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And Management Science

Discussion Paper #1579

Dynamic Managerial Compensation: On the Optimality of Seniority-based Schemes

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Abstract

We study the optimal dynamics of incentives for a manager whose ability to generate cash flows changes stochastically with time and is his private information. We show that, in general, the power of incentives (or "pay for performance") may either increase or decrease with tenure. However, risk aversion and high persistence of ability call for a reduction in the power of incentives later in the relationship. Our results follow from a new variational approach that permits us to tackle directly the "full program," thus bypassing some of the difficulties of working with the "relaxed program" encountered in the dynamic mechanism design literature.

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*This version supersedes an older version circulated under the title "Dynamic Managerial Compensation: A Mechanism Design Approach". For useful comments and suggestions, we thank Dirk Bergemann, anonymous referees, Mike Fishman, Paul Grieco, Igal Hendel, Bill Rogerson, Yuliy Sannikov, and seminar participants at various conferences and workshops where the paper was presented. Meysam Zare provided excellent research assistance. The usual disclaimer applies. Email addresses: dfgarrett@gmail.com (Garrett); alepavan@northwestern.edu (Pavan).

1 Introduction

In dynamic business environments, the ability of top managers to generate profits for their firms is expected to change with time as a result, for example, of changes in the organization, the arrival of new technologies, or market consolidations. A key difficulty is that, while such changes are largely expected, their implications for profitability typically remain the managers' private information. In this paper we ask the following questions: Should managers be induced to work harder at the beginning of their employment relationships or later on? Should the intertemporal variation in the provision of effort be more pronounced for managers of low or of high initial productivity? And, how should "pay for performance" change over the course of the employment relationship to sustain the desired dynamics of effort?

We consider an environment where, at the time of joining the firm, the manager possesses private information about his productivity (i.e., his ability to generate cash flows). This private information originates, for instance, in tasks performed in previous contractual relationships, as well as in personal traits that are not directly observable by the firm. The purpose of the analysis is to examine the implications of such evolving private information for the dynamic provision of incentives.

In the environment described above, a firm finds it expensive to ask a manager to exert more effort for three reasons. First, higher effort is costly for the manager and must be compensated. Second, asking higher effort of a manager with a given productivity requires increasing the compensation promised to all managers with higher productivity. This compensation is required even if the firm does not ask the more productive managers to exert more effort and represents an additional "rent" for these managers. It is needed to discourage them from mimicking the less productive managers by misrepresenting their productivity and reducing their effort. Third, inducing higher effort requires pay to be more sensitive to performance. This, in turn, exposes the manager to more volatility in his compensation. When the manager is risk averse, this increase in volatility reduces his expected payoff, requiring higher compensation by the firm.

The above effects of effort on compensation shape the way the firm induces its managers to respond to productivity shocks over time. In this paper we develop a simple, yet flexible, framework that permits us to investigate the implications of the above trade-offs both for dynamics of effort and the sensitivity of pay to performance under optimal contracts. However, measuring the power of incentives is not straightforward. One difficulty is that a manager may be rewarded for high cash flows both through contemporaneous and future payments. In particular, a manager may be rewarded for his performance in the current period with changes in the responsiveness of incentive payments to firm's cash flows in future periods. How the manager values such changes depends on the manager's future actions and productivity realizations.

To avoid these complications, we suggest a new definition of the "*power of incentives*". In any given period, it is the ratio between the manager's marginal disutility of effort and the marginal

utility of his compensation in that period, evaluated at their equilibrium levels. The rationale for this definition is that, when this ratio is high, the manager’s rewards for generating additional cash flows must also be high. This is either because the manager’s marginal disutility of effort is high, or because his marginal utility of additional pay is low, so that he is difficult to motivate. If one considers compensation schemes that are differentiable in the firm’s cash flows and that depend only on the cash flows generated in the period of compensation, then incentive compatibility requires the derivative be equal to the power of incentives. In this sense, the power of incentives measures the sensitivity of payments to cash flows “locally”. More generally, the definition can be applied to compensation schemes which are non-linear as a function of the firm’s performance, a necessary generalization given that linear schemes need not be optimal when the manager is risk averse. It also applies to compensation schemes that are non-differentiable in cash flows, such as those implemented with a combination of fixed pay, stocks, and options.¹ Importantly, our definition of power of incentives is directly related to the definition of labor "wedges" in the new dynamic public finance literature, which is considered a good measure for distortions in the presence of wealth effects (that is, beyond the quasilinear case).

As is clear from the above definition, when the manager is risk neutral, the manager’s power of incentives coincides with equilibrium effort. This is no longer true when the manager is risk averse. Moreover, it turns out that the dynamics of the power of incentives under optimal contracts are more easily understood than the dynamics of effort choices. Indeed the characterization of optimal effort policies, except for special cases, has been notoriously difficult.² Our analysis identifies certain properties of optimal contracts by applying variational arguments directly to the firm’s “full problem”. That is, we directly account for all of the manager’s incentive constraints. For any incentive-compatible contract, we identify certain “admissible variations”, by which we mean perturbations to the contract which *preserve* incentive compatibility. For a contract to be optimal, these perturbations must not increase the firm’s profits. This requirement implies a new set of Euler conditions that equate the average marginal benefit of higher effort with its average marginal cost. The average marginal benefit is simply the increase in the firm’s expected cash flows. The average marginal cost combines the disutility of effort with the cost of increasing the compensation for higher types to induce them to reveal their private information, and the cost of increasing the volatility of compensation in case the manager is risk averse. Importantly, the admissible variations

¹This flexibility is valuable given that we cannot exclude the possibility that, for certain specifications, the optimal dynamics of effort may be sustained only with non-differentiable schemes.

²In our model, characterizing optimal effort policies is difficult whenever the manager is risk averse. This difficulty is the same one observed by Edmans and Gabaix (2011), who argue that “the full contracting problem is usually intractable as there is a continuum of possible effort choices”. Edmans and Gabaix therefore focus on a particular environment, where a careful balancing of the costs and benefits of additional effort is unnecessary. In their setting, optimal effort is constant over time and equal to the highest feasible level.

that lead to the Euler conditions do not permit us to characterize how effort responds to all possible contingencies. However, they do permit interesting predictions as to how, on average, effort and the power of incentives evolve over time under fully optimal contracts.

The advantage of this approach is that it permits us to bypass some of the difficulties encountered in the literature. The typical approach involves solving for the optimal contract while imposing only a restricted set of incentive constraints, usually referred to as “local” constraints. In other words, one first solves a “relaxed problem”. One then seeks to identify restrictions on the primitive environment that guarantee that the solution to the relaxed problem satisfies the remaining incentive constraints. In our environment (when the manager is risk averse), we are able to validate the “relaxed” approach only on a case-by-case basis.³ On the other hand, when validated, the relaxed approach has the advantage of yielding ex-post predictions about the dynamics of the power of incentives that depend on the *realized productivity history* rather than ex-ante predictions established by *averaging* over histories.

While the approach of tackling directly the full program yields predictions that hold only on average, it has the advantage that the predictions are fairly robust in the sense that they require less stringent primitive conditions. Further, predictions that hold only on average seem important for empirical work, especially given that histories of productivity shocks remain unobservable to the econometrician.⁴

Key results. First, consider the case where managers are risk neutral and where the effect of the initial productivity on future productivity declines with time (recall that, for a risk-neutral manager, the power of incentives coincides with the equilibrium effort).⁵ The concern for reducing the rent left to those managers whose initial productivity is high typically leads the firm to distort downward (relative to the first best) the level of effort asked of those managers whose initial productivity is low. While a similar property has been noticed in previous work (see, among others, Laffont and Tirole, 1986), all existing results have been established for cases where the optimal contract is the solution to the “relaxed program”, whose validation requires assumptions on the monotonicity of the hazard rate of the period-1 distribution. Here we show that this property is true more generally, as long as the effort that the firm asks at each point in time is bounded away from zero from below

³The relaxed approach fails whenever the effort policies that solve the relaxed problem fail to satisfy certain “monotonicity conditions” necessary for incentive compatibility (for the present paper, see Condition (B) in Proposition 1). We refer the reader to Pavan, Segal, and Toikka (2014) for further discussion of how the relaxed approach may fail in quasilinear settings.

⁴The existing empirical literature has taken a reduced-form approach to analyzing how pay-for-performance changes with tenure. For this literature, qualitative predictions about how the power of incentives changes on average may be of greatest interest.

⁵The reason why the firm cannot extract all the surplus by “selling out” the business to the manager is that, as mentioned above, the manager possesses private information about his ability to generate cash flows and hence about his value for the business.

with probability one (that is, except over at most a zero-measure set of productivity histories). We also provide novel primitive conditions for this to be the case that complement those based on the monotonicity of the hazard rate of the period-1 distribution discussed in the literature. Importantly, we show that whenever, on average, period-1 effort is distorted downward relative to the first-best level, the firm asks for higher effort later in the relationship (that is, the manager is given additional “seniority-based” incentives). This is because, when productivity is less than fully persistent, the benefit of distorting the effort of those managers whose initial productivity is low so as to reduce the compensation paid to those managers whose initial productivity is high is greatest early in the relationship.

Next consider the case where the managers are risk averse. Mitigating the volatility of future compensation calls for reducing the power of incentives later in the relationship. The reason is that, viewed from the date the contract is initially agreed, managers face greater uncertainty about their productivity at later dates. Whether the rent effect or the risk effect prevail as the length of the employment relationship grows (i.e., whether the power of incentives increases or decreases with time) then depends on the degree of productivity persistence and on the manager’s degree of risk aversion. In particular, we show that, for low degrees of risk aversion and low degrees of productivity persistence, the dynamics of the power of incentives are the same as in the risk neutral case. When, instead, productivity is perfectly persistent, meaning that shocks to productivity are permanent as in the case of a random walk, then, for any degree of risk aversion, the power of incentives necessarily declines over time.⁶ Subject to certain qualifications, we argue that the same result should also be anticipated for large degrees of persistence (i.e., for persistence levels close to 1). In particular, we argue that the dynamics of the average power of incentives are continuous with respect to the degree of persistence, provided that effort under optimal policies remains bounded.

Implications for empirical work. Our results contribute to the debate about how managerial incentives ought to change over a manager’s tenure, and what explains the observed patterns. The empirical literature often focuses on a measure of “pay for performance” proposed by Jensen and Murphy (1990). This is the responsiveness of CEO pay to changes in shareholder wealth. When applied to compensation schemes which depend only on current-period performance (mentioned above), our definition of the power of incentives mirrors the measures typically used in this literature.

The evidence of how pay-for-performance varies with managerial tenure is mixed. More recent work finds that the sensitivity of pay to performance increases with tenure. Gibbons and Murphy (1992), Lippert and Porter (1997), and Cremers and Palia (2010) support this view, while Murphy (1986) and Hill and Phan (1991) find evidence of the opposite. A number of theories have been

⁶Note that a process that is fully persistent is not necessarily one in which productivity is constant over time. Our result that distortions increase over time in the random walk case, for any strictly concave felicity function, hinges on the fact that future productivity is stochastic.

proposed to explain these patterns. Gibbons and Murphy (1992) provide a model of career concerns to suggest that explicit pay-for-performance ought to increase closer to a manager’s retirement. Edmans et al. (2012) suggest a similar conclusion but based on the idea that, with fewer remaining periods ahead, replacing current pay with future promised utility becomes more difficult to sustain. Arguments for the opposite finding have often centered on the possibility that managers capture the board once their tenure has grown large (see, e.g., Hill and Phan (1991) and Bebchuk and Fried (2004)), while Murphy (1986) proposes a theory based on market learning about managerial quality over time, where the learning is symmetric between the market and the managers.

Our paper contributes to this debate by indicating that the key determinant for whether the power of incentives ought to increase or decrease with tenure may be the manager’s *degree of risk aversion*. Another prediction of our model, although one which is subject to the limitations of the relaxed approach discussed above, is that, under risk neutrality, the increase in the power of incentives over time is most pronounced (equivalently, the decrease is smaller) for those managers whose initial productivity is low.⁷ Because productivity is positively correlated with performance, this result suggests a negative correlation between early performance and the increase in the power of incentives over the course of the employment relationship. This prediction seems a distinctive feature of our theory, albeit one that, to the best of our knowledge, has not been tested yet.

Organization of the paper. The rest of the paper is organized as follows. We briefly review some pertinent literature in the next section. Section 3 then describes the model while Section 4 characterizes the firm’s optimal contract. Section 5 concludes. All proofs are in the Appendix at the end of the manuscript.

2 Related literature

The literature on managerial compensation is obviously too vast to be discussed within the context of this paper. We refer the reader to Prendergast (1999) for an excellent overview and to Edmans and Gabaix (2009) for a survey of some recent developments. Below, we limit our discussion to the papers that are most closely related to our own work.

Our work is related to the literature on “dynamic moral hazard” and its application to managerial compensation. Seminal works in this literature include Lambert (1983), Rogerson (1985), and Spear and Srivastava (1987). These works provide qualitative insights about optimal contracts but do not provide a full characterization. This has been possible only in restricted settings: Phelan and Townsend (1991) characterize optimal contracts numerically in a discrete-time model, while Sannikov (2008) characterizes the optimal contract in a continuous-time setting with Brownian shocks.⁸ In

⁷We expect that this property carries over to settings with risk-averse managers, provided that productivity is less than fully persistent (see Figure 2, for instance, for an example).

⁸See also Sadzik and Stacchetti (2013) for recent work on the relationship between discrete-time and continuous-time

contrast to these works, Holmstrom and Milgrom (1987) show that the optimal contract has a simple structure when (a) the agent does not value the timing of payments, (b) noise follows a Brownian motion, and (c) the agent’s utility is exponential and defined over consumption net of the disutility of effort. Under these assumptions, the optimal contract takes the form of a simple linear aggregator of total profits.

Contrary to the above works, in the current paper we assume that, in each period, the manager observes the shock to his productivity before choosing effort.⁹ In this respect, our paper is closely related to Laffont and Tirole (1986) who first proposed this alternative timing. This timing permits one to use techniques from the mechanism design literature to solve for the optimal contract. The same approach has been recently applied to dynamic managerial compensation by Edmans and Gabaix (2011) and Edmans et al. (2012). Our model is similar in spirit, but with a few key distinctions. First, we assume that the manager is privately informed about his initial productivity before signing the contract; this is what drives the result that the manager must be given a strictly positive share of the surplus. A second key difference is that we characterize how effort and the power of incentives in the optimal contract evolve over time.¹⁰

Our paper is also related to our previous work on managerial turnover in a changing world (Garrett and Pavan, 2012). In that paper we assumed that all managers are risk neutral and focused on the dynamics of retention decisions. In contrast, in the present paper, we abstract from retention (i.e., assume a single manager) and focus instead on the effect of risk aversion on the dynamics of the power of incentives and effort.

A growing number of papers study optimal financial instruments in dynamic principal-agent relationships. For instance, DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), Sannikov (2007),¹¹ and Biais et al. (2010) study optimal financial contracts for a manager who privately observes the dynamics of cash flows and can divert funds from investors to private consumption. In these papers, it is typically optimal to induce the highest possible effort (which is equivalent to no stealing/no saving); the instrument which is then used to create incentives is the probability of terminating the project. One of the key findings is that the optimal contract can often be implemented using long-term debt, a credit line, and equity. The equity component represents a linear component

models, and Cvitanic, Wan and Zhang (2009), Capponi, Cvitanic, and Yolcu (2012), and Sannikov (2014) for the use of variational methods in agency models with hidden actions only.

⁹We abstract from the possibility that performance is affected by transitory noise that occurs after the manager chooses his effort. It is often the case, however, that compensation can be structured so that it continues to implement the desired effort policies even when performance is affected by transitory noise.

¹⁰As mentioned in footnote 2, the above work assumes that it is optimal to induce the highest feasible effort constantly over time.

¹¹As in our work, and contrary to the other papers cited here, Sannikov (2007) allows the agent to possess private information prior to signing the contract. Assuming the agent’s initial type can be either “bad” or “good”, he characterizes the optimal separating menu where only good types are funded.

to the compensation scheme which is used to make the agent indifferent as to whether or not to divert funds to private use. Since the agent's cost of diverting funds is constant over time and output realizations, so is the equity share. In contrast, we provide an explanation for why and how this share may change over time. While these papers suppose that cash-flows are i.i.d., Tchisty (2006) explores the consequences of correlation and shows that the optimal contract can be implemented using a credit line with an interest rate that increases with the balance. As in Tchisty (2006), we also assume that managerial productivity is imperfectly correlated over time.

From a methodological standpoint, we draw from recent results in the dynamic mechanism design literature. In particular, the necessary and sufficient conditions for incentive compatibility in Proposition 1 in the present paper adapt to the environment under examination results in Theorems 1 and 3 in Pavan, Segal, and Toikka (2014). That paper provides a general treatment of incentive compatibility in dynamic settings. It extends previous work by Baron and Besanko (1984), Besanko (1985), Courty and Li (2000), Battaglini (2005), Eso and Szentes (2007), and Kapicka (2013), among others, by allowing for more general payoffs and stochastic processes and by identifying the role of impulse responses as the key driving force for the dynamics of optimal contracts. One of the key properties identified in this literature is that of declining distortions (see, e.g., Baron and Besanko, 1984, Besanko, 1985, and Battaglini, 2005, among others). A contribution of the present paper is to qualify the extent to which this property is robust to the possibility that the agent is risk averse.¹² In this respect, the paper is also related to Farinha Luz (2014) who, in an insurance model with two types, identifies conditions on the utility function that guarantee that distortions decrease over time over all possible paths. Another contribution of the present paper relative to this literature is in the way we identify certain properties of optimal contracts. As explained above, this involves identifying perturbations of the proposed policies that preserve incentive compatibility and then using variational arguments to verify the key properties. To the best of our knowledge, this approach is new to the dynamic mechanism design literature.

The paper is also related to the literature on optimal dynamic taxation (also known as Mirrleesian taxation, or new public finance). Recent contributions to this literature include Battaglini and Coate (2008), Zhang (2009), Golosov, Troshkin, and Tsyvinski (2012) and Farhi and Werning (2013). Our definition of power of incentives mirrors the definition of the labor "wedge" in this literature, which is considered the appropriate measure of distortions in the provision of incentives in the presence of private information. A complication encountered in this literature is that, because of risk aversion, policies solving the relaxed program can only be computed numerically; likewise, the incentive-compatibility of such policies can only be checked with numerical methods. The approach introduced in the present paper may perhaps prove useful for characterizing certain properties of optimal dynamic taxes (as well as optimal contracts for risk-averse agents in other settings) by

¹²For static models with risk aversion, see Salanie (1990), and Laffont and Rochet (1998).

allowing one to bypass this difficulty.

3 The Model

3.1 The environment

Players, actions, and information. The firm’s shareholders (hereafter referred to as the principal) hire a manager to work on a project for two periods.¹³ In each period $t = 1, 2$, the manager receives some private information $\theta_t \in \Theta_t = [\underline{\theta}_t, \bar{\theta}_t]$ about his ability to generate cash flows for the firm (his type). After observing θ_t , he then chooses effort $e_t \in E = \mathbb{R}$. The latter, combined with the manager’s productivity θ_t , then leads to cash flows π_t according to the simple technology $\pi_t = \theta_t + e_t$.

Both $\theta \equiv (\theta_1, \theta_2) \in \Theta \equiv \Theta_1 \times \Theta_2$ and $e \equiv (e_1, e_2) \in \mathbb{R}^2$ are the manager’s private information. Instead, the cash flows $\pi \equiv (\pi_1, \pi_2)$ are verifiable, and hence can be used as a basis for the manager’s compensation.

Payoffs. For simplicity, we assume no discounting.¹⁴ The principal’s payoff is the sum of the firm’s cash flows in the two periods, net of the manager’s compensation, i.e.

$$U^P(\pi, c) = \pi_1 + \pi_2 - c_1 - c_2,$$

where c_t is the period- t compensation to the manager and where $c \equiv (c_1, c_2)$. The function U^P is also the principal’s Bernoulli utility function used to evaluate possible lotteries over (π, c) .

By choosing effort e_t in period t , the manager suffers a disutility $\psi(e_t)$. The manager’s Bernoulli utility function is then given by

$$U^A(c, e) = v(c_1) + v(c_2) - \psi(e_1) - \psi(e_2), \tag{1}$$

where $v : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing, weakly concave, surjective, Lipschitz continuous, and differentiable function.¹⁵ The case where v is linear corresponds to the case where the manager is risk neutral, while the case where v is strictly concave corresponds to the case where he is risk averse. Note that the above payoff specification also implies that the manager has preferences for consumption smoothing. This assumption is common in the dynamic moral hazard (and taxation) literature (a few notable exceptions are Holmstrom and Milgrom (1987) and more recently Edmans and Gabaix

¹³The assumption that there are only two periods is to ease the exposition. All the results extend to an arbitrary number of periods.

¹⁴None of the results hinge on this assumption.

¹⁵The reason for assuming that $v(\cdot)$ is surjective is twofold: (i) it guarantees the existence of punishments sufficient to discourage the agent from not delivering the anticipated cash flows; (ii) it also guarantees that, given any effort policy that satisfies the appropriate monotonicity conditions of Proposition 1 below, one can always construct a compensation scheme that delivers, on path, the utility that is required for the agent to report his productivity truthfully.

(2011)).¹⁶ We denote the inverse of the felicity function by w (i.e., $w \equiv v^{-1}$).

Productivity process. The manager's first-period productivity, θ_1 , is drawn from an absolutely continuous c.d.f. F_1 with density f_1 strictly positive over Θ_1 . His second-period productivity is drawn from an absolutely continuous c.d.f. $F_2(\cdot|\theta_1)$ with density $f_2(\cdot|\theta_1)$ strictly positive over a subset $\Theta_2(\theta_1) = [\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$ of Θ_2 . We will assume θ_t follows an autoregressive process so that $\tilde{\theta}_2 = \gamma\tilde{\theta}_1 + \tilde{\varepsilon}$, with $\tilde{\varepsilon}$ drawn from a continuously differentiable c.d.f. G with finite support $[\underline{\varepsilon}, \bar{\varepsilon}]$.¹⁷

We assume that $\gamma \geq 0$, so that higher period-1 productivity leads to higher period-2 productivity in the sense of first-order stochastic dominance. We will refer to $\gamma = 1$ as to the case of "fully persistent productivity" (meaning that, holding effort fixed, the effect of any shock to period-1 productivity on the firm's average cash flows is constant over time). We will be primarily interested in the case where $\gamma \in [0, 1]$.

Effort disutility. We assume that $\psi(e) = e^2/2$ for all e . That the disutility of effort is quadratic permits us to identify a convenient family of perturbations to incentive-compatible contracts that preserve incentive compatibility. The assumption that effort can take negative values permits us to disregard the possibility of corner solutions. It also guarantees that a manager misreporting his productivity can always adjust his effort to "hide the lie" by generating the same cash flows as the type being mimicked. This property also facilitates the analysis by turning the model de facto into a pure adverse selection one, as first noticed by Laffont and Tirole (1986).

3.2 The principal's problem

The principal's problem is to choose a contract specifying for each period a recommended effort choice along with compensation that conditions on the observed cash flows. It is convenient to think of such a contract as a mechanism $\Omega \equiv \langle \xi, x \rangle$ comprising a recommended *effort policy* $\xi \equiv (\xi_1(\cdot), \xi_2(\cdot))$ and a *compensation scheme* $x \equiv (x_1(\cdot), x_2(\cdot))$.

The effort $\xi_1(\theta_1)$ that the firm recommends in period one is naturally restricted to depend on the manager's self-reported productivities $\theta = (\theta_1, \theta_2)$ only through θ_1 . This property reflects the assumption that the manager learns his period-2 productivity θ_2 only at the beginning of the second period, as explained in more detail below.¹⁸ The effort that the firm recommends in the second period, $\xi_2(\theta)$, depends on the manager's self-reported productivity in each of the two periods, but is independent of the first-period cash flow, π_1 . This property can be shown to be without loss of

¹⁶As is standard, this specification presumes that the manager's period- t consumption c_t coincides with the period- t compensation. In other words, it abstracts from the possibility of secret private saving. The specification also presumes time consistency. This means that, in both periods, the manager maximizes the expectation of U^A , where the expectation depends on all available information.

¹⁷Throughout, we use the superscript "~" to denote random variables.

¹⁸While we naturally restrict ξ_1 to depend on θ only through the period-1 productivity θ_1 , we often abuse notation by writing $\xi_1(\theta)$ whenever this eases the exposition without the risk of confusion.

optimality for the principal, a consequence of the assumptions that (i) cash flows are deterministic functions of effort and productivity (which implies that, on path, π_1 is a deterministic function of θ_1), and (ii) the manager is not protected by limited liability (which implies that incentives for period-1 effort can be provided through the compensation scheme $x_1(\cdot)$ without the need to condition effort in the second period on off-path cash flows). The compensation $x_t(\theta, \pi)$ paid in each period naturally depends both on the reported productivities and the observed cash flows.^{19,20}

Let $\pi_t(\theta) \equiv \theta_t + \xi_t(\theta)$ denote the period- t “equilibrium” cash flows (by “equilibrium,” hereafter we mean under a truthful and obedient strategy for the manager). Note that the compensation scheme x is defined for all possible cash flows $\pi \in \mathbb{R}^2$, not only the equilibrium ones; i.e., each payment $x_t(\theta, \pi)$ is defined also for $\pi \neq \pi(\theta) \equiv (\pi_1(\theta_1), \pi_2(\theta))$. For any $\theta \in \Theta$, we then further define $c_t(\theta) = x_t(\theta, \pi(\theta))$ to be the equilibrium compensation to the manager in state θ and refer to $c \equiv (c_1(\cdot), c_2(\cdot))$ as to the firm’s *compensation policy*. While our focus is on characterizing the firm’s optimal effort and compensation policies, the role of the out-of-equilibrium payments $x_t(\theta, \pi)$ for $\pi \neq \pi(\theta)$ is to guarantee that the manager finds it optimal to follow a truthful and obedient strategy, as will be discussed in detail below.

Importantly, we assume that the firm offers the manager the contract after he is already informed about his initial productivity $\theta_1 \in \Theta_1$. After receiving the contract, the manager then chooses whether or not to accept it. If he rejects it, he obtains an outside continuation payoff which we assume to be equal to zero for all possible types. If, instead, he accepts it, he is then bound to stay in the relationships for the two periods.²¹ He is then asked to report his productivity $\hat{\theta}_1 \in \Theta_1$ and is recommended effort $\xi_1(\hat{\theta}_1)$. The manager then privately chooses effort e_1 which combines with the manager’s productivity θ_1 to give rise to the period-1 cash flows $\pi_1 = \theta_1 + e_1$. After observing the cash flows π_1 , the firm then pays the manager a compensation $x_1(\hat{\theta}_1, \pi_1)$.

The functioning of the contract in period two parallels the one in period one. At the beginning

¹⁹ Again, we abuse notation by writing $x_1(\theta, \pi)$ when convenient, although x_1 is naturally restricted to depend only on (θ_1, π_1) .

²⁰ By reporting his productivity, the manager effectively adjusts his compensation scheme. This seems consistent with the practice of managers proposing changes to their compensation, which has become quite common (see, among others, Bebchuk and Fried, 2004, and Kuhnen and Zwiebel, 2008). However, note that the allocations sustained under the optimal contract as determined below are typically sustainable also without the need for direct communication between the manager and the firm (this is true, in particular, when there is a one-to-one mapping from the manager’s productivity to the equilibrium cash flows).

²¹ We do not expect our results to hinge on the assumption that the manager is constrained to stay in the relationship throughout both periods. For example, when the manager’s period-2 outside option is sufficiently small, the period-2 individual rationality constraints are slack, in which case the solution to the firm’s problem is precisely the one we characterize below. One reason why the outside option in period two may be small is that the manager may anticipate adverse treatment by the labor market in case he leaves the firm prematurely. Fee and Hadlock (2004), for instance, document evidence for a labor market penalty in case a senior executive leaves the firm early, although the size of this penalty depends on the circumstances surrounding departure.

of the period, the manager learns his new productivity θ_2 . He then updates the principal by sending a new report $\hat{\theta}_2 \in \Theta_2$. The contract then recommends effort $\xi_2(\hat{\theta})$ which may depend on the entire history $\hat{\theta} \equiv (\hat{\theta}_1, \hat{\theta}_2)$ of reported productivities. The manager then privately chooses effort e_2 which, together with θ_2 , leads to the cash flows π_2 . After observing π_2 , the firm then pays the manager a compensation $x_2(\hat{\theta}, \pi)$ and the relationship is terminated.

As usual, we restrict attention to contracts that are accepted by all types and that induce the manager to report truthfully and follow the principal's recommendations in each period.²² We will refer to such contracts as *individually rational* and *incentive compatible*.

4 Profit-maximizing Contracts

4.1 Implementable policies

As anticipated above, one can understand the principal's problem as choosing an effort and compensation policy $\langle \xi, c \rangle$ to maximize the firm's profits subject to the policy being *implementable*. By this we mean the following.

Definition 1 *The effort and compensation policies $\langle \xi, c \rangle$ are implementable if there exists a compensation scheme x such that (i) the contract $\Omega = \langle \xi, x \rangle$ is incentive compatible and individually rational, and (ii) the manager's on-path compensation under the contract $\Omega = \langle \xi, x \rangle$ is given by c , i.e. $x_t(\theta, \pi(\theta)) = c_t(\theta)$ for all t , and all θ .*

As explained in the Introduction, we aim at finding effort and compensation policies $\langle \xi, c \rangle$ that maximize the firm's profits among all implementable policies. Our first result thus provides a complete characterization of implementable policies.

Proposition 1 *The effort and compensation policies $\langle \xi, c \rangle$ are implementable if and only if the following conditions jointly hold: (A) for all $\theta \in \Theta$,*

$$v(c_1(\theta_1)) + v(c_2(\theta)) = W(\theta; \xi) + K, \text{ where} \quad (2)$$

$$\begin{aligned} W(\theta; \xi) \equiv & \psi(\xi_1(\theta_1)) + \psi(\xi_2(\theta)) + \int_{\theta_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2 | s} \left[\psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right\} ds \\ & + \int_{\theta_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds - \mathbb{E}^{\tilde{\theta}_2 | \theta_1} \left[\int_{\theta_2}^{\tilde{\theta}_2} \psi'(\xi_2(\theta_1, s)) ds \right], \end{aligned} \quad (3)$$

²²Note that the manager's second-period payoff does not depend directly on his first-period productivity. Hence, the environment is "Markov". This means that restricting attention to contracts that induce the manager to follow a truthful and obedient strategy in period two also after having departed from truthful and obedient behavior in period one is without loss of optimality.

and where $K \geq 0$ is such that

$$\mathbb{E}^{\tilde{\theta}|\theta_1} \left[W(\tilde{\theta}; \xi) - \psi(\xi_1(\tilde{\theta}_1)) - \psi(\xi_2(\tilde{\theta})) \right] + K \geq 0 \quad (4)$$

for all θ_1 ; and (B)(i) for all $\theta_1, \hat{\theta}_1 \in \Theta_1$,

$$\begin{aligned} & \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi' \left(\xi_1(\hat{\theta}_1) + \hat{\theta}_1 - s \right) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\psi' \left(\xi_2(\hat{\theta}_1, \tilde{\theta}_2) \right) \right] \right\} ds \\ & \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi' \left(\xi_1(s) \right) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\psi' \left(\xi_2(s, \tilde{\theta}_2) \right) \right] \right\} ds, \end{aligned} \quad (5)$$

and B(ii) $\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\pi_2(\theta_1, \tilde{\theta}_2) \right]$ is non-decreasing in θ_1 and, for all $\theta_1 \in \Theta_1$, $\pi_2(\theta_1, \theta_2)$ is non-decreasing in θ_2 .

Note that Condition (A) says that the manager's ex-post equilibrium payoff

$$V(\theta) \equiv v(c_1(\theta_1)) + v(c_2(\theta)) - \psi(\xi_1(\theta_1)) - \psi(\xi_2(\theta))$$

in each state of the world θ must be equal to his period-1 expected payoff

$$\mathbb{E}^{\tilde{\theta}|\theta_1} [V(\tilde{\theta})] = \mathbb{E}^{\tilde{\theta}|\theta_1} [V(\tilde{\theta})] + \int_{\underline{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\psi'(\xi_2(s, \tilde{\theta}_2)) \right] \right\} ds \quad (6)$$

augmented by a term

$$\int_{\underline{\theta}_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds - \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(\xi_2(\theta_1, s)) ds \right]$$

that guarantees that the manager has the incentives to report truthfully not only in period-1 but also in period-2, and which vanishes when computed based on period 1's private information. The necessity of this condition is obtained by combining certain period-2 local necessary conditions for incentive compatibility (as derived, for example, in Laffont and Tirole (1986)) with certain period-1 local necessary conditions for incentive compatibility (as derived, for example, in Pavan, Segal and Toikka (2014); see also Garrett and Pavan (2012) for a similar derivation in a model of managerial turnover). Observe that Condition (A) in the proposition implies that the surplus that type θ_1 expects above the one expected by the lowest period-1 type $\underline{\theta}_1$ is increasing in the effort that the firm asks of managers with initial productivities $\theta'_1 \in (\underline{\theta}_1, \theta_1)$ in each of the two periods. This surplus is necessary to dissuade type θ_1 from mimicking the behavior of these lower types. Such mimicry would involve, say, reporting a lower type in the first period and then replicating the distribution of that type's productivity reports in the second period. By replicating the same cash flows expected from a lower type, a higher type obtains the same compensation while working less if the effort asked to the lower type is positive, and more if the effort asked to the lower type is negative.

Also note that, when productivity is only partially persistent (in our autoregressive model, when $\gamma < 1$), then asking for a lower period-1 effort from types $\theta'_1 < \theta_1$ is more effective in reducing type θ_1 's expected surplus than asking for a lower period-2 effort from the same types. The reason is that the amount of effort that type θ_1 expects to be able to save relative to these lower period-1 types (alternatively, the extra effort that he must provide, in case the effort asked to these lower types is negative) is smaller in the second period, reflecting that the initial productivity is imperfectly persistent. As we will see below, this property plays an important role in shaping the dynamics of effort and the power of incentives under optimal contracts.

Finally note that the scalar K in (2) corresponds to the expected payoff $\mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta})]$ of the lowest period-1 type. Using (6), it is easy to see that, when the effort requested is always non-negative, then if the lowest period-1 type finds it optimal to accept the contract, then so does any manager whose initial productivity is higher. This property, however, need not hold in case the firm requests a negative effort from a positive-measure set of types. In this case, the principal may need to leave a strictly positive surplus to the lowest type to guarantee participation by all types.

Next consider Condition (B) in the proposition. Observe that, while Condition (A) imposes restrictions on the compensation that must be paid to the manager, for given effort policy ξ , Condition (B) imposes restrictions on the effort policy that are independent of the manager's felicity function, v . In particular, Condition (B)(ii) combines the familiar monotonicity constraint for the second-period cash flows from static mechanism design (e.g., Laffont and Tirole (1986)) with a novel monotonicity constraint that requires the NPV of the expected cash-flows, weighted by the impulse responses (which here are equal to one in the first period and γ in the second period) to be non-decreasing in period-1 productivity.²³ Finally, Condition (B)(i) is an "integral monotonicity condition," analogous to the one in Theorem 3 of Pavan, Segal and Toikka (2014). That the conditions in the proposition are necessary follows from arguments similar to those in Theorems 2 and 3 in Pavan, Segal, and Toikka (2014), adapted to the environment under examination here. That they are also sufficient follows from the fact, when satisfied, one can construct compensation schemes under which the best a manager can do when mimicking a different type is to replicate the same cash flows of the type being mimicked. This turns the manager's problem into a pure adverse selection one. The conditions in the proposition then guarantee that, at each history, the manager prefers to follow a truthful and obedient strategy in the remaining periods rather than lying and then replicating the cash flows of the type being mimicked, irrespective of past effort, true and reported productivity.

²³Formally, let $\theta_2 = z(\theta_1, \varepsilon)$, where ε is a shock independent of θ_1 . The impulse response of θ_2 to θ_1 is the derivative of z with respect to θ_1 . In the case of a linear autoregressive process $\theta_2 = z(\theta_1, \varepsilon) = \gamma\theta_1 + \varepsilon$, so that the impulse response of θ_2 to θ_1 is equal to the persistence parameter γ .

4.2 Optimal policies

The next step is to use Condition (A) of Proposition 1 to derive an expression for the firm's profits in terms only of the effort policy ξ and the period-1 compensation c_1 . This follows after observing that, given ξ and c_1 , the period-2 equilibrium compensation $c_2(\theta) = x_2(\theta, \pi(\theta))$ is uniquely determined by the need to provide the manager with a lifetime utility of monetary compensation equal to the level required by incentive compatibility, as given by (2). That is,

$$c_2(\theta) = w(W(\theta; \xi) + K - v(c_1(\theta_1))). \quad (7)$$

The following representation of the firm's profits then follows from the result in Proposition 1.

Lemma 1 *Let $\langle \xi, c \rangle$ be implementable effort and compensation policies yielding an expected surplus of K to a manager with the lowest period-1 productivity $\underline{\theta}_1$. The firm's expected profits under $\langle \xi, c \rangle$ are given by*

$$\mathbb{E}[U^P] = \mathbb{E}\left[\tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) + \tilde{\theta}_2 + \xi_2(\tilde{\theta}) - c_1(\tilde{\theta}_1) - w\left(W(\tilde{\theta}; \xi) + K - v(c_1(\tilde{\theta}_1))\right)\right]. \quad (8)$$

Note that, when the manager is risk neutral ($v(y) = w(y)$ for all y), the result in Lemma 1 implies that the firm's payoff is equal to the entire surplus of the relationship, net of a term that corresponds to the surplus that must be left to the manager and which depends only on the effort policy ξ :

$$\mathbb{E}[U^P] = \mathbb{E}\left[\begin{array}{l} \tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) - \psi(\xi_1(\tilde{\theta}_1)) + \tilde{\theta}_2 + \xi_2(\tilde{\theta}) - \psi(\xi_2(\tilde{\theta})) \\ - \frac{1-F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)} \left\{ \psi'(\xi_1(\tilde{\theta}_1)) + \gamma \psi'(\xi_2(\tilde{\theta}_1, \tilde{\theta}_2)) \right\} - K \end{array}\right]. \quad (9)$$

The expression in (9) is what in the dynamic mechanism design literature (where payoffs are typically assumed to be quasilinear) is referred to as “dynamic virtual surplus”.

As one should expect, when instead the manager is risk averse, the firm's payoff depends not only on the effort policy, but also on the way the compensation is spread over time. The value of the result in Lemma 1 comes from the fact that the choice over such compensation can be reduced to the choice over the period-1 compensation. This is because any two compensation schemes implementing the same effort policy ξ must give the manager the same utility of compensation not just in expectation, but ex-post, that is, at each productivity history $\theta = (\theta_1, \theta_2)$. This equivalence result (which is the dynamic analog in our non-quasilinear environment of the celebrated “revenue equivalence” for static quasilinear problems) plays an important role below in the characterization of the optimal policies.²⁴

We now consider the question of which implementable effort and compensation policies maximize the firm's expected profits. As noted in the Introduction, the approach typically followed in the dynamic mechanism design literature to identify optimal policies is the following. First, consider a *relaxed program* that replaces all incentive-compatibility constraints with Condition (2) and all

²⁴See Pavan, Segal, and Toikka (2014) for a more general analysis of payoff-equivalence in dynamic settings.

individual-rationality constraints with the constraint that $K = \mathbb{E}^{\tilde{\theta}|\underline{\theta}_1} [V(\tilde{\theta})] \geq 0$. Then choose policies (ξ_1, ξ_2, c_1) along with a scalar K to solve the unconstrained maximization of the firm's profits as given by (8) and then let $c_2(\cdot)$ be given by (7).²⁵ However, recall that, alone, Condition (2) is necessary but not sufficient for incentive compatibility. Furthermore, when the solution to the relaxed program yields policies prescribing a negative effort over a positive-measure set of types, satisfaction of the participation constraint for the lowest period-1 type $\underline{\theta}_1$ does not guarantee satisfaction of all other participation constraints. Therefore, one must typically identify auxiliary assumptions on the primitives of the problem guaranteeing that the effort and compensation policies (ξ, c) that solve the relaxed program are implementable. Identifying such auxiliary primitive assumptions is not always simple, but it is often possible when the manager is risk neutral. It can be quite difficult when the manager is risk averse.²⁶

The approach we follow here is therefore different. Because the firm's profits under any individually-rational and incentive-compatible contract must be consistent with the representation in (8), we use this expression to evaluate the performance of different contracts. However, not all policies (ξ_1, ξ_2, c_1) , coupled with c_2 as given in (7) for some $K \geq 0$, are implementable (in particular, this may be the case for those policies that maximize (8)). Hence, we do not aim at maximizing this expression directly. Instead, we use variational arguments to identify properties of optimal policies. More precisely, we first identify "admissible variations". By this we mean perturbations to implementable policies such that the perturbed policies remain implementable (i.e., continue to satisfy the conditions of Proposition 1). For the candidate policies to be sustained under an optimal contract, it then must be the case that no admissible variation increases the firm's profits, as expressed in (8).

Natural candidates for admissible variations are obtained by adding functions $\alpha(\theta_1)$ and $\beta(\theta)$ to the original effort policies $\xi_1(\theta_1)$ and $\xi_2(\theta)$, and then adjusting the compensation policy c so that payments continue to satisfy (2). While not all such variations are admissible (in particular, they need not yield effort policies satisfying integral monotonicity), admissible variations are always obtained for constant functions $\alpha(\theta_1) = a > 0$ and $\beta(\theta) = b > 0$, all θ . If the original policies are such that ξ prescribes effort bounded away from zero from below at almost all histories, then we may also take $a, b < 0$. The requirement that such perturbations do not increase the firm's expected

²⁵When the agent is risk neutral, the distribution of payments over time is irrelevant for the agent and hence (8) is independent of $c_1(\cdot)$. In this case, solving the relaxed program means finding an effort policy $\xi = (\xi_1, \xi_2)$ that maximizes (9) and then letting $c = (c_1, c_2)$ be any compensation policy that satisfies (2).

²⁶The additional complexity comes from (a) the difficulty in finding the policies that solve the relaxed program (in our environment, these policies are the ones in Proposition 7 below) along with (b) the difficulty in guaranteeing that such policies are truly implementable. One can, however, proceed numerically on a case-by-case basis. This is the approach taken, for instance, in Farhi and Werning (2012) in the contest of optimal dynamic taxation for risk-averse agents.

profits then yields the following result.²⁷

Proposition 2 *Let $\langle \xi^*, c^* \rangle$ be effort and compensation policies sustained under an optimal contract. Then $\langle \xi^*, c^* \rangle$ must satisfy the following conditions:*

$$\mathbb{E} \left[\psi' \left(\xi_1^*(\tilde{\theta}_1) \right) w' \left(v \left(c_1^*(\tilde{\theta}_1) \right) \right) \right] \geq 1 - \mathbb{E} \left[\frac{\psi'' \left(\xi_1^*(\tilde{\theta}_1) \right)}{f_1(\tilde{\theta}_1)} \int_{\tilde{\theta}_1}^{\bar{\theta}_1} w' \left(v \left(c_1^*(r) \right) \right) f_1(r) dr \right] \quad (10)$$

$$\mathbb{E} \left[\psi' \left(\xi_2^*(\tilde{\theta}) \right) w' \left(v \left(c_2^*(\tilde{\theta}) \right) \right) \right] \geq 1 - \gamma \mathbb{E} \left[\frac{\psi'' \left(\xi_2^*(\tilde{\theta}) \right)}{f_1(\tilde{\theta}_1)} \int_{\tilde{\theta}_1}^{\bar{\theta}_1} w' \left(v \left(c_1^*(r) \right) \right) f_1(r) dr \right] \quad (11)$$

$$- \mathbb{E} \left[\frac{\psi'' \left(\xi_2^*(\tilde{\theta}) \right)}{f_2(\tilde{\theta}_2|\tilde{\theta}_1)} \int_{\tilde{\theta}_2}^{\bar{\theta}_2} \left\{ w' \left(v \left(c_2^*(\tilde{\theta}_1, r) \right) \right) - w' \left(v \left(c_1^*(\tilde{\theta}_1) \right) \right) \right\} f_2(r|\tilde{\theta}_1) dr \right] \\ w' \left(v \left(c_1^*(\theta_1) \right) \right) = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[w' \left(v \left(c_2^*(\theta_1, \tilde{\theta}_2) \right) \right) \right] \quad (12)$$

and

$$c_2^*(\theta) = w \left(W \left(\theta; \xi \right) + K - v \left(c_1^*(\theta_1) \right) \right),$$

where $W \left(\theta; \xi \right) + K$ is the total utility of compensation, as given by (3), with $K = \mathbb{E}^{\tilde{\theta}|\theta_1} \left[V \left(\tilde{\theta} \right) \right]$ denoting the lowest period-1 type's expected payoff under the policies $\langle \xi^*, c^* \rangle$. The effort policy implemented under any optimal contract is essentially unique.²⁸ If v is strictly concave, the compensation policy implemented under any optimal contract is also essentially unique. Lastly, the inequalities in (10) and (11) must hold as equalities if $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\xi_2^*(\tilde{\theta}) \right]$ is bounded away from zero from below with probability one.

Conditions (10) and (11) capture how the firm optimally solves the trade-off between increasing the manager's expected effort on the one hand and reducing the expected payments to the manager on the other. When the manager has preferences for consumption smoothing, his compensation must also be appropriately distributed over time according to Condition (12).

It is worth commenting on where our approach is similar to the one in the existing literature and where it departs. Condition (12) is obtained by considering perturbations to the compensation policy that leave the manager's payoff unchanged. In particular, we consider variations in period-1 compensation coupled with adjustments to the period-2 compensation chosen so that the total utility that the manager derives from his life-time compensation continues to satisfy (2). If the original policies $\langle \xi, c \rangle$ are implementable, so are the perturbed ones $\langle \xi, c' \rangle$. Therefore, under any

²⁷Note that such perturbations also preserve incentive compatibility in environments with more than two periods and richer stochastic processes. Euler conditions analogous to those in Proposition 2 can thus be obtained also for richer environments.

²⁸By essentially unique, we mean except over a zero-measure set of productivity histories.

optimal contract, such perturbations must not increase the firm's expected profits. For this to be the case, the proposed compensation scheme must satisfy Condition (12), which is the same inverse Euler condition

$$\frac{1}{v'(c_1^*(\theta_1))} = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\frac{1}{v'(c_2^*(\theta_1, \tilde{\theta}_2))} \right]$$

first identified by Rogerson (1985). The only novelty relative to Rogerson is that here the total utility from compensation is required to satisfy (2), which is necessary when the manager's productivity is his private information.

The point where our analysis departs from the rest of the literature is in the derivation of Conditions (10) and (11), which link the dynamics of effort to the dynamics of compensation, under optimal contracts. As mentioned above, these conditions are obtained by considering translations of the effort policy ξ that preserve implementability, i.e. that preserve Condition (B) in Proposition 1. Contrary to the perturbations of the compensation policy that lead to Condition (12), these perturbations necessarily change the manager's expected payoff, as one can readily see from (6). For these perturbations not to increase the firm's expected profits, it must be that the original policies satisfy Conditions (10) and (11) in the proposition.

Note that Conditions (10) and (11) hinge on our assumption that the disutility of effort is quadratic. In particular, this assumption is what guarantees that translations of the effort policy ξ continue to satisfy Condition (B)(i) of Proposition 1.²⁹

The next proposition uses an alternative class of perturbations that preserve not only incentive compatibility but also the manager's expected payoff conditional on his period-1 type θ_1 . This is obtained by considering joint perturbations of ξ_1 and ξ_2 of opposite sign. The requirement that such perturbations not increase profits yields another Euler condition that links the effort and compensation policies across the two periods.

Proposition 3 *Let $\langle \xi^*, c^* \rangle$ be effort and compensation policies sustained under an optimal contract. The policies $\langle \xi^*, c^* \rangle$ must satisfy the following condition for almost all $\theta_1 \in \Theta_1$:*

$$\begin{aligned} & \mathbb{E}^{\tilde{\theta}|\theta_1} \left[1 - \psi' \left(\xi_2^*(\tilde{\theta}) \right) w' \left(v(c_2^*(\tilde{\theta})) \right) \right] = \gamma \left[1 - \psi' \left(\xi_1^*(\theta_1) \right) w' \left(v(c_1^*(\theta_1)) \right) \right] \\ & + \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\frac{\psi'' \left(\xi_2^*(\tilde{\theta}) \right)}{f_2(\tilde{\theta}_2|\tilde{\theta}_1)} \int_{\tilde{\theta}_2}^{\bar{\theta}_2} \left\{ w' \left(v(c_2^*(\tilde{\theta}_1, r)) \right) - w' \left(v(c_1^*(\tilde{\theta}_1)) \right) \right\} f_2(r|\tilde{\theta}_1) dr \right]. \end{aligned} \quad (13)$$

Note that the term

$$1 - \psi' \left(e_t \right) w' \left(v(c_t) \right) \quad (14)$$

²⁹One might conjecture, for instance, that our approach could be generalized to disutility functions that are not quadratic as follows: Rather than translating effort by a constant, one could translate the marginal disutility of effort. That is, one could consider the new effort policy given, for some $t \in \{1, 2\}$, by $\psi' \left(\xi_t^\eta \left(\theta \right) \right) = \psi' \left(\xi_t^* \left(\theta \right) \right) + \eta$ for η small, while letting $\xi_s^\eta \left(\theta \right) = \xi_s^* \left(\theta \right)$ for $s \neq t$. Unfortunately, the new effort policy ξ^η typically does not satisfy Condition B(i) of Proposition 1 (even though, by assumption, the original policy ξ^* does satisfy this condition).

is what in the new public finance literature is referred to as the "period- t wedge"; it captures the distortion in the provision of incentives due to the manager's private information (in a first-best world, the latter would be equal to zero at all periods and across all states). The result thus establishes that the expected period-2 wedge (equivalently, the average period-2 distortion) is equal to a fraction $\gamma \in [0, 1]$ of the period-1 wedge, augmented by the same term

$$\mathbb{E}^{\tilde{\theta}|\theta_1} \left[\frac{\psi''(\xi_2^*(\tilde{\theta}))}{f_2(\tilde{\theta}_2|\tilde{\theta}_1)} \int_{\tilde{\theta}_2}^{\bar{\theta}_2} \{w'(v(c_2^*(\theta_1, r))) - w'(v(c_1^*(\theta_1)))\} f_2(r|\theta_1) dr \right]$$

that is present in the Euler Condition (11) and that captures the extra cost of incentives due to the volatility in the second-period compensation (note that this term is equal to zero when the agent is risk neutral). Interestingly, note that Conditions (10) and (11) in Proposition 2 above jointly imply that Condition (13) holds in expectation, but only when the inequalities in (10) and (11) hold as equalities. Thus, an advantage of the perturbations that lead to Proposition 3 is that they permit us to establish (13), without any restriction on the shape of the policies (in particular, these perturbations do not require that $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\tilde{\theta})]$ is bounded away from zero from below with probability one). The new Euler condition (13) will be used below in connection to Conditions (10) and (11) to relate the dynamics of the power of incentives to risk aversion and the persistence of the manager's productivity.³⁰

4.3 Dynamics of the power of incentives

Our next objective is to understand how the power of incentives changes with tenure under optimal contracts. First, we need a workable definition of the "power of incentives".

Definition 2 (Power of incentives) *For each $t = 1, 2$ and each $\theta = (\theta_1, \theta_2)$, the (local) power of incentives under the (incentive-compatible) mechanism $\Omega = \langle \xi, x \rangle$ is the ratio*

$$\frac{\psi'(\xi_t(\theta))}{v'(c_t(\theta))} = \psi'(\xi_t(\theta)) w'(v(c_t(\theta)))$$

between the marginal disutility of effort and the marginal utility of consumption, both evaluated at the equilibrium levels $\xi_t(\theta)$ and $c_t(\theta) = x_t(\theta, \pi(\theta))$.

The rationale for this definition comes from the fact that, when this ratio is high, either because the marginal disutility of effort is high or because the marginal utility of consumption is low, the firm must resort to a high sensitivity of pay to performance to induce the desired level of effort.

To see this more clearly, consider payment schemes x where the payments in each period depend on the history of observed cash flows only through the contemporaneous observations (that is, for all

³⁰Note that Conditions (10) and (11) provide information about the properties of the period- t wedges in addition to the information contained in (13), which establishes only a relationship between wedges across the two periods.

$t = 1, 2$, $x_t(\theta, \pi)$ depends on π only through π_t) and where each payment $x_t(\theta, \pi)$ is differentiable in the contemporaneous cash flows π_t . It is then easy to see that any payment scheme with the above properties implementing the effort and consumption policies $\langle \xi, c \rangle$ must satisfy, for any θ ,

$$\begin{aligned} \frac{\partial x_1(\theta_1, \pi_1)}{\partial \pi_1} \Big|_{\pi_1 = \pi_1(\theta_1)} &= \psi'(\xi_1(\theta_1)) w'(v(c_1(\theta_1))) \\ \frac{\partial x_2(\theta, \pi_2)}{\partial \pi_2} \Big|_{\pi_2 = \pi_2(\theta)} &= \psi'(\xi_2(\theta)) w'(v(c_2(\theta))) \end{aligned}$$

with $\pi_t(\theta) = \theta_t + \xi_t(\theta)$, $t = 1, 2$. Under such schemes, our definition of the power of incentives then coincides with the rate at which the period- t compensation changes with the period- t cash flows, around the target level. This definition parallels the one typically given in the literature that restricts attention to linear schemes (see, e.g., Lazear (2000)), but with two important qualifications. First, our notion is a local measure. This is useful because, in general, a compensation scheme that is linear in the cash flows may fail to implement the optimal policies $\langle \xi^*, c^* \rangle$. Second, our definition does not require the compensation schemes to be differentiable. This is useful because we cannot rule out the possibility that, for certain specifications, the optimal policies $\langle \xi^*, c^* \rangle$ may be implementable only with non-differentiable schemes.³¹ Also note that the above definition of power of incentives parallels the definition of "wedges" in the Mirrleesian taxation literature. As discussed in the context of that literature, such a definition is preferable to the one of marginal taxes precisely because it does not require confining attention to differentiable tax schemes.

With this discussion in mind, hereafter we interpret the left-hand sides of (10) and (11) as the ex-ante expected power of incentives under optimal schemes. Below we examine how these expectations depend on the persistence of the manager's productivity (here captured by γ) and the manager's degree of risk aversion.

Risk neutrality. When the manager is risk neutral (that is, when v is equal to the identity function), the power of incentives is simply the marginal disutility of effort, evaluated at the prescribed effort level $\xi_t^*(\theta)$. In this case, the Euler conditions (10) and (11) describe properties not only of the power of incentives, but also of effort.

³¹However, we conjecture that differentiable schemes can always implement policies which are virtually optimal. By this we mean the following. Let $\langle \xi^*, c^* \rangle$ be fully optimal policies. For any $\varepsilon > 0$ there exist policies $\langle \xi, c \rangle$ and a differentiable compensation scheme x such that the following are true: (i) the contract $\Omega \equiv \langle \xi, x \rangle$ is individually rational and incentive compatible for the manager; (ii) in each state θ , the compensation the manager receives under Ω is given by c ; and (iii) with probability one $\|(\xi(\theta), c(\theta)) - (\xi^*(\theta), c^*(\theta))\| \leq \varepsilon$. In other words, the firm can always implement policies arbitrarily close to the fully-optimal ones using differentiable schemes. Moreover, we conjecture that, when the manager is risk averse, if the policies $\langle \xi, c \rangle$ yield profits arbitrarily close to the ones under the fully optimal policies, then $\langle \xi, c \rangle$ must be arbitrarily close to $\langle \xi^*, c^* \rangle$ in the \mathcal{L}_1 norm. Virtually optimal policies can then be expected to inherit the same dynamic properties discussed below as the fully optimal policies. This is because the key properties discussed below refer to the *expectation* of $\psi'(\xi_t(\theta)) w'(v(c_t(\theta)))$, where the expectation is over all possible productivity histories.

Proposition 4 *Assume the manager is risk neutral (that is, v is the identity function).*

(a) *Suppose that, on average, period-1 effort is distorted downwards relative to the first-best level (that is, $\mathbb{E}[\xi_1^*(\tilde{\theta}_1)] < 1 = e^{FB}$). Then expected effort is higher in the second period than in the first one when $\gamma < 1$ and is the same in both periods when $\gamma = 1$.*

(b) *Suppose the optimal policy ξ^* is such that $\xi_1^*(\theta_1) + \gamma\mathbb{E}^{\tilde{\theta}|\theta_1}[\xi_2^*(\tilde{\theta})]$ is bounded away from zero from below with probability one. Then period-1 effort is distorted downwards relative to the first-best level.*

(c) *Either one of the following two sets of conditions guarantees that $\xi_1^*(\theta_1) + \gamma\mathbb{E}^{\tilde{\theta}|\theta_1}[\xi_2^*(\tilde{\theta})]$ is bounded away from zero from below with probability one: (i) $[1 - F_1(\theta_1)]/f_1(\theta_1)$ is non-increasing and strictly smaller than $(1+\gamma)/(1+\gamma^2)$; (ii) $\sup\{[1 - F_1(\theta_1)]/f_1(\theta_1)\} < (1+\gamma)/(1+\gamma^2) - (\bar{\theta}_1 - \underline{\theta}_1)$ and $F_2(\cdot|\cdot)$ satisfies the monotone-likelihood-ratio property (that is, $f_2(\theta'_2|\theta_1)/f_2(\theta_2|\theta_1)$ is non-decreasing in θ_1 , for all $\theta'_2 \geq \theta_2$).*

The result in Part (a) follows directly from (13) by observing that, under risk neutrality, this equation reduces to

$$\mathbb{E}^{\tilde{\theta}|\theta_1}[1 - \xi_2^*(\tilde{\theta})] = \gamma[1 - \xi_1^*(\theta_1)]. \quad (15)$$

By the law of iterated expectations, we then have that, if on average period-1 effort is distorted downwards relative to the first best, then period-2 effort is, on average, higher in the second period than in the first one when $\gamma < 1$ and is the same in both periods when $\gamma = 1$. The result in part (b) in turn is established by showing that, when $\xi_1^*(\theta_1) + \gamma\mathbb{E}^{\tilde{\theta}|\theta_1}[\xi_2^*(\tilde{\theta})]$ is bounded away from zero from below with probability one, then the participation constraints of all period-1 types except the lowest are slack. In this case, the Euler Conditions (10) and (11) must hold as equalities and reduce to

$$\mathbb{E}[\xi_1^*(\tilde{\theta}_1)] = 1 - \mathbb{E}\left[\frac{1 - F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)}\right] \quad (16)$$

$$\mathbb{E}[\xi_2^*(\tilde{\theta})] = 1 - \gamma\mathbb{E}\left[\frac{1 - F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)}\right]. \quad (17)$$

Note that the left-hand sides of Conditions (16) and (17) represent the expected marginal cost of higher effort, in terms of extra disutility for the manager. The right-hand sides represent the expected marginal benefit for the firm (stemming from the increase in cash flows), less a term which captures the effect of higher effort on the surplus that the firm must leave to the manager to induce him to reveal his productivity (this surplus is over and above the minimal compensation required to compensate the manager for his disutility of effort, as one can see by inspecting (6)). The reason why, in this case, the firm distorts downward the effort asked of those managers whose initial productivity is low is to reduce the rents it must leave to those managers whose initial productivity is high. When productivity is not fully persistent, these distortions are more effective in reducing

managerial rents early in the relationship as opposed to later on. Distortions are therefore smaller at later dates, explaining why the expected power of incentives increases with tenure. The increase is most pronounced when productivity is least persistent. Indeed, as we approach the case where productivity is independent over time (i.e., when γ is close to zero), the expected effort the firm asks of each manager in the second period is close to the first-best level ($e^{FB} = 1$).

Finally Part (c) of the above proposition provides sufficient conditions for the optimal effort policy ξ^* to be such that $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} [\xi_2^*(\tilde{\theta})]$ is bounded away from zero from below. The first condition requires that the hazard rate $f_1(\theta_1)/[1 - F_1(\theta_1)]$ of the period-1 distribution be non-decreasing (as typically assumed in the mechanism design literature) and strictly higher than $\frac{1+\gamma^2}{1+\gamma}$. In this case, the optimal effort policies are those that solve the relaxed program and are given by

$$\xi_1^R(\theta) = 1 - \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \quad (18)$$

$$\xi_2^R(\theta) = 1 - \gamma \frac{1 - F_1(\theta_1)}{f_1(\theta_1)}. \quad (19)$$

That these policies are implementable follows because $f_1(\theta_1)/[1 - F_1(\theta_1)]$ is non-decreasing, which guarantees that (ξ_1^R, ξ_2^R) satisfies the monotonicity conditions B(i) and B(ii) of Proposition 1. It is also worth pointing out that these policies are implementable by payment schemes x which are linear in the cashflows of the firm (the “power of incentives” in each period then coincides with the slope of the linear scheme). Also note that, in this case, effort (as well as the power of incentives) increases over time, not just in expectation, but along any productivity sequence.

Alternatively, one can show that expected effort also increases over time when the hazard rate is large enough (but not necessarily monotone), provided that the conditional distribution $F_2(\cdot|\cdot)$ satisfies MLRP. Note that, when θ_t follows an autoregressive process, as assumed here, the latter requirement is a restriction on the distribution G of the innovation ε . That the conditional distribution $F_2(\cdot|\cdot)$ satisfies MLRP guarantees that, under any optimal contract, period-2 effort is non-increasing in period-2 productivity θ_2 , for almost all θ_1 . That the inverse hazard rate of the period-1 distribution (or, precisely, $\sup\{[1 - F_1(\theta_1)]/f_1(\theta_1)\}$) is small enough in turn guarantees that the policies ξ^R that solve the relaxed program are sufficiently large that, even if they are not implementable, under the optimal policies $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}|\theta_1} [\xi_2^*(\tilde{\theta})]$ continues to be bounded away from zero from below with probability one.

Risk aversion. To understand how risk aversion affects the above conclusions, it is useful to start with the following family of utility functions. Let $(v_\rho)_{\rho \geq 0}$ be a collection of functions $v_\rho : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties: (i) for each $\rho > 0$, v_ρ is surjective, continuously differentiable, increasing, and strictly concave, with $v_\rho(0) = 0$ and $v'_\rho(0) = 1$; (ii) v_0 is the identity function; (iii) v'_ρ converges to one, uniformly over c as $\rho \rightarrow 0$. Hence, $(v_\rho)_{\rho \geq 0}$ captures a family of utility functions such that ρ indexes the level of the manager’s risk aversion and where the manager’s preferences over

compensation converge to the risk-neutral ones as $\rho \rightarrow 0$. Our key finding, however, is Proposition 6 below, which applies to arbitrary utility functions.

Proposition 5 *Suppose there exist $a, b \in \mathbb{R}_{++}$ such that, for almost all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2(\theta_1)$, $a < f_1(\theta_1)$, $f_2(\theta_2|\theta_1) < b$. Fix the level of persistence $\gamma < 1$ of the manager's productivity, and assume that the manager's preferences for consumption in each period are represented by the function $v_\rho(\cdot)$, with the function family $(v_\rho)_{\rho \geq 0}$ satisfying the properties described above. Suppose that, when the manager is risk neutral, then, on average, period-1 effort is distorted downwards relative to the first-best level (recall that Part (c) of Proposition 4 provides restrictions on the primitives that guarantee that this is the case). Then there exists $\bar{\rho} > 0$ such that, for any $\rho \in [0, \bar{\rho}]$, the expected power of incentives under any optimal contract is higher in period two than in period one.*

The result in the proposition thus establishes continuity of the dynamics of the average power of incentives in the degree of risk aversion, around the risk-neutral level. The role of the conditions in the proposition (the uniform bounds on the densities and the assumption of uniform convergence of the derivatives of the v_ρ functions to the derivative in the risk neutral case) is to guarantee that, if the dynamics of the expected power of incentives for small degrees of risk aversion were the opposite of those in the risk neutral case, then one could construct implementable policies that would improve upon the optimal ones, either for $\rho = 0$, or for ρ small enough. Note that the assumptions in the Theorem of Maximum are violated in our setting (in particular, the set of implementable policies need not be compact and continuous in ρ), which explains the need for the additional conditions in the proposition (as well as the length of the proof in the Appendix).

Importantly, also note that while the result in Proposition 5 focuses on the power of incentives, the same properties apply to expected effort levels. That is, under the conditions in the proposition, expected effort is higher at date 2 than at date 1, when ρ is sufficiently small. Intuitively, this is because effort and the power of incentives are identical when the manager is risk neutral, and close to each other, when risk aversion is small.³²

The levels of risk aversion for which the result in Proposition 5 holds (i.e., how large one can take $\bar{\rho}$) should be expected to depend on the persistence of initial productivity γ . For a fixed level of risk aversion, if γ is close to 1, i.e., if the initial productivity is highly persistent, then the above result about the dynamics of the power of incentives is *completely reversed*: the power of incentives declines, on average, over time, as stated in the next Proposition.

Proposition 6 *Fix the productivity distributions F_1 and G and assume that the felicity function v is strictly concave. (a) Suppose that the process governing the evolution of productivity is a random*

³²The result in the proposition extends to the family of iso-elastic felicity functions $v_\rho(c) = \frac{c^{1-\rho}}{1-\rho}$ for $\rho \geq 0$ often considered in the literature, as long as effort under the optimal policies is bounded. In this case, the restrictions on the densities can be dispensed with.

walk (i.e., $\gamma = 1$). Then under any optimal contract, for almost all θ_1 , the expected period-2 power of incentives given θ_1 , $\mathbb{E}^{\tilde{\theta}|\theta_1} \left[\psi' \left(\xi_2^*(\tilde{\theta}) \right) w' \left(v \left(c_2^*(\tilde{\theta}) \right) \right) \right]$, is weakly lower than the period-1 power of incentives $\psi' \left(\xi_1^*(\theta_1) \right) w' \left(v \left(c_1^*(\theta_1) \right) \right)$ (strictly lower, provided that $c_2^*(\theta_1, \cdot)$ varies with θ_2 over a subset of Θ_2 of positive probability under $F_2(\cdot|\theta_1)$)³³. (b) Suppose there exists $b \in \mathbb{R}_{++}$ such that, for all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2(\theta_1)$, $f_2(\theta_2|\theta_1) < b$. Suppose also that there exists $M \in \mathbb{R}_{++}$ and $\gamma' < 1$ such that, for all $\gamma \in [\gamma', 1]$, the optimal effort policy ξ^* is uniformly bounded (in absolute value) by M . Finally, suppose that, for $\gamma = 1$, the ex-ante expected power of incentives at date 2 is strictly lower than at date 1, i.e. $\mathbb{E} \left[\psi' \left(\xi_2^*(\tilde{\theta}) \right) w' \left(v \left(c_2^*(\tilde{\theta}) \right) \right) \right] < \mathbb{E} \left[\psi' \left(\xi_1^*(\tilde{\theta}_1) \right) w' \left(v \left(c_1^*(\tilde{\theta}_1) \right) \right) \right]$.³⁴ Then there exists $\bar{\gamma} \in [\gamma', 1)$ such that, for all $\gamma \in [\bar{\gamma}, 1]$, $\mathbb{E} \left[\psi' \left(\xi_2^*(\tilde{\theta}) \right) w' \left(v \left(c_2^*(\tilde{\theta}) \right) \right) \right] < \mathbb{E} \left[\psi' \left(\xi_1^*(\tilde{\theta}_1) \right) w' \left(v \left(c_1^*(\tilde{\theta}_1) \right) \right) \right]$.

Consider Part (a) of the proposition, which supposes $\gamma = 1$. To see why the result is true, note that, when the manager is risk averse, incentivizing high effort in period two is more costly for the firm. This is because high effort requires high sensitivity of pay to performance. This in turn exposes the manager to volatile compensation as a result of his own private uncertainty about period-2 productivity. Since the manager dislikes this volatility, he must be provided additional compensation by the firm. As a result, the firm lowers the effort asked in period two to save on managerial compensation.

To see Part (a) more formally, let us ease the discussion by supposing that the effort asked by the firm in each period is strictly positive (note that the result in the proposition equally applies to the case where the effort asked to certain types is negative). Recall that, in this case, the Euler conditions (10) and (11) in Proposition 2 must hold as equalities. It is then immediate that the first two terms in the right-hand sides of these equations are identical. The key difference across the two periods then comes from the third term in the right-hand side of (11) which is always negative and captures the effect of the volatility in the period-2 compensation on the surplus that the firm must give to the manager to induce him to participate. This volatility originates in the need to make period-2 compensation sensitive to period-2 performance to incentivize period-2 effort. Such volatility can be reduced by lowering the power of incentives in the second period. Under any optimal contract, the firm thus reduces the power of incentives over time to reduce the manager's exposure to compensation risk.

One further way to understand why the expected power of incentives declines over time when the manager is risk averse and productivity is sufficiently persistent is as follows. Suppose that period-2 effort is restricted to depend only on period-1 productivity (that is, suppose both ξ_1 and ξ_2 depend

³³We expect that this condition holds in all but “knife-edge” cases. A sufficient condition, for instance, is that the hazard rate $\frac{f_1(\theta_1)}{1-F_1(\theta_1)}$ is increasing and that the manager's degree of risk aversion is not too large.

³⁴Note, given Part (a), that this follows if there exists a positive measure set of types θ_1 such that $c_2^*(\theta_1, \cdot)$ varies with θ_2 over a subset of Θ_2 of positive probability under $F_2(\cdot|\theta_1)$.

only on θ_1). The manager's period-2 compensation can then be written as

$$w \left(\begin{array}{l} \psi(\xi_1(\theta_1)) + \psi(\xi_2(\theta_1)) + \int_{\theta_1}^{\theta_1} \{\psi'(\xi_1(s)) + \gamma\psi'(\xi_2(s))\} ds \\ + \left(\theta_2 - \mathbb{E}^{\tilde{\theta}_2|\theta_1}[\tilde{\theta}_2]\right) \psi'(\xi_2(\theta_1)) - v(c_1(\theta_1)) \end{array} \right).$$

It is then easy to see that the volatility of the period-2 compensation is increasing in the period-2 effort $\xi_2(\theta_1)$. When the manager is risk averse, w is strictly convex. By reducing ξ_2 , the firm then reduces the expected period-2 compensation, for any level of the period-1 productivity.³⁵ When $\gamma = 1$, the firm finds it optimal to reduce the power of incentives over time.

Now, consider Part (b) of the proposition. One should expect that whether the power of incentives declines (on average) over time should depend on the persistence parameter γ . The result suggests that the expected power of incentives also declines over time when the persistence parameter γ is sufficiently close to 1. As noted in the Introduction, we obtain this result after assuming that the optimal effort policies in these cases are uniformly (almost surely) bounded. While we believe only mild conditions (such as boundedness of the inverse hazard rate $\frac{1-F_1(\theta_1)}{f_1(\theta_1)}$) are needed to guarantee the existence of a uniform bound, we were unable to find an argument to guarantee it.

4.4 Further discussion of optimal policies

Conditions (10) and (11) were obtained by maximizing the firm's profits over all *implementable* policies. As noted above, an alternative (and more canonical) approach involves maximizing the firm's profits subject only to certain "local incentive constraints". In our environment, this amounts to maximizing (8) over all possible effort and compensation policies, thus ignoring the possibility that policies that maximize (8) need not be implementable by a contract which is individually rational and incentive compatible for the manager. This second approach is called the "relaxed approach". Whether this relaxed approach yields policies that can indeed be implemented under an incentive-compatible contract is something that is verified ex-post, once the solution to the maximization of (8) is in hand. One advantage of this approach is that (when validated) it provides a precise characterization of the optimal policies. In our environment, this means that one can derive conditions analogous to (10) and (11) which hold ex-post, i.e. for each possible productivity history, as opposed to in expectation.

³⁵If we restrict attention to effort policies that depend only on period-1 productivity, then the result in Proposition 6 applies not only to the dynamics of the power of incentives but also to the dynamics of expected effort: i.e., expected effort declines over time under the assumptions of the proposition. When we do not impose this restriction, however, we have been unable to disentangle the effect of risk aversion on expected effort from its effect on the expected power of incentives. This appears difficult because of the need to control for the correlation between second-period compensation and second-period effort, conditional on the period-1 productivity.

Proposition 7 *Suppose that the policies $(\xi_1^R, \xi_2^R, c_1^R)$ maximize (8) and let c_2^R be the period-2 compensation given by (7) for $K = 0$ (note that the effort and compensation policies $\langle \xi^R, c^R \rangle = \langle (\xi_1^R, \xi_2^R), (c_1^R, c_2^R) \rangle$ need not be implementable). Then, with probability one, the policies $\langle \xi^R, c^R \rangle$ must satisfy Condition (12) as well as the following conditions:*

$$\psi'(\xi_1^R(\theta_1)) w'(v(c_1^R(\theta_1))) = 1 - \frac{\psi''(\xi_1^R(\theta_1))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} w'(v(c_1^R(r))) f_1(r) dr, \quad (20)$$

and

$$\begin{aligned} \psi'(\xi_2^R(\theta)) w'(v(c_2^R(\theta))) &= 1 - \gamma \frac{\psi''(\xi_2^R(\theta))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} w'_1(v(c_1^R(r))) f_1(r) dr \\ &\quad - \frac{\psi''(\xi_2^R(\theta))}{f_2(\theta_2|\theta_1)} \int_{\theta_2}^{\bar{\theta}_2} \{w'(v(c_2^R(\theta_1, r))) - w'(v(c_1^R(\theta_1)))\} f_2(r|\theta_1) dr. \end{aligned} \quad (21)$$

The effort policy ξ^R is essentially unique. If v is strictly concave, then the compensation policy c^R is also essentially unique.

Observe that, when the manager is risk neutral, given that the disutility of effort is quadratic, the policy ξ^R is given by conditions (18) and (19) above. Recall from Proposition 4 that these policies also solve the full program (and hence are sustained under optimal contracts) when the hazard rate of the period-1 distribution $f_1(\theta_1)/(1 - F(\theta_1))$ is non-decreasing and strictly above $(1 + \gamma^2)/(1 + \gamma)$. An implication is that managers whose initial productivity is high are offered higher powered incentives than those managers whose initial productivity is low. The reason for this finding relates once again to the effect of effort on managerial rents. When the hazard rate of the period-1 distribution is increasing, the weight the firm assigns to rent extraction relative to efficiency (as captured by the inverse hazard rate $[1 - F_1(\theta_1)]/f_1(\theta_1)$) is smaller for higher types (recall that asking type θ_1 to exert more effort requires increasing the rent of all types $\theta'_1 > \theta_1$). As a result, the firm offers higher powered incentives to those managers whose initial productivity is high. When it comes to the dynamics of the power of incentives, we then have the following comparison across types.

Corollary 1 *Suppose that the manager is risk neutral and that the hazard rate of the period-1 distribution is (weakly) increasing and strictly above $(1 + \gamma^2)/(1 + \gamma)$. Then the increase in the power of incentives over time is larger for those managers whose initial productivity is low.*

The result reflects the fact that period-1 effort is more distorted for those managers whose initial productivity is low, implying that, over time, the correction is larger for those types. The result in the previous corollary thus yields another testable prediction: because productivity is positively correlated with performance, the econometrician should expect to find a negative relationship between early performance and the increase in the power of incentives over time. Note that this prediction

is not shared by the alternative theories (mentioned in the Introduction) which explain increases in the power of incentives over time.

Next, consider the case of a risk-averse manager. In this case, verifying that the policies $\langle \xi^R, c^R \rangle$ that solve the relaxed program are implementable is more difficult. We do so for numerical examples on a case-by-case basis. To illustrate, consider a manager with CRRA preferences with risk aversion parameter equal to one half (meaning that $v(c) = 2\sqrt{c}$). Further assume that θ_1 is uniformly distributed over $[0, 1/2]$ and ε is uniformly distributed over $[0, 1]$. Figures 1 and 2 show, for different levels of productivity persistence ($\gamma = 1$ and $\gamma = 1/2$), how the power of incentives in period 1 and the expected power of incentives in period 2 vary with the initial productivity θ_1 .

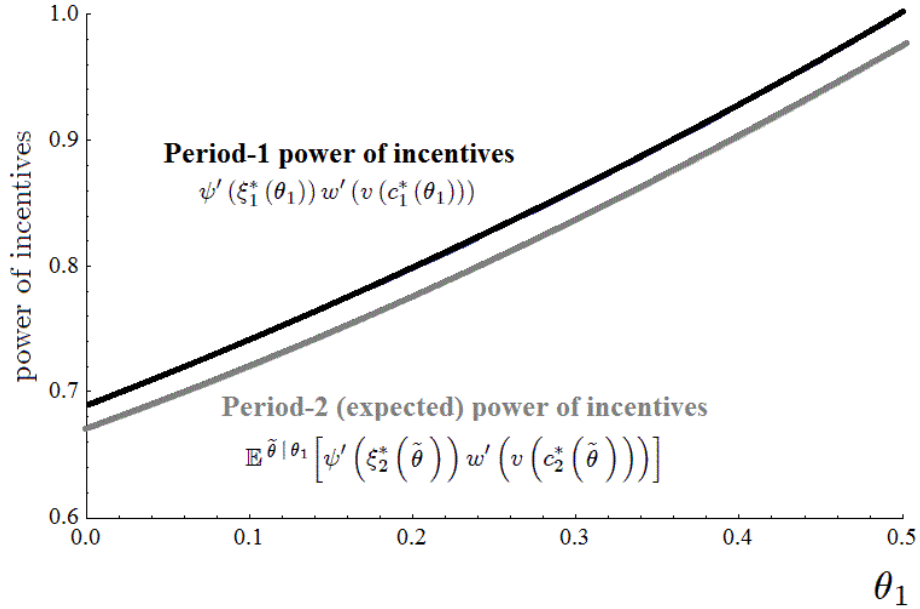


Figure 1: Dynamics of power of incentives for $\gamma = 1$.

When productivity is fully persistent (i.e., for $\gamma = 1$, as in Figure 1), for any θ_1 , the power of incentives in period 1 is higher than the expected power of incentives in period 2, thus illustrating the finding in Part (a) of Proposition 6. For smaller values of γ (e.g., for $\gamma = 1/2$, as in Figure 2), whether the power of incentives increases or decreases over time depends on the initial productivity. For high initial productivities, the power of incentives declines over time, whereas the opposite is true for lower productivity levels. These findings reflect the trade-off between reducing the manager's exposure to risk, which calls for reducing both the power of incentives and effort at later periods, and reducing the manager's expected rents, which calls for low-powered incentives early on followed by higher-powered incentives later in the relationship. The effect of the power of incentives on expected rents is similar across the two periods when either (i) productivity is fully persistent ($\gamma = 1$), or (ii) the initial productivity is high, in which case the effect of effort on rents is negligible. In these cases,

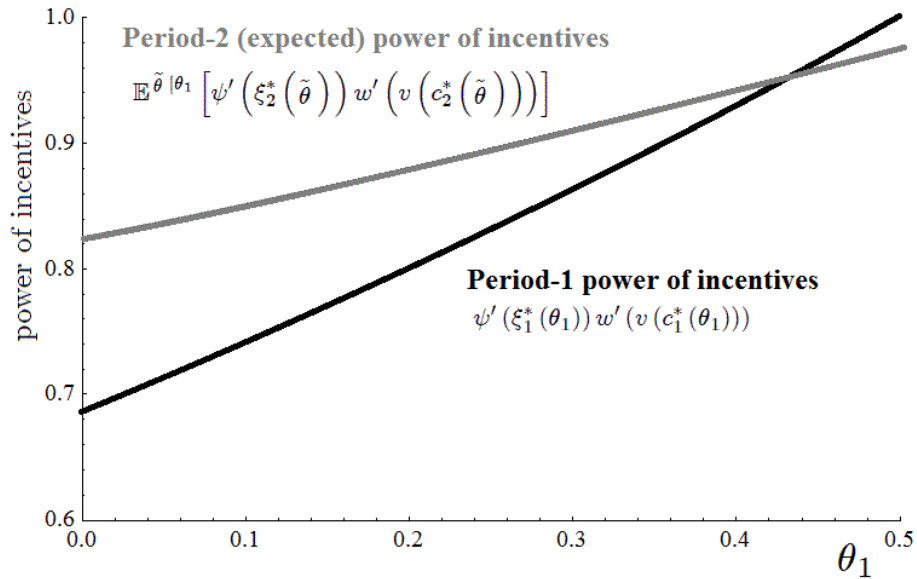


Figure 2: Dynamics of power of incentives for $\gamma = 1/2$.

the firm optimally reduces the power of incentives over time so as to reduce the risk the manager faces when it comes to his future compensation.

Next consider the period-1 effort level and the expected period-2 effort level conditional on the period-1 productivity, as in Figures 3 and 4. Whether effort increases or decreases over time, conditional on the initial productivity, follows a similar pattern as for the power of incentives. However, there can be qualitative differences between the power of incentives and the effort policies. For instance, when $\gamma = 1$, both period-1 effort and expected period-2 effort decline with θ_1 (see Figure 3), whereas the expected power of incentives in each period increases with θ_1 (see Figure 1). To see why this is the case note that high period-1 types must receive a higher compensation than lower period-1 types, even if they work less. Such a higher compensation is necessary to discourage them from mimicking the less productive types. As a result, higher period-1 types are more costly to incentivize. At the optimum, the firm asks a lower effort of such types but, because of the higher compensation, the power of incentives that such types receive remains higher than that of lower period-1 types.

When productivity is less persistent ($\gamma = 1/2$, as in Figure 4), at the optimum, expected period-2 effort declines with the manager's initial productivity. However, period-1 effort increases in the initial productivity. Thus, period-1 effort and period-1 power of incentives share the same monotonicity, whereas expected period-2 effort and expected period-2 power of incentives move in opposite directions. The reason for the latter dynamics is the same as the one discussed above for $\gamma = 1$. A manager with a high initial productivity expects to be paid a lot and is thus more difficult to incentivize, explaining why expected second-period effort declines in this initial productivity. On the

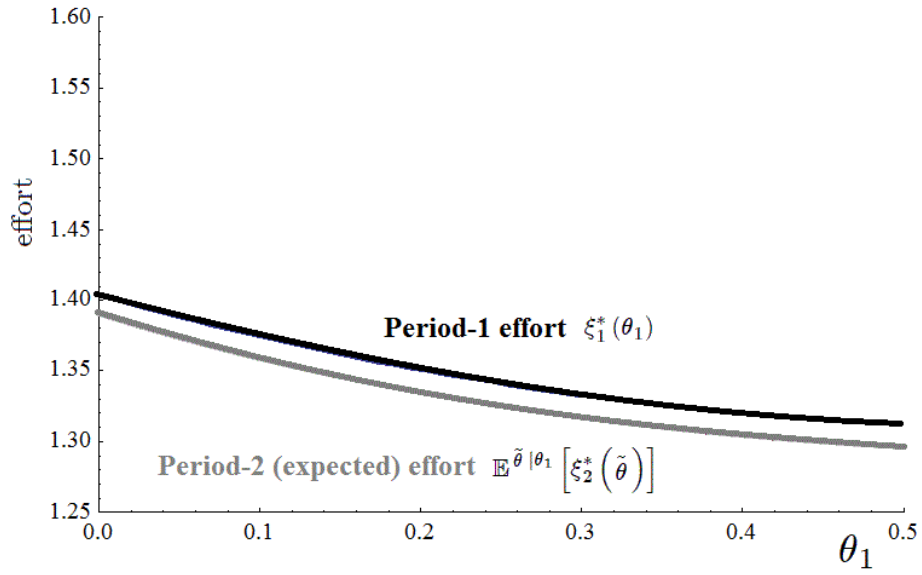


Figure 3: Dynamics of effort for $\gamma = 1$.

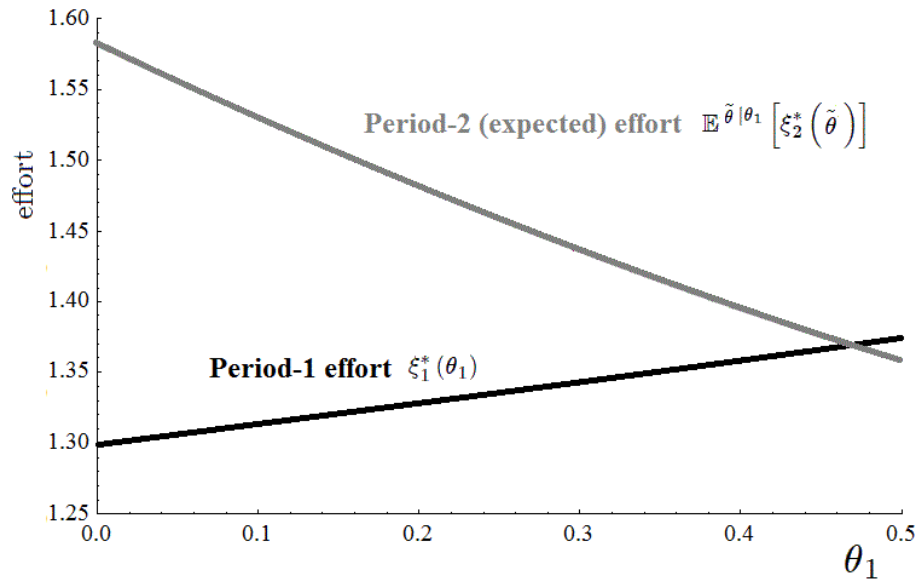


Figure 4: Dynamics of effort for $\gamma = 1/2$.

other hand, precisely because the manager is more difficult to incentivize, higher powered incentives are required to elicit even this diminished level of effort.

5 Conclusions

We investigate the optimal dynamics of incentives for a manager whose ability to generate profits for the firm changes stochastically over time. In doing so, we appeal to a definition of the “power of incentives” which seems appropriate for settings in which payment schemes need not be linear (or even differentiable) in the cash flows.

When the manager is risk neutral, we show that it is typically optimal for the firm to offer a compensation scheme in which the power of incentives increases, on average, over time, thus inducing the manager to exert more effort as his tenure in the firm grows. We then show how the above dynamics can be reversed under risk aversion. In future work, it would be interesting to calibrate the model so as to quantify the relevance of the effects identified in the paper and derive specific predictions about the combination of stocks, options, and fixed pay that implement the optimal dynamics of incentives.

We conclude with a few remarks about the applicability of the approach developed in the present paper (which involves tackling the full program directly) to richer specifications of the contracting problem. First, Euler inequalities like (10) and (11) in Proposition 2 can be obtained for settings with arbitrarily many periods and richer stochastic processes; these inequalities hold as equalities provided that optimal effort is not too small. When the manager is risk neutral, these equalities provide closed-form expressions for expected effort in each period (analogous to Equations (16) and (17) in the paper). Interestingly, these expressions can be obtained without any of the conditions typically imposed in the dynamic mechanism design literature (e.g., log-concavity of the period-1 distribution, monotonicity of the impulse responses of future types to the initial ones). This is because the predictions identified by this approach apply to the “average” dynamics, where the average is over all possible realizations of the type process, as opposed to ex-post. Equations relating the power of incentives across periods, like the one in Proposition 3, can also be obtained for arbitrarily many periods. While no restriction on the shape of the effort policy is needed to establish such condition, the assumption that the process is autoregressive plays a role in the derivation of this condition and is more difficult to relax. This is because such condition is obtained by combining perturbations to the effort policy in one period with perturbations to the effort policy in other periods that preserve incentives, while also leaving the manager’s expected payoff unchanged. Identifying such multi-period perturbations for more general processes appears difficult.

Note also that, while we find the restriction to two periods helpful for drawing conclusions from the aforementioned Euler conditions, we expect our predictions for the dynamics of effort and the

power of incentives to extend to longer horizons. In particular, when the manager is risk neutral, and when the productivity process is imperfectly persistent (e.g., for a persistence parameter less than 1 in the autoregressive setting), we anticipate the expected power of incentives (and effort) to grow over time under any optimal mechanism. Conversely, when the process is highly persistent (say close to a random walk), and when the manager is risk averse, then we anticipate the expected power of incentives to decline over time.³⁶ In this setting, the principal seeks to shield the manager from productivity risk later in the relationship when, from the perspective of the time of contracting, he faces the greatest uncertainty about his productivity. Shielding the manager from risk requires offering lower-powered incentives.

While our approach can be extended to longer relationships and richer stochastic processes, the assumption that the disutility of effort is quadratic is more difficult to relax. This assumption plays no role in the traditional approach (consisting in solving a “relaxed program” and then validating its solution). However, when tackling directly the “full program,” this assumption permits us to identify a simple class of perturbations that preserve incentive compatibility which can be used to arrive at the Euler equations in Propositions 2 and 3. In this respect, this assumption plays in our environment a role similar to that of the linearity of payoffs in Rochet and Choné’s (1998) analysis of multidimensional screening. There are two difficulties with more general effort disutility functions. The first one is in identifying appropriate perturbations of the effort policies that preserve incentive compatibility. Simply translating the marginal disutility of effort by a constant does not guarantee that the new policies preserve integral monotonicity (see footnote 27). The second difficulty is in evaluating the marginal effects of such perturbations on the principal’s payoff. With more general effort disutility functions, the analogs of the Euler-type conditions that we used in the present paper appear less amenable to tractable analysis.

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³⁶A previous version of the paper considered cases with more than two periods, with the analysis tending to confirm these predictions.

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Appendix

Proof of Proposition 1. Given the effort and compensation policies $\langle \xi, c \rangle$, let x be the compensation scheme defined, for each t , by

$$x_t(\theta, \pi) = \begin{cases} c_t(\theta) & \text{if } \pi_t = \pi_t(\theta) \\ -L_t(\theta) & \text{otherwise} \end{cases} \quad (22)$$

with $L_t(\theta) > 0$. It is easy to see that if the policies $\langle \xi, c \rangle$ are implementable, then there exists a compensation scheme x as given by (22) such that (i) the contract $\Omega = \langle \xi, x \rangle$ is incentive compatible and individually rational and (ii) the compensation that the manager receives on-path under x is the one prescribed by the policy c . Hereafter, we thus confine attention to contracts in which the compensation scheme of the form given by (22).

Necessity. Recall that, by definition, if $\langle \xi, c \rangle$ are implementable, then there must exist a compensation contract x such that (i) the contract $\Omega = \langle \xi, x \rangle$ is incentive compatible and individually rational and (ii) the compensation that the manager receives on-path under x is the one prescribed by the policy c . In particular, incentive compatibility of $\Omega = \langle \xi, x \rangle$ requires that a manager of period-1 productivity θ_1 prefers to follow a truthful and obedient strategy in each period than lying about his period-1 productivity by reporting $\hat{\theta}_1$, then adjusting his period-1 effort so as to hide the lie (i.e., choosing effort $e_1 = \pi_1(\hat{\theta}_1) - \theta_1$ so as to generate the same cash flows as the type $\hat{\theta}_1$ being mimicked), and then lying again in period two by announcing, for any true period-2 type $\theta_2 = \gamma\theta_1 + \varepsilon$, a report $\hat{\theta}_2 = \gamma\hat{\theta}_1 + (\theta_2 - \gamma\theta_1)$, and finally adjusting his period-2 effort choice so as to hide again the new lie (i.e., choosing effort $e_2 = \pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \theta_2 - \gamma\theta_1) - \theta_2$ so as to generate the same cash flows as those expected from someone whose true type history is $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$, with $\hat{\theta}_2 = \gamma\hat{\theta}_1 + (\theta_2 - \gamma\theta_1)$). Note that, for any $\theta_1, \hat{\theta}_1 \in \Theta_1$, the expected payoff

$$U_1(\theta_1, \hat{\theta}_1) \equiv \mathbb{E} \left[\begin{array}{c} c_1(\hat{\theta}_1) + c_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \varepsilon) \\ -\psi \left(\pi_1(\hat{\theta}_1) - \theta_1 \right) - \psi \left(\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \varepsilon) - \gamma\theta_1 - \varepsilon \right) \end{array} \right]$$

that the manager obtains from following such a strategy corresponds to the one that the manager obtains by lying in period 1 and then reporting the true shock ε truthfully in period two (and choosing effort in each period so as to generate the same cash flows as the ones expected from the reported type).

Likewise, let

$$U_2(\theta, \hat{\theta}) \equiv c_1(\hat{\theta}_1) + c_2(\hat{\theta}) - \psi \left(\pi_1(\hat{\theta}_1) - \theta_1 \right) - \psi \left(\pi_2(\hat{\theta}) - \theta_2 \right)$$

denote the ex-post payoff of a manager whose true productivity history is $\theta = (\theta_1, \theta_2)$, who reported $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$ and whose effort choices are made to perfectly hide the lies in each period.

The Lemma below establishes monotonicity properties of the equilibrium-cash flows which in turn will permit us to establish that, for any $(\theta_1, \hat{\theta}_1)$, $U_1^\Omega(\theta_1, \hat{\theta}_1)$ is differentiable and equi-Lipschitz

continuous in θ_1 and that, for any $(\theta, \hat{\theta})$, $U_2^\Omega(\theta, \hat{\theta})$ is differentiable and equi-Lipschitz continuous in θ_2 .

Lemma 2 *Suppose that the policies (ξ, c) are implementable and let $\pi_t(\theta)$ be the equilibrium cash flows under such policies. Then necessarily $\pi_1(\theta_1) + \gamma \mathbb{E}[\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon})]$ is non-decreasing in θ_1 and, for any θ_1 , $\pi_2(\theta_1, \theta_2)$ is non-decreasing in θ_2 .*

Proof. That, for any θ_1 , $\pi_2(\theta_1, \theta_2)$ is non-decreasing in θ_2 follows directly from the fact that the manager's flow payoff $c_t - \psi(\pi_t - \theta_t)$ satisfies the increasing difference property with respect to (π_t, θ_t) . That $\pi_1(\theta_1) + \gamma \mathbb{E}[\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon})]$ must be non-decreasing in θ_1 can be seen by combining any pair of IC constraints

$$U_1(\theta_1, \theta_1) \geq U_1(\theta_1, \hat{\theta}_1) \text{ and } U_1(\hat{\theta}_1, \hat{\theta}_1) \geq U_1(\hat{\theta}_1, \theta_1).$$

From these constraints one obtains that

$$\begin{aligned} & \psi(\pi_1(\theta_1) - \hat{\theta}_1) + \mathbb{E} \left[\psi(\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon}) - \gamma\hat{\theta}_1 - \tilde{\varepsilon}) \right] \\ & - \left\{ \psi(\pi_1(\theta_1) - \theta_1) + \mathbb{E} [\psi(\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon}) - \gamma\theta_1 - \tilde{\varepsilon})] \right\} \\ \geq & \psi(\pi_1(\hat{\theta}_1) - \hat{\theta}_1) + \mathbb{E} \left[\psi(\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) - \gamma\hat{\theta}_1 - \tilde{\varepsilon}) \right] \\ & - \left\{ \psi(\pi_1(\hat{\theta}_1) - \theta_1) + \mathbb{E} [\psi(\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) - \gamma\theta_1 - \tilde{\varepsilon})] \right\}. \end{aligned}$$

From the fundamental theorem of calculus, we can rewrite the above inequality as

$$\begin{aligned} & \int_{\hat{\theta}_1}^{\theta_1} \psi'(\pi_1(\theta_1) - y) + \gamma \mathbb{E} [\psi'(\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon}) - \gamma y - \tilde{\varepsilon})] dy \\ \geq & \int_{\hat{\theta}_1}^{\theta_1} \psi'(\pi_1(\hat{\theta}_1) - y) + \gamma \mathbb{E} [\psi'(\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon}) - \gamma y - \tilde{\varepsilon})] dy. \end{aligned}$$

Using the fact that ψ is quadratic, we can in turn rewrite the above inequality as

$$\left(\theta_1 - \hat{\theta}_1 \right) \left(\pi_1(\theta_1) - \pi_1(\hat{\theta}_1) + \gamma \mathbb{E} [\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon}) - \pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon})] \right) \geq 0,$$

which holds only if $\pi_1(\theta_1) + \gamma \mathbb{E}[\pi_2(\theta_1, \gamma\theta_1 + \tilde{\varepsilon})]$ is non-decreasing in θ_1 . ■

The monotonicities of the cash flows in the Lemma, along with the compactness of Θ_1 and Θ_2 , in turn imply that (a), for any $(\theta, \hat{\theta})$, $U_2(\theta, \hat{\theta})$ is differentiable and Lipschitz continuous in θ_2 with Lipschitz constant

$$M_2(\hat{\theta}_1) = \max_{\theta_2 \in \Theta_2} \{ |\pi_2(\hat{\theta}_1, \theta_2)| \} + \max \{ |\gamma \underline{\theta}_1 + \underline{\varepsilon}|, |\gamma \bar{\theta}_1 + \bar{\varepsilon}| \}$$

uniform across $(\theta_2, \hat{\theta}_2)$ and (b) for any $(\theta_1, \hat{\theta}_1)$, $U_1(\theta_1, \hat{\theta}_1)$ is differentiable and Lipschitz continuous in θ_1 with Lipschitz constant

$$M_1 = \max_{\hat{\theta}_1 \in \Theta_1} \{ |\pi_1(\hat{\theta}_1) + \gamma \mathbb{E} [\pi_2(\hat{\theta}_1, \gamma\hat{\theta}_1 + \tilde{\varepsilon})] | \} + \max \{ |\underline{\theta}_1|, |\bar{\theta}_1| \} + \gamma \max \{ |\gamma \underline{\theta}_1 + \underline{\varepsilon}|, |\gamma \bar{\theta}_1 + \bar{\varepsilon}| \}$$

uniform across $(\theta_1, \hat{\theta}_1)$. Using results from the recent dynamic mechanism design literature, one can then show that the following conditions are necessary for incentive compatibility: (1) for any (θ_1, θ_2) , the manager's ex-post equilibrium payoff satisfies

$$V(\theta_1, \theta_2) = V(\theta_1, \underline{\theta}_2) + \int_{\underline{\theta}_2}^{\theta_2} \psi'(\xi_2(\theta_1, s)) ds; \quad (23)$$

and (2) for each θ_1 , the expectation of the equilibrium payoff satisfies (6), where $V(\theta_1, \theta_2) = U_2((\theta_1, \theta_2), (\theta_1, \theta_2))$ and $V_1(\theta_1) \equiv \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta})] = U_1(\theta_1, \theta_1)$. Note that Condition (23) is analogous to the static condition in Laffont and Tirole (1986). The necessity of (6), instead, follows from adapting to the environment under examination the result in Theorem 1 in Pavan, Segal, and Toikka (2014).

Combining (23) with (6), we then obtain that, under any contract that is individually rational and incentive compatible, the equilibrium utility that each manager derives from his lifetime compensation must satisfy Condition (2) for all $\theta = (\theta_1, \theta_2)$, with $K = \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta})] \geq 0$ satisfying Condition (4). This establishes the necessity of Condition (A) in the proposition. The necessity of Condition (B)(ii) follows directly from Lemma 2 above.

Finally, to see that Condition (B)(i) is also necessary, let $\Omega = \langle \xi, x \rangle$ be any contract implementing the effort and compensation policies $\langle \xi, c \rangle$. Then let $V^\Omega(\theta_1, \hat{\theta}_1)$ be the payoff that, under such a contract, a manager whose period-1 productivity is θ_1 obtains when he reports $\hat{\theta}_1$, then chooses period-1 effort $e_1 = \pi_1(\hat{\theta}_1) - \theta_1$ optimally so as to attain the target $\pi_1(\hat{\theta}_1)$, and then behaves optimally in period 2 (which means following a truthful and obedient strategy³⁷). Then observe that

$$\begin{aligned} V^\Omega(\theta_1, \hat{\theta}_1) &= V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \psi(\xi_1(\hat{\theta}_1)) - \psi(\xi_1(\hat{\theta}_1) + \hat{\theta}_1 - \theta_1) \\ &\quad + \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(\xi_2(\hat{\theta}_1, s)) ds \right] - \mathbb{E}^{\tilde{\theta}_2|\hat{\theta}_1} \left[\int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(\xi_2(\hat{\theta}_1, s)) ds \right] \\ &= V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(\hat{\theta}_1) + \hat{\theta}_1 - s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} [\psi'(\xi_2(\hat{\theta}_1, \tilde{\theta}_2))] \right\} ds. \end{aligned} \quad (24)$$

Because the policies $\langle \xi, c \rangle$ implemented under the contract Ω must satisfy (2), we have that

$$V^\Omega(\theta_1, \theta_1) = V^\Omega(\hat{\theta}_1, \hat{\theta}_1) + \int_{\hat{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \mathbb{E}^{\tilde{\theta}_2|s} [\psi'(\xi_2(s, \tilde{\theta}_2))] \right\} ds. \quad (25)$$

A necessary condition for incentive compatibility is that $V^\Omega(\theta_1, \hat{\theta}_1) \leq V^\Omega(\theta_1, \theta_1)$ for all $\theta_1, \hat{\theta}_1 \in \Theta_1$. Using (24) and (25), the latter condition is equivalent to the integral-monotonicity condition (5).

Sufficiency. Suppose that the policies $\langle \xi, c \rangle$ satisfy all the conditions in the proposition. Consider the scheme x given by (22) with $L_t(\theta) > 0$ for each t . Because, for any t , any $\hat{\theta}$, $\pi_t(\hat{\theta})$ is finite

³⁷Note that the optimality of truthful and obedient behavior at *all* period-2 histories follows from the combination of the fact that the environment is Markov along with the fact that, for any θ_1 , (a) the equilibrium cash flows $\pi_2(\theta_1, \cdot)$ are nondecreasing in θ_2 , and (b) the effort and compensation policies satisfy the envelope condition (23), which is implied by (2). The result then follows directly from Laffont and Tirole (1986).

and because Θ_t is bounded, it is easy to see that there exist finite penalties $L_t(\theta)$ such that, faced with the above scheme, for any history of reports $\hat{\theta}$ and any history of true types θ , the period- t optimal choice of effort is $\pi_t(\hat{\theta}) - \theta_t$, irrespective of past effort choices. It is also easy to see that, under such a scheme, the manager finds it optimal to follow a truthful and obedient strategy in the second period, irrespective of his period-1 true and reported type, and irrespective of the effort exerted in period one (the arguments for this result are similar to those in Laffont and Tirole (1986) and hence omitted).

To establish the result, it then suffices to show that, under the proposed scheme, a manager of period-1 productivity θ_1 prefers to follow a truthful and obedient strategy in both periods than lying by reporting $\hat{\theta}_1 \neq \theta_1$ in period one, then optimally choosing effort $e_1 = \pi_1(\hat{\theta}_1) - \theta_1$ so as to attain the target $\pi_1(\hat{\theta}_1)$, and then following a truthful and obedient strategy in period two. Under the scheme x , the payoff that the manager expects from a truthful and obedient strategy in both periods is given by (25), whereas the payoff that he expects by lying in period one and then following the optimal behavior described above is the one in (24). That $V^\Omega(\theta_1, \hat{\theta}_1) \leq V^\Omega(\theta_1, \theta_1)$ for all $\theta_1, \hat{\theta}_1 \in \Theta_1$ then follows from the fact that the policies $\langle \xi, c \rangle$ satisfy the integral-monotonicity condition (5). Q.E.D.

Proof of Proposition 2. The proof is in two steps. Step 1 identifies a family of perturbations that preserve incentive compatibility and then uses this family to identify necessary conditions for the proposed effort and compensation policies $\langle \xi^*, c^* \rangle$ to be sustained under an optimal contract. Step 2 establishes the uniqueness of $\langle \xi^*, c^* \rangle$.

Step 1 (Euler Equations). We want to establish that Conditions (10), (11), and (12) are necessary optimality conditions for the policies ξ^* and c^* . To see this, consider the perturbed effort policy $\xi = (\xi_1^*(\cdot) + a, \xi_2^*(\cdot) + b)$ for some constants $a, b \in \mathbb{R}_+$. Then consider the perturbed compensation policy c given by $c_1(\theta_1) = c_1^*(\theta_1)$ and $c_2(\theta) = w(W(\theta; \xi) + K - v(c_1^*(\theta_1)))$ all θ , where $K = \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta})]$ is the the lowest period-1 type's expected payoff under the original policies $\langle \xi^*, c^* \rangle$.

It is easy to see that, if the policies $\langle \xi^*, c^* \rangle$ are implementable (which, by virtue of Proposition 1, means that they satisfy the conditions in Proposition 1), then so are the perturbed policies $\langle \xi, c \rangle$.

Now consider the firm's expected profits under the perturbed policies. For the original policies $\langle \xi^*, c^* \rangle$ to be optimal, the expected profits must be maximized at $a = b = 0$. Using (8), we have that the right-hand derivative of the firm's expected profits with respect to a , evaluated at $a = b = 0$ is non-positive only if the policies ξ^* and c^* satisfy Condition (10) (to see this, it suffices to take the right-hand derivative of $\mathbb{E}[U^P]$ with respect to a and then integrate by parts). Likewise, the right-hand derivative of $\mathbb{E}[U^P]$ with respect to b , evaluated at $a = b = 0$, is non-positive only if the policies satisfy (11).

Next observe that, when the policy ξ^* is such that $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1}[\xi_2^*(\tilde{\theta})]$ is (almost surely) bounded away from zero from below, then perturbations like the ones described above but with $a, b \in \mathbb{R}_-$, with $|a|$ and $|b|$ small to guarantee that the resulting policies continue to satisfy $\xi_1(\theta_1) +$

$\gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\xi_2(\tilde{\theta}) \right] \geq 0$ for (almost) all θ_1 , also yield implementable policies (that such perturbations preserve integral monotonicity is obvious; the role of the bound on $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\xi_2^*(\tilde{\theta}) \right]$ is to guarantee that such perturbations leave the participation constraints of all types satisfied). Also note that, in this case, the left-hand derivatives of the firm's expected profits with respect to a and b , evaluated at $a = b = 0$ coincide with their right-hand analogs. Optimality of the policies $\langle \xi^*, c^* \rangle$ then requires that such derivatives vanish at $a = b = 0$, which is the case only if the inequalities in (10) and (11) hold as equalities.

The argument for the necessity of (12) is similar. Fix the effort policy ξ^* and consider a perturbation of the period-1 compensation policy so that the new policy satisfies $v(c_1(\theta_1)) = v(c_1^*(\theta_1)) + a\eta(\theta_1)$ for a scalar a and some measurable function $\eta(\cdot)$. In other words, $c_1(\theta_1) = w(v(c_1^*(\theta_1)) + a\eta(\theta_1))$. Then adjust the period-2 compensation so that $c_2(\theta) = w(W(\theta; \xi^*) + K - v(c_1(\theta_1)))$ all θ . It is easy to see that the pair of policies $\langle \xi^*, c \rangle$ continues to be implementable. The firm's expected profits under the perturbed policies are

$$\mathbb{E}[U^P] = \mathbb{E} \left[\begin{array}{l} \tilde{\theta}_1 + \xi_1^*(\tilde{\theta}_1) + \tilde{\theta}_2 + \xi_2^*(\tilde{\theta}) \\ -w(v(c_1^*(\tilde{\theta}_1)) + a\eta(\tilde{\theta}_1)) - w(W(\tilde{\theta}; \xi^*) - v(c_1^*(\tilde{\theta}_1)) - a\eta(\theta_1)) \end{array} \right].$$

Optimality of c^* then requires that the derivative of this expression with respect to a vanishes at $a = 0$ for all measurable functions η . This is the case only if Condition (12) holds.

Step 2 (Uniqueness of the optimal policies). We first show that the optimal effort policy is essentially unique (i.e., unique up to a zero-measure set of productivity histories). Suppose, towards a contradiction, that there exist two pairs of optimal (implementable) policies, $\langle \xi^\#, c^\# \rangle$ and $\langle \xi^{\#\#}, c^{\#\#} \rangle$ respectively, and that $\xi^\#$ and $\xi^{\#\#}$ prescribe different effort levels over a set of productivity histories of strictly positive probability measure. Pick $\alpha \in (0, 1)$ and let $\xi^\alpha \equiv \alpha\xi^\# + (1 - \alpha)\xi^{\#\#}$ be the policy defined by $\xi_t^\alpha(\theta) = \alpha\xi_t^\#(\theta) + (1 - \alpha)\xi_t^{\#\#}(\theta)$ for all θ and $t = 1, 2$. Then let c_1^α be the policy defined, for all θ , by $c_1^\alpha(\theta) \equiv w\left(\alpha v(c_1^\#(\theta)) + (1 - \alpha)v(c_1^{\#\#}(\theta))\right)$. Finally, let c_2^α be the policy defined, for all θ , by $c_2^\alpha(\theta) \equiv w\left(W(\theta; \xi^\alpha) + \alpha K^\# + (1 - \alpha)K^{\#\#} - v(c_1^\alpha(\theta_1))\right)$, where $K^\#$ and $K^{\#\#}$ denote type θ_1 's expected payoff under the policies $\langle \xi^\#, c^\# \rangle$ and $\langle \xi^{\#\#}, c^{\#\#} \rangle$, respectively. Note that the new policies $\langle \xi^\alpha, c^\alpha \rangle$ are implementable (to see this, note that they satisfy the conditions of Proposition 1).

Next, note that (8) is strictly concave in the effort policy ξ (recognizing that the policy ξ enters (8) also through $W(\theta; \xi)$, as defined in (3)) and weakly concave in K and $v(c_1)$.³⁸ This means that the firm's expected profits $\mathbb{E}[U^P]$ under the new policies $\langle \xi^\alpha, c^\alpha \rangle$ are strictly higher than under either $\langle \xi^\#, c^\# \rangle$ or $\langle \xi^{\#\#}, c^{\#\#} \rangle$, contradicting the optimality of these policies.

³⁸By *strict* concavity we mean with respect to the equivalence classes of functions which are equivalent if they are equal almost surely.

Now consider the uniqueness of the compensation policy. Suppose that v is strictly concave and let $\langle \xi^\#, c^\# \rangle$ and $\langle \xi^{\#\#}, c^{\#\#} \rangle$ be two pairs of implementable policies such that $c_1^\#(\theta_1) \neq c_1^{\#\#}(\theta_1)$ over a set of positive probability measure. Then consider the policies $\langle \xi^\alpha, c^\alpha \rangle$ constructed above. Note that such policies yield strictly higher profits than both $\langle \xi^\#, c^\# \rangle$ and $\langle \xi^{\#\#}, c^{\#\#} \rangle$, irrespective of whether or not $\xi^\# \neq \xi^{\#\#}$. This in turn implies that, when v is strictly concave, the optimal compensation policy is also (essentially) unique. Q.E.D.

Proof of Proposition 3. We establish the result by considering perturbations of the effort policy given by

$$\xi_1^\#(\theta_1) = \xi_1^*(\theta_1) + aq(\theta_1) \quad \text{and} \quad \xi_2^\#(\theta) = \xi_2^*(\theta) - \frac{a}{\gamma}q(\theta_1)$$

for some measurable function $q(\theta_1)$. Note that such perturbations leave period-1 expected payoffs unchanged and are implementable. Optimality of the policies $\langle \xi^*, c^* \rangle$ then requires that the derivative of the firm's expected payoff with respect to a , evaluated at $a = 0$ must vanish, for all possible $q(\cdot)$. This leads to the following new Euler equation, for each θ_1 :

$$0 = 1 - \frac{\psi''(\xi_1^*(\theta_1))}{f_1(\theta_1)} \int_{\theta_1}^{\bar{\theta}_1} w'(v(c_1^*(r))) f_1(r) dr - \psi'(\xi_1^*(\theta_1)) w'(v(c_1^*(\theta_1))) \\ - \frac{1}{\gamma} \left(\begin{aligned} & 1 - \gamma \mathbb{E}^{\bar{\theta}|\theta_1} \left[\frac{\psi''(\xi_2^*(\bar{\theta}))}{f_1(\bar{\theta}_1)} \int_{\bar{\theta}_1}^{\bar{\theta}_1} w'(v(c_1^*(r))) f_1(r) dr \right] - \mathbb{E}^{\bar{\theta}|\theta_1} \left[\psi'(\xi_2^*(\bar{\theta})) w'(v(c_2^*(\bar{\theta}))) \right] \\ & - \mathbb{E}^{\bar{\theta}|\theta_1} \left[\frac{\psi''(\xi_2^*(\bar{\theta}))}{f_2(\bar{\theta}_2|\theta_1)} \int_{\bar{\theta}_2}^{\bar{\theta}_2} \left\{ w'(v(c_2^*(\bar{\theta}_1, r))) - w'(v(c_1^*(\bar{\theta}_1))) \right\} f_2(r|\bar{\theta}_1) dr \right] \end{aligned} \right).$$

which is equivalent to (13) in the proposition. Q.E.D.

Proof of Proposition 4. The proof for Parts (a) and (b) follows from the arguments in the main text. Thus consider Part (c)(i). In this case, the optimal effort policies are those that solve the relaxed program, as given in (18) and (19); that is, $\xi_1^*(\theta_1) = 1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)}$ and $\xi_2^*(\theta) = 1 - \gamma \frac{1-F_1(\theta_1)}{f_1(\theta_1)}$. Hence,

$$\begin{aligned} \xi_1^*(\theta_1) + \gamma \mathbb{E}^{\bar{\theta}|\theta_1} \left[\xi_2^*(\bar{\theta}) \right] &= 1 - \frac{1-F_1(\theta_1)}{f_1(\theta_1)} + \gamma \left(1 - \gamma \frac{1-F_1(\theta_1)}{f_1(\theta_1)} \right) \\ &\geq 1 - \frac{1-F_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)} + \gamma \left(1 - \gamma \frac{1-F_1(\underline{\theta}_1)}{f_1(\underline{\theta}_1)} \right) > 0 \end{aligned}$$

where the first inequality follows from the assumption that $[1 - F_1(\theta_1)]/f_1(\theta_1)$ is non-increasing, and where the second inequality from the assumption that $[1 - F_1(\theta_1)]/f_1(\theta_1) < (1 + \gamma)/(1 + \gamma^2)$, for all θ_1 .

Now consider Part (c)(ii). Suppose that $\sup \{ [1 - F_1(\theta_1)]/f_1(\theta_1) \} < (1 + \gamma)/(1 + \gamma^2) - (\bar{\theta}_1 - \underline{\theta}_1)$ and $F_2(\cdot|\cdot)$ satisfies the monotone-likelihood-ratio property (that is, $f_2(\theta'_2|\theta_1)/f_2(\theta_2|\theta_1)$ is non-decreasing in θ_1 , for all $\theta'_2 \geq \theta_2$). We want to show that $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\bar{\theta}_2|\theta_1} \left[\xi_2^*(\bar{\theta}) \right]$ is bounded away from zero from below with probability one. We proceed in two steps. Step 1 establishes four lemmas

that jointly imply that it is without loss of optimality to restrict attention to effort policies such that, for all θ_1 , $\xi_2(\theta_1, \cdot)$ is non-increasing in θ_2 . Step 2 then use this property to establish that, under the conditions in part (c)(ii) in the proposition, if $\langle \xi^*, c^* \rangle$ is such that $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2^*(\tilde{\theta})]$ fails to be bounded away from zero from below with probability one, then there exists another pair of policies $\langle \hat{\xi}, \hat{c} \rangle$ that is also implementable and yields strictly higher profits, thus contradicting the optimality of $\langle \xi^*, c^* \rangle$.

Step 1.

Lemma 3 Consider any function $h : \Theta_2 \rightarrow \mathbb{R}$ such that $h(\theta_2) + \theta_2$ is non-decreasing on Θ_2 . Suppose that h fails to be non-increasing; in particular, there exist $\theta'_2, \theta''_2 \in \Theta_2$, $\theta'_2 < \theta''_2$ such that $h(\theta'_2) < h(\theta''_2)$. Take any $\bar{h} \in (h(\theta'_2), h(\theta''_2))$. There exists $\theta_2^\#, \theta_2^{\#\#} \in \Theta_2$, with $\theta_2^\# \leq \theta_2^{\#\#}$, and $\delta^\#, \delta^{\#\#} > 0$ such that (i) for all $\theta_2 \in (\theta_2^\# - \delta^\#, \theta_2^\#)$, $h(\theta_2) < \bar{h}$, and for all $\theta_2 \in (\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{\#\#})$, $h(\theta_2) > \bar{h}$; and (ii) $\lim_{\theta_2 \downarrow \theta_2^\#} h(\theta_2) \geq \bar{h}$ and $\lim_{\theta_2 \uparrow \theta_2^{\#\#}} h(\theta_2) \leq \bar{h}$.

Proof. It suffices to take

$$\theta_2^\# = \sup \left\{ \theta_2 : h(\tilde{\theta}_2) < \bar{h} \forall \tilde{\theta}_2 \in (\theta'_2, \theta_2) \right\} \text{ and } \theta_2^{\#\#} = \inf \left\{ \theta_2 : h(\tilde{\theta}_2) > \bar{h} \forall \tilde{\theta}_2 \in (\theta_2, \theta''_2) \right\},$$

and then let $\delta^\# = \theta_2^\# - \theta'_2 + \frac{\bar{h} - h(\theta'_2)}{2}$ and $\delta^{\#\#} = \theta''_2 - \theta_2^{\#\#} + \frac{h(\theta''_2) - \bar{h}}{2}$. That $\lim_{\theta_2 \downarrow \theta_2^\#} h(\theta_2)$ exists follows from the fact that $\lim_{\theta_2 \downarrow \theta_2^\#} (h(\theta_2) + \theta_2)$ exists, which in turn follows from the fact that $h(\theta_2) + \theta_2$ is non-decreasing. That $\lim_{\theta_2 \downarrow \theta_2^\#} h(\theta_2) \geq \bar{h}$ follows from the fact that, if this was not true, then there would exist $\hat{\theta}_2 > \theta_2^\#$ such that $h(\tilde{\theta}_2) < \bar{h}$ for all $\tilde{\theta}_2 \in (\theta'_2, \hat{\theta}_2)$, thus contradicting the definition of $\theta_2^\#$. The proof of the fact that $\lim_{\theta_2 \uparrow \theta_2^{\#\#}} h(\theta_2)$ exists and is such that $\lim_{\theta_2 \uparrow \theta_2^{\#\#}} h(\theta_2) \leq \bar{h}$ follows from similar arguments. ■

Lemma 4 Let F_2 be a distribution on Θ_2 . Consider any function $h : \Theta_2 \rightarrow \mathbb{R}$ such that $h(\theta_2) + \theta_2$ is non-decreasing on Θ_2 and suppose that h fails to be non-increasing. Take $\bar{h}, \theta_2^\#, \theta_2^{\#\#}, \delta^\#, \delta^{\#\#}$ as defined in the previous lemma. For arbitrary $\delta^* \in (0, \delta^\#)$ and $\delta^{**} \in (0, \delta^{\#\#})$, define the function $h^*(\theta_2; \delta^*, \delta^{**})$ by $h^*(\theta_2; \delta^*, \delta^{**}) = \bar{h}$ for $\theta_2 \in [\theta_2^\# - \delta^*, \theta_2^\#] \cup [\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**}]$, and $h^*(\theta_2; \delta^*, \delta^{**}) = h(\theta_2)$ otherwise. (i) For any $\delta^* \in (0, \delta^\#)$ and $\delta^{**} \in (0, \delta^{\#\#})$, $h^*(\theta_2; \delta^*, \delta^{**}) + \theta_2$ is non-decreasing over Θ_2 . (ii) There exist $\delta^* \in (0, \delta^\#)$ and $\delta^{**} \in (0, \delta^{\#\#})$ such that $\mathbb{E}^{F_2} \left[h^*(\tilde{\theta}_2; \delta^*, \delta^{**}) \right] = \mathbb{E}^{F_2} \left[h(\tilde{\theta}_2) \right]$ where the expectation is taken under F_2 ; equivalently,

$$\mathbb{E}^{F_2} \left[h(\tilde{\theta}_2) | \tilde{\theta}_2 \in [\theta_2^\# - \delta^*, \theta_2^\#] \cup [\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**}] \right] = \bar{h}.$$

Proof. To prove (i) one need only to verify that $h^*(\theta_2; \delta^*, \delta^{**}) + \theta_2$ is non-decreasing at $\theta_2^\#$ and at $\theta_2^{\#\#}$. By the definition of $\theta_2^\#, \theta_2^{\#\#}$ and of the h^* function, it is easy to see that

$$\lim_{\theta_2 \downarrow \theta_2^\#} (h^*(\theta_2; \delta^*, \delta^{**}) + \theta_2) \geq \theta_2^\# + \bar{h} = \theta_2^\# + h^*(\theta_2^\#; \delta^*, \delta^{**}).$$

and

$$\lim_{\theta_2 \uparrow \theta_2^{\#\#}} (h^*(\theta_2; \delta^*, \delta^{**}) + \theta_2) \leq \bar{h} + \theta_2^{\#\#} = \theta_2^{\#\#} + h^*(\theta_2^{\#\#}; \delta^*, \delta^{**}).$$

The proof for part (ii) follows from the fact that $h^*(\theta_2; \delta^*, \delta^{**}) = h(\theta_2)$ for all $\theta_2 \notin \left[\theta_2^{\#} - \delta^*, \theta_2^{\#} \right] \cup \left[\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**} \right]$, $h^*(\theta_2; \delta^*, \delta^{**}) > h(\theta_2)$ for all $\theta_2 \in \left[\theta_2^{\#} - \delta^*, \theta_2^{\#} \right]$ and $h^*(\theta_2; \delta^*, \delta^{**}) < h(\theta_2)$ for all $\theta_2 \in \left[\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**} \right]$. ■

Now consider any θ_1 for which $\xi_2(\theta_1, \theta_2) + \theta_2$ is non-decreasing in θ_2 , as required by incentive compatibility, but for which $\xi_2(\theta_1, \cdot)$ fails to be non-increasing over Θ_2 . Letting $h(\theta_2) = \xi_2(\theta_1, \theta_2)$, the above two lemmas permit us to establish the following result.

Lemma 5 *Consider any θ_1 for which $\xi_2(\theta_1, \theta_2) + \theta_2$ is non-decreasing in θ_2 , but for which $\xi_2(\theta_1, \cdot)$ fails to be non-increasing over Θ_2 . Suppose that the distribution $F_2(\cdot|\cdot)$ satisfies the MLRP. Then there exists a function $\hat{\xi}_2(\theta_1, \cdot) : \Theta_2 \rightarrow \mathbb{R}$ such that (a) $\mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\hat{\xi}_2(\theta_1, \tilde{\theta}_2) \right] = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\xi_2(\theta_1, \tilde{\theta}_2) \right]$, (b) $\hat{\xi}_2(\theta_1, \theta_2) + \theta_2$ is non-decreasing in θ_2 , (c) for all $s < \theta_1$, $\mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\theta_1, \tilde{\theta}_2) \right] \geq \mathbb{E}^{\tilde{\theta}_2|s} \left[\xi_2(\theta_1, \tilde{\theta}_2) \right]$, while, for all $s > \theta_1$, $\mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\theta_1, \tilde{\theta}_2) \right] \leq \mathbb{E}^{\tilde{\theta}_2|s} \left[\xi_2(\theta_1, \tilde{\theta}_2) \right]$, and (d)*

$$- \int_{\Theta_2} \left(\frac{\hat{\xi}_2(\theta_1, \theta_2)^2}{2} - \frac{\xi_2(\theta_1, \theta_2)^2}{2} \right) dF_2(\theta_2|\theta_1) > 0. \quad (26)$$

Proof. Take any θ_1 for which the properties in the lemma hold. Let $h(\theta_2) = \xi_2(\theta_1, \theta_2)$, and $\hat{\xi}_2(\theta_1, \theta_2) = h^*(\theta_2; \delta^*, \delta^{**})$, where the function h^* (and hence the values \bar{h} , $\theta_2^{\#}$, $\theta_2^{\#\#}$, δ^* and δ^{**}) as defined as in the previous lemma. That $\hat{\xi}_2(\theta_1, \theta_2)$ satisfies properties (a) and (b) follows directly from the above two lemmas.

Next consider property (c). Consider $s > \theta_1$ (the proof for $s < \theta_1$ is symmetric and hence

omitted). We have that

$$\begin{aligned}
& \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\theta_1, \tilde{\theta}_2) \right] - \mathbb{E}^{\tilde{\theta}_2|s} \left[\xi_2(\theta_1, \tilde{\theta}_2) \right] = \int_{(\theta_2^\# - \delta^*, \theta_2^\#)} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) f_2(\theta_2|s) d\theta_2 \\
& + \int_{(\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) f_2(\theta_2|s) d\theta_2 \\
& = \int_{(\theta_2^\# - \delta^*, \theta_2^\#)} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) \frac{f_2(\theta_2|s)}{f_2(\theta_2|\theta_1)} f_2(\theta_2|\theta_1) d\theta_2 \\
& + \int_{(\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) \frac{f_2(\theta_2|s)}{f_2(\theta_2|\theta_1)} f_2(\theta_2|\theta_1) d\theta_2 \\
& \leq \int_{(\theta_2^\# - \delta^*, \theta_2^\#)} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) \frac{f_2(\theta_2^\#|s)}{f_2(\theta_2^\#|\theta_1)} f_2(\theta_2|\theta_1) d\theta_2 \\
& + \int_{(\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**})} \left(\hat{\xi}_2(\theta_1, \theta_2) - \xi_2(\theta_1, \theta_2) \right) \frac{f_2(\theta_2^\#|s)}{f_2(\theta_2^\#|\theta_1)} f_2(\theta_2|\theta_1) d\theta_2 \\
& = \frac{f_2(\theta_2^\#|s)}{f_2(\theta_2^\#|\theta_1)} \left(\mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\hat{\xi}_2(\theta_1, \tilde{\theta}_2) \right] - \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\xi_2(\theta_1, \tilde{\theta}_2) \right] \right) = 0.
\end{aligned}$$

where, for the inequality, we used the fact that, by construction of the function $\hat{\xi}_2(\theta_1, \cdot)$, $\hat{\xi}_2(\theta_1, \theta_2) \geq \xi_2(\theta_1, \theta_2)$ for $\theta_2 \in (\theta_2^\# - \delta^*, \theta_2^\#)$ and $\hat{\xi}_2(\theta_1, \theta_2) \leq \xi_2(\theta_1, \theta_2)$ for $\theta_2 \in (\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**})$, along with the fact that $f_2(\theta_2|s)/f_2(\theta_2|\theta_1)$ is increasing in θ_2 by the MLRP, while, for the equality, we used the property in part (a).

Finally, property (d) follows from Jensen's inequality after noting that, for any

$$\theta_2 \in S \equiv \left(\theta_2^\# - \delta^*, \theta_2^\# \right) \cup \left(\theta_2^{\#\#}, \theta_2^{\#\#} + \delta^{**} \right),$$

$\hat{\xi}_2(\theta_1, \theta_2) = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\xi_2(\theta_1, \tilde{\theta}_2) | \theta_2 \in S \right]$, while $\hat{\xi}_2(\theta_1, \theta_2) = \xi_2(\theta_1, \theta_2)$ for $\theta_2 \notin S$. ■

We then have the following result.

Lemma 6 *Suppose that $F_2(\cdot|\cdot)$ satisfies the MLRP. For any pair of implementable policies $\langle \xi, c \rangle$ such that $\xi_2(\theta_1, \cdot)$ fails to be non-increasing in θ_2 over a positive measure subset of Θ_1 , there exist implementable policies $\langle \hat{\xi}, \hat{c} \rangle$ such that the principal's expected profits under $\langle \hat{\xi}, \hat{c} \rangle$ are strictly higher than under $\langle \xi, c \rangle$.*

Proof. Let $\hat{\xi}_1 = \xi_1$. For any θ_1 such that $\xi_2(\theta_1, \cdot)$ is non-increasing in θ_2 , let $\hat{\xi}_2(\theta_1, \cdot) = \xi_2(\theta_1, \cdot)$, while for any θ_1 for which $\xi_2(\theta_1, \cdot)$ fails to be non-increasing in θ_2 , take $\hat{\xi}_2(\theta_1, \theta_2)$ as in the previous lemma. Then let $\hat{c}_1(\cdot) = c_1(\cdot)$ and for any θ , let $\hat{c}_2(\theta) = W(\theta; \hat{\xi}) + K - \hat{c}_1(\theta_1)$, where $K = \mathbb{E}^{\tilde{\theta}_1|V} [V(\tilde{\theta}); \langle \xi, c \rangle]$ is the lowest period-1's type expected payoff under the original policies $\langle \xi, c \rangle$. From the properties (a)-(c) of $\hat{\xi}_2$ in the previous lemma, it is easy to see that, for each type

θ_1 , $\mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta}); \langle \hat{\xi}, \hat{c} \rangle] = \mathbb{E}^{\tilde{\theta}|\theta_1}[V(\tilde{\theta}); \langle \xi, c \rangle]$, and that the policies $\langle \hat{\xi}, \hat{c} \rangle$ satisfy all the conditions in Proposition 1 and hence are implementable (in particular, note that if (ξ_1, ξ_2) satisfy all the integral monotonicity conditions, so do $(\hat{\xi}_1, \hat{\xi}_2)$). Now, recall that the principal's payoff when the manager is risk neutral is given by the expression in (9). It is then easy to see that, for any θ_1 for which $\xi_2(\theta_1, \cdot)$ fails to be non-increasing in θ_2 , the difference in expected profits under $\langle \hat{\xi}, \hat{c} \rangle$ relative to $\langle \xi, c \rangle$ is given by (26), which is strictly positive. To establish the result it then suffices to note that, for each θ_1 for which the original policy $\xi_2(\theta_1, \cdot)$ fails to be non-increasing in θ_2 , one can choose (δ^*, δ^{**}) , as a function of θ_1 , so as to guarantee that the new policy $\hat{\xi}_2$ remains integrable over $\Theta = \Theta_1 \times \Theta_2$. ■

Step 2. Hence, without loss of optimality, assume that $\xi_2^*(\theta_1, \cdot)$ is non-increasing, for all θ_1 . Next recall that incentive compatibility requires that $\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\pi_2(\theta_1, \tilde{\theta}_2)]$ be non-decreasing in θ_1 ; i.e. $\xi_1^*(\theta_1) + \theta_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2) + \gamma \theta_1]$ must be non-decreasing. Furthermore, from (15), at the optimum, for almost all θ_1 , $\mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] = \gamma \xi_1^*(\theta_1) + 1 - \gamma$. This implies that $\xi_1^*(\theta_1) + \theta_1$ must be non-decreasing. Now suppose that the claim in the proposition is not true. This implies that, for any $\eta > 0$, there is a positive-measure set of θ_1 such that

$$\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] = \xi_1^*(\theta_1) + \gamma [\gamma \xi_1^*(\theta_1) + 1 - \gamma] < \eta, \quad (27)$$

or, equivalently, $\xi_1^*(\theta_1) < [\eta - \gamma(1 - \gamma)] / (1 + \gamma^2)$. We use this observation to show the following.

Lemma 7 *Suppose that $\sup \{[1 - F_1(\theta_1)]/f_1(\theta_1)\} < (1 + \gamma)/(1 + \gamma^2) - (\bar{\theta}_1 - \underline{\theta}_1)$ and that $F_2(\cdot|\cdot)$ satisfies the monotone-likelihood-ratio property. Let*

$$L_1 \equiv -\frac{\gamma(1 - \gamma)}{1 + \gamma^2} + \frac{1}{4} \left(1 - (\bar{\theta}_1 - \underline{\theta}_1) + \frac{\gamma(1 - \gamma)}{1 + \gamma^2} - \sup \left\{ \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right\} \right)$$

and

$$L_2 \equiv 1 - \sup \left\{ \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right\} - \frac{1}{4} \left(1 - (\bar{\theta}_1 - \underline{\theta}_1) + \frac{\gamma(1 - \gamma)}{1 + \gamma^2} - \sup \left\{ \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right\} \right).$$

Suppose that, for any $\eta > 0$, there exists a positive-measure set $\hat{\Theta}_1(\eta) \subset \Theta_1$ such that $\xi_1^(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] < \eta$ for all $\theta_1 \in \hat{\Theta}_1(\eta)$. Then, there exists $\theta_1^\# \in [\underline{\theta}_1, \bar{\theta}_1]$ and $e^\# \in [L_1, L_2]$ such that, for all $\theta_1 < \theta_1^\#$, $\xi_1^*(\theta_1) \leq e^\#$, while for all $\theta_1 > \theta_1^\#$, $\xi_1^*(\theta_1) \geq e^\#$.*

Proof. First note that the assumptions in the lemma imply that there exists a positive-measure set $\Theta'_1 \subset \Theta_1$ such that $\xi_1^*(\theta_1) < L_1$ for all $\theta_1 \in \Theta'_1$. To see this, let

$$\eta = \frac{1 + \gamma^2}{4} \left(1 - (\bar{\theta}_1 - \underline{\theta}_1) + \frac{\gamma(1 - \gamma)}{1 + \gamma^2} - \sup \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} \right)$$

in (27) and note that $\eta > 0$ under the assumptions in the lemma. Next observe that, for ξ^* to be optimal, there must exist a positive-measure set $\Theta''_1 \subset \Theta_1$ such that $\xi_1^*(\theta_1) \geq \xi_1^R(\theta_1) = 1 - \frac{1 - F_1(\theta_1)}{f_1(\theta_1)} >$

L_2 for all $\theta_1 \in \Theta_1''$. If this was not the case, the principal could increase her payoff by increasing $\xi_1^*(\theta_1)$ uniformly across Θ_1 by $\varepsilon > 0$, leaving $c_1(\cdot)$ and $\xi_2(\cdot)$ unchanged, and then adjusting the period-2 compensation c_2 so as to satisfy (2) while continuing to give the lowest period-1 type the same payoff $K = \mathbb{E}^{\tilde{\theta}_1}[V(\tilde{\theta}); \langle \xi^*, c^* \rangle]$ as under the original policies $\langle \xi^*, c^* \rangle$. This would relax the participation constraints (use (6) to see it), would not affect integral monotonicity, and would bring the period-1 policy closer to the one ξ_1^R that maximizes virtual surplus, thus improving the principal's expected payoff, as given by (9).

In what follows, we show that, since $\xi_1^*(\theta_1) + \theta_1$ is non-decreasing, there exists $\theta_1^\# \in [\underline{\theta}_1, \bar{\theta}_1]$ and $e^\# \in [L_1, L_2]$ such that, for all $\theta_1 < \theta_1^\#$, $\xi_1^*(\theta_1) \leq e^\#$, while for all $\theta_1 > \theta_1^\#$, $\xi_1^*(\theta_1) \geq e^\#$. We establish the result by contradiction. Suppose the claim in the lemma is not true. This means that the following result must instead be true:

Claim A: For all $e \in [L_1, L_2]$, all $\theta_1^\# \in [\underline{\theta}_1, \bar{\theta}_1]$, there exists $\theta_1 < \theta_1^\#$ such that $\xi_1^*(\theta_1) > e$, or $\theta_1 > \theta_1^\#$ such that $\xi_1^*(\theta_1) < e$.

Now suppose Claim A is true. Let $[\cdot]^- : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $[a]^- = \max\{-a, 0\}$. Our goal is to construct a partition $\{y_0, y_1, \dots, y_m\}$, $m \in \mathbb{N}$, $\underline{\theta}_1 = y_0 < y_1 < \dots < y_{m-1} < y_m = \bar{\theta}_1$, of Θ_1 such that $\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- > \bar{\theta}_1 - \underline{\theta}_1$, establishing that the negative variation of ξ_1^* over Θ_1 , i.e., the supremum of $\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^-$ over all partitions of Θ_1 , exceeds $\bar{\theta}_1 - \underline{\theta}_1$. We know this to be incompatible with the fact that $\xi_1^*(\theta_1) + \theta_1$ is non-decreasing over Θ_1 , establishing that Claim A must be false.

For any $e \in [L_1, L_2]$, let $\theta_e^\# = \inf\{\theta_1 : \xi_1^*(\bar{\theta}_1) \geq e \text{ for all } \bar{\theta}_1 > \theta_1\}$. By the definition of $\theta_e^\#$, for all $\epsilon > 0$, there must exist $\theta_1 \in (\theta_e^\# - \epsilon, \theta_e^\#)$ such that $\xi_1^*(\theta_1) < e$. Furthermore, again by definition of $\theta_e^\#$, for all $\theta_1 > \theta_e^\#$, $\xi_1^*(\theta_1) \geq e$. Hence, for Claim A to hold, there must exist $\theta_1', \theta_1'' < \theta_e^\#$, $\theta_1' < \theta_1''$, such that $\xi_1^*(\theta_1') > e > \xi_1^*(\theta_1'')$. Now, for each $e \in [L_1, L_2]$, let

$$b_e \equiv \sup\{\xi_1^*(\theta_1) : \theta_1 < \theta_1' \text{ for some } \theta_1' \text{ for which } \xi_1^*(\theta_1') < e\} = \sup\{\xi_1^*(\theta_1) : \theta_1 < \theta_e^\#\}$$

and

$$l_e \equiv \inf\{\xi_1^*(\theta_1) : \text{for all } \epsilon > 0, \theta_1 > \theta_1' \text{ for some } \theta_1' \text{ with } \xi_1^*(\theta_1') > b_e - \epsilon\}.$$

Note that $C = \{(l_e, b_e) : e \in [L_1, L_2]\}$ is an open cover for $[L_1, L_2]$. To see this, note that, for each $e \in [L_1, L_2]$, $l_e < e < b_e$. By the Lindelof property of the real line, there exists a countable sub-cover $D = \{(l_{e_i}, b_{e_i}) : i \in \mathbb{N}\}$ of C , where $(e_i)_{i=1}^\infty$ is a sequence of points in $[L_1, L_2]$. Now let $\lambda(\cdot)$ be the Lebesgue measure. Then $\lambda(\cup_{i=1}^\infty (l_{e_i}, b_{e_i})) \geq L_2 - L_1$ and, for any $\epsilon > 0$, there exists n such that $\lambda(\cup_{i=1}^n (l_{e_i}, b_{e_i})) > L_2 - L_1 - \epsilon$. The following must then also be true.

Property A. Suppose that Claim A is true. Then for any $n \in \mathbb{N}$, any $\epsilon > 0$, there exists a partition $\{y_0, y_1, \dots, y_m\}$, $m \in \mathbb{N}$, $\underline{\theta}_1 = y_0 < y_1 < \dots < y_{m-1} < y_m = \bar{\theta}_1$, of Θ_1 such that

$$\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- \geq \lambda(\cup_{i=1}^n (l_{e_i}, b_{e_i})) - \epsilon. \quad (28)$$

Proof of Property A. Fix $n \in \mathbb{N}$ and $\epsilon > 0$. Note that there is no loss in assuming that the cover D comprises only distinct sets; i.e., $b_{e_i} \neq b_{e_{i'}}$ for all $i \neq i'$. Since n is finite, we can take the values of e_i to be ordered: i.e., $e_1 < \dots < e_{n-1} < e_n$. Let $y_0 = \underline{\theta}_1$. Choose y_1 such that $y_1 < \theta_{e_1}^\#$ and $\xi_1^*(y_1) > b_{e_1} - \epsilon/2n$, together with $y_2 \in (y_1, \theta_{e_1}^\#)$ such that $\xi_1^*(y_2) < l_{e_1} + \epsilon/2n$. It should be clear from the definitions of l_e and b_e that these choices are possible. If the partition has been determined up to y_{2k} , then take $y_{2k+1} \in [\theta_{e_k}^\#, \theta_{e_{k+1}}^\#)$ such that $\xi_1^*(y_{2k+1}) > b_{e_{k+1}} - \epsilon/2n$ and $y_{2k+2} \in (y_{2k+1}, \theta_{e_{k+1}}^\#)$ such that $\xi_1^*(y_{2k+2}) < l_{e_{k+1}} + \epsilon/2n$. Proceeding this way, the partition is determined up to y_{2n} , and we then let $y_{2n+1} = \bar{\theta}_1$ (so that $m = 2n + 1$). Then it is easy to see that

$$\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- \geq \sum_{i=1}^n \left(b_{e_i} - l_{e_i} - \frac{\epsilon}{n} \right) = \sum_{i=1}^n (b_{e_i} - l_{e_i}) - \epsilon \geq \lambda(\cup_{i=1}^n (l_{e_i}, b_{e_i})) - \epsilon.$$

This establishes Property A.

We therefore conclude that, for any $\epsilon > 0$, there exists a partition $\{y_0, y_1, \dots, y_m\}$, $m \in \mathbb{N}$, $\underline{\theta}_1 = y_0 < y_1 < \dots < y_{m-1} < y_m = \bar{\theta}_1$, of Θ_1 such that $\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- > L_2 - L_1 - 2\epsilon$. Because $L_2 - L_1 > \bar{\theta}_1 - \underline{\theta}_1$, there then exists a partition such that $\sum_{k=0}^{m-1} [\xi_1^*(y_{k+1}) - \xi_1^*(y_k)]^- > \bar{\theta}_1 - \underline{\theta}_1$. This shows that the negative variation of ξ_1^* over Θ_1 must be strictly larger than $\bar{\theta}_1 - \underline{\theta}_1$, as desired. ■

Now suppose that, for any $\eta > 0$, there exists a positive-measure set $\hat{\Theta}_1(\eta) \subset \Theta_1$ such that $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] < \eta$ for all $\theta_1 \in \hat{\Theta}_1(\eta)$. The result in the previous lemma implies that there exists $\theta_1^\# \in [\underline{\theta}_1, \bar{\theta}_1]$ and $e^\# \in [L_1, L_2]$ such that, for all $\theta_1 < \theta_1^\#$, $\xi_1^*(\theta_1) \leq e^\#$, while for all $\theta_1 > \theta_1^\#$, $\xi_1^*(\theta_1) \geq e^\#$. It is also easy to see that $\theta_1^\# > \underline{\theta}_1$, and that $\xi_1^*(\theta_1) < e^\#$ for a positive-measure subset of $[\underline{\theta}_1, \theta_1^\#]$ (both properties follow from the fact that, if they were not true, then $\xi_1^*(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)]$ would be bounded away from zero from below with probability one, along with the fact that $\mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] = \gamma \xi_1^*(\theta_1) + 1 - \gamma$). Then consider the alternative effort policy $\hat{\xi}$ defined by

$$\hat{\xi}_1(\theta_1) = \begin{cases} \xi_1^*(\theta_1) & \text{if } \theta_1 > \theta_1^\# \\ e^\# & \text{if } \theta_1 \leq \theta_1^\# \end{cases} \quad \text{and} \quad \hat{\xi}_2(\theta_1, \theta_2) = \begin{cases} \xi_2^*(\theta_1, \theta_2) & \text{if } \theta_1 > \theta_1^\# \\ 1 - \gamma + \gamma e^\# & \text{if } \theta_1 \leq \theta_1^\# \end{cases}$$

along with the compensation policy \hat{c} defined by $\hat{c}_1(\theta_1) = c_1^*(\theta_1)$ all θ_1 , and $\hat{c}_2(\theta_1, \theta_2) = W(\theta; \hat{\xi}) + K - \hat{c}_1(\theta_1)$, where $K = \mathbb{E}^{\tilde{\theta}_1} [V(\tilde{\theta}); \langle \xi^*, c^* \rangle]$ is the lowest period-1 type's expected payoff under the original policies $\langle \xi^*, c^* \rangle$. Now recall that the principal's payoff under any pair of implementable policies is given by (9). Further notice that the expression in (9) is strictly concave in the policies ξ and recall that (9) reaches its maximum at the policy ξ^R given by (18) and (19). Now note that, for all $\theta_1 \leq \theta_1^\#$, $\xi_1^*(\theta_1) \leq \hat{\xi}_1(\theta_1) \leq \xi_1^R(\theta_1)$, with the first inequality strict over a positive measure set of θ_1 . Also, for all $\theta_1 \leq \theta_1^\#$, all θ_2 ,

$$\begin{aligned} \mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\xi_2^*(\theta_1, \tilde{\theta}_2)] &= \gamma \xi_1^*(\theta_1) + 1 - \gamma \leq \hat{\xi}_2(\theta_1, \theta_2) = \gamma \hat{\xi}_1(\theta_1) + 1 - \gamma \\ &\leq \xi_2^R(\theta_1, \theta_2) = \gamma \xi_1^R(\theta_1) + 1 - \gamma, \end{aligned}$$

where, again, the first inequality is strict over a positive measure set of θ_1 . For all $\theta_1 > \theta_1^\#$, instead, $\hat{\xi}_1(\theta_1) = \xi_1^*(\theta_1)$ and $\hat{\xi}_2(\theta_1, \cdot) = \xi_2^*(\theta_1, \cdot)$. It is then clear that, if the policies $\langle \hat{\xi}, \hat{c} \rangle$ are implementable, they lead to higher expected profits than the policies $\langle \xi^*, c^* \rangle$. In what follows we show that indeed, they are implementable. To see this, note that, for all θ_1 ,

$$\mathbb{E}^{\tilde{\theta}|\theta_1} \left[V(\tilde{\theta}); \langle \hat{\xi}, \hat{c} \rangle \right] \geq \mathbb{E}^{\tilde{\theta}|\theta_1} \left[V(\tilde{\theta}); \langle \xi^*, c^* \rangle \right]$$

which implies that $\langle \hat{\xi}, \hat{c} \rangle$ satisfy all the participation constraints. Next observe that, for all θ_1 , $\pi_2(\theta_1, \cdot)$ is non-decreasing in θ_2 and that,

$$\pi_1(\theta_1) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\pi_2(\theta_1, \tilde{\theta}_2) \right] = \theta_1 + \hat{\xi}_1(\theta_1) + \gamma \left\{ 1 - \gamma + \gamma \hat{\xi}_1(\theta_1) + \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\tilde{\theta}_2 \right] \right\}$$

is non-decreasing in θ_1 (these properties follow directly from the way $\hat{\xi}$ is constructed along with the fact that, to be optimal, the policy ξ^* must satisfy the condition $\mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[\xi_2^*(\theta_1, \tilde{\theta}_2) \right] = \gamma \xi_1^*(\theta_1) + 1 - \gamma$). Next, observe that, by construction, the compensation policy \hat{c} satisfies Condition (2). It thus suffices to show that the new effort policy $\hat{\xi}$ satisfies the integral monotonicity constraints of Proposition 1. That is, for all $\theta_1, \hat{\theta}_1 \in \Theta_1$,

$$\int_{\hat{\theta}_1}^{\theta_1} \left\{ \hat{\xi}_1(\hat{\theta}_1) - s + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} ds \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \hat{\xi}_1(s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] \right\} ds.$$

The only two cases which are not immediate are (i) $\hat{\theta}_1 \leq \theta_1^\# < \theta_1$, and (ii) $\theta_1 \leq \theta_1^\# < \hat{\theta}_1$. For Case (i), because $\hat{\xi}_1(\cdot)$ and $\hat{\xi}_2(\cdot)$ are constant over any (θ_1, θ_2) such that $\theta_1 \leq \theta_1^\#$, it is enough to show that, for any $s > \theta_1^\#$,

$$\hat{\xi}_1(s) + s + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] \geq \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right].$$

This follows from the fact that

$$\begin{aligned} \hat{\xi}_1(s) + s + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] &= \hat{\xi}_1(s) + s + \gamma \left(\gamma \hat{\xi}_1(s) + 1 - \gamma \right) \geq \hat{\xi}_1(\theta_1^\#) + \theta_1^\# + \gamma \left(\gamma \hat{\xi}_1(\theta_1^\#) + 1 - \gamma \right) \\ &\geq \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \left(\gamma \hat{\xi}_1(\hat{\theta}_1) + 1 - \gamma \right) = \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right], \end{aligned}$$

where the inequalities follow from the fact that the original policy is such that $\xi_1^*(s) + s$ is non-decreasing, with $\xi_1^*(s) \geq e^\# = \hat{\xi}_1(\hat{\theta}_1)$ for all $s > \theta_1^\#$.

For Case (ii), note first that integral monotonicity requires that

$$\int_{\theta_1}^{\hat{\theta}_1} \left\{ \hat{\xi}_1(\hat{\theta}_1) - s + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} ds \geq \int_{\theta_1}^{\hat{\theta}_1} \left\{ \hat{\xi}_1(s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] \right\} ds.$$

Because the original policy satisfies integral monotonicity, and because $(\hat{\xi}_1(\tilde{\theta}_1), \hat{\xi}_2(\tilde{\theta}_1, \tilde{\theta}_2))$ coincides with the original policy $(\xi_1^*(\tilde{\theta}_1), \xi_2^*(\tilde{\theta}_1, \tilde{\theta}_2))$ for any $(\tilde{\theta}_1, \tilde{\theta}_2)$ such that $\tilde{\theta}_1 > \theta_1^\#$, it suffices to show that

$$\int_{\theta_1}^{\theta_1^\#} \left\{ \hat{\xi}_1(\hat{\theta}_1) - s + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} ds \geq \int_{\theta_1}^{\theta_1^\#} \left\{ \hat{\xi}_1(s) + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] \right\} ds.$$

To see this, it suffices to show that, for any $s < \theta_1^\#$

$$\hat{\xi}_1(s) + s + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] - \left(\hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right) \leq 0. \quad (29)$$

To prove that this is the case, first note that, for all $s < \theta_1^\#$, all $\theta'_1 \geq \theta_1^\#$,

$$\begin{aligned} & \hat{\xi}_1(s) + s + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(s, \tilde{\theta}_2) \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \\ = & e^\# + s + \gamma \left[1 - \gamma + \gamma e^\# \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \\ \leq & \hat{\xi}_1(\theta'_1) + \theta'_1 + \gamma \left[1 - \gamma + \gamma \hat{\xi}_1(\theta'_1) \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \end{aligned}$$

where the inequality follows from the fact that $e^\# + s + \gamma [1 - \gamma + \gamma e^\#] \leq \hat{\xi}_1(\theta'_1) + \theta'_1 + \gamma [1 - \gamma + \gamma \hat{\xi}_1(\theta'_1)]$ and from the fact that $\hat{\xi}_2(\hat{\theta}_1, \cdot)$ is non-increasing, which implies that $\mathbb{E}^{\tilde{\theta}_2|s} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \geq \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right]$.

Finally observe that, because $(\hat{\xi}_1(\cdot), \hat{\xi}_2(\cdot))$ coincides with the original policy $(\xi_1^*(\cdot), \xi_2^*(\cdot))$ for any (θ_1, θ_2) such that $\theta_1 > \theta_1^\#$, the fact that ξ^* satisfies integral monotonicity implies that there must exist a $\theta'_1 \in (\theta_1^\#, \hat{\theta}_1)$ such that

$$\begin{aligned} & \hat{\xi}_1(\theta'_1) + \theta'_1 + \gamma \left[1 - \gamma + \gamma \hat{\xi}_1(\theta'_1) \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \\ = & \hat{\xi}_1(\theta'_1) + \theta'_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\theta'_1, \tilde{\theta}_2) \right] - \left\{ \hat{\xi}_1(\hat{\theta}_1) + \hat{\theta}_1 + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta'_1} \left[\hat{\xi}_2(\hat{\theta}_1, \tilde{\theta}_2) \right] \right\} \leq 0. \end{aligned}$$

We conclude that, for all $s < \theta_1^\#$, the inequality in (29) holds. This completes the proof of the proposition. Q.E.D.

Proof of Proposition 5. Let $\xi_\rho^* \equiv (\xi_{\rho,1}^*, \xi_{\rho,2}^*)$ be the effort policies sustained under any optimal contract, when the manager's preferences over consumption are represented by the function v_ρ , with the function family $(v_\rho)_{\rho \geq 0}$ satisfying the properties described in the main text. Recall, from Proposition 2, that such policies are essentially unique. Next, let K_ρ^* be the expected payoff of the lowest period-1 type (i.e., $\mathbb{E}^{\tilde{\theta}_1|1} [V(\tilde{\theta})]$) under any optimal contract, when the manager's risk-aversion index is ρ . Finally, let $c_\rho^* \equiv (c_{\rho,1}^*, c_{\rho,2}^*)$ be a compensation policy sustained under an optimal contract and recall that, again by virtue of Proposition 2, such a policy is also essentially unique when $\rho > 0$, i.e., when the manager is strictly risk averse. When $\rho = 0$, instead, the distribution of consumption over the two periods is indeterminate, in which case let $c_{0,1}^*(\theta_1) = 0$ for all θ_1 and then let $c_{0,2}^*$ be given by (7).

Proposition 4 implies that the expected power of incentives is strictly higher in period 2 than in period 1, when $\rho = 0$. Our goal is to show the existence of $\bar{\rho} > 0$ such that, for any $\rho \in [0, \bar{\rho}]$, the expected power of incentives under any optimal contract continues to be higher in period two than in period one.

Suppose, for a contradiction, that no such $\bar{\rho}$ exists. Letting w_ρ denote the inverse of the function v_ρ , we then have that the following is instead true.

Claim B. For any $n \in \mathbb{N}$, there exists $\rho_n \in (0, \frac{1}{n})$ such that

$$\mathbb{E} \left[\psi' \left(\xi_{\rho_n,1}^* (\tilde{\theta}_1) \right) w'_{\rho_n} \left(v_{\rho_n} (c_{\rho_n,1}^* (\tilde{\theta}_1)) \right) \right] \geq \mathbb{E} \left[\psi' \left(\xi_{\rho_n,2}^* (\tilde{\theta}) \right) w'_{\rho_n} \left(v_{\rho_n} (c_{\rho_n,2}^* (\tilde{\theta})) \right) \right]. \quad (30)$$

On the other hand, we have that, when $\rho = 0$ (that is, when the manager is risk neutral), $\mathbb{E} \left[\psi' \left(\xi_{0,1}^* (\tilde{\theta}_1) \right) \right] < \mathbb{E} \left[\psi' \left(\xi_{0,2}^* (\tilde{\theta}) \right) \right]$, as established in Proposition 4. Given these observations, below we establish a series of three properties that together imply that Claim B above is false.

Property B1. For any ρ ,

$$\int_{\Theta_1} \int_{\Theta_2(\theta_1)} \{ |\xi_{\rho,1}^*(\theta_1)| + |\xi_{\rho,2}^*(\theta_1, \theta_2)| \} d\theta_2 d\theta_1 \leq 2 \left(\frac{b}{a} \right)^2. \quad (31)$$

Proof of Property B1. Let

$$L_\rho \equiv \int_{\Theta_1} \int_{\Theta_2(\theta_1)} \{ \xi_{\rho,1}^*(\theta_1)^2 + \xi_{\rho,2}^*(\theta_1, \theta_2)^2 \} d\theta_2 d\theta_1. \quad (32)$$

For arbitrary ρ , consider the gain in expected profits from using an optimal policy rather than simply paying the manager a constant wage equal to his outside option (assumed equal to zero), thus eliciting no effort. Given that w_ρ lies nowhere below the identity function, and given the bounds on the densities over Θ , it is easy to see that this gain must be no greater than

$$b^2 \int_{\Theta_1} \int_{\Theta_2(\theta_1)} \{ \xi_{\rho,1}^*(\theta_1) + \xi_{\rho,2}^*(\theta_1, \theta_2) \} d\theta_2 d\theta_1 - \frac{1}{2} a^2 L_\rho.$$

Then note that

$$\int_{\Theta_1} \int_{\Theta_2(\theta_1)} \{ \xi_{\rho,1}^*(\theta_1) + \xi_{\rho,2}^*(\theta_1, \theta_2) \} d\theta_2 d\theta_1 \leq \int_{\Theta_1} \int_{\Theta_2(\theta_1)} \{ |\xi_{\rho,1}^*(\theta_1)| + |\xi_{\rho,2}^*(\theta_1, \theta_2)| \} d\theta_2 d\theta_1 \leq L_\rho^{1/2}, \quad (33)$$

where the second inequality follows from Hölder's inequality. Hence, the expected gain from using the optimal policy is no greater than $b^2 L_\rho^{1/2} - \frac{1}{2} a^2 L_\rho$, which is non-negative only if $L_\rho \leq 4 \left(\frac{b}{a} \right)^4$. The result then follows from (33). ■

Property B2. Let $(\rho_n)_{n=1}^\infty$ be any sequence of real numbers satisfying the property in Claim B (that is, for all $n \in \mathbb{N}$, $\rho_n \in (0, \frac{1}{n})$ is such that Condition (30) holds). There exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\int_{\Theta_1} \int_{\Theta_2(\theta_1)} \left\{ \left| \xi_{\rho_n,1}^*(\theta_1) - \xi_{0,1}^*(\theta_1) \right| + \left| \xi_{\rho_n,2}^*(\theta_1, \theta_2) - \xi_{0,2}^*(\theta_1, \theta_2) \right| \right\} d\theta_2 d\theta_1 > \epsilon, \quad (34)$$

and

$$\left(\int_{\Theta_1} \int_{\Theta_2(\theta_1)} \left\{ \left(\xi_{\rho_n,1}^*(\theta_1) - \xi_{0,1}^*(\theta_1) \right)^2 + \left(\xi_{\rho_n,2}^*(\theta_1, \theta_2) - \xi_{0,2}^*(\theta_1, \theta_2) \right)^2 \right\} d\theta_2 d\theta_1 \right)^{1/2} > \epsilon. \quad (35)$$

Proof of Property B2. For each ρ , let

$$l_\rho \equiv \inf \{l \in \mathbb{R}_+ \cup \{+\infty\} : |w'_\rho(u) - 1| \leq l, \text{ all } u \in \mathbb{R}\}$$

and note that the assumptions on the function family $(v_\rho)_{\rho \geq 0}$ imply that $l_\rho \rightarrow 0$ as $\rho \rightarrow 0$. We start by proving the existence of ϵ and N such that, for all $n \geq N$, (34) holds. Suppose for a contradiction that there are no such ϵ and N . Then there exists a subsequence (ρ_{n_k}) such that

$$\int_{\Theta_1} \int_{\Theta_2(\theta_1)} \left| \xi_{\rho_{n_k},1}^*(\theta_1) - \xi_{0,1}^*(\theta_1) \right| + \left| \xi_{\rho_{n_k},2}^*(\theta_1, \theta_2) - \xi_{0,2}^*(\theta_1, \theta_2) \right| d\theta_2 d\theta_1 \rightarrow 0 \quad (36)$$

as $n_k \rightarrow \infty$. However, then note that

$$\begin{aligned} & \mathbb{E} \left[\begin{aligned} & \psi' \left(\xi_{\rho_{n_k},1}^*(\tilde{\theta}_1) \right) w'_{\rho_{n_k}} \left(v_{\rho_{n_k}} \left(c_{\rho_{n_k},1}^*(\tilde{\theta}_1) \right) \right) - \psi' \left(\xi_{\rho_{n_k},2}^*(\tilde{\theta}) \right) w'_{\rho_{n_k}} \left(v_{\rho_{n_k}} \left(c_{\rho_{n_k},2}^*(\tilde{\theta}) \right) \right) \\ & - \left(\psi' \left(\xi_{0,1}^*(\tilde{\theta}_1) \right) - \psi' \left(\xi_{0,2}^*(\tilde{\theta}) \right) \right) \end{aligned} \right] \\ = & \mathbb{E} \left[\begin{aligned} & \left(\xi_{\rho_{n_k},1}^*(\tilde{\theta}_1) - \xi_{0,1}^*(\tilde{\theta}_1) \right) - \left(\xi_{\rho_{n_k},2}^*(\tilde{\theta}) - \xi_{0,2}^*(\tilde{\theta}) \right) \\ & + \left(w'_{\rho_{n_k}} \left(v_{\rho_{n_k}} \left(c_{\rho_{n_k},1}^*(\tilde{\theta}_1) \right) \right) - 1 \right) \xi_{\rho_{n_k},1}^*(\tilde{\theta}_1) \\ & - \left(w'_{\rho_{n_k}} \left(v_{\rho_{n_k}} \left(c_{\rho_{n_k},2}^*(\tilde{\theta}) \right) \right) - 1 \right) \xi_{\rho_{n_k},2}^*(\tilde{\theta}) \end{aligned} \right] \\ \leq & \mathbb{E} \left[\begin{aligned} & \left| \xi_{\rho_{n_k},1}^*(\tilde{\theta}_1) - \xi_{0,1}^*(\tilde{\theta}_1) \right| + \left| \xi_{\rho_{n_k},2}^*(\tilde{\theta}) - \xi_{0,2}^*(\tilde{\theta}) \right| \\ & + \left| w'_{\rho_{n_k}} \left(v_{\rho_{n_k}} \left(c_{\rho_{n_k},1}^*(\tilde{\theta}_1) \right) \right) - 1 \right| \left| \xi_{\rho_{n_k},1}^*(\tilde{\theta}_1) \right| \\ & + \left| w'_{\rho_{n_k}} \left(v_{\rho_{n_k}} \left(c_{\rho_{n_k},2}^*(\tilde{\theta}) \right) \right) - 1 \right| \left| \xi_{\rho_{n_k},2}^*(\tilde{\theta}) \right| \end{aligned} \right] \\ \leq & b^2 \int_{\Theta_1} \int_{\Theta_2(\theta_1)} \left\{ \left| \xi_{\rho_{n_k},1}^*(\theta_1) - \xi_{0,1}^*(\theta_1) \right| + \left| \xi_{\rho_{n_k},2}^*(\theta_1, \theta_2) - \xi_{0,2}^*(\theta_1, \theta_2) \right| \right\} d\theta_2 d\theta_1 \\ & + b^2 l_{\rho_{n_k}} \int_{\Theta_1} \int_{\Theta_2(\theta_1)} \left\{ \left| \xi_{\rho_{n_k},1}^*(\theta_1) \right| + \left| \xi_{\rho_{n_k},2}^*(\theta_1, \theta_2) \right| \right\} d\theta_2 d\theta_1. \end{aligned}$$

The final expression converges to 0 as $n_k \rightarrow +\infty$, the first integral by (36), and the second by Property B1 above along with the fact that $l_\rho \rightarrow 0$ as $\rho \rightarrow 0$. It follows that, for any $\nu > 0$, there exists N such that, for any $n_k > N$,

$$\begin{aligned} & \mathbb{E} \left[\psi' \left(\xi_{\rho_{n_k},1}^*(\tilde{\theta}_1) \right) w'_{\rho_{n_k}} \left(v_{\rho_{n_k}} \left(c_{\rho_{n_k},1}^*(\tilde{\theta}_1) \right) \right) - \psi' \left(\xi_{\rho_{n_k},2}^*(\tilde{\theta}) \right) w'_{\rho_{n_k}} \left(v_{\rho_{n_k}} \left(c_{\rho_{n_k},2}^*(\tilde{\theta}) \right) \right) \right] \\ & \leq \mathbb{E} \left[\psi' \left(\xi_{0,1}^*(\tilde{\theta}_1) \right) - \psi' \left(\xi_{0,2}^*(\tilde{\theta}) \right) \right] + \nu. \end{aligned}$$

The right-hand side is negative whenever ν is taken sufficiently small, since, as noted above,

$$\mathbb{E} \left[\psi' \left(\xi_{0,1}^*(\tilde{\theta}_1) \right) \right] < \mathbb{E} \left[\psi' \left(\xi_{0,2}^*(\tilde{\theta}) \right) \right].$$

However, this contradicts the assumption that the original sequence (ρ_n) satisfies Condition (30).

Finally, that (35) is also true follows simply from (34) using Hölder's inequality. \blacksquare

Now, for any $\rho \geq 0$, effort policy ξ , first-period consumption policy c_1 , and constant $K \geq 0$, let

$$h_\rho(\xi, c_1, K) \equiv \mathbb{E} \left[\tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) + \tilde{\theta}_2 + \xi_2(\tilde{\theta}) - c_1(\tilde{\theta}_1) - w_\rho \left(W(\tilde{\theta}; \xi) + K - v_\rho(c_1(\tilde{\theta}_1)) \right) \right],$$

which is simply the principal's payoffs, as in (8). The final property we need to prove is the following.

Property B3. *Take any sequence $(\rho_n)_{n=1}^\infty$ such that, for all $n \in \mathbb{N}$, $\rho_n \in (0, \frac{1}{n})$. Then $h_0(\xi_{\rho_n}^*, c_{\rho_n,1}^*, K_{\rho_n}^*) \rightarrow h_0(\xi_0^*, c_{0,1}^*, K_0^*)$ as $n \rightarrow +\infty$.*

Proof of Property B3. Suppose not. Then there must exist a subsequence (ρ_{n_k}) and $\eta > 0$ such that

$$h_0(\xi_{\rho_{n_k}}^*, c_{\rho_{n_k},1}^*, K_{\rho_{n_k}}^*) < h_0(\xi_0^*, c_{0,1}^*, K_0^*) - \eta$$

for all n_k . Then note that, for any (ξ, c_1, K) , any $\rho \geq 0$, $h_\rho(\xi, c_1, K) \leq h_0(\xi, c_1, K)$, since w_ρ lies everywhere above w_0 . Hence, for all n_k ,

$$h_{\rho_{n_k}}(\xi_{\rho_{n_k}}^*, c_{\rho_{n_k},1}^*, K_{\rho_{n_k}}^*) < h_0(\xi_0^*, c_{0,1}^*, K_0^*) - \eta.$$

However, below we show that $h_{\rho_{n_k}}(\xi_0^*, c_{0,1}^*, K_0^*) \rightarrow h_0(\xi_0^*, c_{0,1}^*, K_0^*)$ as $n_k \rightarrow \infty$, which contradicts the optimality of $(\xi_{\rho_{n_k}}^*, c_{\rho_{n_k},1}^*, K_{\rho_{n_k}}^*)$ for some n_k sufficiently large.

To this end, for each ρ , we find a lower bound on $h_\rho(\xi_0^*, c_{0,1}^*, K_0^*) - h_0(\xi_0^*, c_{0,1}^*, K_0^*)$. Recall that, as a consequence of the result in Lemma 2 in the proof of Proposition 1, the function $\psi'(\xi_{0,1}^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}_2 | \theta_1} [\psi'(\xi_{0,2}^*(\theta_1, \tilde{\theta}_2))]$ is bounded uniformly over θ_1 , and hence integrable; similarly, for each θ_1 , $\xi_{0,2}^*(\theta_1, \cdot)$ is uniformly bounded over $\Theta_2(\theta_1)$, and hence integrable. We then have that, for any ρ ,

$$\begin{aligned} & h_\rho(\xi_0^*, c_{0,1}^*, K_0^*) - h_0(\xi_0^*, c_{0,1}^*, K_0^*) \\ & \geq -l_\rho \mathbb{E} \left[\begin{array}{l} \psi(\xi_{0,1}^*(\tilde{\theta}_1)) + \psi(\xi_{0,2}^*(\tilde{\theta})) \\ + K_0^* + \int_{\underline{\theta}_1}^{\tilde{\theta}_1} \left\{ \psi'(\xi_{0,1}^*(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2 | s} [\psi'(\xi_{0,2}^*(s, \tilde{\theta}_2))] \right\} ds \\ + \int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(\xi_{0,2}^*(\tilde{\theta}_1, s)) ds - \mathbb{E}^{\tilde{\theta}_2 | \tilde{\theta}_1} \left[\int_{\underline{\theta}_2}^{\tilde{\theta}_2} \psi'(\xi_{0,2}^*(\tilde{\theta}_1, s)) ds \right] \end{array} \right] \\ & \geq -l_\rho \mathbb{E} \left[\begin{array}{l} \psi(\xi_{0,1}^*(\tilde{\theta}_1)) + \psi(\xi_{0,2}^*(\tilde{\theta})) \\ + K_0^* + \int_{\underline{\theta}_1}^{\tilde{\theta}_1} \left\{ \psi'(\xi_{0,1}^*(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2 | s} [\psi'(\xi_{0,2}^*(s, \tilde{\theta}_2))] \right\} ds \\ + \int_{\underline{\theta}_2(\tilde{\theta}_1)}^{\tilde{\theta}_2} |\psi'(\xi_{0,2}^*(\tilde{\theta}_1, s))| ds + \mathbb{E}^{\tilde{\theta}_2 | \tilde{\theta}_1} \left[\int_{\underline{\theta}_2(\tilde{\theta}_1)}^{\tilde{\theta}_2} |\psi'(\xi_{0,2}^*(\tilde{\theta}_1, s))| ds \right] \end{array} \right] \\ & = -l_\rho \mathbb{E} \left[\begin{array}{l} \psi(\xi_{0,1}^*(\tilde{\theta}_1)) + \psi(\xi_{0,2}^*(\tilde{\theta})) \\ + K_0^* + \frac{1-F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)} \left[\psi'(\xi_{0,1}^*(\tilde{\theta}_1)) + \gamma \psi'(\xi_{0,2}^*(\tilde{\theta})) \right] \\ + 2 \frac{1-F_2(\tilde{\theta}_2 | \tilde{\theta}_1)}{f_2(\tilde{\theta}_2 | \tilde{\theta}_1)} |\psi'(\xi_{0,2}^*(\tilde{\theta}))| \end{array} \right]. \end{aligned}$$

The first inequality follows because, for any $y \in \mathbb{R}$, any ρ , $|w_\rho(y) - y| \leq l_\rho |y|$. The equality follows from integration by parts after noting that

$$\int_{\underline{\theta}_1}^{\tilde{\theta}_1} \left\{ \psi'(\xi_{0,1}^*(s)) + \gamma \mathbb{E}^{\tilde{\theta}_2 | s} [\psi'(\xi_{0,2}^*(s, \tilde{\theta}_2))] \right\} ds$$

is absolutely continuous in θ_1 , and that $\int_{\theta_2(\theta_1)}^{\theta_2} |\psi'(\xi_{0,2}^*(\theta_1, s))| ds$ is absolutely continuous in θ_2 for each θ_1 (these observations follow in turn from the integrability of $\psi'(\xi_{0,1}^*(\theta_1)) + \gamma \mathbb{E}^{\tilde{\theta}_2|\theta_1} [\psi'(\xi_{0,2}^*(\theta_1, \tilde{\theta}_2))]$, and from the integrability of $\xi_{0,2}^*(\theta_1, \cdot)$ for each θ_1). Using the bounds on f_1 and f_2 , we then have that

$$\begin{aligned} & \mathbb{E} \left[\begin{aligned} & \psi(\xi_{0,1}^*(\tilde{\theta}_1)) + \psi(\xi_{0,2}^*(\tilde{\theta})) \\ & + K_0^* + \frac{1-F_1(\tilde{\theta}_1)}{f_1(\tilde{\theta}_1)} \left[\psi'(\xi_{0,1}^*(\tilde{\theta}_1)) + \gamma \psi'(\xi_{0,2}^*(\tilde{\theta})) \right] \\ & + 2 \frac{1-F_2(\tilde{\theta}_2|\tilde{\theta}_1)}{f_2(\tilde{\theta}_2|\tilde{\theta}_1)} |\psi'(\xi_{0,2}^*(\tilde{\theta}))| \end{aligned} \right] \\ \leq & b^2 \int_{\Theta_1} \int_{\Theta_2(\theta_1)} \left[\frac{1}{2} [\xi_{0,1}^*(\theta_1)^2 + \xi_{0,2}^*(\theta_1, \theta_2)^2] + K_0^* \right. \\ & \left. + \frac{1}{a} (|\xi_{0,1}^*(\theta_1)| + \gamma |\xi_{0,2}^*(\theta_1, \theta_2)|) + \frac{2}{a} |\xi_{0,2}^*(\theta_1, \theta_2)| \right] d\theta_2 d\theta_1 \end{aligned}$$

which, using the boundedness of (32) together with (31), is finite. Thus the fact that $l_{\rho_{n_k}} \rightarrow 0$ as $n_k \rightarrow +\infty$ implies $h_{\rho_{n_k}}(\xi_0^*, c_{0,1}^*, K_0^*) - h_0(\xi_0^*, c_{0,1}^*, K_0^*) \rightarrow 0$, which is what we wanted to show. \blacksquare

Now let N and ϵ be the values defined in Property B2. Note that, for any $n > N$, there exists an incentive-compatible mechanism implementing the effort policy $\frac{1}{2}\xi_{\rho_n}^* + \frac{1}{2}\xi_0^*$, under which the manager's period-1 compensation is given by an arbitrary function $c_{\rho_n,1}^\#(\cdot)$, say $c_{\rho_n,1}^\#(\theta_1) = 0$ for all θ_1 , and under which the lowest period-1 type's expected payoff is $\frac{1}{2}K_{\rho_n}^* + \frac{1}{2}K_0^*$ (the existence of such a mechanism follows from Proposition 1; in particular, because ψ is quadratic, $\frac{1}{2}\xi_{\rho_n}^* + \frac{1}{2}\xi_0^*$ satisfies Condition B(i) of this result). For such a mechanism, we have that

$$\begin{aligned} & h_0 \left(\frac{1}{2}\xi_{\rho_n}^* + \frac{1}{2}\xi_0^*, c_{\rho_n,1}^\#, \frac{1}{2}K_{\rho_n}^* + \frac{1}{2}K_0^* \right) - \left[\frac{1}{2}h_0(\xi_{\rho_n}^*, c_{\rho_n,1}^*, K_{\rho_n}^*) + \frac{1}{2}h_0(\xi_0^*, c_{0,1}^*, K_0^*) \right] \quad (37) \\ = & -\frac{1}{2}\mathbb{E} \left[\begin{aligned} & \left(\frac{1}{2}\xi_{\rho_n,1}^*(\tilde{\theta}_1) + \frac{1}{2}\xi_{0,1}^*(\tilde{\theta}_1) \right)^2 - \frac{1}{2}\xi_{\rho_n,1}^*(\tilde{\theta}_1)^2 - \frac{1}{2}\xi_{0,1}^*(\tilde{\theta}_1)^2 \\ & + \left(\frac{1}{2}\xi_{\rho_n,2}^*(\tilde{\theta}) + \frac{1}{2}\xi_{0,2}^*(\tilde{\theta}) \right)^2 - \frac{1}{2}\xi_{\rho_n,2}^*(\tilde{\theta})^2 - \frac{1}{2}\xi_{0,2}^*(\tilde{\theta})^2 \end{aligned} \right] \\ = & \frac{1}{8}\mathbb{E} \left[\left(\xi_{\rho_n,1}^*(\tilde{\theta}_1) - \xi_{0,1}^*(\tilde{\theta}_1) \right)^2 + \left(\xi_{\rho_n,2}^*(\tilde{\theta}) - \xi_{0,2}^*(\tilde{\theta}) \right)^2 \right] \geq \frac{a^2}{8}\epsilon^2. \end{aligned}$$

where the inequality follows from (35) along with the fact that $f_1(\theta_1)f_2(\theta_2|\theta_1) > a^2$ for all $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2(\theta_1)$.

That the property in Claim B is false then follows from the combination of the result in (37) along with Property B3 above, which jointly imply that as $n \rightarrow +\infty$

$$\begin{aligned} & h_0 \left(\frac{1}{2}\xi_{\rho_n}^* + \frac{1}{2}\xi_0^*, c_{\rho_n,1}^\#, \frac{1}{2}K_{\rho_n}^* + \frac{1}{2}K_0^* \right) \geq \frac{1}{2}h_0(\xi_{\rho_n}^*, c_{\rho_n,1}^*, K_{\rho_n}^*) + \frac{1}{2}h_0(\xi_0^*, c_{0,1}^*, K_0^*) + \frac{a^2}{8}\epsilon^2 \\ & > h_0(\xi_0^*, c_{0,1}^*(\theta_1), K_0^*) \end{aligned}$$

thus contradicting the optimality of $(\xi_0^*, c_{0,1}^*, K_0^*)$.

Q.E.D.

Proof of Proposition 6. Let $\xi_\gamma^* \equiv (\xi_{\gamma,1}^*, \xi_{\gamma,2}^*)$ be the (essentially unique) effort policy sustained under any optimal contract, when the persistence of the productivity process is γ . Let K_γ^* be the optimal choice of the expected payoff for the lowest period-1 type (i.e., $\mathbb{E}^{\tilde{\theta}|\theta_1} [V(\tilde{\theta})]$) when the persistence of the process is γ . Finally, let $c_\gamma^* \equiv (c_{\gamma,1}^*, c_{\gamma,2}^*)$ be a compensation policy sustained under an optimal contract and recall that, by virtue of Proposition 2, such a policy is also essentially unique.

Part (a). Consider the case of $\gamma = 1$. From (13), note that, for almost all $\theta_1 \in \Theta_1$,

$$\begin{aligned} \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\psi' \left(\xi_{1,2}^*(\tilde{\theta}) \right) w' \left(v(c_{1,2}^*(\tilde{\theta})) \right) \right] &= \psi' \left(\xi_{1,1}^*(\theta_1) \right) w' \left(v(c_{1,1}^*(\theta_1)) \right) \\ &- \mathbb{E}^{\tilde{\theta}|\theta_1} \left[\frac{\psi'' \left(\xi_{1,2}^*(\tilde{\theta}) \right)}{f_2(\tilde{\theta}_2|\theta_1)} \int_{\tilde{\theta}_2}^{\bar{\theta}_2} \left\{ w' \left(v(c_{1,2}^*(\tilde{\theta}_1, r)) \right) - w' \left(v(c_{1,1}^*(\tilde{\theta}_1)) \right) \right\} f_2(r|\tilde{\theta}_1) dr \right]. \end{aligned} \quad (38)$$

We now establish that, whenever v is strictly concave, then with probability one (that is, for all but a zero-measure set of θ),

$$m(\theta_2; \theta_1) \equiv \int_{\theta_2}^{\bar{\theta}_2} \left\{ w' \left(v(c_{1,2}^*(\theta_1, r)) \right) - w' \left(v(c_{1,1}^*(\theta_1)) \right) \right\} f_2(r|\theta_1) dr \geq 0. \quad (39)$$

To see this, note that, for all θ_1 , almost all θ_2 ,

$$\frac{\partial m(\theta_2; \theta_1)}{\partial \theta_2} = - \left[w' \left(v(c_{1,2}^*(\theta_1, \theta_2)) \right) - w' \left(v(c_{1,1}^*(\theta_1)) \right) \right] f_2(\theta_2|\theta_1). \quad (40)$$

Next, recall from (12) that, with probability one, $w' \left(v(c_{1,1}^*(\theta_1)) \right) = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[w' \left(v(c_{1,2}^*(\theta_1, \tilde{\theta}_2)) \right) \right]$. Moreover, $c_{1,2}^*(\theta_1, \cdot)$ must be non-decreasing (this follows from the fact that incentive compatibility requires that $\pi_2(\theta_1, \cdot)$ be non-decreasing, as established in Proposition 1). Therefore, there exists $\hat{\theta}_2(\theta_1) \in [\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$ such that $c_{1,2}^*(\theta_1, \theta_2) \leq c_{1,1}^*(\theta_1)$ for $\theta_2 \leq \hat{\theta}_2(\theta_1)$ and $c_{1,2}^*(\theta_1, \theta_2) > c_{1,1}^*(\theta_1)$ for $\theta_2 > \hat{\theta}_2(\theta_1)$. Using the property that $w'(v(\cdot))$ is increasing, together with (40), we then have that the function $m(\cdot; \theta_1)$ must be quasi-concave on $[\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$. Finally, note that $w' \left(v(c_{1,1}^*(\theta_1)) \right) = \mathbb{E}^{\tilde{\theta}_2|\theta_1} \left[w' \left(v(c_{1,2}^*(\theta_1, \tilde{\theta}_2)) \right) \right]$, implies that $m(\underline{\theta}_2(\theta_1); \theta_1) = 0$. That $m(\bar{\theta}_2(\theta_1); \theta_1) = 0$, along with the property that $m(\cdot; \theta_1)$ is quasi-concave, establish the claim in (39). Similarly, it is easy to see that the inequality in (39) is strict, unless $c_{1,2}^*(\theta_1, \cdot)$ is constant over $[\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$.

Combining (38) with (39) permits us to conclude that, when $\gamma = 1$, the expected power of incentives is weakly lower in period 2 than in period 1 (strictly lower, unless, with probability one, $c_{1,2}^*(\theta_1, \cdot)$ is constant over $[\underline{\theta}_2(\theta_1), \bar{\theta}_2(\theta_1)]$).

Next consider Part (b). Suppose the result is not true. Then the following must be true.

Claim C. *There exists a sequence $(\gamma_n)_{n=1}^\infty$, with $\gamma_n \geq \gamma'$ all $n \in \mathbb{N}$, converging to 1 from below and such that, for all n ,*

$$\mathbb{E} \left[\psi' \left(\xi_{\gamma_n,1}^*(\tilde{\theta}_1) \right) w' \left(v(c_{\gamma_n,1}^*(\tilde{\theta}_1)) \right) \right] \leq \mathbb{E} \left[\psi' \left(\xi_{\gamma_n,2}^*(\tilde{\theta}_1, \gamma\tilde{\theta}_1 + \tilde{\varepsilon}) \right) w' \left(v(c_{\gamma_n,2}^*(\tilde{\theta}_1, \gamma\tilde{\theta}_1 + \tilde{\varepsilon})) \right) \right]. \quad (41)$$

Below we show that Claim C is inconsistent with the fact that, by assumption, when $\gamma = 1$,

$$\mathbb{E} \left[\psi' \left(\xi_{1,1}^* (\tilde{\theta}_1) \right) w' \left(v(c_{1,1}^* (\tilde{\theta}_1)) \right) \right] > \mathbb{E} \left[\psi' \left(\xi_{1,2}^* (\tilde{\theta}_1, \gamma \tilde{\theta}_1 + \tilde{\varepsilon}) \right) w' \left(v(c_{1,2}^* (\tilde{\theta}_1, \gamma \tilde{\theta}_1 + \tilde{\varepsilon})) \right) \right]. \quad (42)$$

We establish the inconsistency by means of three properties that jointly lead to a contradiction of the claim.

First note that, by assumption, each $\xi_{\gamma_n}^*$ and ξ_1^* are uniformly bounded, with the bound M uniform over n . This last property, along with (2) and (12), in turn imply that there exists $\bar{C} > 0$ such that $|c_{\gamma_n,1}^*|, |c_{1,1}^*|, |c_{\gamma_n,2}^*|, |c_{1,2}^*| \leq \bar{C}$ almost everywhere, and uniformly over n . Furthermore, from the optimality of the policies, one can easily see that there must exist $\bar{K} > 0$ such that $K_{\gamma_n}^* \leq \bar{K}$ for all n .

The following must then be true.

Property C1. *Assume Claim C is true. Then there exist $\epsilon > 0$ and $N \in \mathbb{N}$ such that, for all $n \geq N$, at least one of the following holds:*

$$\begin{aligned} \Pr \left(\left| \xi_{\gamma_n,1}^* (\tilde{\theta}_1) - \xi_{1,1}^* (\tilde{\theta}_1) \right| > \epsilon \right) &\geq \epsilon, \\ \Pr \left(\left| \xi_{\gamma_n,2}^* (\tilde{\theta}_1, \gamma_n \tilde{\theta}_1 + \tilde{\varepsilon}) - \xi_{1,2}^* (\tilde{\theta}_1, \tilde{\theta}_1 + \tilde{\varepsilon}) \right| > \epsilon \right) &\geq \epsilon, \\ \Pr \left(\left| c_{\gamma_n,1}^* (\tilde{\theta}_1) - c_{1,1}^* (\tilde{\theta}_1) \right| > \epsilon \right) &\geq \epsilon, \text{ or} \\ K_{\gamma_n}^* - K_1^* &\geq \epsilon. \end{aligned}$$

Proof of Property C1. Suppose Property C1 is false. Then there exists a subsequence (γ_{n_k}) such that $\left| \xi_{\gamma_{n_k},1}^* (\theta_1) - \xi_{1,1}^* (\theta_1) \right|, \left| \xi_{\gamma_{n_k},2}^* (\theta_1, \gamma_{n_k} \theta_1 + \varepsilon) - \xi_{1,2}^* (\theta_1, \theta_1 + \varepsilon) \right|, \left| c_{\gamma_{n_k},1}^* (\theta_1) - c_{1,1}^* (\theta_1) \right|$, and $\left| c_{\gamma_{n_k},2}^* (\theta_1, \gamma_{n_k} \theta_1 + \varepsilon) - c_{1,2}^* (\theta_1, \theta_1 + \varepsilon) \right|$ all converge in probability to zero, which, given the boundedness of the policies, implies that (41) and (42) are mutually inconsistent. ■

Now, abusing the notation introduced in the proof of Proposition 5, we let

$$h_\gamma (\xi, c_1, K) \equiv \mathbb{E} \left[\tilde{\theta}_1 + \xi_1 (\tilde{\theta}_1) + \tilde{\theta}_2 + \xi_2 (\tilde{\theta}) - c_1 (\tilde{\theta}_1) - w \left(W(\tilde{\theta}; \xi) + K - v(c_1(\tilde{\theta}_1)) \right) \right],$$

denote the firm's expected profits under the policies (ξ, c_1) , when the lowest period-1 type's expected payoff is K . Note that the dependence on γ is both directly through the fact that $\tilde{\theta}_2 = \gamma \tilde{\theta}_1 + \tilde{\varepsilon}$ as well as through the function $W(\cdot; \cdot)$ that, along with (ξ, c_1, K) , determines the period-2 compensation policy $c_2(\cdot)$ according to (7). Note that h_γ is strictly concave in ξ , $v(c_1)$ and K (this follows straightforwardly from the convexity of w and ψ). Strict concavity of $h_1(\xi, c_1, K)$, in particular, implies the following property (the result is obvious and hence the proof omitted).

Property C2. *There exists a function $\kappa : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ satisfying the following property. Take*

any $\epsilon > 0$ and any pair (ξ', c'_1, K') and (ξ'', c''_1, K'') satisfying at least one of the following

$$\begin{aligned} \Pr \left(\left| \xi'_1(\tilde{\theta}_1) - \xi''_1(\tilde{\theta}_1) \right| > \epsilon \right) &\geq \epsilon, \\ \Pr \left(\left| \xi'_2(\tilde{\theta}_1, \tilde{\theta}_1 + \tilde{\epsilon}) - \xi''_2(\tilde{\theta}_1, \tilde{\theta}_1 + \tilde{\epsilon}) \right| > \epsilon \right) &\geq \epsilon, \\ \Pr \left(\left| c'_1(\tilde{\theta}_1) - c''_1(\tilde{\theta}_1) \right| > \epsilon \right) &\geq \epsilon, \text{ or} \\ K' - K'' &\geq \epsilon. \end{aligned}$$

Let (ξ''', c'''_1, K''') be defined by $\xi''' = \frac{1}{2}\xi' + \frac{1}{2}\xi''$, $c'''_1 = w(\frac{1}{2}v(c'_1) + \frac{1}{2}v(c''_1))$, and $K''' = \frac{1}{2}K' + \frac{1}{2}K''$. Then $h_1(\xi''', c'''_1, K''') \geq \kappa(\epsilon)$.

Next, we use the boundedness of the optimal policies to establish the following property.

Property C3. Assume Claim C is true. For all $\theta \in \Theta$, all $n \in \mathbb{N}$, let $\xi'_{\gamma_n,1}(\theta_1) = \xi^*_{\gamma_n,1}(\theta_1)$, $\xi'_{\gamma_n,2}(\theta) = \xi^*_{\gamma_n,2}(\theta_1, \gamma_n\theta_1 + \theta_2 - \theta_1)$, $c'_{\gamma_n,1}(\theta_1) = c^*_{\gamma_n,1}(\theta_1)$, and $K'_{\gamma_n} = K^*_{\gamma_n}$. Then $h_1(\xi'_{\gamma_n}, c'_{\gamma_n,1}, K'_{\gamma_n}) \rightarrow h_1(\xi^*_1, c^*_{1,1}, K^*_1)$ as $n \rightarrow +\infty$.

Proof of Property C3. Our approach to the proof is as follows. We construct, for each n , a policy $(\xi^{\#}_{\gamma_n}, c^{\#}_{\gamma_n,1}, K^{\#}_{\gamma_n})$ which (together with $c^{\#}_{\gamma_n,2}$ defined by (7)) is implementable when $\gamma = \gamma_n$. We choose $(\xi^{\#}_{\gamma_n}, c^{\#}_{\gamma_n,1}, K^{\#}_{\gamma_n})$ in particular so that

$$h_{\gamma_n}(\xi^{\#}_{\gamma_n}, c^{\#}_{\gamma_n,1}, K^{\#}_{\gamma_n}) \rightarrow h_1(\xi^*_1, c^*_{1,1}, K^*_1) \quad (43)$$

as $n \rightarrow \infty$. Similarly, for each n , we construct a policy $(\xi^{\#\#}_{\gamma_n}, c^{\#\#}_{\gamma_n,1}, K^{\#\#}_{\gamma_n})$ which (together with $c^{\#\#}_{\gamma_n,2}$ defined by (7)) is implementable for $\gamma = 1$. Moreover, $(\xi^{\#\#}_{\gamma_n}, c^{\#\#}_{\gamma_n,1}, K^{\#\#}_{\gamma_n})$ is chosen so that

$$h_1(\xi^{\#\#}_{\gamma_n}, c^{\#\#}_{\gamma_n,1}, K^{\#\#}_{\gamma_n}) \rightarrow h_{\gamma_n}(\xi^*_{\gamma_n}, c^*_{\gamma_n,1}, K^*_{\gamma_n}) \quad (44)$$

as $n \rightarrow \infty$. These observations, together with the fact that $h_{\gamma_n}(\xi^{\#}_{\gamma_n}, c^{\#}_{\gamma_n,1}, K^{\#}_{\gamma_n}) \leq h_{\gamma_n}(\xi^*_{\gamma_n}, c^*_{\gamma_n,1}, K^*_{\gamma_n})$ and $h_1(\xi^{\#\#}_{\gamma_n}, c^{\#\#}_{\gamma_n,1}, K^{\#\#}_{\gamma_n}) \leq h_1(\xi^*_1, c^*_{1,1}, K^*_1)$ for each n , then imply that

$$h_{\gamma_n}(\xi^*_{\gamma_n}, c^*_{\gamma_n,1}, K^*_{\gamma_n}) \rightarrow h_1(\xi^*_1, c^*_{1,1}, K^*_1) \quad (45)$$

as $n \rightarrow +\infty$. The result in Property C3 then follows from (45) by considering the functional

$$\begin{aligned} &\hat{h}_{\hat{\gamma}}(\xi, c_1, K; \gamma) \\ = &\mathbb{E} \left[-w \left(\begin{aligned} &\tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) + \hat{\gamma}\tilde{\theta}_1 + \tilde{\epsilon} + \xi_2(\tilde{\theta}_1, \gamma\tilde{\theta}_1 + \tilde{\epsilon}) - c_1(\tilde{\theta}_1) \\ &\psi(\xi_1(\tilde{\theta}_1)) + \psi(\xi_2(\tilde{\theta}_1, \gamma\tilde{\theta}_1 + \tilde{\epsilon})) + \int_{\underline{\theta}_1}^{\theta_1} \{ \psi'(\xi_1(s)) + \hat{\gamma}\mathbb{E}^{\tilde{\epsilon}}[\psi'(\xi_2(s, \gamma s + \tilde{\epsilon}))] \} ds \\ &+ \int_{\underline{\epsilon}}^{\tilde{\epsilon}} \psi'(\xi_2(\tilde{\theta}_1, \gamma\tilde{\theta}_1 + s)) ds - \mathbb{E}^{\tilde{\epsilon}} \left[\int_{\underline{\epsilon}}^{\tilde{\epsilon}} \psi'(\xi_2(\tilde{\theta}_1, \gamma\tilde{\theta}_1 + s)) ds \right] \\ &+ K - v(c_1(\tilde{\theta}_1)) \end{aligned} \right) \right]. \end{aligned}$$

In particular, it follows from observing that $\hat{h}_{\hat{\gamma}}(\xi, c_1, K; \gamma)$ is continuous in $\hat{\gamma}$ uniformly over $\gamma \in [\gamma', 1]$ and over policies (ξ, c_1, K) satisfying the aforementioned bounds, and that $h_{\gamma_n}(\xi_{\gamma_n}^*, c_{\gamma_n,1}^*, K_{\gamma_n}^*) = \hat{h}_{\gamma_n}(\xi_{\gamma_n}^*, c_{\gamma_n,1}^*, K_{\gamma_n}^*; \gamma_n)$ while $h_1(\xi'_{\gamma_n}, c'_{\gamma_n,1}, K'_{\gamma_n}) = \hat{h}_1(\xi_{\gamma_n}^*, c_{\gamma_n,1}^*, K_{\gamma_n}^*; \gamma_n)$.

Our construction of $(\xi_{\gamma_n}^{\#}, c_{\gamma_n,1}^{\#}, K_{\gamma_n}^{\#})$ for each n is as follows. Let $K_{\gamma_n}^{\#} = K_1^* + 2(1 - \gamma_n)M(\bar{\theta}_1 - \theta_1)$. Then, let $\xi_{\gamma_n,1}^{\#}(\theta_1) = \gamma_n \xi_{1,1}^*(\theta_1)$ and $\xi_{\gamma_n,2}^{\#}(\theta_1, \theta_2) = \xi_{1,2}^*(\theta_1, \theta_1 + (\theta_2 - \gamma_n \theta_1))$. Finally, let $c_{\gamma_n,1}^{\#} = c_{1,1}^*$. Note that, since $(\xi_1^*, c_{1,1}^*, K_1^*)$, together with (7), defines an implementable policy when $\gamma = 1$, $(\xi_{\gamma_n}^{\#}, c_{\gamma_n,1}^{\#}, K_{\gamma_n}^{\#})$ also defines an implementable policy when $\gamma = \gamma_n$. This is verified with respect to the conditions in Proposition 1. The only condition that is not immediate to check is B(i), or (using that ψ is quadratic) that, for all $\theta_1, \hat{\theta}_1$,

$$\int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{\gamma_n,1}^{\#}(\hat{\theta}_1) + \hat{\theta}_1 + \gamma_n \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^{\#}(\hat{\theta}_1, \gamma_n s + \tilde{\varepsilon}) \right] \right\} ds \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{\gamma_n,1}^{\#}(s) + s + \gamma_n \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^{\#}(s, \gamma_n s + \tilde{\varepsilon}) \right] \right\} ds,$$

which, substituting for $\xi_{\gamma_n}^{\#}$, we can rewrite as

$$\begin{aligned} & \gamma_n \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{1,1}^*(\hat{\theta}_1) + \hat{\theta}_1 + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{1,2}^*(\hat{\theta}_1, \gamma_n s + \tilde{\varepsilon} + (1 - \gamma_n)\hat{\theta}_1) \right] \right\} ds - (1 - \gamma_n) \int_{\hat{\theta}_1}^{\theta_1} (s - \hat{\theta}_1) ds \\ & \leq \gamma_n \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{1,1}^*(s) + s + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{1,2}^*(s, s + \tilde{\varepsilon}) \right] \right\} ds. \end{aligned} \quad (46)$$

To see that (46) must hold, note that, because ξ_1^* is implementable, $\xi_{1,2}^*(\theta_1, \theta_2) + \theta_2$ is non-decreasing in θ_2 , and so the left-hand side is no greater than

$$\begin{aligned} & \gamma_n \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{1,1}^*(\hat{\theta}_1) + \hat{\theta}_1 + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{1,2}^*(\hat{\theta}_1, s + \tilde{\varepsilon}) \right] + (1 - \gamma_n)(s - \hat{\theta}_1) \right\} ds - (1 - \gamma_n) \int_{\hat{\theta}_1}^{\theta_1} (s - \hat{\theta}_1) ds \\ & = \gamma_n \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{1,1}^*(\hat{\theta}_1) + \hat{\theta}_1 + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{1,2}^*(\hat{\theta}_1, s + \tilde{\varepsilon}) \right] \right\} ds - (1 - \gamma_n)^2 \int_{\hat{\theta}_1}^{\theta_1} (s - \hat{\theta}_1) ds \end{aligned}$$

That (46) holds then follows because condition B(i) of Proposition 1 holds for ξ_1^* , i.e.

$$\int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{1,1}^*(\hat{\theta}_1) + \hat{\theta}_1 + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{1,2}^*(\hat{\theta}_1, s + \tilde{\varepsilon}) \right] \right\} ds \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{1,1}^*(s) + s + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{1,2}^*(s, s + \tilde{\varepsilon}) \right] \right\} ds,$$

since ξ_1^* is an implementable policy.

To see that $(\xi_{\gamma_n}^{\#}, c_{\gamma_n,1}^{\#}, K_{\gamma_n}^{\#})$ satisfies (43), let $\zeta_{\gamma}(\theta_1, \varepsilon) = \xi_2(\theta_1, \gamma\theta_1 + \varepsilon)$ and note that

$$\begin{aligned} h_{\gamma}(\xi, c_1, K) &= \mathbb{E}^{(\tilde{\theta}_1, \tilde{\varepsilon})} \left[-w \begin{pmatrix} \tilde{\theta}_1 + \xi_1(\tilde{\theta}_1) + \gamma\tilde{\theta}_1 + \tilde{\varepsilon} + \zeta_{\gamma}(\tilde{\theta}_1, \tilde{\varepsilon}) - c_1(\tilde{\theta}_1) \\ \psi(\xi_1(\tilde{\theta}_1)) + \psi(\zeta_{\gamma}(\tilde{\theta}_1, \tilde{\varepsilon})) \\ + \int_{\tilde{\theta}_1}^{\theta_1} \left\{ \psi'(\xi_1(s)) + \gamma \mathbb{E}^{\tilde{\varepsilon}} \left[\psi'(\zeta_{\gamma}(s, \tilde{\varepsilon})) \right] \right\} ds \\ + \int_{\tilde{\varepsilon}}^{\tilde{\varepsilon}} \psi'(\zeta_{\gamma}(\tilde{\theta}_1, s)) ds - \mathbb{E}^{\tilde{\varepsilon}} \left[\int_{\tilde{\varepsilon}}^{\tilde{\varepsilon}} \psi'(\zeta_{\gamma}(\tilde{\theta}_1, s)) ds \right] \\ + K - v(c_1(\tilde{\theta}_1)) \end{pmatrix} \right] \quad (47) \\ &\equiv d_{\gamma}(\xi_1, \zeta_{\gamma}, c_1, K). \end{aligned}$$

Let $\zeta_1^*(\theta_1, \varepsilon) = \xi_{1,2}^*(\theta_1, \theta_1 + \varepsilon)$ and, for each n , let $\zeta_{\gamma_n}^\#(\theta_1, \varepsilon) = \xi_{\gamma_n,2}^\#(\theta_1, \gamma_n \theta_1 + \varepsilon)$. Now let $\mathcal{E}_1(M)$ be the space of first-period effort policies ξ_1 bounded by M , and endow this space with the sup norm. Let $\mathcal{Z}(M)$ denote the space of functions $\zeta(\theta_1, \varepsilon)$ (essentially) bounded by M , and let $\mathcal{C}_1(\bar{C})$ denote the space of functions $c_1(\theta_1)$ (essentially) bounded by \bar{C} . Then note that $d_\gamma(\xi_1, \zeta, c_1, K)$ is continuous in (γ, ξ_1, K) uniformly over $[\gamma', 1] \times \mathcal{E}_1(M) \times \mathcal{Z}(M) \times \mathcal{C}_1(\bar{C}) \times [0, \bar{K} + 2(1 - \gamma')M(\bar{\theta}_1 - \underline{\theta}_1)]$. Moreover, by construction, for all n , and for all $(\theta_1, \varepsilon) \in \Theta_1 \times [\underline{\varepsilon}, \bar{\varepsilon}]$, $\zeta_{\gamma_n}^\#(\theta_1, \varepsilon) = \zeta_1^*(\theta_1, \varepsilon)$, and $c_{\gamma_n,1}^\#(\theta_1) = c_{1,1}^*(\theta_1)$. These observations, together with the fact that $(\gamma_n, \xi_{\gamma_n,1}^\#, K_{\gamma_n}^\#)$ converges uniformly to $(1, \xi_{1,1}^*, K_1^*)$, then imply (43).

Next, we construct $(\xi_{\gamma_n}^{\#\#}, c_{\gamma_n,1}^{\#\#}, K_{\gamma_n}^{\#\#})$. Let $\xi_{\gamma_n,1}^{\#\#}(\theta_1) = \frac{1}{\gamma_n} \xi_{\gamma_n,1}^*(\theta_1) + \left(2Mb(1 - \gamma_n) + \frac{1 - \gamma_n}{\gamma_n}\right) \theta_1$ and $\xi_{\gamma_n,2}^{\#\#}(\theta_1, \theta_2) = \xi_{\gamma_n,2}^*(\theta_1, \gamma_n \theta_1 + (\theta_2 - \theta_1))$. Let

$$K_{\gamma_n}^{\#\#} = K_1^* + (\bar{\theta}_1 - \underline{\theta}_1) \left(\frac{1}{\gamma_n} - \gamma_n\right) M.$$

Finally, let $c_{\gamma_n,1}^{\#\#} = c_{1,1}^*$. Note that, since $(\xi_{\gamma_n}^*, c_{\gamma_n,1}^*, K_{\gamma_n}^*)$, together with (7), defines an implementable policy when $\gamma = \gamma_n$, $(\xi_{\gamma_n}^{\#\#}, c_{\gamma_n,1}^{\#\#}, K_{\gamma_n}^{\#\#})$ also defines an implementable policy when $\gamma = 1$. Again, this is verified by considering Proposition 1. The only condition which is not immediate to check is B(i), or (using that ψ is quadratic) that, for all $\theta_1, \hat{\theta}_1$,

$$\int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{\gamma_n,1}^{\#\#}(\hat{\theta}_1) + \hat{\theta}_1 + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^{\#\#}(\hat{\theta}_1, s + \tilde{\varepsilon}) \right] \right\} ds \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{\gamma_n,1}^{\#\#}(s) + s + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^{\#\#}(s, s + \tilde{\varepsilon}) \right] \right\} ds, \quad (48)$$

which, substituting for $\xi_{\gamma_n}^{\#\#}$, we can rewrite as

$$\begin{aligned} & \int_{\hat{\theta}_1}^{\theta_1} \left\{ \frac{1}{\gamma_n} \left(\xi_{\gamma_n,1}^*(\hat{\theta}_1) + \hat{\theta}_1 \right) + 2\hat{\theta}_1 Mb(1 - \gamma_n) + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^*(\hat{\theta}_1, \gamma_n s + \tilde{\varepsilon} + (1 - \gamma_n)(s - \hat{\theta}_1)) \right] \right\} ds \\ & \leq \int_{\hat{\theta}_1}^{\theta_1} \left\{ \frac{1}{\gamma_n} \left(\xi_{\gamma_n,1}^*(s) + s \right) + 2s Mb(1 - \gamma_n) + \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^*(s, \gamma_n s + \tilde{\varepsilon}) \right] \right\} ds. \end{aligned}$$

Then note that, for any $\hat{\theta}_1, s$,

$$\begin{aligned} & \left| \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^*(\hat{\theta}_1, \gamma_n s + \tilde{\varepsilon} + (1 - \gamma_n)(s - \hat{\theta}_1)) \right] - \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^*(\hat{\theta}_1, \gamma_n s + \tilde{\varepsilon}) \right] \right| \\ & = \left| \int_{\min\{\underline{\varepsilon} + (1 - \gamma_n)(s - \hat{\theta}_1), \underline{\varepsilon}\}}^{\max\{\bar{\varepsilon} + (1 - \gamma_n)(s - \hat{\theta}_1), \bar{\varepsilon}\}} \xi_{\gamma_n,2}^*(\hat{\theta}_1, \gamma_n s + \varepsilon) d \left[G(\varepsilon - (1 - \gamma_n)(s - \hat{\theta}_1)) - G(\varepsilon) \right] \right| \\ & \leq 2Mb(1 - \gamma_n) (s - \hat{\theta}_1), \end{aligned} \quad (49)$$

where the inequality follows because the density of $\tilde{\varepsilon}$ is bounded by b (which is equivalent to our requirement that the density $f_2(\theta_2|\theta_1)$ is bounded). The inequality (49), together with

$$\begin{aligned} & \frac{1}{\gamma_n} \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{\gamma_n,1}^*(\hat{\theta}_1) + \hat{\theta}_1 + \gamma_n \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^*(\hat{\theta}_1, \gamma_n s + \tilde{\varepsilon}) \right] \right\} ds \\ & \leq \frac{1}{\gamma_n} \int_{\hat{\theta}_1}^{\theta_1} \left\{ \xi_{\gamma_n,1}^*(s) + s + \gamma_n \mathbb{E}^{\tilde{\varepsilon}} \left[\xi_{\gamma_n,2}^*(s, \gamma_n s + \tilde{\varepsilon}) \right] \right\} ds \end{aligned}$$

(which holds, since ξ_1^* is an implementable effort policy), implies (48).

Finally, (44) follows by arguments analogous to those for (43). \blacksquare

We are now ready to establish that Claim C is false. Let $\xi_{\gamma_n,1}^{1/2}(\theta_1) = \frac{1}{2}\xi'_{\gamma_n,1}(\theta_1) + \frac{1}{2}\xi_{1,1}^*(\theta_1)$, $\xi_{\gamma_n,2}^{1/2}(\theta) = \frac{1}{2}\xi'_{\gamma_n,2}(\theta) + \frac{1}{2}\xi_{1,2}^*(\theta)$, $c_{1,\gamma_n}^{1/2}(\theta_1) = w\left(\frac{1}{2}v\left(c'_{\gamma_n,1}(\theta_1)\right) + \frac{1}{2}v\left(c_{1,1}^*(\theta_1)\right)\right)$ and $K_{\gamma_n}^{1/2} = \frac{1}{2}K'_{\gamma_n} + \frac{1}{2}K_1^*$, where $(\xi'_{\gamma_n}, c'_{\gamma_n,1}, K'_{\gamma_n})$ is defined in Property C3. Recall the construction of $(\xi_{\gamma_n}^{\#\#\#}, c_{\gamma_n,1}^{\#\#\#}, K_{\gamma_n}^{\#\#\#})$ in the proof of Property C3, and recall that this policy is implementable when $\gamma = 1$. Then let $\hat{\xi}_{\gamma_n}^{1/2} = \frac{1}{2}\xi_{\gamma_n}^{\#\#\#} + \frac{1}{2}\xi_1^*$, $\hat{c}_{1,\gamma_n}^{1/2} = w\left(\frac{1}{2}v\left(c_{\gamma_n,1}^{\#\#\#}\right) + \frac{1}{2}v\left(c_{1,1}^*\right)\right)$ and $\hat{K}_{\gamma_n}^{1/2} = \frac{1}{2}K_{\gamma_n}^{\#\#\#} + \frac{1}{2}K_1^*$. Using that $(\hat{\xi}_{\gamma_n}^{1/2}, \hat{c}_{1,\gamma_n}^{1/2}, \hat{K}_{\gamma_n}^{1/2})$ and $(\xi_{\gamma_n}^{1/2}, c_{1,\gamma_n}^{1/2}, K_{\gamma_n}^{1/2})$ are uniformly (essentially) bounded (and hence the continuity of $h_1(\cdot, \cdot, \cdot)$ over the bounded policies), we have

$$h_1\left(\hat{\xi}_{\gamma_n}^{1/2}, \hat{c}_{1,\gamma_n}^{1/2}, \hat{K}_{\gamma_n}^{1/2}\right) - h_1\left(\xi_{\gamma_n}^{1/2}, c_{1,\gamma_n}^{1/2}, K_{\gamma_n}^{1/2}\right) \rightarrow 0 \quad (50)$$

as $n \rightarrow \infty$.

Now note that, if Claim C were true, by virtue of Properties C1 and C2, we have

$$h_1\left(\xi_{\gamma_n}^{1/2}, c_{1,\gamma_n}^{1/2}, K_{\gamma_n}^{1/2}\right) \geq \frac{1}{2}h_1\left(\xi'_{\gamma_n}, c'_{\gamma_n,1}, K'_{\gamma_n}\right) + \frac{1}{2}h_1\left(\xi_1^*, c_{1,1}^*, K_1^*\right) + \kappa(\epsilon) \quad (51)$$

for all $n \geq N$. By the inequality (51), the fact that $\kappa(\epsilon) > 0$, Property C3, and (50), we conclude that, for all large enough n ,

$$h_1\left(\hat{\xi}_{\gamma_n}^{1/2}, \hat{c}_{1,\gamma_n}^{1/2}, \hat{K}_{\gamma_n}^{1/2}\right) > h_1\left(\xi_1^*, c_{1,1}^*, K_1^*\right).$$

However, note that $(\hat{\xi}_{\gamma_n}^{1/2}, \hat{c}_{1,\gamma_n}^{1/2}, \hat{K}_{\gamma_n}^{1/2})$ defines (together with (7)) an implementable policy for $\gamma = 1$ (this follows because both $(\xi_{\gamma_n}^{\#\#\#}, c_{\gamma_n,1}^{\#\#\#}, K_{\gamma_n}^{\#\#\#})$ and $(\xi_1^*, c_{1,1}^*, K_1^*)$ are implementable for $\gamma = 1$, and by the conditions in Proposition 1; in particular, because ψ is quadratic, the convex combination of any two effort policies satisfying condition B(i) in Proposition 1 continues to satisfy this condition). This contradicts the optimality of $(\xi_1^*, c_{1,1}^*, K_1^*)$. That Claim C is false then implies the result in Part (b) in the proposition is true, which concludes the proof. Q.E.D.

Proof of Proposition 7. To establish the necessity of (20) and (21), consider the perturbed effort policy $\xi_1(\theta_1) = \xi_1^R(\theta_1) + a\nu(\theta_1)$ and $\xi_2(\theta) = \xi_2^R(\theta) + b\omega(\theta)$ for scalars a and b and measurable functions $\nu(\cdot)$ and $\omega(\cdot)$. Then differentiate the firm's profits (8) with respect to a and b respectively. A necessary condition for the proposed policy ξ^R to maximize (8) is that these derivatives, evaluated at $a = b = 0$ vanish for all measurable functions $\nu(\cdot)$ and $\omega(\cdot)$. This is true only if ξ^R satisfies (20) and (21) with probability one.

Uniqueness of ξ^R and c^R , as well as the necessity of (12), follow from the same arguments as in the proof of Proposition 2.