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"Monotone Persuasion"

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Abstract

We explore when it is optimal in Bayesian persuasion environments for senders to commit to signal structures which induce the receiver to take higher actions when the underlying state is higher. Building on the literature on monotone comparative statics, we provide a technique to compute the optimal monotone signal structure. We also identify primitive conditions that guarantee that these are optimal among all (possibly non-monotone) signal structures. When the action space is binary, supermodularity of the sender's and receiver's preferences suffices for the optimal signal to have a monotone structure. With a continuum of actions, the conditions are more intriguing. We identify a novel single-crossing condition using a virtual utility representation of the sender's payoff. Applications are given to quadratic loss functions with biases and to credit ratings.

Keywords: Bayesian persuasion, mechanism design, single-crossing property, games of incomplete information

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1 Introduction

A prominent problem in economic literature is that of inducing one party, the "receiver," to take the action desired by another party, the "sender," by influencing the receiver's beliefs. One angle from which to analyze this problem is found in the literature on persuasion, where the sender is given commitment power in designing the signal structure. In an influential paper, Kamenica and Gentzkow (2011) (henceforth KG) show that, when the sender's payoff is convex in the receiver's beliefs and signal design is costless, it is optimal to reveal all information to the receiver; this result was extended by Kamenica and Gentzkow (2014) to environments where signals' costs are increasing in their informativeness. More generally, the optimal signal structure generates a payoff to the sender from the concave closure of his possible payoffs over all possible posteriors for the receiver.

One can think of many environments that fit into the persuasion framework in which the structure of signals is monotone, in the sense that a higher realization of the underlying state leads to a "higher" message being sent in that it induces the receiver to take a higher action. There are two reasons why this may be so. First, as in the signaling literature, there may be complementarities between the underlying state and the action chosen by the receiver. Second, there may just be an exogenous constraint that forces the signal structure to be monotone. For instance, Goldstein and Leitner (2015) consider a model of credit ratings, in the optimal ratings are often not monotone in the creditworthiness of the banks. However, due to legal and/or incentive compatibility constraints, such non-monotone credit ratings may be difficult to enforce. Another environment in which monotone signal structures seem appropriate is that of recommendations: a better candidate for a job receives a better review, in the sense that it leads to the better candidate getting better offers. One may also consider a school administrator placing students in courses of the optimal level of difficulty, where teachers (once the students are placed) set the difficulty level based on their own criteria.[1]

In this paper, we develop a technique of solving for optimal monotone signal structures that applies to both of the scenarios described in the previous paragraph. We take a first-order approach to the persuasion problem, examining the effects on the sender's payoff of "swapping" states between posteriors, by placing slightly more weight on a given state for a given posterior while simultaneously subtracting that weight from a different posterior. By doing so, we can find the boundary conditions at the optimal monotone structure. This first step permits us to characterize optimal signal structures when they are constrained to be monotone. The same approach also enables us to identify appropriate primitive conditions

[1] Though technically not a persuasion problem, this can be analyzed using the same techniques.

that guarantee that the signal structure that maximizes the sender's payoff within the family of monotone structures also maximizes his payoff among all possible ones.

While KG demonstrate the existence of an optimal signal structure, their approach does not address the issue of whether these is monotone. Indeed, by their techniques, it is quite possible that one wish to pool states that are not within a connected interval, or to send multiple possible messages with positive probability for the same state. Furthermore, while the approach of KG is practical for low-dimensional state spaces, it is often intractable when the state space is large. Importantly, the difficulty with the approach in KG arises even for state spaces with at least three elements. Additional methods are often needed in practice to simplify the problem.

Our approach permits one to bypass some of these difficulties by focusing on partitional structures. Specifically, if the state space Ω is a subset of the real line, then each message translates into a posterior that consists of the prior restricted to a subinterval of Ω. We characterize the optimal partition structure via a novel single crossing property: more specifically, if the support for two messages overlap, the single-crossing property we identify implies that it is optimal to "swap" elements of the support from one to the other to remove the overlap. This "swap" consists of a Gâteaux derivative in the direction of the change of the conditional distribution. The resulting method is similar to that in Bergemann and Pesendorfer (2007), who consider the simultaneous problem of providing information and designing an optimal auction for bidders with independent private values. They show that the optimal information structure is generated by partitioning each player's value space into a finite set of intervals by showing that for any non-monotone partition, there exists a local improvement via swapping for some state. Our results can be viewed as a generalization of theirs by identifying a single-crossing condition that guarantees improvements over non-monotone structures via swapping.

It has been noted in the signaling literature that, whenever the sender's and receiver's preferences are supermodular, there is a monotone partitional equilibrium, in which higher types send higher messages, which in turn induce higher actions by the receivers. For instance, the seminal papers of Spence (1973) about costly job-market signalling, and Crawford and Sobel (1982) on cheap talk, hinge on supermodularity for the sender's payoffs between the underlying state on one hand, and the respective actions chosen by the sender and the receiver. More recent papers by Kartik (2009) and Mensch (2016) extend the conditions under which there exists an equilibrium in which the sender's signalling action is increasing in the underlying state. The intuition is that, when there is no commitment, in equilibrium, the receiver's beliefs are specified for any action that the sender may take. A receiver therefore cannot detect that a type has deviated, but only the choice of message, for which the equilibrium specifies her belief. The sender must therefore take the receiver's beliefs and hence her response as given when considering a potential deviation. So, a deviation by the sender does not change the action that the receiver will take, and single-crossing is determined as if the receiver is choosing an action for an exogenously specified belief.

In persuasion environments, single-crossing becomes more complicated. When the receiver has two actions available, supermodularity of the receiver's and sender's preferences in the action and state remains sufficient to guarantee that, under the optimal signal structure, the induced action is higher when the state is higher, though the reasoning is somewhat different. Consider a pair of posteriors in which, conditional on one of them, the receiver takes the higher of the two possible actions, and takes the lower of the two conditional on the other. If the sender increases the weight of a high state in the former posterior, the receiver is even more inclined to take a high action; similarly, by decreasing the weight of a high state conditional on the latter posterior, the receiver will be even more inclined to take a low action. Rather than looking at the change in sender's payoff from shifting an individual state, though (as in the case of no commitment), one must verify that the payoff from pairing a high action with a high state, along with a low action with a low state, is an improvement over pairing the high action with the low state and vice versa.

However, when the receiver has more than two actions available, supermodularity of the sender's payoffs no longer suffices. This is because, when the sender can commit to a certain signal structure, the sender can no longer take the beliefs of the receiver as given when optimizing: since the sender directly chooses the posterior, any deviation will change the beliefs of the receiver. Perturbing the beliefs of the receiver by swapping weight on a state between two posteriors therefore changes what action the receiver takes for those posteriors.[2] Complementarities between the resulting action and the underlying state matter no longer guarantee optimality of a monotone signal structure: one must also consider the effect that changing a signal has on the action chosen by the receiver given that her beliefs have been perturbed.

To account for this difference in the specified beliefs of the receiver, we introduce a novel complementarity condition on the "virtual utility" of the sender: namely, his payoff, adjusted by a term compensating for the requirements of incentive compatibility on the part of the receiver. This notion is analogous to the concept of virtual utility in the mechanism design literature (for example, in the analysis of optimal auctions in Myerson, 1981), in which the

[2]While the beliefs will be perturbed in the same way in the case of two actions, supermodularity still suffices because there is no third action to which the receiver can deviate.

principal adds a handicap to the agent's utility to account for the cost of leaving the agents informational rents necessary to incentivize them to report truthfully.

Interestingly, if we reintroduce supermodularity of the receiver's payoffs between action and state, then it is optimal to fully reveal all information, in which case the optimal signal structure is trivially monotone. Nevertheless, if it is costly to reveal all information (specifically, if the cost is increasing in the number of posteriors), then the optimal signal structure remains monotone, but different from full disclosure.

The first-order technique introduced in this paper allows for characterization of first-order conditions for the optimal signal structure. Specifically, there cannot be an improving swap (given by the virtual utility function) at the boundaries of the intervals. This greatly simplifies the conditions that must be checked for optimization as compared to the general persuasion problem.

In a closely related paper, Ivanov (2015) examines Bayesian persuasion environments in which the optimal signal structure is monotone partitional. There are several key differences between our two papers. First, the approach he uses (Schur convexity and majorization) differs from ours, which is much closer to the monotone comparative statics and mechanism design literatures. Second, the first-order technique in this paper simplifies finding the optimal partitional structure. Third, our approach is usable in environments where the solution is exogenously constrained to be monotone, while his is not. Lastly, he considers a broader range of environments, which allow for incorporation of an additional mediator, as well as commitment to a strategy by the mediator and/or receiver. While this allows additional applicability of his results, it may obscure the intuition in the narrower environment of KG, which we consider here. Due to the differing approaches and goals, our papers should be considered complementary.

We apply our new approach to some of the classic signalling problems which did not assume commitment on the part of the sender. Specifically, we examine the implications of our results for the model of quadratic losses with biases of Crawford and Sobel (1982). We show that the standard examples of this model satisfy the single-crossing conditions identified in this paper, and so it remains optimal to partition the state space even under full commitment. We also provide an example of a constrained solution when signaling is costly. Finally, we solve for a constrained optimal monotone signal structure in the credit ratings environment of Goldstein and Leitner (2015).

2 Setup

For convenience, we use the notation as in KG;[3] this will allow for easy cross-reference and comparison of results. There are two players, a sender (S) and a receiver (R); we will occasionally refer to the sender as "he" and the receiver as "she." There is an underlying compact state space $\Omega \subset \mathbb{R}$, with an associated prior over the states μ_0 taken from the set of Borel probabilities on Ω, which we denote by $(\Delta(\Omega))$, that is common to both parties. Prior to the realization of $\omega \in \Omega$, the sender can choose a signal structure $(\pi, \mathcal{S})$, where $\mathcal{S}$ is a compact metric space representing the set of possible realizations of the signal, and

$$\pi : \, [0,1] \rightarrow \Omega \times \mathcal{S}$$

$$x \rightarrow (\pi_1(x), \pi_2(x))$$

is a measurable function defined so that the realization of $\pi_2(x) = s \in \mathcal{S}$ is correlated with $\pi_1(x) = \omega \in \Omega$. We assume that x is uniformly distributed over $[0,1]$. The interpretation of this structure is as follows. There is a random variable x which signifies the correlation between the signal realization observed ($s \in \mathcal{S}$) and the underlying state variable ω. To explain the correlation, there is a function $\pi_1(\cdot)$ which signifies the underlying state, and $\pi_2(\cdot)$ corresponds to the realized signal. Thus the random variable $s \in \mathcal{S}$ is generated from ω via the probability integral transform as modelled by x through the mapping π.[4]

Example 1: To give an example as to how this would work in the more familiar environment of a binary state space, suppose that μ_0 places equal weight on $\omega = \omega_1$ and $\omega = \omega_2$. To generate two signal realizations $\{s_1, s_2\}$ which each occur with probability $\frac{1}{2}$ and such that s_1 induces conditional probability $\frac{2}{3}$ on ω_1 while s_2 induces conditional probability $\frac{2}{3}$ on ω_2, we would set

$$\pi_1(x) = \begin{cases} \omega_1, & x < 0.5 \\ \omega_2, & x \geq 0.5 \end{cases}$$

$$\pi_2(x) = \begin{cases} s_1, & x \in [0, \frac{1}{3}) \cup [\frac{1}{2}, \frac{2}{3}) \\ s_2, & x \in [\frac{1}{3}, \frac{1}{2}) \cup [\frac{2}{3}, 1] \end{cases}$$

[3]While the published edition of KG deals with environments with a finite number of states, they address the possibility of a continuum of states in their online appendix, and show that the main results extend to such an environemnt as well. We refer the reader there for a discussion of how to extend their proofs to this environment.

[4]This approach, as opposed to the more standard one in which there is a conditional probability $\pi(\cdot|\omega)$ by which the realized signal is generated, is necessary due to measurability issues that arise when the state space is uncountably infinite.

Upon receiving signal realization s, the receiver, using Bayesian reasoning from the signal structure and the prior μ_0, forms conditional probability assessment $\mu(\cdot|s)$. The receiver then chooses some action a from some compact set $A \subset \mathbb{R}$. The utility function for the receiver is given by $u_R(a, \omega)$, which is bounded and measurable in ω. The sender has utility function $v_S(a, \omega, \pi, \mathcal{S}) = u_S(a, \omega) - c(\pi, \mathcal{S})$, where u_S represents the payoff to the sender from the action chosen by the receiver, while $c(\pi, \mathcal{S})$ represents the cost of implementing the signal structure. Thus u_S is the portion of the payoff dependent on a particular realization of the state and action, while c is the portion that depends on implementing the entire signal structure. Assume that u_S is bounded and measurable in ω, and that $c(\cdot, \cdot)$ is increasing in the informativeness of the $(\pi, \mathcal{S})$ in the sense of Blackwell (1951).

Upon observing s, the receiver's problem is then

$$\max_{a \in A} \int u_R(a, \omega) d\mu(\omega|s)$$

Define a distribution of posteriors by $\tau \in \Delta(\Delta((\Omega))$, so that π induces τ, i.e. if $\pi_2(x) \in \mathcal{S}' \subset \mathcal{S}$ for exactly some measurable set $X \subset [0, 1]$, then

$$\int_{\mathcal{S}'} d\tau(\mu(\cdot|s)) = \int_0^1 1[\pi_2(x) \in \mathcal{S}']dx$$

In order to be consistent via Bayes' Theorem, we must have for all measurable $\Psi \subset \Omega$ that

$$\int_{\Delta(\Omega)} \mu(\Psi) d\tau(\mu) = \mu_0(\Psi)$$

Define the choice of action by the receiver, conditional on posterior μ, to be $a^*(\mu)$. The sender's problem is then, since $(\pi, \mathcal{S})$ pins down τ,

$$\max_{(\pi, \mathcal{S})} \int \int u_S(a^*(\mu), \omega) d\mu(\omega|s) d\tau(\mu) - c(\pi, \mathcal{S})$$

$$\text{s.t.} \int_{\Delta(\Omega)} \mu d\tau(\mu) = \mu_0$$

A perfect Bayesian equilibrium of this game will consist of a vector $(\pi, \mathcal{S}, \mu, a^*)$ such that (i) μ is Bayes-consistent with the signal structure $(\pi, \mathcal{S})$, (ii) a^* is optimal for the receiver given μ, and (iii) $(\pi, \mathcal{S})$ is optimal for the sender given a^*. KG showed that in such an environment, the set of perfect Bayesian equilibria is non-empty as long as $c = 0$ for all $(\pi, \mathcal{S})$; Kamenica and Gentzkow (2014) extended this result to environments in which Ω is finite and c is increasing in the informativeness of the signal.

3 Preliminary Definitions

Our central question is to analyze when the optimal signal structure is monotone, in that it partitions the state space into subintervals of Ω. Before we can identify precisely what this will mean, we first introduce several preliminary definitions. Suppose that $|\mathcal{S}|$ is finite;[5] in this case, $\tau(\mu(\cdot|s)) > 0$ for all $s \in \mathcal{S}$. Consider any two conditional distributions $\mu, \mu' \in \Delta(\Omega)$, and assign them the probabilities $\tau(\mu)$ and $\tau(\mu')$, respectively. Then we define an addition operation between μ and μ' so that $\mu + \mu'$ yields a new conditional distribution which is generated with probability $\tau(\mu + \mu') = \tau(\mu) + \tau(\mu')$, where, for any measurable $\Psi \subset \Omega$,

$$(\mu + \mu')(\Psi) = \frac{\mu(\Psi)\tau(\mu) + \mu'(\Psi)\tau(\mu')}{\tau(\mu) + \tau(\mu')}$$

Analogously, we define a subtraction operation which inverts the addition operation, i.e. if $\hat{\mu} = \mu - \mu'$, then $\mu' + \hat{\mu} = \mu$, assuming the posteriors from each such conditional are between 0 and 1 for all such Ψ.

In tandem with the above operations, we allow for scaling of signal realizations. Thus, $\delta \cdot \mu + \epsilon \cdot \mu'$ is defined so that $\tau(\delta \cdot \mu + \epsilon \cdot \mu') = \delta\tau(\mu) + \epsilon\tau(\mu')$

$$(\delta \cdot \mu + \epsilon \cdot \mu')(\Psi) = \frac{\delta\mu(\Psi)\tau(\mu) + \epsilon\mu'(\Psi)\tau(\mu')}{\delta\tau(\mu) + \epsilon\tau(\mu')}$$

In much of the ensuing analysis, we examine the effects to μ that adding a conditional distribution that occurs with probability ϵ which places conditional probability 1 on the state ω'; with a slight abuse of notation, we label this by $\epsilon\omega'$, so that $\tau(\epsilon\omega') = \epsilon$.

Next, we define an ordering over conditional distributions. This will allow for comparisons to show that, if a signal structure that is not monotone, there exists a signal structure that is "more monotone" in the sense of the ordering that will be a local improvement.

Definition 1: A binary relation $\succeq_\sigma$ is an *ordering over conditional distributions* if the following two properties hold:

(i) *Completeness*: For any μ_1, μ_2, either $\mu_1 \succeq_\sigma \mu_2$ or $\mu_2 \succeq_\sigma \mu_1$

(ii) *Transitivity*: For any μ_1, μ_2, μ_3, if $\mu_1 \succeq_\sigma \mu_2$ and $\mu_2 \succeq_\sigma \mu_3$, then $\mu_1 \succeq_\sigma \mu_3$. Furthermore, if one of the orderings is strict, then $\mu_1 \succ_\sigma \mu_3$.

The ordering of the conditional distributions will be useful to essentially "summarize" the information contained therein. In particular, we will be interested in the case where higher

[5] By Proposition 4 in the online appendix of KG, such $\mathcal{S}$ is guaranteed to be optimal whenever $\min\{|A|, |\Omega|\} < \infty$.

distributions induce higher actions on the part of the receiver.

Assumption 1 (Action Monotonicity): $\mu_1 \succeq_\sigma \mu_2 \iff a^*(\mu_1) \geq a^*(\mu_2)$.

While this assumption may seem restrictive, one should note that for any game that fits into our model, there will always exist such an ordering $\succeq_\sigma$: one can simply define the ordering to be that induced by the action chosen by the receiver upon believing μ.

To relate Assumption 1 to the underlying preferences of the receiver, we introduce the following property that posteriors may hold.

Definition 2: The ordering $\succeq_\sigma$ is *FOSD-consistent* if $\mu_1((-\infty, \omega]) \leq \mu_2((-\infty, \omega])$ for all $\omega \in \Omega$, then $\mu_1 \succeq_\sigma \mu_2$.

The following lemma characterizes the relationship between action monotonicity and the FOSD-consistency condition. If the induced beliefs from μ_1 first-order stochastically dominated those from μ_2, we write this as $\mu_1 \succeq_{FOSD} \mu_2$. This will be useful for our later results, since there will be occasions where we will need to construct an improvement over a given signal structure through a multi-step perturbation, each step of which preserves first-order stochastic dominance. We will then be able to preserve ordering of the signals through each step.

Lemma 1: *Suppose that there is a unique optimal action $a^*(\mu)$, and that u_R is supermodular in (a, ω). Let $\succeq_\sigma$ be a complete and transitive ordering. Then it is FOSD-consistent if it satisfies action monotonicity (strictly so if u_R is strictly supermodular and either $a^*(\mu_1) \in$ int(A) or $a^*(\mu_2) \in$ int(A)).*

The proofs of all lemmas and propositions are contained in the appendix.

The above result shows that, when the receiver has supermodular preferences, it is only possible to have the optimal action by the receiver be increasing in the conditional distribution if the ordering ranks the induced distribution higher when it first-order stochastically dominates the other. In this sense, the ordering $\succeq_\sigma$ is a completion of the ordering induced by first-order stochastic dominance. There are many possible methods of ranking that fall into this latter category; an important one is by the conditional mean.

Example 2: Suppose that signal realizations are ranked by the conditional mean of the distribution, i.e. $\mu_1 \succeq_\sigma \mu_2 \iff E_{\mu_1}[\omega] \geq E_{\mu_2}[\omega]$. This is a complete and transitive ordering. To see that it is FOSD-consistent, note that $E_\mu[\omega] = \int \omega d\mu(\omega)$. Since $f(\omega) = \omega$ is an increasing function, it follows that if $\mu_1 \succeq_{FOSD} \mu_2$, then $E_{\mu_1}[\omega] \geq E_{\mu_2}[\omega]$. $\square$

We can see from here how changing a conditional distribution affects its ordering. Specifically, if we consider adding more weight to the highest portion of the support of a distribution, then it will increasing its ranking in the ordering $\succeq_\sigma$. Similarly, subtracting weight from the highest portion will decrease the distribution.

Now that we have defined an ordering over conditional distributions, we can present our definition of monotone signal structures.

Definition 3: A signal structure $(\pi, \mathcal{S})$ is *monotone* if, for all $\omega_1, \omega_2 \in \Omega$, if $\omega_1 > \omega_2$ and $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$, then

$$\omega_1 \in \text{supp}(\mu(\cdot|s_2)) \implies \omega_2 \notin \text{supp}(\mu(\cdot|s_1))$$

The following lemma will be useful for our results, stating that without loss of generality, it is possible to merge realizations which are ranked equally under $\succeq_\sigma$.[6] The intuition is that they result in the same action chosen by the receiver, and so the sender is indifferent to combining the two realizations into one. We can therefore assume without loss of generality that, for every $s_1, s_2 \in \mathcal{S}$, either $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$ or $\mu(\cdot|s_2) \succ_\sigma \mu(\cdot|s_1)$.

Lemma 2: *Suppose that the optimal signal $(\pi, \mathcal{S})$ generates some measurable family of subsets $\{\mathcal{S}^*(a)\} \subset \mathcal{S}$, where $|\mathcal{S}^*(a)| \geq 2$ and $\int_{\bigcup\{\mathcal{S}^*(a)\}} d\tau(\mu(\cdot|s)) > 0$ such that, for all $s_1, s_2 \in \mathcal{S}^*(a)$, $a^*(\mu(\cdot|s_1)) = a^*(\mu(\cdot|s_2)) = a$. Then the signal $(\pi', \mathcal{S}')$ is also optimal, where $\mathcal{S} = \mathcal{S}'$ except that each $\mathcal{S}^*(a)$ is replaced by a single realization $s'(a)$, i.e.*

1. $\mathcal{S} \setminus \bigcup\{\mathcal{S}^*(a)\} = \mathcal{S}' \setminus \bigcup\{s'(a)\}$;
2. If $\pi_2(x) \in \mathcal{S} \setminus \bigcup\{\mathcal{S}^*(a)\}$, then $\pi_2'(x) = \pi_2(x)$;
3. If $\pi_2(x) \in \mathcal{S}^*(a)$, then $\pi_2'(x) = s'(a)$; and
4. $\pi_1(x) = \pi_1'(x)$, $\forall x$

Using the previous lemma, it is easy to see that any monotone signal structure partitions the state space into intervals (though there may be some overlap in supports at the boundaries if there is an atom at a particular ω in the prior μ_0). We formalize this in the following proposition.

Proposition 1: *If a signal structure is monotone, then:*

(a) For any two states $\omega_1, \omega_2 \in$ supp$(\mu(\cdot|s))$, any $\omega' \in \Omega \cap (\omega_1, \omega_2)$ is contained in supp$(\mu(\cdot|s))$.

(b) If $\omega \in$ supp$(\mu(\cdot|s))$ and there exist $\omega_1 < \omega$ and $\omega_2 > \omega$ such that $\omega_1, \omega_2 \in$ supp$(\mu(\cdot|s))$ (i.e. ω is in the "interior" of the support of s), then for all s' such that either $\mu(\cdot|s') \succ_\sigma \mu(\cdot|s)$ or $\mu(\cdot|s) \succ_\sigma \mu(\cdot|s')$, $\omega \notin$ supp$(\mu(\cdot|s'))$.

[6] This is analogous to Proposition 1 in KG.

(c) $|supp(\mu(\cdot|s)) \cap (supp(\mu(\cdot|s'))| \leq 1$.

If the signal structure is monotone, then it is quite easy to define the posteriors $\mu(\cdot|s)$ using Bayes' theorem: they will just be the prior μ_0 restricted to the interval of states in the support of μ given s. If there is an atom at some ω that is at the boundary of the support of the posterior induced by s, then we may need to scale it by the probability that ω generates s. If there is just a single ω in the support of s, then $\mu(\omega|s) = 1$.

4 Two Actions

There are many relevant situations in which a sender must convince a receiver to take one of two actions. In many cases, there is complementarity between the state and the decision to be taken. For instance, in the well-known example of KG, a prosecutor wishes to convince a judge to convict a defendant. The judge is more inclined to convict the defendant if there is a greater likelihood that he is guilty, and so there is complementarity between the state (the guilt of the defendant) and the action (the verdict). As shown in their two-state example, the optimal structure is monotone: one realization has only in its support one of the states (in their case, that the defendant is innocent), while the other is supported by both of the states (that the defendant is guilty, as well as, with some probability, that he is innocent). We extend this reasoning more generally to environments with complementarities.

Consider an environment in which $A = \{a_1, a_2\}$, with $a_1 < a_2$. By Lemma 2, one can restrict attention to $|\mathcal{S}| \leq 2$, and so $\tau(\mu(\cdot|s)) > 0$ for each $s \in \mathcal{S}$.

Suppose that the preferences of both the sender and the receiver are weakly supermodular in (a, ω). Assuming that the induced action a does not change for sufficiently small ϵ, the marginal change in payoff to the sender from adding $\epsilon\mu'$ to μ in the limit as $\epsilon \to 0$ is

$$d(\mu; \mu') \equiv \lim_{\epsilon \to 0} \frac{\tau(\mu) \int u_S(a^*(\mu), \omega) d\mu + \epsilon\tau(\mu') \int u_S(a^*(\mu), \omega) d\mu' - \tau(\mu) \int u_S(a^*(\mu), \omega) d\mu}{\epsilon\tau(\mu')}$$

$$= \int u_S(a^*(\mu), \omega) d\mu' \tag{1}$$

This is the Gâteaux derivative of the payoff of the sender from conditional distribution μ in the direction of μ'. If we define a posterior μ' and assign it the value $\tau(\mu')$, then at the optimal signal structure $(\pi, \{s_1, s_2\})$, if it is feasible to subtract $\epsilon \cdot \mu'$ (for some small enough $\epsilon > 0$) from $\mu(\cdot|s_1)$ and add it to $\mu(\cdot|s_2)$, then

$$d(\mu(\cdot|s_2); \mu') - d(\mu(\cdot|s_1); \mu') \leq 0 \tag{2}$$

If not, then for small enough ϵ, there would exist an improvement from such a swap, i.e. subtracting $\epsilon \cdot \mu'$ from $\mu(\cdot|s_1)$ and adding it to $\mu(\cdot|s_2)$. This enables us to characterize a condition for monotone persuasion in the case of binary action spaces.

Definition 4: The *restriction* of a signal realization s to some measurable subset $\Psi \subset \mathbb{R}$ such that $\Psi \cap \Omega \neq \emptyset$, s_Ψ, assigns

$$\mu(B|s_\Psi) = \frac{\mu(B \cap \Psi|s)}{\mu(\Psi|s)}$$

for any measurable $B \subset \Omega$, and

$$\tau(\mu(\cdot|s_\Psi)) = \tau(\mu(\cdot|s)) \cdot \mu(\Psi|s)$$

The concept of a restriction will allow us to find a "swap" that forms an improvement from a non-monotone signal structure.

Theorem 2: *If both u_R and u_S are supermodular in (a, ω), then a monotone signal structure is optimal.*

The intuition for Theorem 2 is straightforward: due to the complementarity between actions and states, the sender would like to align the higher states with higher actions. If a signal structure is not monotone, he can always increase the level of alignment by swapping some of the support from the lower conditional distribution to the higher. This will not affect incentive compatibility of the receiver, as now the receiver is even more inclined to choose the higher action conditional on the higher signal realization.

One might be tempted to extend Theorem 2 to environments in which the sender's payoff is submodular to show that it is optimal to reveal no information. Similarly, one might be tempted to show that a monotone signal structure is optimal when $|A| > 2$ as well. However, this cannot be assumed, as once the posteriors are modified, the incentive compatibility constraints no longer need hold. Thus the critical condition that the changes in the conditional distributions do not change the induced actions by the receiver will not necessarily hold, either. For instance, when u_S is submodular, transferring a high state ω to the distribution of the lower of the two signal realizations, $\mu(\cdot|s_1)$, will increase the latter's conditional distribution, and so may induce the receiver to instead take action a_2. This may be bad for the sender (especially if he always prefers a_1). Similarly, for $|A| > 2$, increasing $\mu(\cdot|s_2)$ may lead to the receiver taking an even higher action, which may or may not be beneficial to the sender. In Section 5, we will extend our analysis to an environment in which A is a continuum (specifically, an interval) in which a different but related condition will allow for monotonicity.

In tandem with the inequality (2), one can observe that the optimal cutoff state ω^* for the signal structure, such that $\omega > \omega^* \in \text{supp}(\mu(\cdot|s_2))$ and $\omega < \omega^* \in \text{supp}(\mu(\cdot|s_1))$, must satisfy the following first-order condition.

Corollary 3: *Let $(\pi, \mathcal{S})$ be an optimal (monotone) signal structure such that $\mu(\cdot|s_2) \succ_\sigma \mu(\cdot|s_1)$. Then at least one of the following conditions must hold:*

(a) $\int u_R(a_1, \omega) d\mu(\omega|s_1) = \int u_R(a_2, \omega) d\mu(\omega|s_1)$

(b) $\int u_R(a_1, \omega) d\mu(\omega|s_2) = \int u_R(a_2, \omega) d\mu(\omega|s_2)$

(c) If $\bar{\omega}_1 \equiv \sup\{\omega \in \textit{supp}(\mu(\cdot|s_1))\} < \inf\{\omega \in \textit{supp}(\mu(\cdot|s_2))\} \equiv \underline{\omega}_2$*, then* $u_S(a_2, \underline{\omega}_2) \geq u_S(a_1, \underline{\omega}_2)$ *and* $u_S(a_2, \bar{\omega}_1) \leq u_S(a_1, \bar{\omega}_1)$*. In particular, if u_S is continuous in ω and $\bar{\omega}_1 = \underline{\omega}_2 \equiv \omega^*$, then $u_S(a_2, \omega^*) = u_S(a_1, \omega^*)$.*

Note that the first two scenarios of Corollary 3 are incentive compatibility constraints for the receiver: one must check to see if, from a small perturbation in the posterior, the receiver would jump to a different action. If they are not binding, then on the margin, the sender can shift some weight at the margin to the sender's preferred action at state ω^*. Thus the problem to find the optimal signal structure is simple: one checks the incentive compatibility constraints, and if they are not binding, one finds the value of ω^* are which the sender is indifferent between the actions taken.

5 Continuum of Actions

We now explore the case where A is an interval, $[\underline{a}, \bar{a}]$. As alluded to above, one may have the intuition to attempt to align the higher states with higher actions as in the environment with a binary action space. However, one must consider here an additional effect: by perturbing the posteriors, one also changes the action that the receiver takes. Thus, one must incorporate this effect as well when checking if such an alignment is an improvement for the sender.

To proceed with our analysis, we make a few additional smoothness assumptions. Assume that u_R is twice continuously differentiable with $\frac{\partial^2 u_R}{\partial a^2} < 0$, so that there will exist a unique optimal action for the receiver for each posterior μ. Assume that u_S is continuously differentiable in a and both u_S and $\frac{\partial u_S}{\partial a}$ are continuous in ω.

As indicated in the introduction, we take a first-order approach to the optimization problem. Suppose that there are two signal realizations s_1, s_2 such that $\mu(\cdot|s_1) \succeq_\sigma \mu(\cdot|s_2)$ that have overlapping distributions, so that there exists ω^* such that $\mu((-\infty, \omega^*]|s_1) > 0$ but

$\mu((\omega^*, \infty)|s_2) > 0$ as well. We want to show that there exists a local improvement by swapping some weight of the distribution $\mu(\cdot|s_2)$ above ω^* from the support of $\mu(\cdot|s_2)$ with some of the distribution below ω^* from that of $\mu(\cdot|s_1)$; since such a swap is possible whenever there are overlapping conditional distributions, this will imply that the signal structure was not optimal. To precisely define what such a swap would entail, define the marginal change in payoff from adding two conditional distributions μ and μ' for which $\tau(\mu)$ and $\tau(\mu')$ are positive by $D(\mu, \mu')$ (if we subtract μ' from μ, this change is $D(\mu, -\mu')$). Thus, if one adds μ and $\epsilon \cdot \mu'$, $D(\cdot, \cdot)$ is defined to be

$$D(\mu, \epsilon \cdot \mu') \equiv \frac{1}{\epsilon \tau(\mu')} \{\tau(\mu)[\int u_S(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu - \int u_S(a^*(\mu), \omega) d\mu]$$

$$+\epsilon \tau(\mu') \cdot \int u_S(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu\} \tag{3}$$

This illustrates two effects of adding μ' to μ on the payoff of the sender. First, this will perturb μ, and so perturb the optimal action for the receiver, $a^*(\mu)$. Second, there will now be an additional measure of state ω' which will now have the action associated with μ chosen for it as well.

For the case where we add $\epsilon\omega'$ to μ, the marginal change in payoff to the sender can be written as

$$D(\mu, \epsilon\omega') = \frac{1}{\epsilon} \{\tau(\mu)[\int u_S(a^*(\mu + \epsilon\omega'), \omega) d\mu - \int u_S(a^*(\mu), \omega) d\mu] + \epsilon \cdot u_S(a^*(\mu + \epsilon\omega'), \omega')\} \tag{4}$$

Before this can be applied directly to form the single-crossing condition, one must note that, in order to maintain Bayes-consistency, adding some weight $\epsilon \cdot \mu'$ to μ_1 requires subtracting the same amount from the other posteriors. It is not immediately clear that these effects are symmetric, i.e. subtracting $\epsilon\mu'$ from μ yields the same marginal change in the sender's payoff as adding it to μ, with the signs of course reversed. Fortunately, due to the assumptions on the preferences of the receiver, the best-reply $a^*(\mu)$ is well-behaved. To see the intuition for this symmetry, note that if the receiver were completely certain of the state, then by taking the second derivative with respect to ω, we have

$$\frac{\partial^2 u_R}{\partial a \partial \omega}(a^*(\omega), \omega) + \frac{\partial^2 u_R}{\partial a^2}(a^*(\omega), \omega) \cdot \frac{da^*(\omega)}{d\omega} = 0$$

and so the implicit function $a^*(\omega)$ is differentiable with derivative

$$\frac{da^*(\omega)}{d\omega} = -\frac{\frac{\partial^2 u_R}{\partial a \partial \omega}(a^*(\omega), \omega)}{\frac{\partial^2 u_R}{\partial a^2}(a^*(\omega), \omega)}$$

Note that the marginal effects of increasing or decreasing the underlying state are the same; since it is the receiver's reaction that affects the payoff of the sender, the marginal effect is symmetric for the sender as well. A similar idea holds in showing that the effect of shifting the conditional distribution up or down is symmetric in its marginal effects on the payoff of the sender. To do so, as in the previous section, we define

$$d(\mu; \mu') \equiv \lim_{\epsilon \to 0} D(\mu, \epsilon \cdot \mu')$$

to be the Gâteaux derivative of the payoff of the sender from conditional distribution μ in the direction of μ'.

Lemma 3: *(a) For all μ such that $a^*(\mu) \in (\underline{a}, \bar{a})$,*

$$d(\mu; \mu') = -\frac{\int \frac{\partial u_S}{\partial a}(a^*(\mu), \omega) d\mu \cdot \int \frac{\partial u_R}{\partial a}(a^*(\mu), \omega) d\mu'}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu} + \int u_S(a^*(\mu), \omega) d\mu' \tag{5}$$

Otherwise, if $a^*(\mu) = a^*(\mu + \epsilon\mu') \in \{\underline{a}, \bar{a}\}$ for small enough ϵ, then $d(\mu; \mu') = \int u_S(a^*(\mu), \omega) d\mu'$.

(b) Suppose that, in either case described in (a), it is feasible to subtract $\epsilon \cdot \mu'$ from μ for small enough ϵ. Then

$$d(\mu; -\mu') = -d(\mu; \mu')$$

Lemma 3 provides a formula for the effects of adding an infinitessimally small amount of μ' to μ. For the special case of adding $\epsilon\omega'$ to μ in the limit as $\epsilon \to 0$, this is given by

$$d(\mu; \omega') = u_S(a^*(\mu), \omega') - \frac{\frac{\partial u_R}{\partial a}(a^*(\mu), \omega') \int \frac{\partial u_S}{\partial a}(a^*(\mu), \omega) d\mu}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu), \omega) d\mu} \tag{6}$$

Hence we can write $d(\mu; \mu')$ as

$$d(\mu; \mu') = \int d(\mu; \omega') d\mu' \tag{7}$$

Interestingly, $d(\mu; \mu')$ does not depend on τ, so we do not need to take account of the probability of a given posterior occuring in our ordering. This is due to the inverse proportionality of the size of the effects from perturbing the distribution: if $\tau(\mu)$ is larger, then the posterior will change less from adding $\epsilon \cdot \mu'$, and so the induced action will shift less; on the other hand, there is a greater proportion of states (with respect to the prior) that are affected by this change in posterior. These two effects will exactly cancel out.

An economic interpretation of the expression for $d(\mu;\omega')$ is in order. As mentioned above, there are two effects from changing the posterior. First, the action taken at the particular states will change, since the action taken at the old posterior is now taken at the new posterior. Second, there is an effect of changing the action taken at the old posterior, since the receiver will change his action as the posterior has changed. For marginal changes in the distribution, only these two first-order effects matter. Thus, the first term in the expression, $u_S(a^*(\mu),\omega')$, is the effect of having action $a^*(\mu)$ taken when the state is ω', since that is how the distribution was perturbed. The second term is the marginal effect on the payoff of the sender from the receiver changing her action from that taken at posterior μ. Thus $\int \frac{\partial u_S}{\partial a}(a^*(\mu),\omega)d\mu$ is the marginal utility of the sender from the change in action of the receiver at posterior μ, while $-\frac{\frac{\partial u_R}{\partial a}(a^*(\mu),\omega')}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu),\omega)d\mu}$ is the marginal amount by which $a^*(\mu)$ is perturbed by placing more weight on ω^*. These two effects are combined additively for a readily interpretable expression.

In this sense, the expression $d(\mu;\omega')$ can be viewed in a similar spirit to the "virtual utility" concept found in the mechanism design literature, such as in Myerson (1981). In mechanism design, the principal wants to maximize the surplus that is available to be extracted, but is constrained by incentive compatibility to leave a certain amount to the agent. Here, the sender maximizes the surplus (his utility) subject to the incentive compatibility of the agent (the receiver). To show this, we must incorporate a second term that accounts for these incentive compatibility constraints.

The definition of d will extend naturally to distributions μ which do not arise as posteriors from $(\pi,\mathcal{S})$, as long as we assign such μ a positive value of τ. As seen in the previous section, if $(\pi,\mathcal{S})$ is optimal, then for any $s_1,s_2\in\mathcal{S}$ for which $\tau(\mu(\cdot|s_1)),\tau(\mu(\cdot|s_2))>0$ such that it is feasible to subtract $\epsilon\cdot\mu'$ from $\mu(\cdot|s_1)$, it must be that

$$d(\mu(\cdot|s_2);\mu') - d(\mu(\cdot|s_1);\mu') \leq 0 \tag{8}$$

or else there would exist an improvement by swapping $\epsilon\cdot\mu'$ from $\mu(\cdot|s_1)$ to $\mu(\cdot|s_2)$. This is also in the same spirit as in mechanism design, where the problem of the principal is equivalent to maximizing the virtual utility.

To illustrate how the first-order approach can be used to find the optimal signal structure, we present the following simple example.

Example 3: Let $u_R(a,\omega) = -\frac{1}{2}a^2+2a\omega-a\omega^2$ and $u_S = a\omega$. The state space is $\Omega=\{\frac{1}{2},1,\frac{3}{2}\}$.

Then

$$d(\mu;\omega') = a(\mu)\omega' + (\int \omega d\mu)(-a + 2\omega' - (\omega')^2)$$

Note that

$$a(\mu) = \mu(1) + \frac{3}{4}(\mu(\frac{1}{2}) + \mu(\frac{3}{2})) = \frac{1}{4}\mu(1) + \frac{3}{4}$$

$$\int \omega d\mu = \mu(1) + \frac{1}{2}\mu(\frac{1}{2}) + \frac{3}{2}\mu(\frac{3}{2})$$

We use equation (8) to find the optimal signal structure. First, for $\mu' \succ_\sigma \mu$, $d(\mu'; \frac{3}{2}) - d(\mu'; \frac{1}{2}) > d(\mu; \frac{3}{2}) - d(\mu; \frac{1}{2})$. Therefore it will not be the case that in the optimal signal structure, $\frac{1}{2} \in \text{supp}(\mu')$ and $\frac{3}{2} \in \text{supp}(\mu)$, as the sender can always increase his payoff by transferring some weight on either $\omega = \frac{1}{2}$ or $\omega = \frac{3}{2}$ from one posterior to the other.

Next, suppose that $\mu(\omega|s) > 0$ for all ω for a given realization s. Consider an alternative signal structure $(\pi', \mathcal{S}')$ which is the same as $(\pi, \mathcal{S})$ except that s is replaced with s_1, s_2, so that $\pi'_1(x) = \pi_1(x)$ but, whenever $\pi_2(x) = s$, we set $\pi'_2(x)$ equal to either s_1 or s_2. Note that the receiver's choice of action a^* only depends on $\mu(\omega = 1)$. Thus we can split into two posteriors $\mu(\cdot|s_1), \mu(\cdot|s_2)$ such that $\mu(1|s_i)$ is the same in both, but $\mu(\frac{3}{2}|s_1) > \mu(\frac{3}{2}|s_2)$ while inducing

$$a^*(\mu(\cdot|s_1)) = a^*(\mu(\cdot|s)) = a^*(\mu(\cdot|s_2))$$

But then $d(\mu(\cdot|s_1); 1) > d(\mu(\cdot|s_2); 1)$, so this cannot be optimal, and so there is no such μ in the optimal signal structure.

Suppose there is a posterior with $\mu(\frac{3}{2}|s) = 0$. Then $d(\mu(\cdot|s); \frac{1}{2}) = \frac{3}{8} - \frac{1}{8}(\mu(1|s))^2$ which is decreasing in $\mu(1|s)$. So, one could split this posterior into two identical posteriors μ_1, μ_2 and add weight $\epsilon > 0$ of $\omega = \frac{1}{2}$ from μ_1 to μ_2. By the fundamental theorem of calculus, this results in a change in payoff of

$$\int_0^\epsilon (d(\mu_2 + \beta \cdot \omega; \omega) - d(\mu_1 - \beta \cdot \omega; \omega))d\beta$$

where $\omega = \frac{1}{2}$. Since $\mu_1 - \beta \cdot \omega \succ_\sigma \mu_2 + \beta \cdot \omega$ (and so the conditional probability of $\omega = 1$ is higher given the former), this is strictly positive, and so this cannot be optimal. Hence without loss of generality, we can assume that $\omega = \frac{1}{2}$ is placed in its own posterior, i.e. there exists some s such that $\mu(\frac{1}{2}|s) = 1$ and for all other $s' \in \mathcal{S}$, $\mu(\frac{1}{2}|s') = 0$.

Lastly, suppose that $\frac{1}{2} \notin \text{supp}(\mu(\cdot|s)) \cup \text{supp}(\mu(\cdot|s'))$, but $\mu(1|s') > \mu(1|s)$. Then since $d(\mu(\cdot|s); \frac{3}{2}) = \frac{9}{8} + \frac{1}{8}(\mu(1|s))^2$ is increasing in $\mu(1|s)$, this cannot be optimal since $\frac{3}{2} \in \text{supp}(\mu|s)$. Hence the optimal signal structure pools $\omega = 1$ and $\omega = \frac{3}{2}$ in one signal realization, and reveals $\omega = \frac{1}{2}$ in the other. □

This first-order approach will greatly ease our definitions for our single-crossing conditions, which we now present.

Definition 5: The payoffs for the sender are *d-supermodular* if, for $\omega' > \omega$ and $\mu' \succ_\sigma \mu$,

$$d(\mu';\omega') - d(\mu';\omega) > d(\mu;\omega') - d(\mu;\omega) \tag{9}$$

Similarly, the payoffs to the sender are *d-submodular* if

$$d(\mu';\omega') - d(\mu';\omega) < d(\mu;\omega') - d(\mu;\omega) \tag{10}$$

Definition 6: The payoffs for the sender are *d-quasisupermodular* if

$$d(\mu';\omega) - d(\mu;\omega) \geq 0 \implies d(\mu;\omega') - d(\mu;\omega') > 0 \tag{11}$$

The payoffs to the sender are *d-quasisubmodular* if

$$d(\mu';\omega) - d(\mu;\omega) \leq 0 \implies d(\mu;\omega') - d(\mu;\omega') < 0 \tag{12}$$

The intuition for these single-crossing conditions is that, if the sender could choose which action to induce once the state was realized, he would want to choose a higher action when the state was higher if his preferences were quasisupermodular; this follows from the standard analysis of Milgrom and Shannon (1994). This justifies the reasoning in signaling games where the sender cannot precommit to a signal structure (such as cheap talk), since in a perfect-Bayesian equilibrium, the sender takes the receiver's beliefs conditional on any message as given, and so the sender will not change the induced action by deviating to a particular message. In our environment, though, the sender *will* change the action taken following a particular message (i.e. when the posterior is perturbed), since a different action will now be optimal for the receiver. Thus we must combine this effect as well into the problem of the sender, and only then check for single-crossing in the posterior (and hence the action) that the sender wants to induce given the state.

We now present our main results for this section.

Theorem 4: *Suppose that the sender's preferences are d-quasisupermodular. Then there exists an optimal signal structure that is monotone.*

The basic intuition for Theorem 4 is simple: if there were a (finite) signal structure that was non-monotone, then there would exist an improvement through swapping some weight on states between two posteriors that violated monotonicity. That is, one could add some weight on higher states to the higher posterior by subtracting from the lower posterior, or add weight on lower states to the lower posterior by subtracting from the higher posterior. By d-quasisupermodularity, one of these operations must be an improvement. To extend the result to signal structures with $|\mathcal{S}| = \infty$, one shows that one can approximate such signal structures by a sequence of finite ones $\{(\pi^k, \mathcal{S}^k)\}_{k=1}^{\infty}$, and then take the limit to show that monotone signal structures remain optimal.

Remark: It is possible to weaken the conditions in Theorem 4 as follows. The argument in the proof shows that, under d-quasisupermodularity, there always exists an improvement through the method described in the previous paragraph. However, this is stronger than needed, since it implies that swapping any two states in which the higher of the two is in the support of the higher posterior, and the lower is in the support of the lower, is an improvement. It would be sufficient to show that, for any two such posteriors, there exists such a pair of states. Since this would be complicated to analyze and distract from the main point, we leave this observation as a remark. □

One can get even stronger results when one assumes complementarity in the receiver's preferences. This is natural in many applications, and is a feature of both the models of Spence (1973) and Crawford and Sobel (1982). It turns out that complementarity leads to extremal information structures: it is either optimal to reveal all or no information.

Theorem 5: *Let $c = 0$ for all $(\pi, \mathcal{S})$ and $\frac{\partial^2 u_R}{\partial a \partial \omega} \geq 0$.*

(i) If the sender's preferences are d-quasisupermodular, then it is optimal for the sender to reveal all information.

(ii) Conversely, if his preferences are d-quasisubmodular, then it is optimal to reveal no information, i.e. the posterior beliefs for the receiver will just be μ_0.

The intuition for the proof of Theorem 5(i) is that, if there is a posterior μ with more than one state in the support, then we can duplicate the posterior to form μ_1, μ_2, and perturb them so that a little more weight is placed at the top of the support of, say, μ_2, or a little less weight at the bottom, while the opposite is done for μ_1. After this perturbation, μ_2 will be ranked higher than μ_1 due to FOSD-consistency, and so it will be a strict improvement to swap even more of the support. Since this initial perturbation can be arbitrarily small, it turns out that there will be a strict improvement over this initial posterior. Thus there cannot be any posteriors which have more than one state in the support, implying that full separation is optimal.

The intuition for Theorem 5(ii) is that, if there were two posteriors $\mu_2 \succ_\sigma \mu_1$ that induced different actions, then by FOSD-consistency, there must be some pair of states for which a higher action $a^*(\mu_2)$ is taken for the higher of the two states, while $a^*(\mu_1)$ is taken for the latter. It will then be an improvement to swap either the higher state or the lower state, and so this could not be optimal. All realizations must therefore lead to the same action in the optimal signal structure, and so by Lemma 2 one can just as well reveal nothing.

A key element of the proof of Theorem 4 was the absence of costs of implementing a signal. However, in many cases of interest, this assumption is not reasonable. This may arise from physical costs of providing additional information, which may make it optimal to only partially reveal the information to the receiver. A notable paper in the literature in which similar costs arise is that of Cremer, Garicano and Prat (2007), in which they examine the breadth of words in languages to describe a given situation. On the one hand, words that cover more states are more difficult to parse; on the other, given their bounded rationality assumptions which limit the ability of the sender to describe the situation, there is a maximum number of words that can be used. Their model has more recenty been applied by Sobel (2015) to settings in which one would expect an "interval property" to apply to the language used, which, as we have seen, is a consequence of monotone persuasion. We therefore include the following result for the case where it is not optimal to reveal all information.

Theorem 6: *Assume that $c(\pi, \mathcal{S})$ is increasing in $|\mathcal{S}|$ and $\lim_{|\mathcal{S}|\to\infty} c(\pi, \mathcal{S}) = \infty$.*

(i) Suppose that $c(\pi, \mathcal{S})$ is solely a (continuous) function of the vector of probabilities $\{\tau(\mu)\}$ irrespective of the posteriors that they induce. That is, consider any two signal structures $(\pi, \mathcal{S}), (\pi', \mathcal{S}')$ which induce respective posteriors μ, μ' and distributions over posteriors τ and τ'; there exists a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{S}'$ so that if $s \in \mathcal{S}$ corresponds to $s' \in \mathcal{S}'$, then $c(\pi, \mathcal{S}) = c(\pi', \mathcal{S}')$. Then if the sender's payoffs are d-supermodular, then the optimal signal structure is monotone.

(ii) If $c(\pi, \mathcal{S})$ is solely a function of the number of posteriors generated (i.e. $|\mathcal{S}|$) and the sender's payoffs are d-quasisupermodular, then the optimal signal structure is monotone.

Moreover, if $|\Omega| = \infty$, then full disclosure is not optimal.

In addition to the obvious economic interpretation of it being costly to the sender to generate the signal, one can apply this result to the provision of incentives in a non-persuasion context. Recall that in Lemma 2, we saw that without loss of generality, we could merge signal realizations if they resulted in the same outcome. Suppose that the receiver will choose to implement N actions, and the sender can only influence what those N actions are. One possible instance in which this might apply is in tracked courses: a school may offer a fixed number of tracked classes in a certain subject, and teach them at a difficulty level determined by the students in the respective classes. The school principal's optimal course difficulty level may differ from that of the teachers, as the must take into account the preferences of the students' parents; however, the teachers are the ones who have control over their classrooms. The principal's problem would then be to optimize the distribution of students so as to induce the most favorable incidence of student quality and course difficulty level. Alternatively, one could view the capacity as regarding class size: the school can only fit a fixed number of students in each class, and so the sender must optimize with the constraint that the vector of class sizes remains constant. Assuming that the number of students is large, this can be viewed as fixing the proportion of students in each class. We summarize these results in the following corollary.

Corollary 7: *Let* $c(\pi, \mathcal{S}) = 0$*.*

(i) Suppose that the receiver has a capacity constraint on the number of actions that it can implement, so that it is costless for the receiver to implement up to N *different actions, but for any* $n > N$*, it is infinitely costly for the receiver. Then if the sender's preferences are d-quasisupermodular, the optimal signal structure is monotone.*

(ii) Suppose that the receiver has a fixed capacity $\hat{\tau}(n)$ *for each action* $a_n \in A$*, where* $n \in \{1, ..., N\}$ *such that* $\sum_{n=1}^{N} \hat{\tau}(n) = 1$ *(i.e. each of the* N *actions taken must be taken with a fixed probability, but which action to choose is up to the receiver) Then if the sender's preferences are d-supermodular, the optimal signal structure is monotone.*

Remark: As we saw in Theorem 5, d-quasisupermodularity combined with complementarity in the receiver's preferences leads to the sender's optimal signal structure revealing all information, or revealing nothing. Given that KG also provide sufficient conditions for which it is optimal to either reveal all information, or reveal nothing (the former when the sender's payoff is convex in the posterior, the latter when it is concave), it is natural to consider the relationship between these conditions. A simple example based on the presence of costs as described in Theorem 6 demonstrates that these conditions are distinct. Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, with $\omega_1 < \omega_2 < \omega_3 < \omega_4$ and μ_0 assigning equal probability to each state. Convexity implies that the signal structure $(\pi, \{s_1, s_2\})$ in which $\text{supp}(\mu(\cdot|s_1)) = \{\omega_1, \omega_3\}$ and $\text{supp}(\mu(\cdot|s_2)) = \{\omega_2, \omega_4\}$ is at least as good for the sender as revealing no information, whereas this is not necessarily the case in the presence of either d-supermodularity or d-quasisupermodularity. On the other hand, there is some monotone signal structure $(\pi', \{s'_1, s'_2\})$ which is preferable to $(\pi, \{s_1, s_2\})$ under d-quasisupermodularity, which is not necessarily the case under convexity. □

The characterization of the signal structure as monotone greatly eases the search for the optimal structure, as on must now only search over a countable number of possible cutoff points to describe the intervals over which the signals are supported. This also allows the construction of first-order conditions in looking for the optimal structure. Suppose, as in case (ii) of Theorem 6, that the cost structure only depends on the number of signal realizations. Then one cannot be better off by switching the marginal state of some interval of support to another, and vice versa. This is true whether the optimization problem is constrained to monotone solutions or not. We describe these incentive conditions in the following result.

Proposition 8: *Suppose that $c(\pi, \mathcal{S})$ is solely a function of $|\mathcal{S}|$. Let $(\pi, \mathcal{S})$ be an optimal (monotone) signal structure. Consider $s_1, s_2 \in \mathcal{S}$ such that $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$ and there does not exist $s_3 \in \mathcal{S} \setminus (s_1, s_2)$ such that $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_3) \succ_\sigma \mu(\cdot|s_2)$. If*

$$\bar{\omega}_2 \equiv \sup\{\omega \in supp(\mu(\cdot|s_2))\} < \inf\{\omega \in supp(\mu(\cdot|s_1))\} \equiv \underline{\omega}_1$$

then $d(\mu(\cdot|s_1); \underline{\omega}_1) \geq d(\mu(\cdot|s_2); \underline{\omega}_1)$ and $d(\mu(\cdot|s_1); \bar{\omega}_2) \leq d(\mu(\cdot|s_2); \bar{\omega}_2)$. In particular, if $\bar{\omega}_2 = \underline{\omega}_1 \equiv \omega^*$, then $d(\mu(\cdot|s_1); \omega^*) = d(\mu(\cdot|s_2); \omega^*)$.

We now present further applications of our results.

6 Additional Applications

6.1 Quadratic loss functions with biases

One of the classic models of signalling is the cheap talk model of Crawford and Sobel (1982). In their model, they show that in any perfect Bayesian equilibrium, the sender's types are partitioned, so that a higher type sends a higher message. They also show that when types are uniformly distributed, and

$$u_S(a, \omega) = -(a - \omega - b)^2$$

$$u_R(a, \omega) = -(a - \omega)^2$$

then if it were possible for the sender to commit to reveal his type, he would receive a higher payoff than the equilibrium from their model. However, they do not explore whether this would indeed be optimal overall, nor do they analyze other distributions of types.

KG explored a more general class of sender preferences in the persuasion environment, and showed that if the sender's preferences are given by

$$u_S(a, \omega) = -(a - (b_1 + b_2\omega))^2 \tag{13}$$

for $[\underline{a}, \bar{a}] = \Omega = [0, 1]$, then it is optimal to reveal all information if $b_2 > \frac{1}{2}$, and reveal nothing if $b_2 < \frac{1}{2}$.

We extend the analysis here further to look at the optimal structure when full separation is not possible. When the sender's preferences are given as in (13), we find that, since $a^*(\mu) = E_\mu[\omega]$,

$$d(\mu; \omega') = -(a^*(\mu) - (b_1 + b_2\omega'))^2 + 2(a^*(\mu) - \omega')(\int (a^*(\mu) - (b_1 + b_2)\omega)d\mu$$

$$= -(a^*(\mu) - (b_1 + b_2\omega'))^2 + 2(a^*(\mu) - \omega')(a^*(\mu) - b_1 - b_2a^*(\mu))$$

To check for d-supermodularity, we check the sum of the terms involving both μ and ω', which is

$$2b_2a^*(\mu)\omega' - 2a^*(\mu)\omega' + 2b_2a^*(\mu)\omega'$$

Because $a^*(\mu)$ is increasing in μ, this above expression is supermodular in (μ, ω') if $b_2 > \frac{1}{2}$, and submodular if $b_2 < \frac{1}{2}$. Thus it is optimal to reveal all information in the former case, and reveal none in the latter, as anticipated by KG. Furthermore, if costs are as in Theorem 6, it is optimal to have a monotone signal structure.

To give a concrete example of a finite partition, let us look at an optimal partition when $|\mathcal{S}| = 2$, $A = [0, 1]$, $b_2 = 1$, and μ_0 induces the uniform distribution over $[0, 1]$. Consider the optimal signal structure $(\pi, \{s_1, s_2\})$, with $\mu(\cdot|s_2) \succ_\sigma \mu(\cdot|s_1)$. Since the signal structure is monotone, there will exist ω^* such that $\mu(\cdot|s_1)$ induces the uniform distribution on $[0, \omega^*]$, while $\mu(\cdot|s_2)$ induces the uniform distribution on $[\omega^*, 1]$. Thus

$$a^*(\mu(\cdot|s_1)) = \frac{\omega^*}{2}$$

$$a^*(\mu(\cdot|s_2)) = \frac{1 + \omega^*}{2}$$

and so, by Proposition 8, it must be that

$$-(\frac{\omega^*}{2} - (b + \omega^*))^2 + 2(\frac{\omega^*}{2} - \omega^*)(\frac{\omega^*}{2} - b - \frac{\omega^*}{2}) = -(\frac{1 + \omega^*}{2} - (b + \omega^*))^2 + 2(\frac{1 + \omega^*}{2} - \omega^*)(\frac{1 + \omega^*}{2} - b - \frac{1 + \omega^*}{2})$$

Simplifying this expression yields

$$-(\frac{\omega^*}{2} - (b + \omega^*))^2 - 2b(-\frac{\omega^*}{2}) = -(\frac{1 + \omega^*}{2} - (b + \omega^*))^2 - 2b(\frac{1 - \omega^*}{2})$$

$$b = 2b\omega^* \implies \omega^* = \frac{1}{2}$$

Thus it is optimal to divide the interval $[0, 1]$ into two equal pieces. The intuition for this is that, were the intervals to be uneven (say, with the upper interval being larger and the bias b being low), the concavity of the sender's payoff would imply that there would be a greater loss from high states (i.e. ω near 1) being farther from the induced action of the receiver than there would a benefit of moving the higher states of the lower interval (i.e. ω close to ω^*) closer to the induced action of the receiver.

6.2 Credit ratings

In a recent paper, Goldstein and Leitner (2015) apply the persuasion model of KG to the framework of credit ratings. In their framework, a social planner sets criteria by which a bank is given a certain rating; his preferences are those of the banks from an ex-ante perspective (described below). Conditional on that rating, a buyer offers to purchase an asset sold by that bank at the expected value of the asset. Banks aim to maintain sufficient reserves for a potential crisis, and so being able to sell their assets ensures that they will have sufficient reserves, as long as they receive a high enough score. However, if the offer is too low, they would rather not sell since the offered price does not provide sufficient insurance. The reader is directed to their paper for additional discussion of this environment.

A key feature of their results is that the optimal ratings criteria are often non-monotone, in that a bank with higher underlying value can be assigned a worse rating. This is often unrealistic, in that there may be legal ramifications to the ratings agency if they were to do so. Moreover, there may be incentive-compatibility issues, in that it may give banks perverse incentives to undermine their own credit.

We find the optimal monotone signal structure in the model of Goldstein and Leitner. Formally, the state representing the quality of asset is located in the interval $[\underline{\omega}, \bar{\omega}]$, where $\underline{\omega} < 1 < \bar{\omega}$, and is distributed with full support according to F and continuous density f. The uncertainty over the final value of the bank is represented by the variable ϵ, which is distributed independently of ω according to the continuous function $G(\epsilon)$ with $E[\epsilon] = 0$. For all states, $1 - G(1 - \omega) > 0$ and $G(1 - \omega) > 0$. The banks care about having sufficient amounts of cash on hand when ϵ is realized, and so have payoff $\omega + \epsilon$ if they do not sell and $\omega + \epsilon < 1$; otherwise they have payoff $\omega + \epsilon + r$ for some $r > 0$. If they sell, their payoff is a if $a < 1$, and $a + r$ otherwise. The sender (a social planner) cares about the ex-ante welfare of the banks. The receiver (a buyer) takes action equal to $E[\omega|s]$, and so can be viewed as having utility function $u_R(a, \omega) = -(a - \omega)^2$.

The sender's payoff in state ω is thus when the receiver chooses action $a \geq 1$ is $a+r$; otherwise, he declines to participate and gets expected payoff $\omega + r(1 - G(1 - \omega))$. The banks know their own state, and can decline to sell if the price offered is too low. As shown in Goldstein and Leitner (2015), they will always do so if $a < 1$, since the banks at least have positive chance of ex-post assets $\omega + \epsilon > 1$ if they reject. This imposes an incentive compatibility constraint on the signal structures that can be implemented: it must be optimal for the bank to sell at the price offered if the social planner wants the signal realized to induce the bank to accept the offer.

We say that resources are *scarce* if it is impossible to ensure that the asset is always sold.

Proposition 9: *If resources are scarce, in the optimal monotone signal structure, there exists some signal realization s^* in which $E[\omega|s^*] = \omega^* + r(1 - G(1 - \omega^*))$, where $\omega^* \equiv \sup\{\omega \in supp(\mu(\cdot|s^*))\}$. Banks with assets with state $\omega > \omega^*$ sell the asset, while banks with assets with $\omega < \omega' \equiv \inf\{\omega \in supp(\mu(\cdot|s^*))\}$ do not.*

Moreover, either

$$E[\omega|s^*] = 1 \tag{14}$$

or

$$r(1 - G(1 - \omega^*))f(\omega^*) = 1 \tag{15}$$

The idea here is that the high-credit assets do sell, while the ones with lower credit prefer to take the risk rather than guarantee that they have insufficient funds. To determine what the cutoff is, one must balance the incentive compatibility of the highest states in subsidizing the lower states with the ability to subsidize more states when the expected value of the asset given the critical credit rating s^* is higher. This creates a hidden adverse selection problem that is resolved by either equation (14), the corner solution, or (15), the interior solution.

7 Conclusion

There is a strong intuition in communication games that in circumstances with complementarities (such as supermodularity) between the state and the action, the action eventually chosen by the receiver will be increasing in the state. This has been reinforced by the standard approach of the literature, in which the sender cannot commit to a strategy ex-ante, whether or not signalling is costly. Our paper serves to show that this approach breaks down when the sender can commit, and that one must take account of the additional effects of changing the receiver's beliefs, and hence their actions, when optimizing. The virtual utility characterization of these effects allows one to view the persuasion problem in the context of the mechanism design literature.

Appendix: Proofs

Proof of Lemma 1: Suppose that $\mu_1 \succeq_{FOSD} \mu_2$. Since (strict) supermodularity is preserved by first-order stochastic dominance, $\mu_1 \succeq_{FOSD} \mu_2$ only if, for any $a_1 > a_2$,

$$\int u_R(a_1,\omega)d\mu_1 - \int u_R(a_2,\omega)d\mu_1 \geq \int u_R(a_1,\omega)d\mu_2 - \int u_R(a_2,\omega)d\mu_2$$

where the inequality is strict if $\mu_1 \succ_{FOSD} \mu_2$, by Theorem 7 of Milgrom and Shannon (1994). By Theorem 5 of Milgrom and Shannon (1994), this implies that $a^*(\mu_1) \geq a^*(\mu_2)$ (strictly so if $\mu_1 \succ_{FOSD} \mu_2$ and either $a^*(\mu_1)$ or $a^*(\mu_2)$ is located in the interior of A by Theorem 1 of Edlin and Shannon (1998)). $\square$

Proof of Lemma 2: Clearly the payoff from any realization $s \notin \mathcal{S}^*(a)$ is the same, so we only need to check that the payoff from s' is equal to that from $\bigcup\{\mathcal{S}^*(a)\}$. The payoff from the former is

$$\int_{\bigcup\{s'(a)\}} \int u_S(a^*(\mu(\cdot|s')),\omega)d\mu(\omega|s')d\tau(\mu(\cdot|s'))$$

while from the latter, it is

$$\int_{\bigcup\{S^*(a)\}} \int u_S(a^*(\mu(\cdot|s)),\omega)d\mu(\omega|s)d\tau(\mu(\cdot|s))$$

Since $a^*(\mu(\cdot|s)) = a$ for all $s \in \mathcal{S}^*(a)$, and action a is feasible conditional on s' being realized, we conclude that $a^*(\mu(\cdot|s')) = a$, and so the two signals generate the same payoffs. $\square$

Proof of Proposition 1: To show (a) and (b), by Bayes' Theorem, there must exist s' such that $\omega' \in \text{supp}(\mu(\cdot|s'))$. Suppose that $s' \neq s$. By Lemma 2, we can assume that $\mu(\cdot|s') \nsim_\sigma \mu(\cdot|s)$. If $\mu(\cdot|s') \succ_\sigma \mu(\cdot|s)$, then monotonicity would be violated since $\omega_1 \in \text{supp}(\mu(\cdot|s))$. On the other hand, if $\mu(\cdot|s') \prec_\sigma \mu(\cdot|s)$, then monotonicity would be violated by ω_2. We therefore conclude that $s' = s$.

For (c), if $\omega_1, \omega_2 \in \text{supp}(\mu(\cdot|s)) \cap \text{supp}(\mu(\cdot|s'))$, where $\omega_1 \neq \omega_2$, monotonicity is violated since the higher of $\{\omega_1,\omega_2\}$ is in the support of the lower distribution, while the lower is in the support of the higher distribution. $\square$

Proof of Theorem 2: In the case where the sender's preferences are supermodular, consider any optimal signal structure $(\pi,\mathcal{S})$. If $|\mathcal{S}| = 1$, then the signal structure is trivially monotone, and we are done. Thus we can assume without loss of generality that $|\mathcal{S}| = 2$ and that there exist $s_1, s_2 \in \mathcal{S}$ such that $a^*(\mu(\cdot|s_1)) = a_1$ and $a^*(\mu(\cdot|s_2)) = a_2$. Suppose that the $(\pi, \mathcal{S})$ is not monotone. Then there exist $\omega_1 \in \text{supp}(\mu(\cdot|s_1))$ and $\omega_2 \in \text{supp}(\mu(\cdot|s_2))$ such that $\omega_1 > \omega_2$. Without loss of generality, let $\omega_1 = \sup\{\omega : \omega \in \text{supp}(\mu(\cdot|s_1))\}$ and $\omega_2 = \inf\{\omega : \omega \in \text{supp}(\mu(\cdot|s_2))\}$. For any $\delta > 0$, let s_2^δ be the restriction of s_2 to $[\omega_2, \omega_2 + \delta)$, and s_1^δ be the restriction of s_1 to $[\omega_1 - \delta, \omega_1]$. Let $\omega^* \in [\omega_2, \omega_1]$ be the supremum of the values of ω at which $\tau(\mu(\cdot|s_2^{\omega^*-\omega_2})) \leq \tau(\mu(\cdot|s_1^{\omega_1-\omega^*}))$. Then for some $\tau(\mu(\cdot|s_i))\mu(\omega^*|s_i) \geq \nu_i \geq 0$, we can define $\bar{\mu}_1 = \mu(\cdot|s_1^{\omega_1-\omega^*}) + \nu_1\omega^*$ and $\bar{\mu}_2 = \mu(\cdot|s_2^{\omega^*-\omega_2}) + \nu_2\omega^*$ such that $\tau(\bar{\mu}_1) = \tau(\bar{\mu}_2)$.

Since u_S is supermodular in (a, ω), it follows that

$$d(\mu(\cdot|s_2); \bar{\mu}_1) - d(\mu(\cdot|s_2); \bar{\mu}_2) \geq d(\mu(\cdot|s_1); \bar{\mu}_1) - d(\mu(\cdot|s_1); \bar{\mu}_2) \tag{16}$$

as long as the induced action a^* does not change. Note that for all $\eta \in (0, 1]$,

$$\mu(\cdot|s_2) - \eta\bar{\mu}_2 + \eta\bar{\mu}_1 \succ_{FOSD} \mu(\cdot|s_2) \tag{17}$$

and so by single-crossing this implies that

$$a^*(\mu(\cdot|s_2) - \eta\bar{\mu}_2 + \eta\bar{\mu}_1) \geq a^*(\mu(\cdot|s_2)) = a_2$$

Similarly, since

$$\mu(\cdot|s_1) \succ_{FOSD} \mu(\cdot|s_1) + \eta\bar{\mu}_2 - \eta\bar{\mu}_1 \tag{18}$$

by single-crossing,

$$a^*(\mu(\cdot|s_1) + \eta\bar{\mu}_2 - \eta\bar{\mu}_1) \leq a^*(\mu(\cdot|s_1)) = a_1$$

Moreover, by (17) and (18), (16) remains true if we replace $\mu(\cdot|s_2)$ with $\mu(\cdot|s_2) - \eta\bar{\mu}_2 + \eta\bar{\mu}_1$ and $\mu(\cdot|s_1)$ with $\mu(\cdot|s_1) + \eta\bar{\mu}_2 - \eta\bar{\mu}_1$. By the fundamental theorem of calculus, it follows that

$$\begin{aligned}
&\tau(\mu(\cdot|s_2)) \int u_S(a_2, \omega) d(\mu(\cdot|s_2) - \bar{\mu}_2 + \bar{\mu}_1) + \tau(\mu(\cdot|s_1)) \int u_S(a_1, \omega) d(\mu(\cdot|s_1) + \bar{\mu}_2 - \bar{\mu}_1) \\
&= \tau(\mu(\cdot|s_2)) \int u_S(a_2, \omega) d\mu(\cdot|s_2) + \tau(\mu(\cdot|s_1)) \int u_S(a_1, \omega) d\mu(\cdot|s_1) \\
&+ \int_0^1 [d(\mu(\cdot|s_2) - \eta\bar{\mu}_2 + \eta\bar{\mu}_1; \bar{\mu}_1) - d(\mu(\cdot|s_2) - \eta\bar{\mu}_2 + \eta\bar{\mu}_1; \bar{\mu}_2)] d\eta \\
&- \int_0^1 [d(\mu(\cdot|s_1) + \eta\bar{\mu}_2 - \eta\bar{\mu}_1; \bar{\mu}_1) - d(\mu(\cdot|s_1) + \eta\bar{\mu}_2 - \eta\bar{\mu}_1; \bar{\mu}_2)] d\eta \\
&\geq \tau(\mu(\cdot|s_2)) \int u_S(a_2, \omega) d\mu(\cdot|s_2) + \tau(\mu(\cdot|s_2)) \int u_S(a_1, \omega) d\mu(\cdot|s_1)
\end{aligned}$$

Hence adding $\bar{\mu}_1 - \bar{\mu}_2$ to $\mu(\cdot|s_2)$ and adding $\bar{\mu}_2 - \bar{\mu}_1$ to $\mu(\cdot|s_1)$ results in a weak improvement over $(\pi, \mathcal{S})$. Note that the resultant signal structure is now monotone, as there is no $\omega < \omega^*$ in the support of $\mu(\cdot|s_2) + \bar{\mu}_1 - \bar{\mu}_2$, and no $\omega > \omega^*$ in the support of $\mu(\cdot|s_1) + \bar{\mu}_2 - \bar{\mu}_1$. Since $(\pi, \mathcal{S})$ was optimal, there also exists a monotone signal structure which is optimal. □

Proof of Corollary 3: Suppose that neither (a) nor (b) hold. Then for any $\omega \in \Omega$ and for small enough ϵ, $a^*(\mu(\cdot|s_i) + \epsilon\omega) = a^*(\mu(\cdot|s_i))$. For each of the signals s_1, s_2, we define the signal realizations s_1^δ, s_2^δ which restrict s_1, s_2 to the open interval within δ of $\underline{\omega}_2$ and $\bar{\omega}_1$, respectively. For every $\delta > 0$, if $(\pi, \mathcal{S})$ is optimal, then $d(\mu(\cdot|s_1); \mu(\cdot|s_1^\delta)) \geq d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta))$ and $d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta)) \geq d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta))$. Note that as $\delta \to 0$, $\mu(\cdot|s_1^\delta) \to \mu(\cdot|\bar{\omega}_1)$ in the weak-* topology. Hence

$$\lim_{\delta \to 0} d(\mu(\cdot|s_i); \mu(\cdot|s_1^\delta)) = u_S(a_i, \bar{\omega}_1)$$

which is precisely the value of $d(\mu(\cdot|s), \epsilon\bar{\omega}_1)$. Similarly, we find that

$$\lim_{\delta \to 0} d(\mu(\cdot|s_i); \mu(\cdot|s_2^\delta)) = u_S(a_i, \underline{\omega}_2)$$

Suppose that $u_S(a_2, \underline{\omega}_2) < u_S(a_1, \underline{\omega}_2)$. Then for sufficiently small δ, $d(\mu(\cdot|s_1); \mu(\cdot|s_2^\delta)) > d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta))$. Thus there would be a feasible improvement from shifting a a weight of ϵ of $\mu(\cdot|s_2^\delta)$ from $\mu(\cdot|s_2)$ to $\mu(\cdot|s_1)$, and so $(\pi, \mathcal{S})$ could not be optimal. Similarly, if $u_S(a_2, \bar{\omega}_1) > u_S(a_1, \bar{\omega}_1)$, then there exists an improvement by shifting (for small enough δ and ϵ) a weight of ϵ of $\mu(\cdot|s_1^\delta)$ from $\mu(\cdot|s_1)$ to $\mu(\cdot|s_2)$. □

Proof of Lemma 3: For the receiver, the change in payoff from adding $\epsilon \cdot \mu'$ to μ, for a given choice of $a \in [\underline{a}, \bar{a}]$, is

$$\int u_R(a, \omega) d(\mu + \epsilon \cdot \mu') - \int u_R(a, \omega) d\mu = \frac{\epsilon\tau(\mu')}{\tau(\mu) + \epsilon\tau(\mu')} \cdot \int u_R(a, \omega) d\mu'$$

Hence the receiver's problem is now

$$\max_a \frac{\tau(\mu)}{\tau(\mu) + \epsilon\tau(\mu')} \int u_R(a, \omega) d\mu + \frac{\epsilon\tau(\mu')}{\tau(\mu) + \epsilon\tau(\mu')} \cdot \int u_R(a, \omega) d\mu'$$

and so the first-order condition becomes

$$\frac{\tau(\mu)}{\tau(\mu) + \epsilon\tau(\mu')} \int \frac{\partial u_R}{\partial a}(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu(\omega|s) + \frac{\epsilon\tau(\mu')}{\tau(\mu) + \epsilon\tau(\mu')} \cdot \int \frac{\partial u_R}{\partial a}(a^*(\mu + \epsilon \cdot \mu'), \omega) d\mu' = 0$$

Note that the second term of the above equation vanishes as $\epsilon \to 0$. By Berge's maximum theorem, $\lim_{\epsilon\to 0} a^*(\mu+\epsilon\cdot\mu') = a^*(\mu)$. Combining these cases and dividing by $\epsilon\tau(\mu')$, we have

$$\frac{\tau(\mu)}{\epsilon\tau(\mu')(\tau(\mu)+\epsilon\tau(\mu'))}\int \frac{\partial u_R}{\partial a}(a^*(\mu+\epsilon\cdot\mu'),\omega)d\mu + \frac{\epsilon\tau(\mu')}{\epsilon\tau(\mu')(\tau(\mu)+\epsilon\tau(\mu'))}\cdot\int \frac{\partial u_R}{\partial a}(a^*(\mu+\epsilon\cdot\mu'),\omega)d\mu'$$

$$-\frac{1}{\epsilon\tau(\mu')}\int \frac{\partial u_R}{\partial a}(a^*(\mu),\omega)d\mu = 0$$

Taking the limit as $\epsilon \to 0$,

$$-\frac{1}{\tau(\mu)}\int \frac{\partial u_R}{\partial a}(a^*(\mu),\omega)d\mu' = \lim_{\epsilon\to 0}\frac{1}{\epsilon\tau(\mu')}\{\int \frac{\partial u_R}{\partial a}(a^*(\mu+\epsilon\cdot\mu'),\omega)d\mu - \int \frac{\partial u_R}{\partial a}(a^*(\mu),\omega)d\mu\}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon\tau(\mu')}\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu),\omega)d\mu\cdot(a^*(\mu+\epsilon\cdot\mu')-a^*(\mu)) \tag{19}$$

and so

$$\lim_{\epsilon\to 0}\frac{a^*(\mu+\epsilon\cdot\mu')-a^*(\mu)}{\epsilon\tau(\mu')} = \frac{-\frac{1}{\tau(\mu)}\int \frac{\partial u_R}{\partial a}(a^*(\mu),\omega)d\mu'}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu),\omega)d\mu}$$

Substituting this into the sender's problem gives

$$\lim_{\epsilon\to 0} D(\mu,\epsilon\cdot\mu') =$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon\tau(\mu')}\{\tau(\mu)[\int \frac{\partial u_S}{\partial a}(a^*(\mu+\epsilon\cdot\mu'),\omega)d\mu]\cdot[a^*(\mu+\epsilon\cdot\mu')-a^*(\mu)]+\epsilon\tau(\mu')\cdot\int u_S(a^*(\mu+\epsilon\cdot\mu'),\omega)d\mu'\}$$

$$= -\frac{\int \frac{\partial u_S}{\partial a}(a^*(\mu),\omega)d\mu\cdot\int \frac{\partial u_R}{\partial a}(a^*(\mu),\omega)d\mu'}{\int \frac{\partial^2 u_R}{\partial a^2}(a^*(\mu),\omega)d\mu} + \int u_S(a^*(\mu),\omega)d\mu'$$

Note that if if $a^*(\mu) = a^*(\mu+\epsilon\omega) \in \{\underline{a},\bar{a}\}$ for small enough $\epsilon > 0$, then the first term vanishes, and we are left with $\lim_{\epsilon\to 0} D(\mu,\epsilon\cdot\mu') = \int u_S(a^*(\mu),\omega)d\mu'$.

Since this is true regardless of the sign of ϵ,

$$\lim_{\epsilon\to 0^+} D(\mu,\epsilon\cdot\mu') = -\lim_{\epsilon\to 0^-} D(\mu,-\epsilon\cdot\mu')$$

whenever such a subtraction is feasible. □

Proof of Theorem 4: By Lemma 2, without loss of generality, we can assume that for any two signal realizations s_i, s_j, $\mu(\cdot|s_i) \succ_\sigma \mu(\cdot|s_j)$. If $(\pi,\mathcal{S})$ is non-monotone, then there exists $\omega_i \in \text{supp}(\mu(\cdot|s_i))$ and $\omega_j \in \text{supp}(\mu(\cdot|s_j))$ such that $\omega_i < \omega_j$.

Suppose that $|\mathcal{S}| < \infty$. Fix $\delta < \frac{1}{2}(\omega_j - \omega_i)$, and define s_i^δ to be the restriction of s_i to

$[\omega_i - \delta, \omega_i + \delta]$. By d-quasisupermodularity, either for all $\omega'_j \in \text{supp}(\mu(\cdot|s^\delta_j))$,

$$d(\mu(\cdot|s_i); \omega'_j) - d(\mu(\cdot|s_j); \omega'_j) > 0$$

or for all $\omega'_i \in \text{supp}(\mu(\cdot|s^\delta_i))$,

$$d(\mu(\cdot|s_j); \omega'_i) - d(\mu(\cdot|s_i); \omega'_i) > 0$$

and so either

$$d(\mu(\cdot|s_i); \mu(\cdot|s'_j)) - d(\mu(\cdot|s_j); \mu(\cdot|s'_j)) > 0$$

or

$$d(\mu(\cdot|s_j); \mu(\cdot|s'_i)) - d(\mu(\cdot|s_i); \mu(\cdot|s'_i)) > 0$$

In either case, there exists a local improvement, in the former case by transfering sufficiently small ϵ weight of $\mu(\cdot|s^\delta_2)$ from $\mu(\cdot|s_2)$ to $\mu(\cdot|s_1)$, or in the latter case from transferring ϵ of $\mu(\cdot|s^\delta_1)$ from $\mu(\cdot|s_1)$ to $\mu(\cdot|s_2)$, and so the non-monotone signal structure cannot be optimal. This is sufficient whenever $|\Omega| < \infty$.

To extend to the case where $|\mathcal{S}| = \infty$, recall that by Proposition 1, the optimal signal structure partitions the state space into intervals. Using a probability integral transform, we can express this partition by partitioning the interval $[0, 1]$ into subintervals, and then translating those intervals into the state space Ω. That is, there will exist some nondecreasing function

$$\tilde{\omega} : [0, 1] \to \Omega$$

$$\alpha \to \omega$$

For any K, let the boundary points of the optimal signal structure conditional on $|\mathcal{S}| = K$ be $\{(l_k, h_k)\}_{k=1}^K$, where $0 \le l_k \le h_k \le 1$ and $l_{k+1} = h_k$. The probability integral transform operates so that, for any $\alpha \in [0, 1]$, the corresponding state $\tilde{\omega}(\alpha) \in \Omega$ is the lowest state for which $\mu_0((-\infty, \omega]) \ge \alpha$.

Let $l_K(\alpha), h_K(\alpha)$ be functions which describe the upper- and lower- bounds of the interval to which α belongs in the optimal structure when $|\mathcal{S}|$ is constrained to equal K. Both of these are increasing functions, and so by Helly's selection theorem (Kolmogorov and Fomin, p. 373), there exists a convergent subsequence of $\{l_K(\alpha), h_K(\alpha)\}_{K=1}^\infty$, which we call $\{l_{K_m}(\alpha), h_{K_m}(\alpha)\}_{m=1}^\infty$, to some monotone functions $l(\alpha), h(\alpha)$ which respect the interval properties of each of the pair of functions (l_K, h_K). Moreover, the posteriors implied by these intervals must converge in the weak-* topology almost everywhere, since there can only be a discontinuity where $l(\alpha) = h(\alpha)$ and $\tilde{\omega}(\alpha)$ is discontinuous, which, since the latter is monotone itself, can only occur at a countable number of values of α.

To conclude the argument for the convergence of the payoffs for this convergent subsequence of monotone signal structures, we must show that the actions taken by the receiver at almost every value of α converge. Define the action taken at α when $|\mathcal{S}| = K$ to be $\tilde{a}_K(\alpha)$. By definition of the monotonicity of signal structures, the action taken by the receiver at higher values of ω is increasing, and so it is increasing in the corresponding value of α as well. Hence by Helly's selection theorem, there exists a convergent subsequence of $\{\tilde{a}_{K_m}(\alpha)\}_{m=1}^{\infty}$ to some $\tilde{a}(\alpha)$ almost everywhere. Since beliefs converge in the weak-* topology almost everywhere and u_R is continuous in (a, ω), $\tilde{a}(\alpha)$ is optimal for the receiver almost everywhere as well by Berge's maximum theorem.

To show that the payoff to the sender implied by the signal structure in which the collection of intervals is given by $\{l(\alpha), h(\alpha)\}$ is optimal, we must show that it is at least as good as any other signal structure. Consider any optimal signal structure $(\pi, \mathcal{S})$. We approximate it for $|\mathcal{S}| = 2^K$ by dividing A into 2^K intervals of equal length, and merging any set of signal realizations $\{s\} \subset \mathcal{S}$ such that $a^*(\mu(\cdot|s))$ fall in the same interval.

Let $a^*_K(\mu(\cdot|\{s\}_K))$ be the action taken at signal realization s after it has been merged as described in the previous paragraph with other signal realizations that lead to actions in the same interval. We show that $\lim_{K\to\infty} a^*_K(\mu(\cdot|\{s\}_K)) = a^*(\mu(\cdot|s))$ for every s in this set. Since $\frac{\partial^2 u_R}{\partial a^2} < 0$ and is continuous in (a, ω), and both A and Ω are compact, we find that $\int \frac{\partial^2 u_R}{\partial a^2}(a, \omega) d\mu < 0$ and is uniformly continuous over the set of all possible posteriors μ. Fix $\epsilon > 0$. For a given K, define an interval $[a_k, a_{k+1})$, where $k < 2^K$. There exists δ such that if $a \notin [a_k - \delta, a_{k+1} + \delta]$, then $\int [u_R(a^*(\mu(\cdot|s), \omega)) - u_R(a, \omega)] d\mu(\omega|s) > \epsilon$ if $a^*(\mu(\cdot|s)) \in [a_k, a_{k+1})$ by the strict concavity of u_R. Moreover, as $\epsilon \to 0$, the infimum of such δ for which this is true also must converge to 0 by the uniform continuity of $\int \frac{\partial^2 u_R}{\partial a^2}(a, \omega) d\mu$. However, for sufficiently high K, for any $a \in [a_k, a_{k+1})$, $\int [u_R(a^*(\mu(\cdot|s)), \omega) - u_R(a, \omega)] d\mu(\omega|s) < \epsilon$ by the continuity of u_R. Thus $a \notin [a_k - \delta, a_{k+1} + \delta]$ cannot be optimal.

Since $a^*_K(\mu(\cdot|\{s\}_K))$ converges, the payoff from this finite approximation of $(\pi, \mathcal{S})$ also converges to that from $(\pi, \mathcal{S})$. But we saw that a monotone signal structure was optimal over all finite signal structures, and so the limit monotone structure as $K \to \infty$ must give at least as good a payoff to the sender as $(\pi, \mathcal{S})$. $\square$

Proof of Theorem 5: In both cases, we first proceed for the case in which Ω is finite, and then extend to the infinite case through successive finite approximations of Ω. By Proposition 4 in the online appendix of KG, this implies that $|\mathcal{S}|$ is finite, and so $\tau(\mu(\cdot|s)) > 0$ for all $s \in \mathcal{S}$.

(i) Suppose that $(\pi, \mathcal{S})$ is optimal and there exists some signal $s \in \mathcal{S}$ such that $\mu(\omega|s) \notin \{0,1\}$ for some ω. We show that there exists an improvement over $(\pi, \mathcal{S})$. Consider the alternative signal structure $(\pi', \mathcal{S}')$ which is identical to $(\pi, \mathcal{S})$ except that it duplicates s into s_1 and s_2 such that $\mu(\omega|s) = \mu(\omega|s_1) = \mu(\omega|s_2)$ and $\tau(\mu(\cdot|s_1)) = \tau(\mu(\cdot|s_2)) = \frac{1}{2}\tau(\mu(\cdot|s))$. For shorthand, we write $\mu(\cdot|s_i) = \mu_i$.

Suppose that $a^*(\mu(\cdot|s)) \in (\underline{a}, \bar{a})$. Let $\underline{\omega} \equiv \min\{\omega : \omega \in \text{supp}(\mu(\cdot|s))\}$ and $\bar{\omega} \equiv \max\{\omega : \omega \in \text{supp}(\mu(\cdot|s))\}$. Then for $\nu, \eta \geq 0$ (with at least one strictly so), by Lemma 1,

$$\mu_2 + \nu\bar{\omega} - \eta\underline{\omega} \succ_\sigma \mu_1 - \nu\bar{\omega} + \eta\underline{\omega} \tag{20}$$

Moreover, by d-quasisupermodularity, either

$$d(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega}; \bar{\omega}) - d(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega}; \bar{\omega}) > 0 \tag{21}$$

or

$$d(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega}; \underline{\omega}) - d(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega}; \underline{\omega}) > 0 \tag{22}$$

Let $\delta = \frac{1}{4}\tau(\mu)\min\{\mu(\underline{\omega}|s), \mu(\bar{\omega}|s)\}$ For any $\beta, \gamma \in (0, \delta)$, by the fundamental theorem of calculus, there exist $\delta \geq \nu \geq \beta$ and $\delta \geq \eta \geq \gamma$ (with either $\nu > \beta$ or $\eta > \gamma$) such that

$$\begin{aligned} &\tau(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega}) \int u_S(a^*(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega}), \omega) d(\mu_1 - \nu\bar{\omega} + \eta\underline{\omega})(\omega) \\ &+\tau(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega}) \int u_S(a^*(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega}), \omega) d(\mu_2 + \nu\bar{\omega} - \eta\underline{\omega})(\omega) \\ &> \tau(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega}) \int u_S(a^*(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega}), \omega) d(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega})(\omega) \\ &+\tau(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}) \int u_S(a^*(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}), \omega) d(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}) \end{aligned} \tag{23}$$

and so we can without loss of generality set $\max\{\nu, \eta\} = \delta$, as there will exist an improvement by incrementing the weight on either $\bar{\omega}$ or $\underline{\omega}$ for the cases where $\nu < \delta$ and $\eta < \delta$ since the payoffs is continuous in (ν, η) by Lemma 3. Note that (β, γ) were arbitrary, and that

$$\begin{aligned} &\lim_{(\beta,\gamma)\to(0,0)} \tau(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega}) \int u_S(a^*(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega}), \omega) d(\mu_1 + \gamma\underline{\omega} - \beta\bar{\omega})(\omega) \\ &+\tau(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}) \int u_S(a^*(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega}), \omega) d(\mu_2 - \gamma\underline{\omega} + \beta\bar{\omega})(\omega) \\ &= \tau(\mu) \int u_S(a^*(\mu), \omega) d\mu(\omega) \end{aligned} \tag{24}$$

Hence we can take a sequence of values $\{(\beta_k, \gamma_k, \nu_k, \eta_k)\}_{k=1}^{\infty}$ such that $(\beta_k, \gamma_k) \to (0,0)$ and (without loss of generality) $\nu_k = \delta$ and $\eta_k \leq \delta$ for all k such that each $(\beta_k, \gamma_k, \nu_k, \eta_k)$ satisfies (23). Since the interval $[0, \delta]$ is compact, there exists a convergent subsequence of $\{\eta_k\}_{k=1}^{\infty}$ to some $\eta_\infty \in [0, \delta]$ (without loss of generality, the sequence itself). By Lemma 3, the payoff to the sender is continuous in (β, γ, ν), and so in the limit, we find from (24) that

$$\tau(\mu_1 - \delta\bar{\omega} + \eta_\infty\underline{\omega}) \int u_S(a^*(\mu_1 - \delta\bar{\omega} + \eta_\infty\underline{\omega}), \omega) d(\mu_1 - \delta\bar{\omega} + \eta_\infty\underline{\omega})(\omega)$$

$$+\tau(\mu_2 + \delta\bar{\omega} - \eta_\infty\underline{\omega}) \int u_S(a^*(\mu_2 + \delta\bar{\omega} - \eta_\infty\underline{\omega}), \omega) d(\mu_2 + \delta\bar{\omega} - \eta_\infty\underline{\omega})(\omega)$$

$$\geq \tau(\mu) \int u_S(a^*(\mu), \omega) d\mu(\omega) \tag{25}$$

Moreover,

$$\mu_2 + \delta\bar{\omega} - \eta_\infty\underline{\omega} \succ_{FOSD} \mu_2$$

$$\sim_{FOSD} \mu_1 \succ_{FOSD} \mu_1 - \delta\bar{\omega} + \eta_\infty\underline{\omega}$$

and so $\mu_2 + \delta\bar{\omega} - \eta_\infty\underline{\omega} \succ_\sigma \mu_1 - \delta\bar{\omega} + \eta_\infty\underline{\omega}$ by Lemma 1. There then exists a local improvement from either adding small enough weight ϵ of $\bar{\omega}$ from $\mu_1 - \delta\bar{\omega} + \eta_\infty\underline{\omega}$ to $\mu_2 + \delta\bar{\omega} - \eta_\infty\underline{\omega}$, or small enough ϵ of $\underline{\omega}$ from $\mu_2 + \delta\bar{\omega} - \eta_\infty\underline{\omega}$ to $\mu_1 - \delta\bar{\omega} + \eta_\infty\underline{\omega}$, due to inequalities (21) and (22), respectively. Hence there exists an improvement over $(\pi, \mathcal{S})$, and so it cannot be optimal.

For the case in which $a^*(\mu(\cdot|s)) \in \{\underline{a}, \bar{a}\}$, the above method does not quite work, since it may be that $a^*(\mu_2 + \delta\bar{\omega} - \eta_\infty\underline{\omega}) = a^*(\mu_1 - \delta\bar{\omega} + \eta_\infty\underline{\omega})$. To circumvent this issue, we take a closer look at the support of $\mu(\cdot|s)$. Suppose that $a^*(\mu(\cdot|s)) = \bar{a}$. If, for all $\omega \in \text{supp}(\mu(\cdot|s))$, $a^*(\omega) = \bar{a}$, then splitting s into separate signals for each state does not change the payoff, and so it will again be optimal to reveal all information. Otherwise, by supermodularity of u_R in (a, ω), it must be that $a^*(\underline{\omega}) < \bar{a}$. Since a^* is continuous in μ, there exists $\beta^* \in (0, 1]$ for which $a^*(\beta^* \cdot \mu(\cdot|s) + (1 - \beta^*)\underline{\omega}) = \bar{a}$, but for all $\beta \in (0, \beta^*)$, $a^*(\beta \cdot \mu(\cdot|s) + (1 - \beta)\underline{\omega}) < \bar{a}$. It will then follow that for $\eta, \nu \geq 0$ (with one strict),

$$\beta^* \cdot \mu(\cdot|s) + (1 - \beta^*)\underline{\omega} \succ_\sigma \beta^* \cdot \mu(\cdot|s) + (1 - \beta^*)\underline{\omega} + \eta\underline{\omega} - \nu\bar{\omega}$$

Thus we can split s into two signals s_1, s_2 such that, for some $c > 0$,

$$\mu_1 = c\beta^* \cdot \mu(\cdot|s) + c(1 - \beta^*)\underline{\omega}$$

and

$$\mu_2 = \mu(\cdot|s) - (c\beta^* \cdot \mu(\cdot|s) + c(1 - \beta^*)\underline{\omega})$$

This will not change the payoff, as in both cases $a^*(\mu(\cdot|s_i)) = \bar{a}$. The argument then proceeds as above, where now $d(\mu_1;\omega)$ is given by (5), but $d(\mu_2;\omega) = u_S(\bar{a}_2,\omega)$. The case where $a^*(\mu(\cdot|s)) = \underline{a}$ is symmetric, and so is omitted.

(ii) Suppose that $(\pi, \mathcal{S})$ is optimal, and there exist $s_1, s_2 \in \mathcal{S}$ such that (without loss of generality) $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$; define these respective distributions as μ_1 and μ_2, respectively. By Lemma 1, $\succ_\sigma$ is FOSD-consistent, and so there must exist $\omega_1 \in \text{supp}(\mu(\cdot|s_1))$ and $\omega_2 \in \text{supp}(\mu(\cdot|s_2))$ such that $\omega_1 > \omega_2$. By d-quasisubmodularity, either

$$d(\mu_2;\omega_1) - d(\mu_1;\omega_1) > 0$$

or

$$d(\mu_1;\omega_2) - d(\mu_2;\omega_2) > 0$$

In either case, an improvement exists, and so $(\pi, \mathcal{S})$ can only be optimal if for all $s_1, s_2 \in \mathcal{S}$, $\mu_1 \sim_\sigma \mu_2$. By Lemma 2, merging all such realizations generates the same payoff. Hence the signal structure which generates the posterior beliefs identical to the prior is optimal.

Now consider the case where Ω is infinite. Consider any information structure $(\pi, \mathcal{S})$. Let $\bar{\omega} = \sup\{\omega \in \Omega\}$ and $\underline{\omega} = \inf\{\omega \in \Omega\}$. Consider a sequence of approximating state spaces $\{\Omega_k\}$ defined by dividing the interval $[\underline{\omega}, \bar{\omega}]$ into subintervals of length $\frac{\bar{\omega}-\underline{\omega}}{2^k}$, and so that all states ω in the same interval I_k^n (indexed by $n \in \{1, ..., 2^k\}$) are mapped to the same value, ω_k^n; the prior distribution μ_0^k is then defined by assigning to ω_k^n the probability

$$\mu_0^k(\omega_k^n) = \int_{I_k^n} d\mu_0(\omega)$$

where $I_k^n = [\frac{n(\bar{\omega}-\underline{\omega})+\underline{\omega}}{2^k}, \frac{(n+1)(\bar{\omega}-\underline{\omega})+\underline{\omega}}{2^k})$, with the interval closed for $n = 2^k$. We correspondingly approximate $(\pi, \mathcal{S})$ by $(\pi_k, \mathcal{S}_k)$ in which for any $s \in \mathcal{S}$, there is a signal realization $s_k \in \mathcal{S}_k$ such that

$$\mu^k(\omega_k^n|s_k) = \int_{I_k^n} d\mu(\omega|s)$$

and

$$d\tau(\mu^k(\cdot|s_k)) = d\tau(\mu(\cdot|s))$$

Since π is measurable, this construction is well-defined. Note that this may lead to some duplicates, i.e. there may be multiple values of s_k which lead to the same μ^k; if so, these realizations can be merged by Lemma 2 without changing the payoff from the resultant signal structure.

As we have already shown, when the sender's payoffs are d-quasisubmodular it is optimal to reveal no information when the state space is Ω_k with prior μ_0^k, and so

$$\int u_S(a^*(\mu_0^k),\omega)d\mu_0^k(\omega) \geq \int\int u_S(a^*(\mu^k),\omega)d\mu^k(\omega)d\tau(\mu^k)$$

Since u_R is continuous and both $\mu^k \to \mu$ and $\mu_0^k \to \mu_0$ in the weak-* topology, it follows that $a^*(\mu^k) \to a^*(\mu)$ and $a^*(\mu_0^k) \to a^*(\mu_0)$ by Berge's maximum theorem. By the continuity of u_S, this implies that

$$\int u_S(a^*(\mu_0),\omega)d\mu_0(\omega) \geq \int\int u_S(a^*(\mu),\omega)d\mu(\omega)d\tau(\mu)$$

and so the signal structure which reveals no information provides at least as good a payoff to the sender as $(\pi, \mathcal{S})$.

Similarly, when the sender's payoffs are d-quasisupermodular, it is optimal to reveal all information, and so

$$\int u_S(a^*(\omega),\omega)d\mu_0^k(\omega) \geq \int\int u_S(a^*(\mu^k),\omega)d\mu^k(\omega)d\tau(\mu^k)$$

Taking the limit as $\mu^k \to \mu$ and $\mu_0^k \to \mu_0$ yields

$$\int u_S(a^*(\omega),\omega)d\mu_0(\omega) \geq \int\int u_S(a^*(\mu),\omega)d\mu(\omega)d\tau(\mu)$$

and so it is optimal to reveal all information. □

Proof of Theorem 6: For both (i) and (ii), we show that there is a local improvement for any $(\pi, \mathcal{S})$ that is not monotone. Without loss of generality, by Lemma 2 we can assume that for all $s_1, s_2 \in \mathcal{S}$, either $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$ or $\mu(\cdot|s_2) \succ_\sigma \mu(\cdot|s_1)$. Now suppose that $\mu(\cdot|s_1) \succ_\sigma \mu(\cdot|s_2)$, but there exists $\omega_2 \in \text{supp}(\mu(\cdot|s_2))$ and $\omega_1 \in \text{supp}(\mu(\cdot|s_1))$ such that $\omega_2 > \omega_1$.

To show that an optimal signal structure exists, note that since $\lim_{|S|\to\infty} c(\pi, \mathcal{S}) = \infty$, there must exist some N^* such that optimal structure must have $|\mathcal{S}| < N^*$. We then check for the optimal signal structure conditional on $|\mathcal{S}| = N < N^*$, and optimize over N. For fixed N, the function

$$\sum_{s\in\mathcal{S}} \tau(\mu(\cdot|s)) \int_\Omega u_S(a^*(\mu),\omega)d\mu(\omega|s) - c(\pi, \mathcal{S})$$

is continuous in $(\mu, \tau(\mu))$, since $a^*(\mu)$ is continuous in μ and $c(\cdot,\cdot)$ is continuous in $\{\tau(\mu)\}$. The set of such μ is compact under the weak-* topology, while $\{\tau(\mu)\}$ is an N-dimensional Euclidean vector contained in the N-dimensional simplex (a compact set, as long as we allow duplicate signal realization that induce the same posterior μ). A complication may arise in that, when there is a duplicate signal realization, the two realizations can be merged and therefore reduce the cost; however, this only serves to increase the sender's payoff, and so $c(\pi, \mathcal{S})$ is lower-semicontinuous in π. Therefore an optimal $(\pi, \mathcal{S})$ exists.

To show that a monotone signal structure is optimal, fix $\delta < \frac{1}{2}(\omega_2 - \omega_1)$, and define s_1^δ to be the restriction of s_1 to $[\omega_1 - \delta, \omega_1 + \delta]$, and s_2^δ to be the restriction of s_2 to $[\omega_2 - \delta, \omega_2 + \delta]$.

(i) By d-supermodularity, for all $\omega_2' \in \text{supp}(\mu(\cdot|s_2^\delta))$ and $\omega_1' \in \text{supp}(\mu(\cdot|s_1^\delta))$, we have that

$$d(\mu(\cdot|s_1); \omega_2') - d(\mu(\cdot|s_1); \omega_1') > d(\mu(\cdot|s_2); \omega_2') - d(\mu(\cdot|s_2); \omega_1')$$

and so

$$d(\mu(\cdot|s_1); \mu(\cdot|s_2^\delta)) - d(\mu(\cdot|s_1); \mu(\cdot|s_1^\delta)) > d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta)) - d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta))$$

Hence there exists an improvement (excluding costs) by adding $(\epsilon \frac{\tau(\mu(\cdot|s_1^\delta))}{\tau(\mu(\cdot|s_2^\delta))}) \cdot \mu(\cdot|s_2^\delta)$ (where ϵ is sufficiently small) to $\mu(\cdot|s_1)$ and subtracting an equal weight from $\mu(\cdot|s_2)$, while subtracting $\epsilon \cdot \mu(\cdot|s_1^\delta)$ from $\mu(\cdot|s_1)$ and adding it to $\mu(\cdot|s_2)$. Moreover,

$$\tau(\mu(\cdot|s_1) + \epsilon \frac{\tau(\mu(\cdot|s_1^\delta))}{\tau(\mu(\cdot|s_2^\delta))} \cdot \mu(\cdot|s_2^\delta) - \epsilon \mu(\cdot|s_1^\delta)) = \tau(\mu(\cdot|s_1))$$

$$\tau(\mu(\cdot|s_2) + \epsilon \cdot \mu(\cdot|s_1^\delta) - \epsilon \frac{\tau(\mu(\cdot|s_1^\delta))}{\tau(\mu(\cdot|s_2^\delta))} \cdot \mu(\cdot|s_2^\delta)) = \tau(\mu(\cdot|s_2))$$

and so the vector of probabilities $\{\tau\}$ (and hence c) is unaffected. Therefore $(\pi, \mathcal{S})$ is not optimal.

(ii) Similarly, if the sender's payoffs are d-quasisupermodular, either

$$d(\mu(\cdot|s_1); \omega_2') - d(\mu(\cdot|s_2); \omega_2') > 0$$

or

$$d(\mu(\cdot|s_2); \omega_1') - d(\mu(\cdot|s_2); \omega_1') > 0$$

and so either

$$d(\mu(\cdot|s_1); \mu(|s_2^\delta)) - d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta)) > 0$$

or

$$d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta)) - d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta)) > 0$$

In either case, ignoring costs, there exists a local improvement, in the former case by transfering sufficiently small ϵ weight of $\mu(\cdot|s_2^\delta)$ from $\mu(\cdot|s_2)$ to $\mu(\cdot|s_1)$, or in the latter case from transferring ϵ of $\mu(\cdot|s_1^\delta)$ from $\mu(\cdot|s_1)$ to $\mu(\cdot|s_2)$. Let the new signal structure be $(\pi', \mathcal{S}')$. Since $|\mathcal{S}'| = |\mathcal{S}|$, the cost c is unaffected. Hence $(\pi, \mathcal{S})$ is not optimal. □

Proof of Corollary 7: We show that, in both cases, the problem reduces to that described in Theorem 6.

(i) By Lemma 2, $|\mathcal{S}| \leq N$; moreover, if it is optimal to set $|\mathcal{S}| < N$, it is also optimal to set $|\mathcal{S}| = N$, as one can always construct duplicate signals with the same posterior. We can therefore rewrite the problem as one in which

$$c(\pi, \mathcal{S}) = \begin{cases} 0, & |\mathcal{S}| \leq N \\ \infty, & \text{otherwise} \end{cases}$$

(ii) As shown in part (i), the signal structure must have $|\mathcal{S}| \leq N$. Let the optimal signal structure then be $(\pi, \mathcal{S})$ (such a structure will exist because the set of signal structures with $|\mathcal{S}| \leq N$ is compact). Suppose that the set of actions taken is $\{a_n^*\}_{n=1}^N$, and let the posterior distribution of states conditional on each a_n^* be $\nu(\cdot|a_n^*)$ (this may be achieved through randomization by the receiver). Then the signal structure $(\pi', \mathcal{S}')$ in which $\mathcal{S}' = \{s_n\}_{n=1}^N$, $\mu(\cdot|s_n) = \nu(\cdot|a_n^*)$, and $\tau(\mu(\cdot|s_n)) = \hat{\tau}(n)$ will also be optimal.[7] To see this, note that the receiver to take the same set of actions $\{a_n^*\}_{n=1}^N$, as any other set of actions $\{\hat{a}_n^*\}_{n=1}^\infty$ with associated posterior distributions of states $\hat{\nu}(\cdot|\hat{a}_n^*)$ that would be optimal under $(\pi', \mathcal{S}')$ would also be feasible under $(\pi, \mathcal{S})$ since $\hat{\nu}(\cdot|\hat{a}_n^*) = \sum_{n=1}^N \alpha_n \nu(\cdot|a_n^*)$ for some $\{\alpha_n\}_{n=1}^N$ such that $\sum_{n=1}^N \alpha_n \tau(\mu(\cdot|s_n)) = \hat{\tau}(n)$. Thus the problem of the sender can be rewritten as one in which

$$c(\pi, \mathcal{S}) = \begin{cases} 0, & \{\tau(\mu)\} = \{\hat{\tau}(n)\} \\ \infty, & \text{otherwise} \end{cases}$$

In both cases (i) and (ii), the result then follows from Theorem 6. □

Proof of Proposition 8: For each of the signals s_1, s_2, we define the signal realizations s_1^δ, s_2^δ which restrict s_1, s_2 to within δ of $\underline{\omega}_1$ and $\bar{\omega}_2$, respectively. For every $\delta > 0$, if $(\pi, \mathcal{S})$ is optimal, then $d\mu(\cdot|s_1); \mu(\cdot|s_1^\delta)) \geq d(\mu(\cdot|s_2); \mu(\cdot|s_1^\delta))$ and $d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta)) \geq d(\mu(\cdot|s_2); \mu(\cdot|s_2^\delta))$. Note that as $\delta \to 0$, $\mu(\cdot|s_1^\delta) \to \mu(\cdot|\underline{\omega}_1)$ in the weak-* topology. Hence if we take the limit as $\delta \to 0$,

[7] A similarity can be seen here to the revelation principal, in that instead of implementing this outcome indirectly via some randomization on the part of the receiver, the sender can implement this outcome directly by creating posteriors that are the same as the post-randomization distribution.

we find that

$$\lim_{\delta\to 0} d(\mu(\cdot|s);\mu(\cdot|s_1^\delta)) = -\int \frac{\partial u_S}{\partial a}(a^*(\mu(\cdot|s)),\omega)d\mu(\omega|s)]\cdot\frac{\frac{\partial u_R}{\partial a}(a^*(\mu(\cdot|s)),\underline{\omega}_1)}{\int\frac{\partial^2 u_R}{\partial a^2}(a^*(\mu(\cdot|s)),\omega)d\mu(\omega|s)}+u_S(a^*(\mu(\cdot|s)),\underline{\omega}_1)$$

which is precisely the value of $d(\mu(\cdot|s),\underline{\omega}_1)$. Similarly, we find that

$$\lim_{\delta\to 0} d(\mu(\cdot|s);\mu(\cdot|s_2^\delta)) = -\int \frac{\partial u_S}{\partial a}(a^*(\mu(\cdot|s)),\omega)d\mu(\omega|s)]\cdot\frac{\frac{\partial u_R}{\partial a}(a^*(\mu(\cdot|s)),\bar{\omega}_2)}{\int\frac{\partial^2 u_R}{\partial a^2}(a^*(\mu(\cdot|s)),\omega)d\mu(\omega|s)}+u_S(a^*(\mu(\cdot|s)),\bar{\omega}_2)$$

Suppose that $d(\mu(\cdot|s_1);\bar{\omega}_2) > d(\mu(\cdot|s_2);\bar{\omega}_2)$. Then for sufficiently small δ, $d(\mu(\cdot|s_1);\mu(\cdot|s_2^\delta)) > d(\mu(\cdot|s_2);\mu(\cdot|s_2^\delta))$. For any fixed $\epsilon > 0$, there would be a feasible improvement from shifting any weight of no more than ϵ of $\mu(\cdot|s_2^\delta)$ from $\mu(\cdot|s_2)$ to $\mu(\cdot|s_1)$; this is possible due to the fact that A and Ω are compact, and so $d(\cdot;\cdot)$ is uniformly continuous. If there is not an atom at $\bar{\omega}_2$, one can then choose δ so that $\tau(\mu(\cdot|s_2^\delta)) < \epsilon$, and so thereby maintain monotonicity after this swap; this is important if there is an exogenous monotonicity constraint. If there is an atom at $\bar{\omega}_2$, then one can simply add weight on $\bar{\omega}_2$ from $\mu(\cdot|s_2)$ to $\mu(\cdot|s_2)$. Thus $(\pi,\mathcal{S})$ could not be optimal. Similarly, if $d(\mu(\cdot|s_2);\underline{\omega}_1) > d(\mu(\cdot|s_1);\underline{\omega}_1)$, then there exists an improvement by shifting (for small enough δ and ϵ) a weight of ϵ of $\mu(\cdot|s_1^\delta)$ from $\mu(\cdot|s_1)$ to $\mu(\cdot|s_2)$. □

Proof of Proposition 9: As shown in Goldstein and Leitner (2015), it must be that any state $\omega \geq 1$ must sell; otherwise, the sender can just design a new signal in which the conditional distribution just contains that state; the receiver then offers price $a = \omega$ and the sender accepts, thereby improving the sender's payoff.

Without loss of generality, one can consider a finite $|\mathcal{S}|$. One can merge all signal realizations s for which $E[\omega|s] < 1$, since the bank will not sell in those cases anyway and so the signal is irrelevant; call this signal $\underline{s}$ which induces posterior $\underline{\mu}$. For s such that $E[\omega|s] > 1$, one can always merge a collection of signals $\{s\}$ such that

$$\sup\{\omega \in \bigcup_{\{s\}}\text{supp}(\mu(\cdot|s))\} - \inf\{\omega \in \bigcup_{\{s\}}\text{supp}(\mu(\cdot|s))\} < rG(1-\bar{\omega})$$

since all such ω will still accept an offer to sell at the expectation of the merged signals rather than risk having insufficient reserves. Lastly, there need be at most one signal realization at $E[\omega|s] = 1$ by Lemma 2.

To be incentive-compatible, clearly $E[\omega|s^*] \geq \omega^* + r(1-G(1-\omega^*))$. If $E[\omega|s^*] > \omega^* + r(1-G(1-\omega^*))$ then for some $\delta > 0$ and resources are scarce, there will exist some interval of states $[\omega'-\delta,\omega']$ such that if $\underline{\mu}_\delta$ is the posterior generated by the restriction of μ_0 to $[\omega'-\delta,\omega']$, then $E_{\mu(\cdot|s^*)+\underline{\mu}_\delta}[\omega] > \omega^* + r(1 - G(1-\omega^*))$. This would be an improvement since

$$\lim_{\delta\to 0} d(\mu^*;\underline{\mu}_\delta) - d(\underline{\mu};\underline{\mu}_\delta) = \lim_{\delta\to 0} E[\omega|s^*] + r - (E[\omega|s^*] - E_{\underline{\mu}_\delta}[\omega]) - (E_{\underline{\mu}_\delta}[\omega] + r(1 - E_{\underline{\mu}_\delta}[G(1-\omega)]))$$

$$= rG(1-\omega') > 0$$

Therefore, by Proposition 8, $E[\omega|s^*] = \omega^* + r(1 - G(1-\omega^*))$.

Let $\bar{\mu}_\delta$ be the restriction of μ_0 to $[\omega^*, \omega^* + \delta]$. The marginal amount by which a^* increases from adding $\bar{\mu}_\delta$ is $-(E[\omega|s^*] - E_{\bar{\mu}_\delta}[\omega])$, so the total amount it increases per increase in δ is $-(E[\omega|s^*] - E_{\bar{\mu}_\delta}[\omega]) \cdot \frac{1}{\delta}\int_0^\delta f(\omega^* + x)dx$. In the limit as $\delta \to 0$, this becomes $r(1 - G(1 - \omega^*))f(\omega^*)$. Hence the marginal amount by which the price offered increases by increasing ω^* must be $r(1-G(1-\omega^*))f(\omega^*)$. Now suppose that this is greater than 1. Then by shifting ω^* up, the incentive compatibility constraint becomes slack for a small enough increase, and so one can add more of $\omega < \omega'$ to s^*. Otherwise, if it is less than 1, then by decreasing ω^* one also slackens the incentive compatibility constraint, and so can add more of $\omega < \omega'$. Hence either $E[\omega|s^*] = 1$ or $r(1 - G(1 - \omega^*))f(\omega^*) = 1$. □

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