with respect to the probability vector $p^q$. From Lemma 2 applied to $p^q$, we have

$$E_q[f(X)|X_i = 1] - E_q[f(X)|X_i = 0] \leq \varepsilon \quad (5)$$

for all $i$, except for a subset $N_q$ of $\{1, \ldots, n\}$ which contains at most $N(\varepsilon, K, \eta) = K^2/(2\pi\varepsilon^2\eta^2)$ elements. Since the set $\{1, \ldots, n\}$ is finite, there exists an infinite subsequence of $\{N_q\}_{q \geq 1}$ which is constant, equal to some set $N$ of cardinality at most $N(\varepsilon, K, \eta)$.

The conditional expectations entering (5) are polynomial and, hence, continuous in $q$. Therefore, $E_q[f(X)|X_i = 1] - E_q[f(X)|X_i = 0]$ converges to $E[f(X)|X_i = 1] - E[f(X)|X_i = 0]$ for all $i$ as $q \to +\infty$. Applying this observation to $i \in N$ for the converging subsequence proves the result.
References