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Correlation of Types in Bayesian Games

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July 2012

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: dependence of types, pure strategy equilibrium existence, affiliation, games with incomplete information, quasi-supermodular games, revenue ranking of auctions.

Correlation of Types in Bayesian Games*

Luciano De Castro[†]

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Abstract

Despite their importance, games with incomplete information and dependent types are poorly understood; only special cases have been considered and a general approach is not yet available. In this paper, we propose a new condition (named *richness*) for correlation of types in (asymmetric) Bayesian games. Richness is related to the idea that "beliefs do not determine preferences" and that types should be modeled with two explicit parts: one for payoffs and another for beliefs. With this condition, we are able to provide the first pure strategy equilibrium existence result for a general model of multi-unit auctions with correlated types. We then focus on a special case of richness, called "grid distributions," and establish necessary and sufficient conditions for the existence of a symmetric monotonic pure strategy equilibrium in first-price auctions with general levels of correlation. We also provide a polynomial-time algorithm to verify this existence and suggest, using simulations, that the revenue superiority of English auctions may not hold for positively correlated types in general.

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[‡]This paper supersedes de Castro (2008): "Grid Distributions to Study Single Object Auctions".

Contents

1 Introduction

Suppose that you want to auction off the assets of a bankrupt company or public licenses (e.g. for spectrum, offshore drilling, etc.). An important question is what auction format you should use. As you look for guidance in the economic literature, you find out that a fundamental result—the revenue equivalence theorem—suggests that such a format does not matter. Unfortunately you are not relieved, because you also find out that such a result hinges on the rather restrictive conditions of symmetry of bidders and independence of their private information. As you quickly realize, the potential buyers of your assets are not symmetric and most likely do not have statistically independent assessments. What to do?

This question is certainly not new; it goes back to the origins of auction theory itself, with Vickrey (1961). Early efforts to tackle this question were made by Wilson (1969), (1977) and Milgrom and Weber (1982), generating a huge literature. Unfortunately, this literature is yet unable to successfully deal with statistical dependence of private information (types), specially under asymmetries.[1] Even the basic problem of pure strategy equilibria existence when types are correlated remains unsettled.

This paper suggests that the origin of the difficulties is the "usual approach" to modeling dependence in private information settings. By "usual approach" we mean the practice of assuming that a single variable v_i conveys two different pieces of information: player i's information about her payoff and her beliefs about other players' parameters. These beliefs are usually defined through the conditional probabilities $\gamma(\cdot|v_i)$ of a common prior γ.[2] This usual approach or model is widely adopted, and its main justification seems to be its simplicity; see section 7.2. An implication of our results is that the theory becomes more tractable if we *do not* make this simplifying assumption.

We suggest to depart from this usual model and explicitly describe each type t_i as composed of two separate parts: a preference parameter v_i and a belief parameter δ_i.[3] For instance, player i can receive a bidimensional signal $t_i = (v_i, \delta_i)$, such that the first dimension v_i determines the value

[1] See section 7.1 for a further discussion of the received literature.

[2] Although nowadays collapsing players' beliefs and tastes in this way seems very natural, when Harsanyi (1967-8) introduced the idea of types, he was careful to maintain different parameters for tastes and beliefs.

[3] Actually, our theory also covers some interesting cases of the usual model; see section 2.1.1. In particular, it covers the case of unidimensional grid distributions, defined below. See also discussion in section 7.2.

of the objective for herself and the second signal δ_i is correlated with a "state of the world" variable ω, which is unknown but affects every player's payoffs. Actually, from the perspective of universal type spaces introduced by Mertens and Zamir (1985), types can always be seen as having two parts: a payoff type and a belief type. Therefore, the explicit consideration of the beliefs goes in the direction of a more general model.[4]

A simplistic summary of this paper is the following: as long as we model players' information as composed of these two separated parts, we can efficaciously tackle the study of dependence. Moreover, we can study dependence using tools and techniques that have been so far restricted to models with independence. Of course, this rough summary is incomplete and requires more details and formalization. The main contribution of this paper is the introduction of the following condition on type spaces:[5]

Richness: Any non-null set of types contains two strictly ordered types sharing the same belief.

The remainder of this introduction discusses what richness means, describes the main results that it allows us to prove, and comments on other technical contributions of this paper.

What richness means

First, observe that the most important aspect in *richness* is the fact that the two types share the same belief. Indeed, the order implicitly assumed to exist by the condition can be constructed from the payoff part of each type. For instance, if the payoff part V_i refers to the values of the objects in a multi-unit auction, we could order its elements v_i using the standard coordinate-wise order of euclidean spaces. Then, any set with positive measure will always contain two strictly ordered signals if the measure is nonatomic—see details in section 2.1. In this sense, the belief part is the most important restriction and hereafter we will focus on this aspect. Let us begin by describing situations where it is satisfied:

(a) A usual setting with independence. Indeed, if types are independent, then any signals v_i, v_i' imply the same (conditional probability or) belief

[4] It seems that Neeman (2004) was the first to explicitly argue that beliefs and preferences should be seen as "causally independent" from one another. As we discuss in more detail in section 7.2, our idea is very related to his "beliefs do not determine preferences" assumption. See also Heifetz and Neeman (2006).

[5] The condition implicitly requires that types spaces are ordered, as we further discuss below.

$\delta_i = \gamma(\cdot|v_i) = \gamma(\cdot|v_i')$ about other players' types. That is, any pair of ordered types share exactly the same beliefs.

(b) A usual setting where there is only a countable number of different beliefs $\delta_i = \gamma(\cdot|v_i)$. Notice that independence is just a special case of this, where there is just *one* belief.

(c) A setting where each player receives a two dimensional signal $t_i = (v_i, \delta_i)$, where v_i affects directly her utility (her private value component), and δ_i is a unidimensional signal correlated with ω, the state of nature (common value component). The utility is $u_i(t_i, \omega, a)$, where a is the profile of actions. The s_i's are independent, while the δ_i's have any kind of non-degenerated dependence with ω.

(d) T_i is (a subset of) $V_i \times \Delta(T_{-i})$ and the measure on T_i is absolutely continuous with respect to the product of its marginals over V_i and $\Delta(T_{-i})$.[6]

These examples will be fully specified and justified in section 2.1.

Of course, it is also useful to know settings where richness is not satisfied. A simple example is a usual setting with a uniform distribution γ over the triangle defined by $0 \leqslant v_1 \leqslant v_2 \leqslant 1$. In this case, if player 1 has signal v_1, her belief about player 2's signal is the uniform distribution over $[v_1, 1]$. Therefore, richness cannot be satisfied because there are not two different signals sharing the same belief. Fortunately, however, for every usual setting where richness is not satisfied, there exists another setting sufficiently "close" that satisfies richness. Indeed, assume that with probability $1 - \varepsilon$, the types and beliefs are just as described and, with probability $\varepsilon > 0$, when receiving the signal v_i, player i believes that other players' signals are uniformly distributed on $[0, 1]$. As we further discuss in section 2, this is "ε-close" to the original model and satisfies richness. Therefore, richness does not add any significant extra restriction than that already imposed by the game model itself.

Having clarified some aspects about richness, it is time to describe the main results in this paper and other technical contributions, which could be of interest by themselves. But before proceeding, it is convenient to explain why richness can be useful at all. In allowing us to work with sets of types that have the same beliefs, richness captures the main simplifying aspect of *independence*, without being as restrictive. Indeed, when moving

[6] This is just a generalization of the previous case.

across types that share the same beliefs, we maintain the measure with respect to which we integrate other players' types fixed. This property is the single most important aspect in using richness to obtain our results.

1.1 Main Results

The main results of the paper can be summarized as follows.

1. For Bayesian games with (possibly infinite dimensional) action spaces satisfying some weak assumptions (details in section 3), utility functions satisfying a generalization of supermodularity and increasing differences and type spaces satisfying richness, we show that every best reply to any mixed strategies is pure. That is, we give conditions under which strictly mixed strategies are never best replies. From this, it follows that *any* equilibrium must be in pure strategies. See Theorem 4.3 in section 4.

2. Building on this first result, we turn to a more particular setting: that of multi-unit auctions studied by Jackson and Swinkels (2005). In this setting, we show that if richness is satisfied, there exists a monotonic pure strategy equilibrium. This is the first result in the literature that establishes existence in pure strategies for general multi-unit auctions out of independence. See Theorem 5.1 in section 5.

3. Although the equilibrium mentioned in 2 above is in "monotonic" strategies, this monotonicity depends on a special order on types, which may be different from standard ones. For example, it may different from the standard real order in single-unit auctions with unidimensional signals. This raises the question whether the pure strategy equilibria shown to exist will also be in monotonic strategies taking in account the standard order on real-valued signals. Considering a symmetric first price auction in the setting of grid distributions described in point (iv) in section 1.2 below—, we establish *necessary and sufficient* conditions for the existence of a monotonic equilibrium. Moreover, since the setting is specially suitable for numerical simulations, we are able to establish an algorithm for checking when there is or there is not an equilibrium. The algorithm is surprisingly fast. While the best known algorithms for finding mixed strategy equilibria in finite games run in exponential time, our algorithm requires only

$O(k^2)$ manipulations in an auction with 2 players and k intervals.[7] See Theorems 6.2 and 6.5. Section 6.3 shows how numerical methods could establish some results about the revenue ranking of first and second-price auctions with general kinds of correlation. Moreover, we show in section 6.5 that results obtained in this framework satisfy approximate results in any usual setting.

As this list of results reveals, the results of this paper are not restricted to pure strategy equilibrium existence. In particular, it is not our main purpose to offer alternative methods for proving pure strategy equilibrium existence, as Reny (1999), Athey (2001), McAdams (2003) and Reny (2011) do. Rather, the main objective of the paper is to propose a new approach to the correlation of types in Bayesian games, establish its foundations and provide some venues for applications.

1.2 Other Contributions

Besides the ideas in the proofs of the above mentioned results, we introduce some technical contributions that can be of interest by themselves. These are the following:

(i) Our basic monotonicity theorem requires quasi-supermodularity and strict single-crossing, a condition that some games do not exactly satisfy. However, we show that a simple perturbation technique allows to approximate such games with games that do satisfy such conditions. This technique is at the heart of the proof of pure strategy equilibrium existence in the private value multi-unit auctions (Theorem 5.1) and extends the usefulness of our approach.

(ii) In the context of multi-unit auctions, we define a condition on tie-breaking rules (Assumption 5.2) that allows us to prove the modularity of allocations (Proposition 5.4) and payments (Corollary 8.19). This condition is satisfied by a specific tie-breaking rule used by McAdams (2003) and Reny (2011) for uniform-price and discriminatory price auctions, but includes other potentially relevant rules and allows general auction formats. These results are instrumental for our proof of Theorem 5.1.

[7] The number k of intervals is part of the definition of grid distributions; see point (iv) in section 1.2 below and formal definition in section 2.1.2.

(iii) We also introduce a condition (Assumption 3.2) about the interplay between the order and the metric of the action space that allows us to state our more general results for compact metric space, instead of just euclidean spaces. This condition is automatically satisfied in euclidean spaces with the usual coordinate-wise order, but also in many standard function spaces and other infinitely dimensional spaces. We are not aware of any similar condition being considered in the literature.

(iv) Finally, we introduce a special class of distributions, which we call "grid" distributions or "very simple" distributions; see section 2.1.2. Here, it is enough to define these distributions in the simple case of single dimensional types and just two players. For this, assume that the signals support is $[0, 1]$ and divide this interval into k equal pieces. A very simple distribution is any distribution defined by a density function which is constant in the squares thus formed. Figure 1 below illustrates this construction for $k = 2$ divisions of $[0, 1]$. To see that this satisfies richness, even without explicit mention to the beliefs, notice that once the player learns v_1, she also knows the interval where her value is, and every other signal in that interval shares exactly the same belief about the other bidder's valuation. Our results about the complete characterization of monotonic equilibrium existence in first price auctions (point 3 in section 1.1 above) is given in this setting. These distributions prove to be very convenient for theoretical and numerical manipulations, besides being dense in the set of all distributions—which allows approximation to any usual setting.

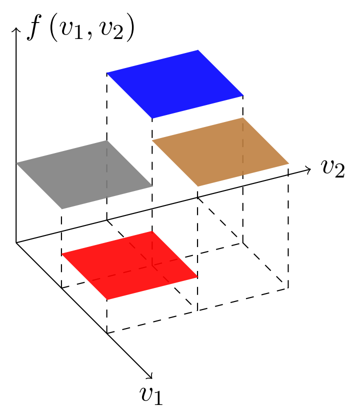

Figure 1: The density function of a grid distribution.

2 Correlated Types

In this section we introduce our model of correlated types and the *richness* condition.

Let $I \equiv \{1, ..., N\}$ be the set of players. Eventually we will consider a non-strategic player, numbered 0, but this player does not need to have types and can be thought as of "Nature." Each individual $i \in I$ has a type $t_i \in T_i$, where $(T_i, \mathcal{T}_i)$ is a measurable space. Let $T \equiv \times_{i=1}^{N} T_i$ and $T_{-i} \equiv \times_{j \neq i} T_j$. The product σ-algebra on T and T_{-i} are denoted $\mathcal{T}$ and $\mathcal{T}_{-i}$, respectively. For a measurable space $(X, \mathcal{X})$, let $\Delta(X)$ denote the set of probability measures on $(X, \mathcal{X})$. Notice that we do not impose any topological assumptions in this setup.[8]

Player i's type determines the beliefs of player i about other players. This will be given by a map $\widehat{\delta}_i : T_i \to \Delta(T_{-i})$. Finally, we denote by $\Delta_i \subset \Delta(T_{-i})$ the set of player i's possible beliefs about other players' types, that is, $\Delta_i \equiv \widehat{\delta}_i(T_i)$.

For some results, it will be convenient to assume that the types $t_i \in T_i$ are generated by a measure τ_i. If there is a common prior on $(T, \mathcal{T})$, denoted τ, we can define τ_i as T_i-marginal of τ, that is:

$$\tau_i(B) \equiv \tau(T_1 \times ... \times T_{i-1} \times B \times T_{i+1} \times ... \times T_N), \forall B \in \mathcal{T}_i.$$

In this case, the belief map $\widehat{\delta}_i$ could be defined as a regular conditional probability of τ given t_i. However, the existence of a common prior is not strictly necessary; it is enough to consider directly a measure τ_i on T_i.[9]

We assume that T_i is endowed with a partial order $\geqslant_i$.[10] Eventually, we will use just $\geqslant$ instead of $\geqslant_i$ if there is no possibility of confusion. As stated in the introduction, richness requires that any set of positive measure contains a pair of types that are strictly ordered and have the same belief. The following repeats this condition with symbols:

> **Richness:** If $E \in \mathcal{T}_i$ has positive measure, i.e., $\tau_i(E) > 0$, then $\exists t_i, t_i' \in E$ such that $t_i <_i t_i'$ and $\widehat{\delta}_i(t_i) = \widehat{\delta}_i(t_i')$.[11]

[8] Heifetz and Samet (1998) study type spaces without topology.

[9] The measure τ_i is used in the definition of richness, but analogous results can be stated even without it. See section 4.3.

[10] A binary relation $\geqslant$ on a set X is a partial order if t $\geqslant$ is transitive, reflexive and anti-symmetric. A binary relation $\geqslant$ is: *transitive* if for any $x, y, z \in X$ $x \geqslant y$ and $y \geqslant z$ implies $x \geqslant z$; *reflexive* if for any $x \in X$, $x \geqslant x$; and *anti-symmetric* if for any $x, y \in X$, $x \geqslant y$ and $y \geqslant x$ implies $x = y$. In this case, we also say that $(X, \geqslant)$ is a partially ordered set or poset.

[11] For any poset $(X, \geqslant)$ we will write $x > y$ if $x \geqslant y$ but it is not the case that $y \geqslant x$ (that is, $x \geqslant y$ and $y \neq x$). We will also write $x \leqslant y$ and $x < y$ if $y \geqslant x$ and $y > x$, respectively.

Richness is somewhat more demanding than "beliefs do not determine types," introduced by Neeman (2004). As Neeman's assumption, richness requires that there are at least two types sharing the same belief, that is, $\widehat{\delta}_i(t_i) = \widehat{\delta}_i(t_i')$. However, it also requires that those types are ordered, while Neeman (2004) considers no order. Moreover, richness requires that we can find such a pair of ordered types in every set of positive measure.

The requirement that there are ordered types in any set of positive measure is not strong in continuous type spaces. In this setting, it just guarantees that the partial order is not too restricted.[12]

The requirement that every set of positive measure contains two (strictly ordered) types rules out atoms in τ_i. If the type spaces have atoms, some games have equilibria only in strictly mixed strategies, which precludes our primary objective of investigating existence in pure strategies. On the other hand, assuming that types are atomless is far from sufficient to guarantee equilibrium in pure strategies. For instance, the type spaces in Jackson and Swinkels (2005) are atomless, but they are able to prove equilibrium existence only in mixed strategies when some kind of correlation between the types is possible. Moreover, Radner and Rosenthal (1982) and Khan, Rath, and Sun (1999) provide examples of games with atomless types and without pure strategy equilibria.

In section 2.1 below, we describe natural settings where richness holds.

2.1 Sufficient conditions and examples for richness

Let $V = \times_{i=1}^{N} V_i$, where $v_i \in V_i$ will represent the payoff or preference part of player i's type. For instance, v_i could represent the values of the objects to player i in a multi-unit private value auction. In this case, $V_i \subset \mathbb{R}_+^n$ has a natural order: the coordinate-wise order of euclidean spaces. Thus, we may assume that we have a natural partial order on V_i, which we will denote by $\succcurlyeq_i$. We can use this order to define a natural order on the type spaces as follows. Let $\widehat{V}_i : T_i \to V_i$ specify for each type $t_i \in T_i$, the preference (payoff) parameter $\widehat{V}_i(t_i) \in V_i$ that is known by t_i. We can then define the order on

[12] This condition fails in very restricted partial orders. For example, it fails for the partial order that specifies that two points are "ordered" only if they are equal. Indeed, in this case there are no points t_i, t_i' such that $t_i <_i t_i'$.

T_i by:[13]

$$t_i' \geqslant_i t_i \iff \widehat{V}_i(t_i') \succcurlyeq_i \widehat{V}_i(t_i). \tag{1}$$

For all constructions in this section, we will assume the order on T_i defined by (1). We will consider two classes of examples: based on the usual approach, in which the beliefs are given by common priors, and settings where T_i is a (subset of) $V_i \times \Delta(T_{-i})$.

2.1.1 Sufficient conditions on the usual setting

As we described in the introduction, we call the usual setting one in which the types (or signals, actually) are given directly on $V = \times_{i=1}^N V_i$ and the beliefs about other players' signals are given by a joint distribution γ on V. Of course, there is a σ-algebra Ξ such that (V, Ξ, γ) is a probability space. This space is the primitive in this model; we define a type space from this primitive as follows. Let T_i be defined as the following subset of $V_i \times \Delta(V_{-i})$:

$$T_i \equiv \{(v_i, \delta) \in V_i \times \Delta(V_{-i}) : \delta(\cdot) = \gamma(\cdot|v_i)\},$$

where $\gamma(\cdot|v_i)$ denotes the γ-conditional probability on v_{-i} given v_i. Note that this T_i corresponds to the original signal space, described in the language of type of parameter spaces and beliefs. To complete the definition of the type space $(T_i, \mathcal{T}_i, \tau_i)$, let γ_i denote the marginal of γ on V_i,[14] and define: $\mathcal{T}_i \equiv \widehat{V}_i^{-1}(\Xi_i)$ and $\tau_i(E) \equiv \gamma_i(\widehat{V}_i(E))$, for every $E \in \mathcal{T}_i$.

To state our results, we need to introduce some notation. Let Δ_i denote the set of possible beliefs, that is,

$$\Delta_i \equiv \{\delta \in \Delta(V_{-i}) : \exists v_i \in V_i \text{ such that } (v_i, \delta) \in T_i\}.$$

Also, let $\Gamma : V_i \rightrightarrows \Delta_i$ be the correspondence defined by

$$\Gamma(v) \equiv \widehat{\delta}_i\left(\widehat{V}_i^{-1}(v)\right) = \{\delta \in \Delta_i : \exists t_i \in T_i \text{ s.t. } \widehat{V}_i(t_i) = v \text{ and } \widehat{\delta}_i(t_i) = \delta\}. \tag{2}$$

We first show that if there is a countable set of beliefs that "corresponds" to almost all parameters, then richness holds if the parameters are "sufficiently" ordered. More precisely, we have the following:

[13] An alternative definition, sometimes convenient, is the following: $t_i' \geqslant_i t_i$ if and only if $\widehat{V}_i(t_i') \succcurlyeq_i \widehat{V}_i(t_i)$ and $\widehat{\delta}_i(t_i') = \widehat{\delta}_i(t_i)$, that is, we restrict (1) to be valid only for types with the same beliefs. All of our results remain valid without change under this alternative definition (some of them could be actually simplified). We use this alternative definition in our result about equilibrium existence on the JS auction. See (9).

[14] That is, $\gamma_i(E) \equiv \gamma(V_1 \times \cdots \times V_{i-1} \times E \times V_{i+1} \times \cdots \times V_N)$, for any measurable $E \subset V_i$.

Lemma 2.1 *The following conditions imply richness:*

(i) *For any measurable $E \subset V_i$, such that $\gamma_i(E) > 0$, there exist $v, v' \in E$ such that $v' \succ v$;*[15]

(ii) *There exists a countable set $\Delta_i' \subset \Delta_i$ such that for almost all $v \in V_i$, $\Gamma(v) \cap \Delta_i' \neq \emptyset$.*

It is useful to discuss Lemma 2.1's conditions. Condition (i) is satisfied if the measure on V_i is positive in the σ-algebra generated by sets of the form $[c, d] \equiv \{x \in T_i : c \leqslant_i x \leqslant_i d\}$. An example where this condition holds is the following: T_i is an euclidean space, the Lebesgue measure is absolutely continuous with respect to τ_i and $\leqslant_i$ is the standard coordinate-wise order.[16]

Condition (ii) is trivially satisfied if Δ_i is countable because $\Gamma(V_i) \subset \Delta_i$. In particular, if γ implies independent types, then Δ_i is unitary and condition (ii) is satisfied. However, condition (ii) does not require Δ_i to be countable, only that there is a countable set of beliefs that corresponds to every preference parameter. This possibility can be used to approximate usual type spaces where richness does not hold by type spaces where it does hold. We illustrate this construction as follows.

Fix some $\delta_0 \in \Delta(V_{-i})$ and establish that with probability $\varepsilon > 0$, player i with parameter type $v_i \in V_i$ has belief δ_0 (instead of $\gamma(\cdot|v_i)$) and with probability $1-\varepsilon$, she has the original belief $\delta(\cdot) = \gamma(\cdot|v_i)$. The belief δ_0 could be thought of as an "ignorant" or default belief. In this case, $\Delta_i' = \{\delta_0\}$ would satisfy condition (ii) of Lemma 2.1 and, therefore, richness would hold.[17]

Another class of usual models where richness holds is given by the grid distributions defined in section 2.1.2 below. This class of models can also approximate (in a strong sense) any usual model, as we show in section 6.5. The approximation results discussed in this section and in section 6.5 have

[15] We write $x \succ y$ if $x \succcurlyeq y$ but not $y \succcurlyeq x$.

[16] By itself, condition (i) is weaker than Reny (2011)'s assumption G3, which requires that there is a countable subset T_i^0 of T_i such that every set in $\mathcal{T}_i$ assigned positive probability by τ_i contains two points between which lies a point in T_i^0. As Reny (2011, Lemma A.21, p. 546) shows, in the case T_i is a separable metric space, Reny's assumption G3 is equivalent to the requirement that every atomless set with positive measure contains two strictly ordered points, that is, Lemma 2.1's condition (i).

[17] The new type space thus formed would be $T_i^\varepsilon = T_i \cup (V_i \times \{\delta_0\})$. The definition of $\mathcal{T}_i^\varepsilon$ would be the same as before, just using the map $\widehat{V}_i^\varepsilon$ instead of $\widehat{V}_i$, where $\widehat{V}_i^\varepsilon : T_i^\varepsilon \to V_i$ is defined in the obvious way as a projection. The definition of τ_i^ε would take in account the ε probability above described.

an important implication for applied work: richness cannot be refuted with any amount of finite data. Therefore, it does not add any significant extra restriction than that already imposed by the game model itself.

2.1.2 Very Simple (or Grid) Distributions

Perhaps the above discussion is yet too abstract for the taste of more applied readers, who are familiar with the usual approach and would like to see general, robust examples where our theory could be used. For this, we offer a concrete class of distributions where richness holds in the usual approach. This setting is our suggested framework for applied works. Section 6 illustrates the convenience of working with it.

For simplicity, assume that $V_i = [0,1]$ for all $i \in I$. Grid distributions for multidimensional signals are formally defined in the appendix; see section 9.1. Also, assume that the common prior γ on $V = [0,1]^N$ is defined by a density function $f : [0,1]^N \to \mathbb{R}_+$. Consider the division of each $V_i = [0,1]$ into k intervals of the form $\left(\frac{m-1}{k}, \frac{m}{k}\right]$, for $m = 1, ..., k$. We say that f defines a grid distribution if f is constant in each of the cubes thus formed. More precisely:

Definition 2.2 *A function $f : [0,1]^N \to \mathbb{R}_+$ is a k-very simple density function or k-grid density function (and defines a k-very simple distribution or k-grid distribution), if for each $\boldsymbol{m} = (m_1, ..., m_N) \in \{1, ..., k\}^N$, f is constant on $\mathbb{I}_{\boldsymbol{m}}$, where*

$$\mathbb{I}_{\boldsymbol{m}} \equiv \left(\frac{m_1 - 1}{k}, \frac{m_1}{k}\right] \times ... \times \left(\frac{m_N - 1}{k}, \frac{m_N}{k}\right]. \tag{3}$$

The set of k-grid density functions is denoted by $\mathcal{D}^k$. We say that f is a grid density function if k is not important in the context. The set of all grid density functions is denoted by $\mathcal{D}^\infty \equiv \cup_{k\in\mathbb{N}} \mathcal{D}^k$.

For instance, if $k = 3$ and we have $N = 2$ players, we can describe the density function f by a matrix, as shown in the figure below.

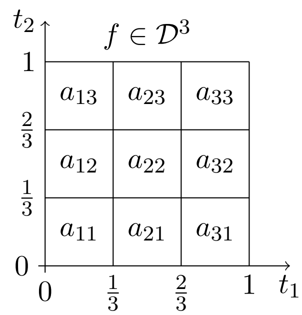

Figure 2: A density $f \in \mathcal{D}^k$ can be represented by a matrix $A = (a_{ij})_{k\times k}$.

It is easy to see that this class of distributions satisfies richness. Indeed, if $v_i, v' \in \left(\frac{m-1}{k}, \frac{m}{k}\right]$ for some $m \in \{1, ..., k\}$, then

$$f(v_{-i}|v_i) = \frac{f(v_i, v_{-i})}{f(v_i)} = \frac{f(v_i', v_{-i})}{f(v_i')} = f(v_{-i}|v_i'), \quad \forall v_{-i} \in V_{-i},$$

because f assumes the the same values for v_i and v_i', no matter what v_{-i} is. In other words, any two signals in one of the intervals imply the same beliefs about the signals of the opponents. Since there is a finite number of intervals, there is a finite number of different beliefs. Therefore, condition (ii) of Lemma 2.1 is satisfied.

As discussed in section 6.5, we can approximate any continuous distribution in the usual approach with grid distributions, in a strong and useful sense.

2.1.3 Product structure

Now assume that each T_i has a product structure. By this, we mean that there is a homeomorphism m between T_i and (a subset of) $V_i \times \Delta(T_{-i})$. In this case, the space $(T_i, \mathcal{T}_i, \tau_i)$ is taken as primitive and the maps $\widehat{V} : T_i \to V_i$ and $\widehat{\delta}_i$ are defined by $m \circ p^{V_i}$ and $m \circ p^{\Delta_i}$, respectively, where $p^{V_i} : V_i \times \Delta(T_{-i}) \to V_i$ and $p^{\Delta_i} : V_i \times \Delta(T_{-i}) \to \Delta_i$ are the natural projection maps. This implies that τ_i defines measures $\gamma_i \equiv \tau_i \circ \widehat{V}_i^{-1}$ and $\nu_i \equiv \tau_i \circ \widehat{\delta}_i^{-1}$ on V_i and Δ_i, respectively. We slightly abuse terminology by saying that τ_i is absolutely continuous with respect to the product $\gamma_i \times \nu_i$ if for any measurable set $E \subset V_i \times \Delta(T_{-i})$ such that $\gamma_i \times \nu_i(E) = 0$, then $\tau_i(m^{-1}(E)) = 0$.

This allows us to establish the following:

Lemma 2.3 *Assume that T_i has the product structure as described above and that condition (i) of Lemma 2.1 is satisfied. Moreover, assume that:*

(ii)' τ_i is absolutely continuous with respect to the product $\gamma_i \times \nu_i$.

Then, richness is satisfied.

All these results suggest that richness is a reasonable condition in models with correlated types.

3 Bayesian Games: setting and assumptions

We now describe our general model of games of incomplete information. Each player $i \in I$ chooses actions in a set A_i, which satisfies the following:

Assumption 3.1 *For each i, (A_i, ρ_i) is a compact metric space and $\leqslant_i$ is a partial order on A_i.*

Notice that we do not assume that A_i is a lattice.[18] For an example of a partial order that does not lead to a lattice, see footnote 29. Besides this, there are other examples of important partial orders used in economics that do not lead to lattices structures. For instance, Muller and Scarsini (2006) shows that some standard stochastic orders fail to be lattices.

However, supermodularity and quasi-supermodularity are defined only if A_i is a lattice. For this reason, in the body of the paper we do assume that A_i is a lattice. In the appendix, we define two properties introduced by de Castro (2012) that generalize supermodularity and quasi-supermodularity (ID- and SC-monotonicity properties, respectively). These properties allow working with action spaces A_i that are only partially ordered sets. All of our results remain valid in this more general setting.

A_i is endowed with its Borel σ-algebra $\mathcal{A}_i$. The metric ρ_i and the binary relation $\leqslant_i$ are related by the following property:

Assumption 3.2 *For every $i \in I$ and every $a_i, a_i', \underset{\sim}{a}_i, \widetilde{a}_i \in A_i$, we have:*

$$\underset{\sim}{a}_i \leqslant_i a_i, a_i' \leqslant_i \widetilde{a}_i \Rightarrow \rho_i(a_i, a_i') \leqslant \rho_i(\underset{\sim}{a}_i, \widetilde{a}_i). \tag{4}$$

[18] A partial order set $(X, \leqslant)$ generates (or is) a lattice if for any $x, y \in X$ there is lowest upper bound $x \vee y$ and a greatest lower bound $x \wedge y$ for any pair $x, y \in X$. The lowest upper bound (l.u.b.) for $x, y \in X$ is $z \in X$ such that $z \geqslant x$ and $z \geqslant y$ and for any $w \in X$ satisfying $w \geqslant x$ and $w \geqslant y$ we have $w \geqslant z$. The greatest lower bound (g.l.b.) for $x, y \in X$ is $z \in X$ such that $z \leqslant x$ and $z \leqslant y$ and for any $w \in X$ satisfying $w \leqslant x$ and $w \leqslant y$ we have $w \leqslant z$. It is easy to see that if they exist, g.l.b. and l.u.b. are unique.

The above assumption is trivially satisfied in euclidean spaces with the standard coordinatewise partial order. It is also satisfied if A_i is a space of real-valued functions and $\leqslant_i$ is the coordinatewise order, as long as the distance $\rho_i(a_i, a_i')$ is obtained through the function $x \mapsto |a_i(x) - a_i'(x)|$, as it would be the case for the sup or the L^p-metrics. However, it may fail in some ordered spaces; for instance, it fails if $A_i = \mathbb{R}^2$ and $\leqslant_i$ is the lexicographic order.[19]

The product space $A \equiv \times_{i \in I} A_i$ is endowed with the sum metric ρ, that is, for $a = (a_i, a_{-i})$ and $a' = (a_i', a_{-i}')$.

$$\rho(a, a') \equiv \sum_{i \in I} \rho_i(a_i, a_i').$$

Given a profile of types $t = (t_1, ..., t_N)$ and a profile of actions $a = (a_1, ..., a_N)$ played by each individual, player i receives the payoff $u_i(t, a)$. We assume the following:

Assumption 3.3 *For each $i \in I$, the function $u_i : T \times A \to \mathbb{R}$ is bounded and measurable.*

Let $\mathcal{F}_i$ denote the set of measurable functions from T_i to A_i. The strategy adopted by player i will be a function $s_i \in \mathcal{F}_i$. Let $\mathcal{F}_{-i}$ denote $\times_{j \neq i} \mathcal{F}_j$. Given a profile of strategies $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_N) \in \mathcal{F}_{-i}$, player i's interim payoff when she is of type t_i and plays action a_i is given by:[20]

$$\Pi_i(t_i, a_i, s_{-i}) \quad \equiv \quad \int_{T_{-i}} u_i(t_i, t_{-i}, a_i, s_{-i}(t_{-i}))\ \tau(dt_{-i}|t_i). \tag{5}$$

Similarly, player i ex ante utility when strategies (s_i, s_{-i}) are played is:

$$U_i(s_i, s_{-i}) \equiv \int u_i(t, s(t))\ d\tau = \int u_i(t, s(t)) f(t)\ \tau(dt), \tag{6}$$

where we assume that $t \mapsto u_i(t, s(t))$ is measurable, hence integrable.

As usual, a profile $s = (s_1, ..., s_N)$ is a (Bayesian pure strategy) equilibrium if $U_i(s_i, s_{-i}) \geqslant U_i(s_i', s_{-i})$ for all i and $s_i' \in \mathcal{F}_i$.

Although we will focus primarily on pure strategies, some of our results consider mixed strategies. For this, we will define mixed (behavioral) strategies.[21] A behavioral strategy for player i is a Markov kernel

[19] Indeed, we could have $\underline{a}_i = (1, 0) \leqslant_i a_i = (1, 1) \leqslant_i a_i' = (1, 2) \leqslant_i \bar{a}_i = (1.1, 0)$ and $\rho_i(a_i, a_i') = 1 > 0.1 = \rho_i(\underline{a}_i, \bar{a}_i)$.

[20] In this version, we will restrict our definitions to the case with a common prior; these definitions can be easily extended to cases without common prior. See for instance, van Zandt (2010).

[21] Behavioral and distributional strategies are equivalent. See, for instance, Rustichini (1993).

$\mu_i : T_i \times \mathcal{A}_i \to [0,1]$. Let S_i denote the set of behavioral strategies by player i. Following Balder (1988), we define product $\boldsymbol{\mu} = \mu_1 \otimes ... \otimes \mu_m$ by:

$$\boldsymbol{\mu}\left((t_1, ..., t_N), B_1 \times ... \times B_N\right) \equiv \prod_{i \in I} \mu_i(t_i, B_i)$$

for rectangles and extend it for measurable sets by standard arguments. (See details in Balder (1988).) For each distributional strategy σ_i, corresponds a behavioral strategy μ_i. Therefore, the ex ante utility function can be given by:

$$U_i(\mu_1, ..., \mu_N) \equiv \int_T \left[\int_A u_i(t,a)\boldsymbol{\mu}(t, da)\right] \tau(dt). \tag{7}$$

Recall that a behavioral strategy μ_i is *pure* if $\mu_i(t_i, \cdot)$ is a Dirac measure (has just one point in its support) for almost all t_i. Player i's interim payoff when she is of type t_i and plays action a_i is given by the function $\Pi_i : T_i \times A_i \times S_{-i} \to \mathbb{R}$ defined by:

$$\Pi_i(t_i, a_i, \mu_{-i}) \quad \equiv \quad \int_{T_{-i}} P_i(t_i, t_{-i}, a_i, \boldsymbol{\mu}_{-i})\ \tau(dt_{-i}|t_i),$$

where $P_i : T \times A_i \times S_{-i} \to \mathbb{R}$ is defined by:

$$P_i(t, a_i, \boldsymbol{\mu}_{-i}) \equiv \int_{A_{-i}} u_i(t, a_i, a_{-i})\boldsymbol{\mu}_{-i}(t_{-i}, da_{-i}).$$

4 Basic Results

We now present the results for the general framework described above.

4.1 Monotonic Best Replies, without richness

We present two versions of our basic result, which does not assume richness. The first one contains assumptions on the ex post utility function, which are in general easier to verify. For each $\delta \in \Delta_i \equiv \widehat{\delta}_i(T_i)$, define $T_{i\delta} \equiv \{t_i \in T_i : \widehat{\delta}_i(t_i) = \delta\}$. Let S_{-i} denote the set of profiles of mixed strategies played by players $j \neq i$. Given $\mu_{-i} \in S_{-i}$ and $t_i \in T_i$, let $BR(t_i, \mu_{-i}) \subset A_i$ denote the set of best replies by player i of type t_i to the mixed strategies μ_{-i}.

Theorem 4.1 *Let Assumptions 3.1, 3.2 and 3.3 hold and fix $\delta \in \Delta_i$. Let A_i be a lattice. Assume that $u_i : T_{i\delta} \times T_{-i} \times A \to \mathbb{R}$ is supermodular in A_i and satisfies increasing differences in $T_{i\delta} \times A_i$.*[22] *Let $\mu_{-i} \in S_{-i}$. The following holds:*

1. *If $t_i, t_i' \in T_{i\delta}$, $t_i < t_i'$, $a_i \in BR(t_i, \mu_{-i})$, $a_i' \in BR(t_i', \mu_{-i})$ then $a_i \leqslant a_i'$.*
2. *If μ_i be a better reply strategy to μ_{-i}, then the set of player i's types in $T_{i\delta}$ who possibly play mixed strategies under μ_i is a denumerable union of antichains.*[23,24]

This result can be stated with (slightly more general) assumptions in the interim payoff function.

Theorem 4.2 *Let Assumptions 3.1, 3.2 and 3.3 hold and fix $\delta \in \Delta_i$. Let A_i be a lattice. Assume that $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$ is quasi-supermodular in A_i and satisfies the strict single-crossing property in $T_{i\delta} \times A_i$. Let $\mu_{-i} \in S_{-i}$. Then the two conditions of Theorem 4.1 hold.*

Both Theorems 4.1 and 4.2 apply to any type space. For instance, the set of types can be *finite*, continuous or even a universal type space without common priors. Note that richness *was not* assumed in either theorem. We also remark that, although the first implication in each theorem is easier to understand and appreciate, the second implication is perhaps the most useful for the rest of this paper, because it allows us to characterize strategies as pure, as it will be clear below.

It is useful to observe that the beliefs do not play a specific role in the first implication in Theorem 4.2, because this result is stated on the interim payoff function, where beliefs about other players' types do not play a role. That is, the first implication remains true if we replace $T_{i\delta}$ by any subset T_i' of T_i. However, we do not have this freedom under the assumptions of Theorem 4.1, which are given in the ex post utility function.

We prove (slightly more general versions of) Theorems 4.1 and 4.2 in section 8.2 of the appendix. The proofs can be summarized as follows. First, we reduce Theorem 4.1 to Theorem 4.2. This step is accomplished in by the observation that supermodularity and increasing differences of the

[22] See section 8.1.1 in the appendix for definitions of supermodularity, quasi-supermodularity, increasing differences and single-crossing properties.

[23] By denumerable we mean either finite or countable.

[24] An antichain in a partially ordered set $(X, \leqslant)$ is a subset of X that contains no two ordered points.

ex post utility function are preserved under integration if the set of beliefs are held constant. Therefore, the assumptions of Theorem 4.1 imply those of Theorem 4.2.

The proof of Theorem 4.2 can be divided in two parts. The first and easiest part is to establish the monotonicity of best actions; see Lemma 8.13 in the appendix. This comes from a monotonicity property introduced by de Castro (2012) that generalizes quasi-supermodularity and strict single-crossing. The argument for the second claim in the theorem is more involved. First, we need to establish a mathematical result about chains (Lemma 8.14).[25] Namely, if a chain has at least three elements, then we can divide it in three sets such that for every pair of points in one of the sets, we can find a third point in the other two sets which are strictly between these points.[26] The proof of this fact requires Zorn's Lemma and some non-trivial constructions. This fact is used in conjunction with Assumptions 3.1 and 3.2 to show that any chain of types that have two best reply actions at least $\frac{1}{n}$ apart must have a given finite number of elements (Lemma 8.15). Then, we use a result in the theory of partially ordered sets (Dilworth's Theorem, see Lemma 8.16) to argue that this implies that the number of antichains of types with two best reply actions at least $\frac{1}{n}$ apart is finite. This implies the conclusion stated in (2).

4.2 Pure strategy equilibria with richness

Now, we will introduce our main assumption richness on the type space and show that all best replies are in pure strategies.

Theorem 4.3 *Let richness and the assumptions of Theorem 4.1 or 4.2 hold. Let μ_{-i} be any mixed strategy played by player i's opponents and let μ_i be a player i's best reply to μ_{-i}. Then, μ_i is pure. Moreover, μ_i is essentially unique: if μ_i' is also a best reply to μ_{-i} then $\mu_i'(t_i, \cdot) = \mu_i(t_i, \cdot)$ for almost all $t_i \in T_i$.*

Therefore, under these assumptions, if the game has an equilibrium, it has a pure strategy equilibrium. Moreover, all equilibria are in pure strategies.

Theorem 4.3 gives conditions under which all equilibria must be in pure strategies; more than that, they show that every best reply, even to mixed strategies, is pure.

[25] A chain is a totally ordered subset of a partially ordered set.

[26] If the chain is finite, we can divide it in just two sets with the mentioned property.

The two closest results available in the literature are Maskin and Riley (2000, Proposition 1) and Araujo and de Castro (2009, Theorem 1). There are two main differences between these results and Theorem 4.3: first, both papers restrict attention to unidimensional auction games and second, both assume independence of types. Therefore, even in the more familiar setting of independence, Theorem 4.3 presents a novel result because it allows more general action spaces and utility functions.

In the remainder of the paper, we will show how Theorem 4.3 leads to new equilibrium existence results in games of incomplete information. But before discussing these results, we would like to comment on how richness and Theorem 4.3 could be extended to a setting without measure τ_i.

4.3 Insignificant sets

The use of a measure τ_i on the statement of richness may be considered undesirable, given that in principle the type spaces T_i could be considered only measurable spaces. In this section, we show that richness could be rephrased without mentioning a measure τ_i, using instead a notion of "insignificant sets."

Definition 4.4 *A collection $\mathcal{I}_i$ of subsets of T_i is a collection of insignificant sets if: (i) $\mathcal{I}_i \subset \mathcal{T}_i$; (ii) $\mathcal{I}_i$ is closed for countable unions.*

In this case, any set in $\mathcal{T}_i \setminus \mathcal{I}_i$ is called a significant set.

For the discussion below, fix a collection of insignificant sets $\mathcal{I}_i$. Richness can be rephrased as follows:

> **Richness*:** If $E \in \mathcal{T}_i$ is a significant set, then there exist $t_i, t'_i \in E$ such that $t_i <_i t'_i$ and $\widehat{\delta}_i(t_i) = \widehat{\delta}_i(t'_i)$.

The proof of Theorem 4.3 allows us to conclude the following:

Proposition 4.5 *Let richness* and the assumptions of Theorem 4.1 or 4.2 hold. Let μ_{-i} be any mixed strategy played by player i's opponents and let μ_i be i's best reply to μ_{-i}. Then, the set of types $t_i \in T_i$ which plays strictly mixed strategies according to μ_i is an insignificant set.*

5 PSE existence in Private Value Auctions

In this section, we describe the first application of our theory: a general pure strategy equilibrium result for correlated types in private value auctions. We begin by describing the type structure and then describe the model for the game, which follows very closely that of Jackson and Swinkels (2005), henceforth JS.

5.1 Multi-unit Private Value Auctions

The description of the model is organized in two subsections: values and types (5.1.1) and the game itself (5.1.2).

5.1.1 Multidimensional values and belief types

A useful instance of our main framework is now described. Assume that the set of player i's types is $T_i \subset E_i \times \tilde{V}_i \times \Delta_i$, where:[27]

- $E_i \equiv \{0, 1, \ldots, \ell\}$ is the set of player i's possible initial endowments;
- $\tilde{V}_i = [\underline{v}_i, \overline{v}_i]^\ell \subset \mathbb{R}^\ell$ denotes the vector of player i's valuations for objects in a multi-unit auction;[28]
- Δ_i is the set of parameters defining i's beliefs about other players' types.

A typical type is therefore $t_i = (e_i, v_i, \delta_i)$, where:

- $e_i \in \{0, 1, \ldots, \ell\}$ denotes the number of units that player i is endowed with;
- $v_i = (v_{i1}, \ldots, v_{i\ell})$ is the vector of player i's valuations, meaning that i has marginal value v_{ih} for the hth object, satisfying nonincreasing marginal valuations, that is, $v_{ih} \geqslant v_{i,h+1}$ for all i, h; and
- $\delta_i \in \Delta_i$ is the parameter that defines i's beliefs about other bidders' valuations.

The values $\underline{v}_i$ and $\overline{v}_i$ are finite for all i. The objects are indivisible and identical. The vector of endowments is $e = (e_0, ..., e_N) \in E \equiv \{0, 1, ..., \ell\}^{N+1}$;

[27] Recall that T_i is the support of τ_i.

[28] Observe that $V_i \equiv E_i \times \tilde{V}_i$ corresponds to the "preference" part of the type in the notation of section 2.1.

the vector of values is $v = (v_1, ..., v_N) \in \tilde{V} \equiv \times_{i=1}^{N} [\underline{v}_i, \overline{v}_i]^{\ell}$. Let $\Delta \equiv \times_{i=0}^{N} \Delta_i$, so that $T = E \times \tilde{V} \times \Delta$ and $t = (e, v, \delta) \in T$ denotes a profile of types.

Observe that given that δ_i determines i's beliefs about other players' types, then for $t_i = (e_i, v_i, \delta_i)$ and $t'_i = (e'_i, v'_i, \delta_i)$, we must have:

$$\Pr\left[t_{-i} \in E | t_i\right] = \Pr\left[t_{-i} \in E | \delta_i\right] = \Pr\left[t_{-i} \in E | t'_i\right].$$

We also define an order in $V_i = E_i \times \tilde{V}_i$. Actually, we define the order exclusively on $\tilde{V}_i = [\underline{v}_i, \overline{v}_i]^{\ell}$ as follows. First, define $v_i <_i v'_i$ by:

$$v_i <_i v'_i \iff v_{ih} < v'_{ih}, \forall h = 1, ..., \ell. \tag{8}$$

Then, define $v_i \leqslant_i v'_i$ iff $v_i <_i v'_i$ or $v_i = v'_i$. Notice that this is a partial order, but it does not generate a lattice.[29] On the other hand, it is not difficult to see that this order satisfies condition (i) of Lemma 2.1 if the measure on V_i is absolutely continuous with respect to the Lebesgue measure.

This order on $\tilde{V}_i$ defines an order on T_i by (1). More concretely, for $t_i = (e_i, v_i, \delta_i)$ and $t'_i = (e'_i, v'_i, \delta'_i)$, we define

$$t_i <_i t'_i \iff v_i < v'_i \text{ and } \delta_i = \delta'_i, \tag{9}$$

and $t_i \leqslant_i t'_i$ iff $t_i <_i t'_i$ or $t_i = t'_i$. Note that (9) is a variation of definition (1) given in section 2.1.

Although we will use this order below, we should emphasize that this is not the only case where our techniques apply. Actually, any order that is stronger than (9) could be automatically used. For instance, in games with risk aversion, one could consider an adaptation of the order defined by Reny (2011).

Additionally, we assume either condition (ii) of Lemma 2.1 or condition (ii)' of Lemma 2.3. That is, that τ_i is absolutely continuous with respect to the product $\gamma_i \times \nu_i$, where γ_i be the marginal of τ_i over $\tilde{V}_i$ and ν_i, the marginal over Δ_i. As discussed in section 2.1, these conditions are sufficient for implying richness. Richness is essentially the only extra assumption that we require with respect to JS.[30]

The assumptions and description above are maintained for our results in this section, even without explicit reference.

[29] For a definition of lattices, see footnote 18. To see that the order above is not a lattice, it is enough to consider $\ell = 2$. Consider the points $x = (1, 0)$ and $y = (0, 1)$. Then $x, y \leqslant z^{\varepsilon} \equiv (1 + \varepsilon, 1 + \varepsilon)$ for any $\varepsilon > 0$ but $\neg(x \leqslant z^0)$ and $\neg(y \leqslant z^0)$. Therefore, there is no lowest upper bound for $\{x, y\}$.

[30] We do require an extra technical assumption—see assumption 5.1 below. However, this is done more for simplicity and it does not exclude any conventional auction game.

5.1.2 Game description

We will now describe the general private value auction model introduced by JS. It will be useful to explicitly consider the non-strategy player 0. Each bidder places a bid $b_i \in A_i \subseteq [\underline{b}, \overline{b}]^\ell$. The order on the action space is the standard coordinate-wise order on $\mathbb{R}^\ell$. Let $A = A_0 \times ... \times A_N$. The set of allocations is $\Omega \equiv \{0, 1, ..., \ell\}^{N+1}$. Given types $t \in T \subset E \times \tilde{V} \times \Delta$, bids $b \in A$ and allocation $\omega = (h_0, ..., h_N) \in \Omega$, player i's ex post utility is:[31],[32]

$$u_i(t, b, \omega) \equiv \sum_{j=0}^{h_i} v_{ij} - p_i(h_i, e, b), \tag{10}$$

where $p_i : \{0, ..., \ell\} \times \Omega \times A \to \mathbb{R}$ is player i's payment function, which can depend not only on how many objects she gets, but also on everybody's bids and endowments. We will introduce restrictions on the payment functions p_i below.

The outcome correspondence $O : \Omega \times A \to \Omega$ is defined by:

$$\begin{aligned} O(e, b) \equiv \{m \in \Omega : & \sum_{i=0}^{N} m_i = \sum_{i=0}^{N} e_i, \text{ and} \\ & \big(b_{jh'} > b_{ih} \text{ and } m_i \geqslant h\big) \implies m_j \geqslant h'\}. \end{aligned} \tag{11}$$

The first condition above is just the requirement that objects are not created nor destroyed. The second condition amounts to requiring that higher bids are given priority over lower bids in allocating objects. A tie-breaking rule is just a selection of this correspondence.[33] More formally, a tie-breaking rule will be denoted by a profile of functions $h^* = (h_i^*)_{i=0}^N$ such that $h^*(e, b) \in O(e, b)$. Note that the only points were there is some freedom for the value of h^* ($O(e, b)$ is not a singleton) occur when there is some "relevant tie," that is, a tie that occurs exactly at the number of units $\sum_{i=0}^N e_i$ available for negotiation.

When there is a tie, (11) still determines maximum and minimum values for the number of units $h_i^*(e, b)$ that player i can receive. We will denote these values by $\underline{h}_i(e, b)$ and $\overline{h}_i(e, b)$. For simplicity of notation, we will

[31]Define $v_{i0} = 0$ for all i.

[32] For simplicity, we do not consider risk aversion, although our techniques could be extended to this setting as well.

[33]The formal definition given by JS is not exactly this, as they allow for randomizations on the allocations. They also allow for "omniscient" tie-breaking rules, that is, rules that may vary depending on the values. However, they prove that the actual tie-breaking rule is not important in their setting. Therefore, this formulation is enough for our purposes.

constantly omit e in the argument of the functions h_i^*, $\underline{h}_i$ and $\overline{h}_i$ below. No confusion should arise.

We assume the following for the payment rule:

Assumption 5.1 *The payment function is given by:*[34]

$$p_i(h, e, b_i, b_{-i}) = \sum_{j=1}^{h} p_{ij}(h, b_{ij}) + q_i(h, b_{-i}) + \underline{r}_i(h, \underline{p}) + \overline{r}_i(h, \overline{p}), \tag{12}$$

where $\underline{p}$ denotes the highest losing bid and $\overline{p}$ denotes the lowest winning bid, if there is no competitive tie; if there is a competitive tie at β, $\underline{p} = \overline{p} = \beta$. Moreover, all these functions are nondecreasing.

The first term in the sum above corresponds to "pay-your-bid" elements and will not be zero in discriminatory auctions. The second term, $q_i(h, b_{-i})$ depends exclusively (and arbitrarily) on the bids of other players. This allows us to cover Vickrey auctions, for instance. The other two terms, $\underline{r}_i(h, \underline{p})$ and $\overline{r}_i(h, \overline{p})$, allow to capture the two distinct formats of uniform price auctions: the ones where the clearing price is the highest losing bid $\underline{p}$ or the lowest winning bid $\overline{p}$. Note that these values are not determined exclusively by b_{-i} and, therefore, cannot be captured only on $q_i(h, b_{-i})$. Note also that all terms can vary with the total number of allocated objects h. Therefore, variations and combinations of the payment rules of standard auctions are allowed.

5.2 Pure Strategy Equilibrium Existence

Consider the framework described in subsections 5.1.1 and 5.1.2, which is essentially the same as JS', with their assumptions 1-9. We have the following:

Theorem 5.1 *Let richness and assumption 5.1 hold, together with JS' assumption 10.*[35] *Then, there exists a monotonic pure strategy equilibrium in undominated* strategies with a zero probability of competitive ties, which is an equilibrium under any omniscient and effectively trade-maximizing tie-breaking rule, including the standard tie-breaking rule.*

[34] If $h = 0$, the term $\sum_{j=1}^{h} p_{ij}(h, b_{ij})$ should be understood as zero.

[35] JS' assumption 10 specifies a technical measurability condition to ensure the existence of equilibrium in undominated* strategies. For a definition of this class of strategies and an explicit statement and discussion of JS' assumption 10, we refer the reader to that paper.

Note that the main difference from Theorem 5.1 above and JS' Corollary 14 is that they assume that the distribution of types is independent.

The proof of Theorem 5.1 goes as follows. We first establish that the auction is modular, using Proposition 5.4 below and Assumption 5.1.

The first difficulty in proving Theorem 5.1 comes from the fact that auctions do not satisfy in general increasing differences in $T_i \times A_i$, although they satisfy nondecreasing differences. Therefore, the assumptions of Theorems 4.1 or 4.2 may fail to hold. To circumvent this problem, we consider modified auctions (n-auctions), which are auctions where with a probability $\frac{1}{n}$, a non-strategic player bids uniformly on $[\underline{b}, \overline{b}]$. Actually, the n-modified auction also includes a modification of the tie-breaking rule. Since the discussion about the tie-breaking rule modification requires many more details, we discuss this issue in section 5.3 below.

We show that supermodularity and increasing differences hold in each of the n-modified auctions. This allows us to conclude that each n-modified auction has an equilibrium in pure strategies which are monotonic when restricted to the set of types sharing the same beliefs.

When $n \to \infty$, there is a pointwise convergent subsequence of strategies because the set of monotonic strategies is compact. Pointwise convergence leads (through Lebesgue theorem) to convergence of the (interim) payoffs, both for strategies in the optimum, and for deviating strategies. In fact, the argument needs to be more careful here, because of the possibility of ties with positive probability. For this, we need to define a special tie-breaking rule that guarantees convergence of the payoffs. After this, we are able to argue that the tie-breaking rule does not matter. Finally, we show that if there is a deviating strategy that does better, it would also do better along the sequence, which is a contradiction. The appendix details this argument.

Remark 5.2 *Theorem 5.1 states the existence of a monotonic pure strategy equilibria. This is possible because order (9) is considered. If we use instead (1), then the equilibrium could fail to be monotonic. This is just to highlight that there is a subtlety when we talk about "monotonicity" in multi-dimensional spaces. Since we may have more than one "reasonable" order in some spaces, we may have different notions of monotonicity.*[36] *When we do have a clear unique candidate for the types' order (for instance, in uni-dimensional settings), then the monotonicity question is important. For this reason, section 6 address the existence of monotonic equilibrium in first price auctions.*

[36] Actually, if we have complete freedom in defining the types' order, any pure strategy equilibrium $a = (a_i, a_{-i})$ can be monotonic: just define $t_i \leqslant t'_i$ if and only if $a_i(t_i) \leqslant a_i(t'_i)$.

Remark 5.3 *Theorem 5.1 is stated only for private value auctions, as in the JS setting. However, the existence of pure strategy equilibrium can be extended to interdependent values if we are willing to allow special kinds of tie-breaking rules, as Jackson and Swinkels (2005) and Araujo and de Castro (2009) did. In other words, Theorem 5.1 is restricted to private values to prevent us from dealing with special tie-breaking rules. As Jackson, Simon, Swinkels, and Zame (2002) show, the need for special tie-breaking rules is unavoidable in general interdependent values settings. See also Jackson (2009). It should be emphasized that the need for special tie-breaking rules is related to the interdependence of values, not to the statistical dependence of types. Indeed, the need for special tie-breaking rules may arise even when types are independent as in the Jackson, Simon, Swinkels, and Zame (2002)'s example, but it does not arise if values are private, even with correlation, as Jackson and Swinkels (2005) have shown.*

5.3 Tie-breaking rule for the modified auction

In order to establish the assumptions of Theorems 4.1 or 4.2, we will need an assumption on the tie-breaking rule. This assumption generalizes a rule that was introduced by McAdams (2003) to establish equilibrium existence for the uniform price auction with interdependent values and risk neutral bidders. The same rule was also used by Reny (2011) to prove analogous results for uniform and discriminatory auctions with risk averse bidders. Reny (2011, footnote 47, p. 518) describes the rule as follows: "Bidders are ordered randomly and uniformly. Then one bidder at a time according to this order—each bidder's total remaining demand (i.e., his number of bids equal to p) or as much as possible—is filled at price p per unit until supply is exhausted." A formalization of the rule is given by McAdams (2003, p.1198) as follows:[37]

> Each bidder is assigned at least $\underline{h}_i(b)$ and randomly ordered into a ranking ρ to ration the remaining quantity $r \equiv K - \sum_{i=1}^{N} \underline{h}_i(b)$. If $r = 0$, stop. Else the first bidder in order, $i_1 = \rho(1)$, receives $q^*_{i_1} = \underline{h}_i(b) + \min\{\overline{h}_i(b) - \underline{h}_i(b), r\}$. Decrement r by $\overline{h}_i(b) - \underline{h}_i(b)$ and repeat this process with bidder $i_2 = \rho(2)$ and so on until all quantity has been assigned.

It is not difficult to see that this rule satisfies the following:

[37] We adapted his notation to ours.

Assumption 5.2 *Let $\tilde{b}_i$, $\hat{b}_i \in A_i$ and fix $b_{-i} \in A_{-i}$ and $h \in \{1, ..., \ell\}$. The following holds:*

1. *If $\tilde{b}_i \leqslant_i \hat{b}_i$ then $h_i^*(\tilde{b}_i, b_{-i}) \leqslant h_i^*(\hat{b}_i, b_{-i})$;*
2. *If $h_i^*(\tilde{b}_i, b_{-i}) \geqslant h-1$, $h_i^*(\hat{b}_i, b_{-i}) \geqslant h-1$ and $\tilde{b}_{ih} = \hat{b}_{ih} = s_{ih}$, then:*

$$h_i^*(\tilde{b}_i, b_{-i}) \geqslant h \iff h_i^*(\hat{b}_i, b_{-i}) \geqslant h. \tag{13}$$

The first requirement in Assumption 5.2 is just a mild monotonicity condition: no bidder can receive *more* units by bidding less. Most tie-breaking rules satisfy this condition. The second condition is a little bit more restrictive. It requires that a bidder wins unit h-th irrespective of what are his bids for the $(h+1)$-th unit and above, and also it depends on her bids for units below h only through the fact of winning or not winning those units. It is easy to see that McAdams' rule satisfies Assumption 5.2. Another rule that satisfies Assumption 5.2 is the following. Let bidders be ordered in some arbitrary fashion. If there is a tie, give one object to the first bidder in the tie, according to this order; then give the second unit to the second one in the order, and so on. If all bidders in the tie receive one object, but there are still unassigned objects, repeat the process, until no object is left unassigned. Since the allocation of the h-th unit does not depend on the bids after the h-th, it is easy to see that this rule also satisfies Assumption 5.2 . On the other hand, consider the following rule: in the case of a tie, bidder i receives n_i objects, where n_i is the integer closest to the product of the number of objects still available at the tie and the ratio between bidder i's number of bids at the tie and the total number of bids at the tie. Any difference between the number of objects thus allocated and the number of available objects is then randomly adjusted.[38] This rule does not satisfy Assumption 5.2 and also fails to have the property described in the following:[39]

---

[38]This is similar to the pro-rata on the margin rule that is used in some real auctions. See Kremer and Nyborg (2004).

[39] This rule does not obey Assumption 5.2 because a bidder could increase the number of units he receives in a tie at h by raising (to the tie level) the bids she places for units $h+1$ and beyond, since this increases her number of units in the tie. This violates (13). For an example of the failure of Proposition 5.4, assume that we have two bidders, four units, bidder 2 bids 2 for all units and $b_i^1 = (4, 3, 1, 1)$, $b_i^2 = (2, 2, 2, 2)$. Then, $b_i^1 \wedge b_i^2 = (2, 2, 1, 1)$, $b_i^1 \vee b_i^2 = (4, 3, 2, 2)$, $h_i^*(b_i^1, b_{-i}) = h_i^*(b_i^2, b_{-i}) = 2$, but $h_i^*(b_i^1 \wedge b_i^2, b_{-i}) = 1$ and $h_i^*(b_i^1 \vee b_i^2, b_{-i}) = 3$.

Proposition 5.4 *Let Assumption 5.2 hold. Let b_i^1, $b_i^2 \in A_i$ and assume (w.l.o.g.) that $h_i^*(b_i^1, b_{-i}) \leqslant h_i^*(b_i^2, b_{-i})$. Then,*[40]

$$h_i^*(b_i^1, b_{-i}) = h_i^*(b_i^{1\wedge 2}, b_{-i}) \text{ and } h_i^*(b_i^2, b_{-i}) = h_i^*(b_i^{1\vee 2}, b_{-i}).$$

Proposition 5.4's proof is simpler and more direct than McAdams' argument.

As a step in the proof of Theorem 5.1, we show in the appendix (see Lemma 8.17 and Corollary 8.19) that Assumption 5.2 and (12) lead to the following property for any i, b_i^1, b_i^2 and b_{-i}:

$$p_i(h, e, b_i^1, b_{-i}) - p_i(h, e, b_i^{1\wedge 2}, b_{-i}) = p_i(h, e, b_i^{1\vee 2}, b_{-i}) - p_i(h, e, b_i^2, b_{-i}). \quad (14)$$

Condition (14) may be interpreted as modularity of payments. This condition is the crucial step in establishing that u_i is modular and, hence, super-modular (see Lemma 8.20). It is also useful in establishing that u_i satisfies increasing differences with positive probability (see Lemma 8.21). This last result is enough for applying Theorem 4.3, which is used in the proof of Theorem 5.1.

It should be noted that condition (14) is more complex than the analogous result in McAdams (2003), because he focused exclusively on the uniform price auction, with its simpler payment rule.

6 Existence of symmetric monotonic pure strategy equilibria

As observed in remark 5.2, when types are unidimensional the order considered in Theorem 5.1 may be different from the standard order on real numbers. This implies that a "monotonic" strategy in Theorem 5.1 may fail to be monotonic in the conventional sense. Therefore, it is natural to ask whether (and when) it is possible to establish existence of equilibrium in conventional monotonic pure strategies. In this section we give a complete answer to this question in a more particular setting, that of very simple or grid distributions, defined in section 2.1.2.

To consider the relevant issues, first recall the standard result of auction theory on symmetric monotonic pure strategy equilibria (SMPSE) in private value auctions: if there is a differentiable symmetric increasing equilibrium,

[40] Hereafter, $b_i^{1\wedge 2}$denotes $b_i^1 \wedge b_i^2$ and $b_i^{1\vee 2}$ denotes $b_i^1 \vee b_i^2$.

it satisfies the differential equation (see Krishna 2002 or Menezes and Monteiro 2005):

$$b^{'}(v) = \frac{v - b(v)}{F(v|v)} f(v|v).$$

If f is Lipschitz continuous, one can show that this equation has a unique solution. Under some assumptions (affiliation or, a little bit more generally, Property VI' in de Castro (2011)), it is possible to ensure that this solution is, in fact, equilibrium. Now, if the distribution is very simple ($f \in \mathcal{D}^{\infty}$), the right hand side of the above equation is not continuous, and one cannot directly apply standard techniques. We proceed as follows.

First, we show that if there is a symmetric increasing equilibrium b, under mild conditions (satisfied by $f \in \mathcal{D}^{\infty}$), b is continuous. We also prove that b is differentiable at the points where f is continuous. Thus, for $f \in \mathcal{D}^{\infty}$, b is continuous everywhere and differentiable everywhere but, possibly, at the points of the form $\frac{m}{k}$. See Figure 3.

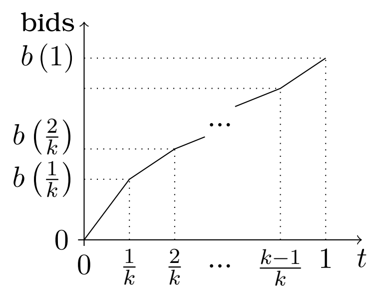

Figure 3: Bidding function for $f \in \mathcal{D}^k$.

With the initial condition $b(0) = 0$ and the above differential equation being valid for the first interval $\left(0, \frac{1}{k}\right)$, we have uniqueness of the solution on this interval and, thus, a unique value of $b\left(\frac{1}{k}\right)$. Since b is continuous, this value is the initial condition for the interval $\left(\frac{1}{k}, \frac{2}{k}\right)$, where we again obtain a unique solution and the uniqueness of the value $b\left(\frac{2}{k}\right)$. Proceeding in this way, we find that there is a unique b which can be a symmetric increasing equilibrium for an auction with $f \in \mathcal{D}^{\infty}$.

To formalize this result, assume that we have a first price auction with n symmetric players, such that if player i with signal $v_i \in V_i = [\underline{v}, \overline{v}]$ wins the object with the bid b_i, her utility will be $u(v_i - b_i)$. Types are distributed according to the density function $f : [\underline{v}, \overline{v}]^N \to \mathbb{R}_+$. We have the following:

Theorem 6.1 *Assume that u is twice continuously differentiable, $u' > 0$, $f \in \mathcal{D}^k$, f is symmetric and positive ($f > 0$).*[41] *If $b : [\underline{v}, \overline{v}] \to \mathbb{R}$ is a symmetric monotonic pure strategy equilibrium (SMPSE), then b is continuous in $(0,1)$ and is differentiable almost everywhere in $(0,1)$.*[42] *Moreover, b is the unique symmetric increasing equilibrium. If $u(x) = x^{1-c}$, for $c \in [0,1)$, b is given by*

$$b(x) = x - \int_{\underline{v}}^{x} \exp\left[-\frac{1}{1-c}\int_{\alpha}^{x} \frac{f(s \mid s)}{F(s \mid s)} ds\right] d\alpha. \tag{15}$$

Proof. *See the supplement to this paper.* ■

Having established the uniqueness of the candidate for equilibrium, our task is reduced to verifying whether this candidate is, indeed, an equilibrium. We begin with the two bidders case and then, generalize it to the n bidders case. Although it is possible to extend our results for risk averse bidders, specially if $u(x) = x^{1-c}$ for $c > 0$, we focus below on the case of risk neutrality $u(x) = x$.

6.1 Two risk neutral players case

Theorem 6.1 establishes the uniqueness of the candidate for symmetric increasing equilibrium for $f \in \mathcal{D}^{\infty} = \cup_{k=1}^{\infty} \mathcal{D}^k$. We have now only to check if the unique candidate is indeed equilibrium. In economics, this is usually done by checking the second order condition. In auction theory, it is more common to appeal to monotonicity arguments based on a single crossing condition (see, among others, Milgrom and Weber (1982) and Athey (2001)). These methods give sufficient conditions for equilibrium, but these conditions are, in general, not necessary. Sufficient and necessary conditions would not only provide grounds to understand what really entails equilibrium existence, but they would also allow working with the most general possible setup that yields equilibrium existence. In order to give necessary and sufficient conditions for equilibrium existence, we propose a method that is different from the one established by Theorem 5.1, although the main advantage of richness (captured by the interval of fixed beliefs) also plays a major role. The method consists in directly checking the equilibrium conditions, as we explain next.

[41] In fact, it is not necessary that f has full support. See the supplement of the paper for details.

[42] b may be non-differentiable only in the points $\frac{m}{k}$, for $m = 0, 1, ..., k$.

Let $b(\cdot)$, given by (15) with $c = 0$, denote the candidate for equilibrium. Let $\Pi(y, b(x)) = (y - b(x)) F(x|y)$ be the interim payoff of a player with type y who bids as type x, when the opponent follows $b(\cdot)$. Let $\Delta(x, z)$ represent the expected interim payoff of a player of type x who bids as a type z, that is, $\Delta(x, z) \equiv \Pi(x, b(z)) - \Pi(x, b(x))$. It is easy to see that $b(\cdot)$ is equilibrium if and only if $\Delta(x, y) \leq 0$ for all x and $z \in [0, 1]^2$. In other words, the equilibrium condition requires checking an *inequality for an infinite pair of points*. Of course, it is not possible to check an inequality at an infinite number of points.

If $\Delta(x, z)$ is continuous, then an approximation algorithm could check the inequality only at some points and, with some confidence, ensure equilibrium existence. Of course, this method would not be exact in the sense that approximation errors are inherent to the algorithm. However, for $n = 2$ players, the next theorem shows that when $f \in \mathcal{D}^k$ there is an *exact* algorithm, that does not introduce errors, and it is fast because it requires only a small number of comparisons.

Theorem 6.2 *Consider Symmetric Risk Neutral Private Value Auction with two players with $f \in \mathcal{D}^\infty = \cup_{k\geq 1} \mathcal{D}^k$. There exists an algorithm that decides in finite time if there is or not a symmetric monotonic pure strategy equilibrium for this auction. For $f \in \mathcal{D}^k$, the algorithm requires less than $3\left(k^2 + k\right)$ comparisons. The algorithm is exact, in the sense that errors can occur only in elementary operations.*[43]

Theorem 6.2 gives *necessary and sufficient* conditions for SMPSE existence. This is what we mean by "decides" above. Unfortunately, these necessary and sufficient conditions are long. Theorem 9.5 below contains an explicit statement of those conditions.

The use of the term "algorithm" above should not be confused with a complex procedure for deciding equilibria. On the contrary, the theorem reduces the verification of equilibrium to a set of simple conditions that can be explicitly given. Thus, we can state Theorem 6.2 just through this reduction and even avoid the use of the word "algorithm." This is exactly what Theorem 9.5 does.

Remark 6.3 *It is useful to compare this result with the best algorithms for solving simpler games as bimatrix games (see Savani and Von Stengel (2006)). While best known algorithms for bimatrix games require operations*

[43] By elementary operations we mean sums, multiplications, divisions, comparisons and square roots.

that grow exponentially with the size of the matrix, the number of our comparisons increases with k^2. We do not state that the algorithm runs in polynomial time because our problem is in continuous variables, not in discrete ones. "Polynomial time" would be slightly vague here. However, the algorithm would be polynomial in the number of continuous variables needed to describe the setting (distribution). Also, as stated, the possible errors are elementary and require a small number of operations. This allows one to realize the important benefits of working with continuous variables but density functions in $\mathcal{D}^k$, as we propose. The characterization of the strategies obtained through differential equations allows one to drastically reduce the computational effort, by reducing the equilibrium candidates to one. The fact that we work on $\mathcal{D}^k$ allows us to precisely characterize a small number of points to be tested for the equilibrium condition. The speed of the method allows auction theorists to run simulations for a big number of trials and get a good figure of what happens in general. From this, conjectures for theoretical results can also be derived.

The proof of this theorem is long and complex, because $\Delta(x, y)$ is not monotonic in the squares $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Indeed, the main part of the proof is the analysis of the non-monotonic function $\Delta(x, y)$ in the sets $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$ and the determination of its maxima for each of these sets. It turns out that we need to check a different number of points (between 1 and 5) for some of these squares. Section 9.3 in the appendix outlines the proof. The complete proof is given in a supplement to this paper.

Remark 6.4 *If there is no SMPSE for a $f \in \mathcal{D}^k$, we still know that there is a pure strategy equilibrium which is monotonic in each grid-interval. This follows from Theorem 5.1. Thus, a direction for future research would be to characterize such equilibria. The advantage of using grid distribution in this case is that we can still take advantage of the first order conditions to obtain such characterizations.*

6.2 Equilibrium results for N players

The ideas in the proof of Theorem 6.2 generalize from 2 to N players with minor complications. Thus, one can use grid distributions (those in $\mathcal{D}^\infty$) to study auctions in a more general setup.

As before, the equilibrium candidate is unique and we have an expression for it. Thus, SMPSE will be established if and only if $\Delta(x, z) =$

$\Pi(x, b(z)) - \Pi(x, b(x))$ is non-positive. We can test the signal of $\Delta(x, z)$ for $(x, z) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for m, $p \in \{1, ..., k\}$. This is simplified to check non-positiveness of a polynomial over $[0, 1]^2$. The only difference from the $N = 2$ case is that in this last case, the polynomial is of degree 3 and we can analytically solve it. For $N > 2$, the polynomial (in the two variables, x and z) has a degree of at least $N + 1$ and we have to rely on numerical methods for finding minimal points. The following establishes the existence of an algorithm that solves for this, that only makes errors in numerical approximations:[44]

Theorem 6.5 *Consider a symmetric risk neutral private value auction with n players with $f \in \mathcal{D}^\infty$. There exists an algorithm that decides in finite time if there is or not a symmetric monotonic pure strategy equilibrium for this auction. Errors are committed in finding roots of polynomials and in elementary operations.*

Note that we did not make statements about the speed of the method. This is just because this speed depends on the numerical method used to find roots of polynomials. We were unable to find good characterizations of the running time of solutions to this problem.

Remark 6.6 *Grid distributions are also useful to study asymmetric auctions. The method described above can be adapted to this case, but since an explicit expression for $b(\cdot)$ is not available in the asymmetric case, we need to rely on more approximations and depart further from the exact algorithm contained in Theorem 6.2. To be more precise: in Theorem 6.2 we used the explicit solution to the differential equation obtained from the first order condition and were able to pin down exactly what points should be checked. In Theorem 6.5 above, we also have the explicit solution to the differential equation and were able to repeat the same procedures as before. However, with $N > 2$ players, we need to use numerical approximations for finding roots of polynomials. In the asymmetric case, another layer of numerical methods is needed, because we do not have an explicit solution to the system of differential equations that are obtained from the first order conditions. Therefore, we will have to obtain a numerical approximation for $\Delta(x, y)$ and check whether it is non-positive everywhere using also numerical methods. In some sense, this is simpler than what we have done, requiring much less theoretical effort. On the other*

[44] The proof of this theorem is contained in the supplement to this paper.

hand, this difference impacts the speed and accuracy of the method (it is not possible to state a "if and only if" result anymore). However, as long as these numerical problems are well understood, the method does not present particular difficulties.

6.3 The Revenue Ranking of Auctions

As a further illustration of our approach, we show how grid distributions can be used to address the problem of revenue ranking of the first price and second price auctions. Let us denote by R_2^f the expected revenue (with respect to $f \in \mathcal{D}^k$) of the second price auction.[45] Similarly, R_1^f denotes the expected revenue (with respect to $f \in \mathcal{D}^k$) of the first price auction. When there is no need to emphasize the pdf $f \in \mathcal{D}^k$, we write R_1 and R_2 instead of R_1^f and R_2^f. Below, μ refers to the natural measure defined over $\mathcal{D}^\infty = \cup_{k=1}^{\infty} \mathcal{D}^k$, as further explained in the supplement to this paper.

The following theorem gives the expression of the expected revenue difference $\Delta_R^f \equiv R_2^f - R_1^f$ between the second and the first price auctions and it is *not* restricted to densities in $f \in \mathcal{D}^\infty$.

Theorem 6.7 *Assume that f has a SMPSE in the first price auction. The revenue difference between the second and the first price auction is given by*

$$\int_0^1 \int_0^x b'(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

where $b(\cdot)$ is the first price equilibrium bidding function, or by

$$\int_0^1 \int_0^x \left[\int_0^y L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \quad (16)$$

where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$.

Proof. *See the appendix.* ■

In order to make a relative comparison, we define $r \equiv \frac{R_2^f - R_1^f}{R_2^f}$, for each f. Generating a uniform sample of $f \in \mathcal{D}^k$, we can obtain the probabilistic distribution of Δ_R^f or of r. The procedure to generate $f \in \mathcal{D}^k$ uniformly

[45]Remember that, since we are working with private value, second price auctions are equivalent to English auctions.

is described in the supplement to this paper. The results are shown in subsection 6.4 below.

In this fashion, we are able to obtain previsions based on simulations and also theoretical results. One possible objection to this approach is that it considers too equally the pdf's in the sets $\mathcal{D}^k$. But this is just because we are not assuming any specific information about the context where the auction runs—in some sense, this is a "context-free"approach. If one has information on the environment where the auction takes place, so that one can restrict the set of suitable pdf's, then the uniform measure should be substituted by the empirical measure obtained from this environment. Obviously, the method can be easily adapted to this, once one has such "empirical measure" of the possible distributions.

Now, we present the numerical results that one can obtain using this approach.

6.4 Numerical results

The method developed in this paper allows us to compare the revenue ranking of first and second price auction with general dependence. In the figure below, we show the histograms of $r \equiv \frac{R_2^f - R_1^f}{R_2^f}$ where $f \in \mathcal{D}^k$ is drawn from a uniform distribution among the distributions in $\mathcal{D}^k$, for $k = 3, ..., 10$. By this we mean that each $f \in \mathcal{D}^k$ is equally likely to be drawn in our simulation. The generation of a uniform distribution over $\mathcal{D}^k$ is not trivial. We base our development on Devroye (1986); see the supplement of this paper for more details.

The results remain qualitatively the same if, instead of the uniform distribution in $\mathcal{D}^k$, we choose some normal distribution in $\mathcal{D}^k$ with peak centered around the independent and symmetric $f \in \mathcal{D}^k$ (i.e., $f(x, y) \equiv 1, \forall (x, y)$).[46] In the figure, two curves are shown in the graph of each k: one corresponds to the histogram for all f such that there is a SMPSE in the first price auction; the other curve corresponds to all f such that there is a SMPSE in the first price auction *and* the correlation implied by f is non-negative. As the reader can see, this restriction does not change significantly the results, that is, the difference between the two histograms is small.

[46] We use the euclidean metric for defining a distance between densities in $\mathcal{D}^k$. Then, we use this distance from the $f \equiv 1$ as the parameter of a unidimensional normal with zero average and some positive variance. We tested many variances and the results do not change much. When the variance is big, the numbers become very similar to the previous case, where all distributions in $\mathcal{D}^k$ are equally likely.

The main finding is that in most of the (grid) distributions that have a pure strategy equilibrium, the first-price auction tends to give higher expected revenue than the second-price auction. This finding remains the same irrespective of restricting or not to cases with positive correlation. This result seems “surprising” to us, since affiliation implies that the revenue ranking is always in the reverse direction, that is, the second price gives higher revenue.

Interestingly, if we do not consider the restriction that there is a pure strategy equilibrium, the histogram for r becomes centered around zero.[47] This suggests that some kind of weak version of the revenue equivalence principle may be true on average, without the existence constraint. On the other hand, this also suggests a reason for why first price auction tends to give higher revenue. In short, it seems that the existence of equilibrium in first price auction is more likely in cases where this format also gives higher revenue. The pure strategy equilibrium existence seems to be a condition that favors higher revenue for the first-price auction. These observations may direct future theoretical formalizations.

6.5 Approximation

Another advantage of working with grid distributions is that they have nice approximation properties, as the following result illustrates.

Theorem 6.8 *The set of grid distributions is dense in $\Delta(V)$, endowed with its natural (weak*) topology. Also, the set $\mathcal{D}^\infty$ of grid densities is dense in:*

(i) $\mathcal{D}$, the set of densities $f : T \to \mathbb{R}_+$, endowed with the pointwise topology.

(ii) L^p, the set $\mathcal{D}$ endowed with the L^p-norm (w.r.t. Lebesgue measure).

(iii) $\mathcal{C}$, the set of continuous densities, endowed with the sup-norm.[48]

Theorem 6.8 shows that by restricting the set of distributions on types that we consider to grid distributions, we do not lose much: we can approximate any measure as well as we want.

[47]The integral expressions for the revenue difference (16) are meaningful even without SMPSE existence.

[48]This is a slight abuse of terminology, since that $\mathcal{D}^\infty$ is not contained in $\mathcal{C}$. What we show is that for each $f \in \mathcal{C}$ and $\epsilon > 0$, there exists $k \in \mathbb{N}$ and $f^k \in \mathcal{D}^k$ such that $\|f - f^k\| \equiv \sup_t |f(t) - f^k(t)| < \epsilon$.

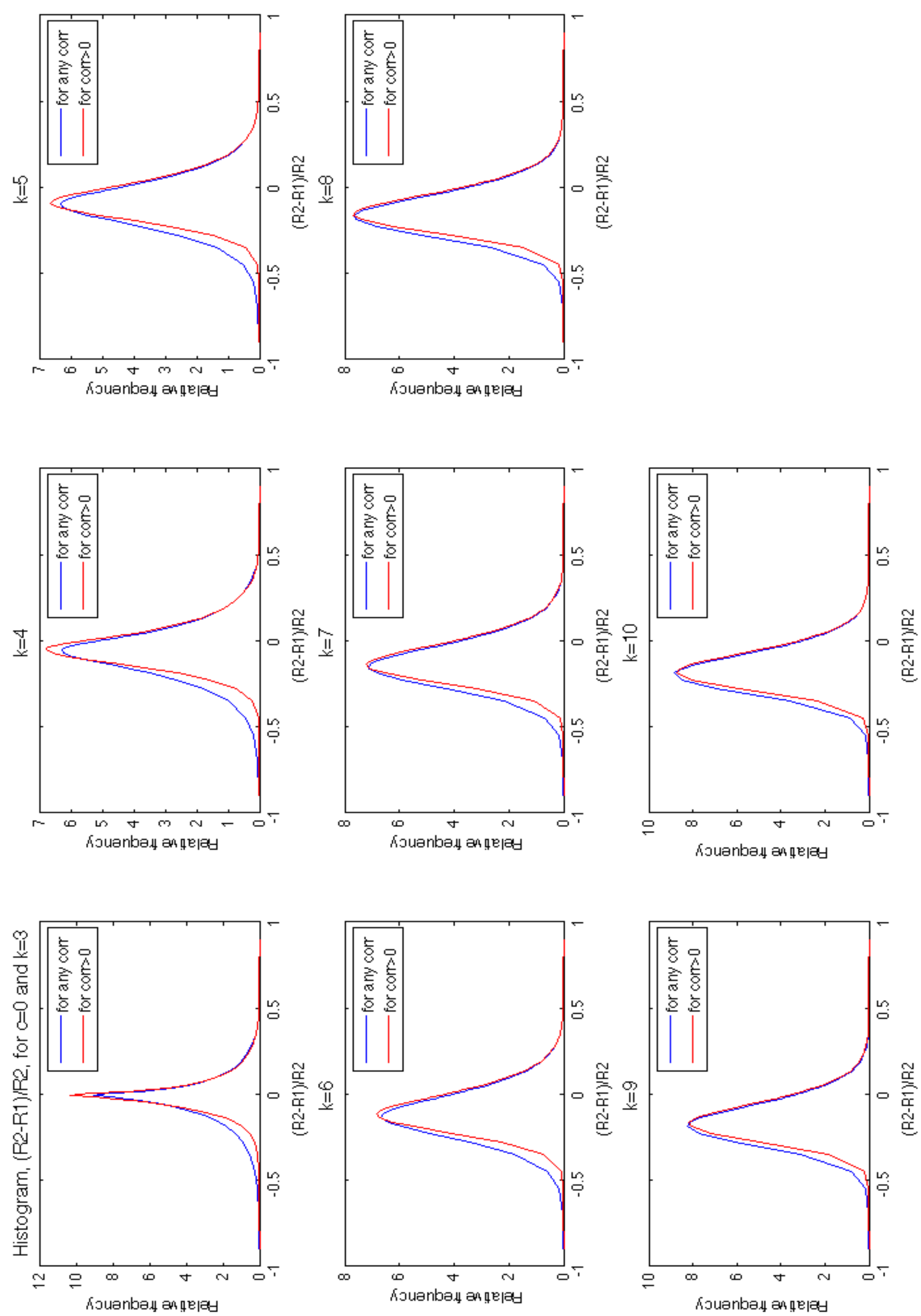

Figure 4: Histogram of Revenue Differences

The transformation $\mathbb{T}^k$ is stable for some conditions like affiliation.[49] Indeed, let $\mathcal{A} \subset \mathcal{D}$ denote the set of affiliated density functions; we have the following:

Proposition 6.9 *Assume that f is continuous. Then f is affiliated if and only if for all k, $\mathbb{T}^k(f)$ also is. In mathematical notation: $f \in \mathcal{A} \Leftrightarrow \mathbb{T}^k(f) \in \mathcal{A}$, $\forall k \in \mathbb{N}$ or yet, $\mathcal{A} = \cap_{k\in\mathbb{N}} \left(\mathbb{T}^k\right)^{-1}\left(\mathcal{A} \cap \mathcal{D}^k\right)$.*

More importantly, the approximation given by $\mathbb{T}^k$ preserves pure strategy equilibria for all continuous Bayesian games.[50]

Proposition 6.10 *Assume that f and the ex post utility $u_i : T \times A \to \mathbb{R}$ are continuous, $\forall i \in I$. Assume that there is a sequence $\{k_n\}$ such that $\mathbb{T}^{k_n}(f)$ has an equilibrium $s^{k_n} = (s_1^{k_n}, ..., s_N^{k_n})$ for all n and the sequence $\{s^{k_n}\}$ converges pointwise to $s \in \mathcal{F}$. Then, s is a pure strategy equilibrium under f.*

Although a complete converse of the above result is not possible, because the equilibrium inequality can be approximated by above, the following provides a partial converse.

Proposition 6.11 *Let $f \in \mathcal{D}$ have an equilibrium $s \in \mathcal{F}$ and assume that u_i is continuous for every i. Then for each $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that s is a ε-equilibrium for $\mathbb{T}^k(f)$, for all $k \geqslant k_\varepsilon$.*

The above result is satisfactory for numerical applications, because numerical calculation will involve errors anyway.

7 Discussion

This section discusses the received literature and clarifies the connections between this paper and Neemans's "beliefs do not determine preferences" (section 7.2). Section 7.3 is a short conclusion.

[49] We say that $f \in \mathcal{D}$ is affiliated if for all $x, y \in T$, $f(x)f(y) \leqslant f(x \wedge y)f(x \vee y)$.

[50] Whenever discontinuities occur in zero probability sets, as in the private value auctions covered by Theorem 5.1, the conclusion of Propositions 6.10 and 6.11 still hold.

7.1 Received literature

After the pioneering work of Wilson (1969) and (1977), one of the most important contributions to the study of dependence in games was made by Milgrom and Weber (1982) when they introduced affiliation. This became a central assumption in the study of correlated types and it is, to this date, a hallmark of economic theory. However, some of the implications of affiliation are not robust to other forms of dependence—see de Castro (2011).

Even with affiliation, however, progress in the study of Bayesian games with correlated types has been slow and difficult.[51] The case of first price auction with interdependent values with two players was obtained by Athey (2001) and Lizzeri and Persico (2000). Later, Reny and Zamir (2004) extended this to n players. In the case of multi-dimensional auctions, pure strategy equilibrium existence (PSEE) is established exclusively for independent types; Reny (1999), McAdams (2003), Jackson and Swinkels (2005), Reny (2011). There is not a single general result of PSEE with affiliation for multi-unit auctions. Indeed, McAdams (2007) provides a counterexample for the existence of monotonic equilibria in a uniform price auction with affiliated types. Departing from the question of equilibrium existence, affiliation is also not enough to imply the linkage principle in multi-unit auctions (see Perry and Reny (1999)). This suggests a difficult tension: it seems that we need stronger conditions, but at the same time we would like to have a more flexible framework. But what do we know outside of affiliation?

The more general papers that allow correlation of types—Jackson, Simon, Swinkels, and Zame (2002) or Jackson and Swinkels (2005)—prove equilibrium existence only in mixed strategies.[52] The question is whether pure strategy equilibria exist. Some general results have been established by van Zandt and Vives (2007) and van Zandt (2010), but these papers require assumptions that are not valid for some important classes of Bayesian games.[53] Out of affiliation, we do not have general pure strategy equilibria existence results even for games as simple as symmetric private value first price auctions.[54] Thus, PSEE in Bayesian games out of independent types

[51] We will just highlight some of the standing gaps in the literature. For a more complete survey, see de Castro and Karney (2011).

[52] As mentioned above, Jackson and Swinkels (2005) also establish pure strategy equilibrium existence assuming independence.

[53] The utility functions are required to have increasing differences in all actions, a property that does not hold in auctions, for instance.

[54] de Castro (2007) and Monteiro and Moreira (2006) present partial results in this particular setting.

is an old, but still open question, although important progress has been made by Athey (2001), McAdams (2003) and Reny (2011), as we discuss below.

7.1.1 Relation to Athey-McAdams-Reny's approach

The most general and successful approach to establish pure strategy equilibria in Bayesian games has been initiated by Athey (2001) and further developed by McAdams (2003), Reny and Zamir (2004) and Reny (2011). Our main contribution is not a substitute for this approach, but it can be seen as a complement, as we clarify below.

These papers consider the best reply of each player to monotonic strategies played by the opponents. Since they are interested in proving only the existence of monotonic equilibria, this is enough and convenient. They work with different (increasingly weaker) assumptions on *interim* payoff functions when the opponents play monotonic strategies and use different forms of single crossing conditions to obtain their equilibrium results. Their conditions are then checked on specific games, leading to pure strategy equilibrium results for those games. Assumptions on distributions are in general not necessary at the interim level, so they appear only at the examples given. These papers contain a number of results that expanded the set of results where we could establish equilibrium existence in Bayesian games. For instance, Athey (2001, Theorem 7) contains many new results for different Bayesian games, including first price auctions with interdependent asymmetric values, affiliated types and two bidders. Reny and Zamir (2004) extended Athey's result and obtain a similar result for the case of n bidders. McAdams (2003) established pure strategy equilibrium existence for uniform price auction with multi-unit demand, interdependent values, risk neutrality, n bidders and independent types. Reny (2011) generalizes this last result by establishing equilibrium existence for risk averse bidders, both for the uniform and discriminatory auctions. Note that with respect to the applications to multi-unit auctions, all of these papers assume that types are independent.

Since their main model is on the interim stage and no correlation assumption is needed at this level, these authors seem to aim results that do not require independence. Therefore, it is very understandable that they did not consider properties that seem to crucially depend on independence. Namely, they did not explore the property that if types are independent, then a best reply to *any strategy*—not only to monotonic ones—needs to be monotonic (under standard monotonicity assumptions on the utility functions). This property was first established for first price auction by Maskin and Riley (2000) and later generalized by Araujo and de Castro (2009). Theorem 4.3, which generalizes this result, seems to be the first of its kind for general action spaces (see also Theorem 8.11 in the appendix).[55] Note that this property leads to a much stronger kind of result: every equilibrium, if it exists, is in pure strategies. Also, the best replies are unique.

Thus, our paper differs from Athey-McAdams-Reny's approach in the following aspects. While they provide specific examples where dependence plays a role, their main objective is to provide general methods to prove equilibrium existence in Bayesian games in a level where dependence is not important (interim stage). In contrast, we focus on the dependence of types, including but not exclusively focusing on equilibrium existence. Although we did not work out the details, it is conceivable (and actually expected) that richness could also be used in conjunction with their method to establish pure strategy equilibrium existence. In this sense, our contributions could be viewed as complementary. On the other hand, since their assumptions allow multiple best replies, their fixed point arguments require to deal with the issue of convexity of the set of best replies or, in the case of Reny (2011), contractibility. Actually, this issue is important if not central in all these papers. In our case, the set of best replies is unitary (at least in the perturbed games that we consider), so that convexity is trivial.

7.1.2 Other relevant papers

The grid distributions discussed here appeared first in de Castro (2008) and were used by de Castro and Paarsch (2010) to test for affiliation.

Fang and Morris (2006) study the revenue ranking of first and second price auctions in a model with finite correlated types, who share some beliefs. They obtain equilibrium in mixed strategies because they work with finite types, but one of the characterizations of the equilibrium (monotonicity of best replies) could be obtained using our Theorem 8.11. Notice that they also write the types as consisting of two parts: the preference part v_i and the belief part δ_i (in our notation).

Barelli and Duggan (2011) investigated the question of pure strategy equilibrium existence from the point of view of the Lyapunov theorem. Quah and Strulovici (2009, 2012) studied extensions of the single-crossing condition and its properties for Bayesian games. Their extensions explore di-

[55]The generalization for infinite-dimensional action spaces is not trivial. Among other things, it requires some form of restriction on the metric space considered, as Assumption 3.2. Also, our argument is quite different from previous results.

mensions somewhat different from ours. For a more complete discussion, see de Castro (2012).

7.2 Usual Approach and the two parameters alternative

In the introduction, we called "usual approach" the practice of using a single variable v_i to convey two different pieces of information: player i's information about her payoff and her beliefs about other players' parameters. Milgrom and Weber (1982, Footnote 14, p. 1097) justify the usual approach as follows: "To represent a bidder's information by a single real-valued signal is to make two substantive assumptions. Not only must his signal be a sufficient statistic for all of the information he possesses concerning the value of the object to him, it must also adequately summarize his information concerning the signals received by the other bidders. The derivation of such a statistic from several separate pieces of information is in general a difficult task (...). It is in the light of these difficulties that we choose to view each X_i as a "value estimate," which may be correlated with the "estimates" of others but is the only piece of information available to bidder i."

As we can see, simplicity seem to be the main motivation for the usual approach. As attractive as this simplicity may seem, it carries with it an important drawback. Namely, it opens the door for the (counter-intituitive and unrealistic) possibility of full-extraction of surplus, as established by Crémer and McLean (1985). Neeman (2004) and Heifetz and Neeman (2006) show that full extraction of surplus requires that "beliefs determine preferences." That is, it requires a setting where almost all types have different beliefs; if types share beliefs, full extraction is not possible. This suggests that models where beliefs do not determine preferences (non-BDP) are more realistic and desirable from a mechanism design point of view. This paper argues essentially the same, but from the point of view of applications to general Bayesian games.

As mentioned before, richness is stronger than their non-BDP in two aspects. First, while it is enough for them that a positive mass of types share beliefs, we require that every set with a positive mass contains types sharing beliefs. Second, richness requires that types are strictly ordered, while they do not need any order on types. Despite these technical differences, which are necessary to establish our results, we view the general message of both works as very similar: we should give more attention to models where beliefs do not determine preferences.

The economic intuition for why this is reasonable was given by Neeman (2004, p. 67-8) in the following terms:

> Issues of generality notwithstanding, preferences and beliefs have traditionally been considered to be independent of one another in both economic and decision theory. This tradition presumably reflects the idea that the processes that generate utilities and beliefs are cognitively distinct and causally "independent," or at least should be treated as such.

There is a natural way of interpreting that preferences and beliefs are "independent," as Neeman puts it: that types are formed as in a product structure of preferences and beliefs. In this interpretation, one should require that the measure on types is absolutely continuous with respect to the product of the marginals over preferences and types. As established by Lemma 2.3, this implies richness. Therefore, richness also captures their intuition.

This discussion leads to the following rough summary: if we insist on working with models where richness fails, then we are essentially in the world of full-extraction of surplus and the difficulty in establishing pure strategy equilibrium existence. On the other hand, if we accept richness, full-extraction of surplus is not possible and we can prove pure strategy equilibrium existence and study its properties. This model choice can also be discussed from the point of view of simplicity vs. generality. The more general point of view is to allow payoffs and beliefs to be "independent," thus leading to richness. If we emphasize simplicity and want to stick to the usual approach, then we can go to the setting of very simple distributions, which is simpler and still satisfy richness.

7.3 Conclusion

In this paper we introduced a special class of distributions that allows general dependence and asymmetry in general games with incomplete information. This class of distribution can be studied both with theoretical and numerical methods. We illustrated the potential applications with theoretical results and computer experiments (simulations). It is shown that a fast algorithm exists for determining symmetric pure strategy equilibrium existence in auctions with n players. We also proved the existence of pure strategy equilibrium in Bayesian games with a new monotonicity condition, richness. Although we focused mainly (but not exclusively) on equilibrium existence results, so that most of our results are deeply related to equilibrium existence, our main contribution is to offer a new framework for working with correlation of types.

Appendix

8 Appendix A: Main Proofs

8.1 Proofs for Section 2.1

Proof of Lemma 2.1: Given a measurable $E \subset V_i$ satisfying $\gamma_i(E) > 0$, for each $\delta \in \Delta_i$ define $E^\delta \equiv \{v \in E : \delta \in \Gamma(v)\}$ and $V_i^\delta \equiv \{v \in V_i : \delta \in \Gamma(v)\}$. Let $E' \equiv \cup_{\delta \in \Delta_i'} E^\delta \subset \cup_{\delta \in \Delta_i} E^\delta = E$. Since the set $V_i' \equiv \cup_{\delta \in \Delta_i'} V_i^\delta$ has full measure by (ii), $\gamma_i(E') = \gamma_i(E \cap V_i') > 0$. Since Δ_i' is countable and $\sum_{\delta \in \Delta_i'} \gamma_i(E^\delta) \geqslant \gamma_i(E') > 0$, for at least one $\delta \in \Delta_i'$, we have $\gamma_i(E^\delta) > 0$. By (i), there exist $v, v' \in E^\delta$ such that $v' \succ v$. Since $\delta \in \Gamma(v) \cap \Gamma(v')$, there exist t_i and t_i' such that $\widehat{V}_i(t_i) = v, \widehat{\delta}_i(t_i) = \delta$, $\widehat{V}_i(t_i') = v'$ and $\widehat{\delta}_i(t_i') = \delta$. By the order definition (1), the existence of these t_i, t_i' establishes richness. ∎

Proof of Lemma 2.3: For a set $E \in \mathcal{T}_i$, let E_δ denote projection of E over V_i, that is, $E_\delta \equiv \{v \in V_i : (v, \delta) \in m(E)\}$. Let π_i denote the product measure $\gamma_i \times \nu_i$. By Fubini's Theorem, $\pi_i(m(E)) = \int_{\Delta_i} \gamma_i(E_\delta)\nu_i(\delta)$. Thus, if $\tau_i(E) > 0$, which implies $\pi_i(m(E)) > 0$, the set of $\delta \in \Delta_i$ such that $\gamma_i(E_\delta) > 0$ has ν_i-positive measure. Fix any δ in this set. Since $\gamma_i(E_\delta) > 0$, by condition (i) of Lemma 2.1, there exist $v, v' \in E_\delta$ such that $v' \succ v$. By the definition (1) of $\geqslant_i$, we have that $t_i \equiv (v, \delta), t_i' \equiv (v', \delta)$ satisfy richness. ∎

An inspection of the above proof shows that we actually need less than absolute continuity with respect to $\pi_i = \gamma_i \times \nu_i$. It is enough that $\tau_i(E) > 0$ imply the existence of a δ such that $\gamma_i(E_\delta) > 0$. But this property is essentially just a restatement of richness for this particular setting with a product structure.

8.1.1 Monotonicity properties—definitions and proofs

Let X and Y be partially ordered sets and Z an arbitrary set. Consider a function $g : X \times Y \times Z \to \mathbb{R}$. It is useful to recall some definitions.

Definition 8.1 (Supermodularity) *If Y is a lattice, g is supermodular in Y if for any $x \in X$ and $z \in Z$ and any pair $y, y' \in X$, we have:*

$$g(x, y, z) - g(x, y \wedge y', z) \quad \leqslant \quad g(x, y \vee y', z) - g(x, y', z).$$

Also, we say that g is strictly supermodular if the inequality above is strict whenever y and y' are incomparable.

It is well-known and easy to verify that supermodularity is preserved under integration. It is also easy to see that supermodularity implies quasi-supermodularity:

Definition 8.2 (Quasi-supermodularity) *Assume that Y is a lattice and consider the following implication, implicitly supposed to hold $\forall x \in X, \forall y, y' \in Y, \forall z \in Z$:*

$$g(x,y,z)\ \mathbf{R}\ \ g(x,y \wedge y',z) \quad \Longrightarrow \quad g(x,y \vee y',z)\ \mathbf{R}'\ \ g(x,y',z). \tag{17}$$

We say that g is:

- *weak quasi-supermodular in Y if (17) holds with $(\mathbf{R},\mathbf{R}') = (\geqslant,\geqslant)$;*
- *strictly quasi-supermodular in Y if (17) holds with $(\mathbf{R},\mathbf{R}') = (\geqslant,>)$;*
- *quasi-supermodular in Y if (17) holds both with $(\mathbf{R},\mathbf{R}') = (\geqslant,\geqslant)$ and $(\mathbf{R},\mathbf{R}') = (>,>)$.*

Although x does not a play a role in supermodularity and quasi-supermodularity, it is important for nondecreasing differences and single-crossing properties.

Definition 8.3 (Nondecreasing differences) *g satisfies the non-decreasing differences property in $X \times Y$ if for any $x, x' \in X$ $y, y' \in Y'$ and $z \in Z$, such that $x < x'$ and $y < y'$,*

$$g(x,y',z) - g(x,y,z) \leqslant g(x',y',z) - g(x',y,z). \tag{18}$$

It satisfies increasing differences in $X \times Y$ if (18) holds with $<$ instead of $\leqslant$.

As it is well known (and easy to see), the nondecreasing differences implies single-crossing, while increasing differences implies strict single-crossing.

Definition 8.4 (Single-crossing property) *Consider the following implication, supposed to hold for any $x, x' \in X$ $y, y' \in Y'$ and $z \in Z$, such that $x < x'$ and $y < y'$:*

$$g(x,y',z)\ \mathbf{R}\ g(x,y,z) \implies g(x',y',z)\ \mathbf{R}'\ g(x',y,z); \tag{19}$$

We say that g satisfies:

- *weak single-crossing in $X \times Y$ if (19) holds with $(\mathbf{R},\mathbf{R}') = (\geqslant,\geqslant)$;*
- *strict single-crossing in $X \times Y$ if (19) holds with $(\mathbf{R},\mathbf{R}') = (\geqslant,>)$;*
- *single-crossing in $X \times Y$ if (19) holds both with $(\mathbf{R},\mathbf{R}') = (\geqslant,\geqslant)$ and $(\mathbf{R},\mathbf{R}') = (>,>)$.*

The following property is introduced and discussed in de Castro (2012):

Definition 8.5 (ID-Monotonicity Property) *g satisfies the ID-monotonicity property in $X \times Y$ if for any $x, x' \in X$ and $y, y' \in Y$ such that $x' > x$ and $\neg(y' \geqslant y)$,*[56] *there exists $\underset{\sim}{y}, \widetilde{y} \in Y$ satisfying the following:*[57]

$$g(x, y, z) - g(x, \underset{\sim}{y}, z) < g(x', \widetilde{y}, z) - g(x', y', z), \forall z \in Z. \tag{20}$$

de Castro (2012) establishes the following:

Proposition 8.6 *Assume that Y is a lattice and that $g : X \times Y \times Z \to \mathbb{R}$ is supermodular in Y and satisfies increasing differences in $X \times Y$ or, alternatively, it is strictly supermodular in Y and satisfies non-decreasing differences in $X \times Y$. Then g satisfies the ID-monotonicity property in $X \times Y$.*

Proposition 8.6 is proved below for reader's convenience.

Proof of Proposition 8.6: Let $x, x' \in X$, such that $x < x'$; $y, y' \in Y$ such that $\neg(y' \geqslant y)$, and $z \in Z$. Since Y is a lattice, there exists $\widetilde{y} = \overline{y} \equiv y \vee y'$ and $\underset{\sim}{y} = \underline{y} \equiv y \wedge y'$. From supermodularity, we have:

$$g(x, y, z) - g(x, \underset{\sim}{y}, z) \quad \leqslant \quad g(x, \widetilde{y}, z) - g(x, y', z). \tag{21}$$

Since $\neg(y' \geqslant y)$, $\widetilde{y} \neq y'$, that is , $\widetilde{y} > y'$. By increasing differences, we have

$$g(x, \widetilde{y}, z) - g(x, y', z) < g(x', \widetilde{y}, z) - g(x', y', z). \tag{22}$$

Then, (21) and (22) imply (20). In the case of strict supermodularity and non-decreasing differences, (21) holds with a strict inequality sign while (22) holds with weak inequality, still giving (20). ∎

An interesting characteristic of the ID-monotonicity property is that it is preserved under integration. Recall that S_{-i} is the set of behavioral strategies by players $j \neq i$. Fix $\delta \in \Delta_i \equiv \widehat{\delta}_i(T_i)$ and define $T_{i\delta} \equiv \{t_i \in T_i : \widehat{\delta}_i(t_i) = \delta\}$.

Lemma 8.7 *If $u_i : T_{i\delta} \times A_i \times T_{-i} \times A_{-i} \to \mathbb{R}$ satisfies the ID-monotonicity property in $T_{i\delta} \times A_i$, so does $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$.*

Proof. Consider $t_i, t_i' \in T_{i\delta}$ such that $t_i < t_i'$, and $a_i, a_i' \in A_i$ such that $\neg(a_i' \geqslant a_i)$. Then, there exists $\underline{a}_i, \overline{a}_i \in A_i$ such that $\overline{a}_i \neq a_i'$ and

$$u_i(t_i, a_i, \cdot) - u_i(t_i, \underline{a}_i, \cdot) < u_i(t_i', \overline{a}_i, \cdot) - u_i(t_i', a_i', \cdot), \tag{23}$$

[56] As usual, the symbol $\neg$ means negation.

[57] Whenever we consider functions $g : X \times Y \to \mathbb{R}$, just ignore z and Z in (20).

where "·" stands for t_{-i}, a_{-i}. Now, integrating with respect to a mixed strategy $\mu_{-i} \in S_{-i}$ and the belief $\delta \in \widehat{\delta}_i(T_i) \subset \Delta(T_{-i})$, we obtain:

$$\begin{aligned}
& \Pi(t_i, a_i) - \Pi(t_i, \underline{a}_i) \\
= \; & \int_{T_{-i}} \int_{A_{-i}} [u_i(t_i, a_i, t_{-i}, a_{-i}) - u_i(t_i, \underline{a}_i, t_{-i}, a_{-i})] \, \mu_{-i}(t_{-i}, da_{-i}) \delta(dt_{-i}) \\
< \; & \int_{T_{-i}} \int_{A_{-i}} [u_i(t_i', \overline{a}_i, t_{-i}, a_{-i}) - u_i(t_i', a_i', t_{-i}, a_{-i})] \, \mu_{-i}(t_{-i}, da_{-i}) \delta(dt_{-i}) \\
= \; & \Pi(t_i', \overline{a}_i) - \Pi(t_i', a_i'), \qquad (24)
\end{aligned}$$

as we wanted to show. ∎

Therefore, if the ex post utility function satisfies it, the interim payoff function also does. This is useful because it is in general much easier to check a condition on the ex post payoff function. If we are willing to make assumptions directly on the interim function, then the increasing difference form of the monotonicity property can be relaxed to a single-cross form:

Definition 8.8 (SC-Monotonicity Property) *g satisfies the SC-monotonicity property in $X \times Y$ if for any $x, x' \in X$ and $y, y' \in Y$, such that $x' > x$ and $\neg(y' \geqslant y)$, there exists $\underset{\sim}{y}, \widetilde{y} \in Y$ satisfying the following:*

$$g(x, y, z) \geqslant g(x, \underset{\sim}{y}, z) \implies g(x', \widetilde{y}, z) > g(x', y', z), \forall z \in Z. \qquad (25)$$

It is clear that ID-monotonicity implies SC-monotonicity. Also, Proposition 8.6 has a parallel version for the SC-monotonicity property (see de Castro (2012)):

Proposition 8.9 *Assume that Y is a lattice and that $g : X \times Y \times Z \to \mathbb{R}$ is weak quasi-supermodular in Y and satisfies strict single crossing in $X \times Y$ or, alternatively, it is strictly quasi-supermodular in Y and satisfies single crossing in $X \times Y$. Then g satisfies the SC-monotonicity property in $X \times Y$.*

Once more, we include the proof for readers' convenience.

Proof of Proposition 8.9: Let $x, x' \in X$, such that $x' > x$; $y, y' \in Y$ such that $\neg(y' \geqslant y)$, and $z \in Z$. Since Y is a lattice, there exists $\widetilde{y} = \overline{y} \equiv y \vee y'$ and $\underset{\sim}{y} = \underline{y} \equiv y \wedge y'$. From weak quasi-supermodularity, we have:

$$g(x, y, z) \geqslant \; g(x, \underset{\sim}{y}, z) \quad \implies \quad g(x, \widetilde{y}, z) \geqslant \; g(x, y', z). \qquad (26)$$

Since $\neg(y' \geqslant y)$, $\widetilde{y} > y'$. By strict single crossing, we have

$$g(x, \widetilde{y}, z) \geqslant \; g(x, y', z) \quad \implies \quad g(x', \widetilde{y}, z) > \; g(x', y', z); \qquad (27)$$

Then, (26) and (27) imply (25). In the case of strict quasi-supermodularity and single crossing, (26) holds with a strict inequality sign on the left, while (27) holds with strict inequalities in both sides, still giving (25). ∎

Eventually, we want to establish the monotonicity property of Π_i without having the exact monotonicity property on u_i. The following Lemma can be useful in this case.[58]

Lemma 8.10 *Assume that* $u_i : T_{i\delta} \times A_i \times T_{-i} \times A_{-i} \to \mathbb{R}$ *satisfies the following:*

(i) it is supermodular in A_i*;*

(ii) it satisfies non-decreasing differences in $T_{i\delta} \times A_i$*;*

(iii) for any $t_i^1, t_i^2 \in T_{i\delta}$ *and* $a_i^1, a_i^2 \in A_i$ *and* $\mu_{-i} \in S_{-i}$ *satisfying* $t_i^1 < t_i^2$, $a_i^1 <_i a_i^2$, *we have* $\delta(T'_{-i}) > 0$, *where* T'_{-i} *is the set defined by those* $t_{-i} \in T_{-i}$ *for which*

$$\begin{aligned} & \int_{A_{-i}} \left[u_i(t_i^1, a_i^2, t_{-i}, a_{-i}) - u_i(t_i^1, a_i^1, t_{-i}, a_{-i})\right] \mu_{-i}(t_{-i}, da_{-i}) \\ < & \int_{A_{-i}} \left[u_i(t_i^2, a_i^2, t_{-i}, a_{-i}) - u_i(t_i^2, a_i^1, t_{-i}, a_{-i})\right] \mu_{-i}(t_{-i}, da_{-i}). \end{aligned}$$

Then $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$ *has the ID-monotonicity property in* $T_{i\delta} \times A_i$.

Proof. Using (i) and (ii), we can repeat the proof of Proposition 8.6, where both (21) and (22) are with weak inequalities. Now, following the proof of Lemma 8.7, we have (23) holding with weak inequality, which also implies (24) with weak inequality. However, because of (iii), (24) actually holds with strict inequality. This concludes the proof. ∎

8.2 Proofs for Section 4

Instead of Theorems 4.1 and 4.2, we will actually prove the following, respectively:

Theorem 8.11 *Let Assumptions 3.1, 3.2 and 3.3 hold. Assume that for each* $\delta \in \Delta_i$, $u_i : T_{i\delta} \times T_{-i} \times A \to \mathbb{R}$ *satisfies the ID-monotonitiy property in* $T_{i\delta} \times A_i$. *Let* $\mu_{-i} \in S_{-i}$ *. The following holds:*

1. *If* $t_i, t'_i \in T_{i\delta}$, $t_i < t'_i$, $a_i \in BR(t_i, \mu_{-i})$, $a'_i \in BR(t'_i, \mu_{-i})$ *then* $a_i \leqslant a'_i$.

2. *If* μ_i *be a better reply strategy to* μ_{-i}, *then for each* $\delta \in \Delta_i$, *the set of player* i*'s types in* $T_{i\delta}$ *who possibly play mixed strategies under* μ_i *is a denumerable union of antichains.*[59,60]

Theorem 8.12 *Let Assumptions 3.1, 3.2 and 3.3 hold. Assume that for each* $\delta \in \Delta_i$, $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$ *satisfies the SC-monotoniticy property in* $T_{i\delta} \times A_i$. *Let* $\mu_{-i} \in S_{-i}$. *Then the two conditions of Theorem 8.11 hold.*

[58] It is actually used in our proof of Theorem 5.1.

[59] By denumerable we mean either finite or countable.

[60] An antichain in a partially ordered set $(X, \leqslant)$ is a subset of X that contains no two ordered points.

8.2.1 Proof of Theorems 8.11 and 8.12

By Lemma 8.7, ID-monotonicity on u_i implies ID-monotonicity on Π_i which trivially implies SC-monotonicity on Π_i. Therefore, Theorem 8.11 is implied by Theorem 8.12 and it is enough to prove this last one.

The proof will be divided in a number of lemmas. In all results below, we fix $\delta \in \Delta_i \equiv \widehat{\delta}_i(T_i)$ and define $T_{i\delta} \equiv \{t_i \in T_i : \widehat{\delta}_i(t_i) = \delta\}$.

Lemma 8.13 *Assume that Π_i satisfies the SC-monotonicity property in $T_i' \times A_i$, where $T_i' \subset T_i$. Let $BR_i : T_i' \to A_i$ denote the correspondence of i's best replies to μ_{-i}. Then any selection of BR_i is monotone nondecreasing, that is, for any $t_i, t_i' \in T_i$ such that $t_i' > t_i$ and $a_i \in BR_i(t_i)$, $a_i' \in BR_i(t_i')$, we have $a_i' \geqslant a_i$.*

Proof. This is proved by de Castro (2012). We reproduce the proof here for reader's convenience. Fix $t_i < t_i'$, $a_i \in BR_i(t_i)$, $a_i' \in BR_i(t_i')$ and assume that $\neg(a_i' \geqslant a_i)$. By the SC-monotonicity property, there exists $\underset{\sim}{a_i}, \widetilde{a_i} \in Y$ satisfying the following:

$$\Pi_i(t_i, a_i) \geqslant \Pi_i(t_i, \underset{\sim}{a_i}) \implies \Pi_i(t_i', \widetilde{a_i}) > \Pi_i(t_i', a_i'). \tag{28}$$

Notice that the fact that $a_i \in BR_i(t_i)$ implies that the inequality on the left holds and, therefore, so does the one on the right. However, this contradicts the fact that $a_i' \in BR_i(t_i')$. The contradiction establishes the implication. ∎

The following technical result about chains will be used below.

Lemma 8.14 *Let C be a chain in a partially ordered set $(X, \geqslant)$ with at least three elements. Then there exist disjoint sets C_1, C_2 and C_3 such that $C_1 \cup C_2 \cup C_3 = C$ and for any two points $x, y \in C_i$ with $x > y$, for $i = 1, 2, 3$, there exists $z \in C \setminus C_i$ such that $x > z > y$.*

Proof. Let $\mathcal{E}$ denote the class of pair of sets (C_1, C_2) such that $C_1 \cap C_2 = \emptyset$, $C_1, C_2 \subset C$ and satisfying the following:

($\bullet$) for $i = 1, 2$ and any $x, y \in C_i$ with $x > y$, there exists $z \in C_{3-i}$ such that $x > z > y$.

Order $\mathcal{E}$ by inclusion, that is, $(E_1, E_2) \succcurlyeq (C_1, C_2)$ if $E_i \supset C_i$ for $i = 1, 2$.

Consider a chain $\{(C_1^\lambda, C_2^\lambda)_{\lambda \in \Lambda}\}$ in $\mathcal{E}$. Define $D_1 = \cup_{\lambda \in \Lambda} C_1^\lambda$ and $D_2 = \cup_{\lambda \in \Lambda} C_2^\lambda$. We claim that $(D_1, D_2) \in \mathcal{E}$. Indeed, trivially $D_1, D_2 \subset C$. If there exists $x \in D_1 \cap D_2$, then $x \in C_1^\lambda$ and $x \in C_2^{\lambda'}$ for some $\lambda, \lambda' \in \Lambda$, but since Λ determines a chain, either $(C_1^\lambda, C_2^\lambda) \succcurlyeq (C_1^{\lambda'}, C_2^{\lambda'})$ or $(C_1^{\lambda'}, C_2^{\lambda'}) \succcurlyeq (C_1^\lambda, C_2^\lambda)$. Thus, we have either $x \in C_1^\lambda \cap C_2^\lambda \supset C_1^\lambda \cap C_2^{\lambda'}$ or $x \in C_1^{\lambda'} \cap C_2^{\lambda'} \supset C_1^\lambda \cap C_2^{\lambda'}$, but this contradicts $C_1^\lambda \cap C_2^\lambda = C_1^{\lambda'} \cap C_2^{\lambda'} = \emptyset$. Thus, $D_1 \cap D_2 = \emptyset$. Finally, we observe that (D_1, D_2) satisfies ($\bullet$). Indeed, fix $i = 1$ or 2 and pick $x, y \in D_i$ with $x > y$. Similarly to the previous argument, there exists $\lambda \in \Lambda$, such that $x, y \in C_i^\lambda$. But because $(C_1^\lambda, C_2^\lambda) \in \mathcal{E}$, then there exists $z \in C_{3-i}^\lambda$ such that $x > z > y$. But then $z \in D_{3-i}$, which completes the proof of the claim.

Trivially, (D_1, D_2) is an upper bound for the chain $\{(C_1^\lambda, C_2^\lambda)_{\lambda\in\Lambda}\}$. Therefore, by the Zorn's Lemma, there exists a maximal element (C_1, C_2) on $\mathcal{E}$. We will need to establish some facts about such maximal element.

Suppose that there exists $x \in C \setminus (C_1 \cup C_2)$. Since (C_1, C_2) is maximal in $\mathcal{E}$, we must have $(C_1 \cup \{x\}, C_2) \notin \mathcal{E}$ and $(C_1, C_2 \cup \{x\}) \notin \mathcal{E}$. This means that $(\bullet)$ fails for both pairs, that is,

$$(\star)_i\text{: } \exists y_i \in C_i \text{ such that } x > y_i \text{ and } C_{3-i} \cap [y_i, x] = \emptyset \text{ or } y_i > x \text{ and } C_{3-i} \cap [x, y_i] = \emptyset^{61}$$

holds for both $i = 1$ and $i = 2$. Fix $y_1 \in C_1$ and $y_2 \in C_2$ satisfying $(\star)_1$ and $(\star)_2$ respectively. Since C is a chain, we may assume, without loss of generality, that $y_1 > y_2$. If $x > y_1 > y_2$ then $(\star)_2$ would be false; therefore, we must have $y_1 > x$. Similarly, if $y_2 > x$, we would have $y_1 > y_2 > x$, contradicting $(\star)_1$. Therefore, we must have $y_1 > x > y_2$. Observe that $C_2 \cap (x, y_1) = \emptyset$ and $C_1 \cap (y_2, x) = \emptyset$. Also, $C_1 \cap (x, y_1) = \emptyset$; otherwise, let $y' \in C_1 \cap (x, y_1)$. Since $(C_1, C_2) \in \mathcal{E}$, by $(\bullet)$ there would exist $z \in C_2$ such that $y_1 > z > y'$ which would imply $z \in C_2 \cap (x, y_1)$, an absurd. Similarly, we have $C_2 \cap (y_2, x) = \emptyset$. Therefore, $(C_1 \cup C_2) \cap (y_2, y_1) = \emptyset$. Actually, we claim that more is true, namely, $C \cap (y_2, y_1) = \{x\}$.

To see this, suppose on the contrary that $x' \in C \cap (y_2, y_1)$ for some $x' \neq x$. Since $(C_1 \cup C_2) \cap (y_2, y_1) = \emptyset$, it must be the case that $x' \in C \setminus (C_1 \cup C_2)$. Assume $x' > x$—the case $x > x'$ is analogous, switching x and x'. We will prove that $(C_1', C_2') \equiv (C_1 \cup \{x\}, C_2 \cup \{x'\}) \in \mathcal{E}$. To see this, it is enough to verify $(\bullet)$. Indeed, let $u, v \in C_1'$ be such that $u > v$. If $u, v \in C_1$ there is nothing to prove. If $u = x$, since $C_1 \cap (y_2, y_1) = \emptyset$, then $y_2 > v$, which implies that $x > y_2 > v$, that is, $(\bullet)$ is satisfied. On the other hand, if $v = x$, then $u \geqslant y_1 > x' > x$ with $x' \in C_2'$ and $(\bullet)$ is also satisfied. The argument for $u, v \in C_2'$ is analogous. Therefore, $(C_1', C_2') \in \mathcal{E}$, but this contradicts the fact that (C_1, C_2) is maximal on $\mathcal{E}$. Therefore, $C \cap (y_2, y_1) = \{x\}$.

Let $\{x^\lambda\}_{\lambda\in\Lambda}$ denote the set $C_3 \equiv C \setminus (C_1 \cup C_2)$. By the above reasoning, for each $x^\lambda \in C_3$, there exist $\underline{y}^\lambda$ and $\overline{y}^\lambda$ such that (i) $\overline{y}^\lambda > x > \underline{y}^\lambda$; (ii) $C \cap (\underline{y}^\lambda, \overline{y}^\lambda) = \{x^\lambda\}$ and (iii) $\underline{y}^\lambda \in C_i$ and $\overline{y}^\lambda \in C_{3-i}$ for either $i = 1$ or $i = 2$.

Let $x^\lambda, x^{\lambda'} \in C_3$ be such that $x^{\lambda'} > x^\lambda$. Thus, $x^{\lambda'} > \underline{y}^{\lambda'} \geqslant \overline{y}^\lambda > x^\lambda$. Therefore, there exists $\overline{y}^\lambda \in C_1 \cup C_2$ between $x^{\lambda'}$ and x^λ. This shows that the C_1, C_2 and C_3 just defined satisfy the required property. ∎

Given a set $S \subset A_i$, its diameter will be denoted $diam(S)$ and defined by:

$$diam(S) \equiv \sup_{a_i, a_i' \in S} \rho_i(a_i, a_i').$$

[61] Here, $[w, z] \equiv \{u \in X : w \leqslant u \leqslant z\}$. Observe that since $y_1, x \notin C_2$, $C_2 \cap [y_1, x] = C_2 \cap (y_1, x)$, where $(w, z) \equiv \{u \in X : w < u < z\}$.

For each $n, j \in \mathbb{N}$ and $i \in I$, let $T^n_{i\delta}$ denote the set $\{t_i \in T_{i\delta} : diam\,(BR_i(t_i)) > \frac{1}{n}\}$. Recall that a set $C \subset \mathbb{R}^\ell$ is a chain if $a, b \in C$ implies $a \geqslant b$ or $b > a$. Also, $|C|$ denotes the number of elements of the set C, with $|C| = \infty$ if C is not finite.

Lemma 8.15 *There is a number L_n such that if $C \subset T^n_{i\delta}$ is a chain, then C has at most L_n elements.*[62]

Proof. Let us denote the elements of the chain $C \subset T^n_{i\delta}$ by $\{t^k_i\}_{k\in K}$, where K is an arbitrary set. For each $k \in K$, fix a pair $a^k_i, \tilde{a}^k_i \in BR_i(t^k_i)$ such that

$$\rho_i(a^k_i, \tilde{a}^k_i) > \frac{1}{n}. \tag{29}$$

If C has less than three elements, there is nothing to prove. Otherwise, by Lemma 8.14, we have three disjoint sets K_1, K_2 and K_3 such that $K_1 \cup K_2 \cup K_3 = K$ and for $j = 1, 2$ and any $k, k' \in K_j$ such that $t^{k'}_i > t^k_i$, there exists $k'' \in K \setminus K_j$ such that $t^{k'}_i > t^{k''}_i > t^k_i$. In this case, by Lemma 8.13, $a^k_i, \tilde{a}^k_i \leqslant a^{k''}_i, \tilde{a}^{k''}_i \leqslant a^{k'}_i, \tilde{a}^{k'}_i$.[63] Therefore, by Assumption 3.2 and (29), we have $\rho_i(a^k_i, a^{k'}_i) > \frac{1}{n}$; $\rho_i(\tilde{a}^k_i, a^{k'}_i) > \frac{1}{n}$; $\rho_i(a^k_i, \tilde{a}^{k'}_i) > \frac{1}{n}$ and $\rho_i(\tilde{a}^k_i, \tilde{a}^{k'}_i) > \frac{1}{n}$. That is, any point in the set $B_j \equiv \cup_{k\in K_j}\{a^k_i, \tilde{a}^k_i\}$, for $j = 1, 2, 3$, has a distance of at least $\frac{1}{n}$ to any other point in the same set.

Now, consider the open cover of A_i by balls of radius $\frac{1}{2n}$, with center in each of the points of A_i. Since A_i is ρ_i-compact by Assumption 3.1, it is covered by just $\ell_n \in \mathbb{N}$ of these balls. Since $\rho_i(x, y) > \frac{1}{n}$ for any $x, y \in B_j$, $j = 1, 2, 3$, there are no two points of B_j in the same ball. Therefore, there are at most ℓ_n points in each B_j, which shows that there are at most $L_n = 3\ell_n$ points in K. ∎

The height of a partially ordered set (poset) $(X, \geqslant)$ is the number of elements in the highest chain contained in X. Recall that a set A is an antichain if $a, b \in A$ implies that $a \not> b$ and $b \not> a$ (of course, it could be $a = b$). The following result is due to Dilworth (1950); for this version of the result, see Trotter (1992).

Lemma 8.16 (Dilworth's Theorem) *If $(X, \geqslant)$ is a poset with height m, then there exists a partition $X = A_1 \cup ... \cup A_m$, where A_i is an antichain for $i = 1, ..., m$.*

Proof of Theorem 8.12: The first part of Theorem 8.12 was proved in Lemma 8.13. By Lemma 8.15, for each δ and n, $T^n_{i\delta}$ has height L_n. By Lemma 8.16, $T^n_{i\delta}$ is formed by the union of L_n antichains. The set of those $t_i \in T_{i\delta}$ that can possibly play mixed strategies, that is, those t_i for which $BR(t_i)$ contains more than one point is contained in $\cup_{n\in\mathbb{N}} T^n_{i\delta}$ and, therefore, this is a denumerable union of antichains. ∎

[62] Note that this is stronger than to say that any chain has a finite number of elements.

[63] Of course $x, y \leqslant z, w$ is an abbreviation for four inequalities.

8.2.2 Proof of Theorem 4.3

Proof of Theorem 4.3: By Lemmas 8.15 and 8.16, for each δ and n, $T^n_{i\delta}$ is formed by the union of L_n antichains, that is, we can write $T^n_{i\delta} = \cup_{m=1}^{L_n} T^{n,m}_{i\delta}$, where $T^{n,m}_{i\delta}$ is an antichain. By defining $T^{n,m}_{i\delta} = \emptyset$ for $m > L_n$, we can consider $T^{n,m}_{i\delta}$ well defined for all $m \in \mathbb{N}$ and write $T^n_{i\delta} = \cup_{m\in\mathbb{N}} T^{n,m}_{i\delta}$.

Now, let E denote the set of types $t_i \in T_i$ which play strictly mixed strategies under the best reply μ_i. This set is included in $\cup_{n\in\mathbb{N}} T^n_i$, where T^n_i denotes the set $\{t_i \in T_i : diam\,(BR_i(t_i)) > \frac{1}{n}\}$. Since $\tau_i(\cup_{n\in\mathbb{N}} T^n_i) \leqslant \sum_{i\in\mathbb{N}} \tau_i(T^n_i)$, all claims in the Theorem will be established if we show that $\tau_i(T^n_i) = 0$ for every $n \in \mathbb{N}$. Actually, it is enough to prove that $\tau_i(T^{n,m}_i) = 0$, where $T^{n,m}_i \equiv \cup_{\delta\in\Delta_i} T^{n,m}_{i\delta}$.

To see that $\tau_i(T^{n,m}_i) = 0$, assume otherwise; that is, there exist n, m such that $\tau_i(T^{n,m}_i) > 0$. By richness, there exist $\delta \in \Delta_i$ and $t_i, t'_i \in T^{n,m}_i \cap \widehat{\delta}_i^{-1}(\delta) = T^{n,m}_{i\delta}$ such that $t_i < t'_i$, but this is an absurd, because $T^{n,m}_{i\delta}$ is an antichain. ∎

8.3 Proofs for Section 5

8.3.1 Tie-breaking rule

Hereafter, let $b_i^{1\wedge 2}$ be an abbreviation for $b_i^1 \wedge b_i^2$ and $b_i^{1\vee 2}$, for $b_i^1 \vee b_i^2$.

Proof of Proposition 5.4: By the first part of Assumption 5.2 , $h_i^*(b_i^1, b_{-i}) \geqslant h_i^*(b_i^{1\wedge 2}, b_{-i})$ and $h_i^*(b_i^2, b_{-i}) \leqslant h_i^*(b_i^{1\vee 2}, b_{-i})$. For a contradiction, suppose that $h_i^*(b_i^1, b_{-i}) > h_i^*(b_i^{1\wedge 2}, b_{-i})$. Let $h - 1 = h_i^*(b_i^{1\wedge 2}, b_{-i})$. By Assumption 5.2 , this implies that either (i) $b_{ih}^{1\wedge 2} \neq s_{ih}$ or that (ii) $b_{ih}^1 \neq s_{ih}$. We need the following two observations. First, since $h_i^*(b_i^{1\wedge 2}, b_{-i}) < h$, then $b_{ih}^{1\wedge 2} \leqslant s_{ih}$. Also, since $h_i^*(b_i^1, b_{-i}) \geqslant h$, we have $b_{ih}^1 \geqslant s_{ih}$. Therefore, $b_{ih}^1 \geqslant s_{ih} \geqslant b_{ih}^{1\wedge 2}$. This implies that in case (i) we must have $b_{ih}^{1\wedge 2} < s_{ih}$ and in case (ii), we must have $b_{ih}^1 > s_{ih}$. In any case, $b_{ih}^1 > b_{ih}^{1\wedge 2}$, which implies that $b_{ih}^{1\wedge 2} = b_{ih}^2$. Since $h_i^*(b_i^{1\wedge 2}, b_{-i}) < h \leqslant h_i^*(b_i^1, b_{-i}) \leqslant h_i^*(b_i^2, b_{-i})$, and $b_{ih}^{1\wedge 2} = b_{ih}^2 \leqslant s_{ih}$, another application of Assumption 5.2 leads to $b_{ih}^2 < s_{ih}$. However, this would imply $h_i^*(b_i^2, b_{-i}) < h$, a contradiction. The proof that $h_i^*(b_i^2, b_{-i}) = h_i^*(b_i^{1\vee 2}, b_{-i})$ is analogous. ∎

8.3.2 Payment

Before we establish the proof of (14), it will be useful to introduce some notation. For this, fix (e, b) and let $m = h^*(e, b)$. If $m_i < e_i$, player i has sold $e_i - m_i$ units in the auction, while she has bought $m_i - e_i$ if $m_i > e_i$. No negotiation is made by i if $m_i = e_i$. Define $K \equiv \sum_{i=0}^N e_i$ and let $s_i = (s_{i,1}, s_{i,2}, ..., s_{i,K})$ be the profile of the K-highest bids made by players $j \neq i$, such that $s_{i,1} \leq s_{i,2} \leq ... \leq s_{i,K}$. In other words, s_i denotes the (inverse) residual supply curve facing bidder i; $s_{i,K}$ is the highest of the bids by players $j \neq i$, $s_{i,K-1}$ is the second highest and so on. Thus, for getting (for sure) at least one unit, bidder i's highest bid must be above $s_{i,1}$, that is, $b_{i,1} > s_{i,1}$. For bidder i winning at least two units for sure, it is necessary $b_{i,2} > s_{i,2}$ and so on. Figure 5 below illustrates this.

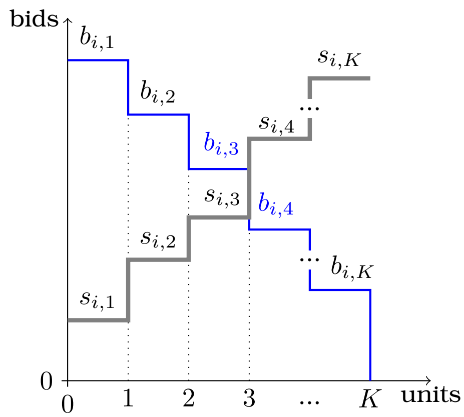

Figure 5: Bid (b_i) and supply (s_i) curves for bidder i. In the situation displayed, bidder i receives three units, because $b_{i,3} > s_{i,3}$ but $b_{i,4} < s_{i,4}$.

Given $b \in A$ and $i \in I$, let $h = \overline{h}_i(b) \equiv \max\{j : b_{i,j} \geqslant s_{i,j}\}$. Define $\underline{p}(b) \equiv \max\{s_{i,h}, b_{i,h+1}\}$ and $\overline{p}(b) \equiv \min\{b_{i,h}, s_{i,h+1}\}$. These definitions do not depend on i. Indeed, first notice that $\underline{p}(b) \leqslant \overline{p}(b)$. If there is a competitive tie, that is, $b_{ih} = s_{ih}$, then $\underline{p}(b) = \overline{p}(b)$ and both are equal to the tying bid (thus, they do not depend on i). Consider now the case with non-competitive ties, that is, $b_{ih} > s_{ih}$ and $b_{i,h+1} < s_{i,h+1}$. In this case, $\underline{p}(b)$ is the highest losing bid and $\overline{p}(b)$ is the lowest winning bid and, as such, both $\underline{p}(b)$ and $\overline{p}(b)$ do not depend on i. Finally, note that the definition would not have changed if we have used $h = h_i^*(b)$ instead of $\overline{h}_i(b)$, because these two quantities are different only when there is a competitive tie and, in this case, even if $h = h_i^*(b) < \min\{j : b_{i,j} = s_{i,j}\}$, that is, i does not receive any object in the tie, we still have $b_{i,h+1} = s_{i,h+1}$, which is sufficient to imply that both $\underline{p}(b)$ and $\overline{p}(b)$ are equal to the competitive tie.

Lemma 8.17 *Fix b_i^1, b_i^2, b_{-i} and let $\underline{p}^k = \underline{p}(b_i^k, b_{-i})$ and $\overline{p}^k = \overline{p}(b_i^k, b_{-i})$, for $k = 1, 2, 1 \wedge 2$ and $1 \vee 2$. Without loss, assume that $h^1 = h_i^*(b_i^1, b_{-i}) \leqslant h^2 = h_i^*(b_i^2, b_{-i})$. We have the following: if $h^1 < h^2$, then*

$$\underline{p}^{1\vee 2} = \underline{p}^2; \underline{p}^{1\wedge 2} = \underline{p}^1; \overline{p}^{1\vee 2} = \overline{p}^2; \text{ and } \overline{p}^{1\wedge 2} = \overline{p}^1; \tag{30}$$

and if $h^1 = h^2$, $b_{ih}^{1\vee 2} = b_{ih}^k$ and $b_{ih}^{1\wedge 2} = b_{ih}^{k'}$ for $(k, k') = (1, 2)$ or $(2, 1)$, then $\overline{p}^{1\vee 2} = \overline{p}^k$; and $\overline{p}^{1\wedge 2} = \overline{p}^{k'}$; and a similar condition holds for $\underline{p}$.

Proof. By Proposition 5.4, $h_i^*(b_i^{1\wedge 2}, b_{-i}) = h^1$ and $h_i^*(b_i^{1\vee 2}, b_{-i}) = h^2$.

Let's consider first the case that $h^1 < h^2$. This means that $b_{i,h^2}^{1\vee 2} = b_{i,h^2}^2$, otherwise, $b_{i,h^2}^2 < b_{i,h^2}^{1\vee 2} = b_{i,h^2}^1$, which would imply $b_{i,h^2}^1 \geqslant s_{i,h^2}$. In this case, Assumption 5.2 would imply $h^1 \geqslant h^2$, a contradiction. Therefore, $\overline{p}^{1\vee 2} = \overline{p}^2$. Next, we want to show that $\overline{p}^{1\wedge 2} = \min\{b_{i,h^1}^{1\wedge 2}, s_{i,h^1+1}\} = \min\{b_{i,h^1}^1, s_{i,h^1+1}\} = \overline{p}^1$.

Suppose otherwise, that is, $\min\{b^{1\wedge 2}_{i,h^1}, s_{i,h^1+1}\} < \min\{b^1_{i,h^1}, s_{i,h^1+1}\}$. Then, $b^{1\wedge 2}_{i,h^1} < b^1_{i,h^1}$ and $b^{1\wedge 2}_{i,h^1} = b^2_{i,h^1} < s_{i,h^1+1}$. But this implies $s_{i,h^2} \geqslant s_{i,h^1+1} > b^2_{i,h^1} \geqslant b^2_{i,h^2}$, which contradicts $h^2 = h^*_i(b^2_i, b_{-i})$. This shows that $\overline{p}^{1\wedge 2} = \overline{p}^1$.

Analogously, suppose that $\underline{p}^{1\wedge 2} = \max\{b^{1\wedge 2}_{i,h^1+1}, s_{i,h^1}\} < \max\{b^1_{i,h^1+1}, s_{i,h^1}\} = \underline{p}^1$. This implies that $b^2_{i,h^1+1} = b^{1\wedge 2}_{i,h^1+1} < b^1_{i,h^1+1} \leqslant s_{i,h^1+1}$, which contradicts $h^2 > h^1 \Rightarrow b^2_{i,h^1+1} \geqslant b^2_{i,h^2} \geqslant s_{i,h^2} \geqslant s_{i,h^1+1}$. Thus, $\underline{p}^{1\wedge 2} = \underline{p}^1$.

Now, assume $\underline{p}^{1\vee 2} = \max\{b^{1\vee 2}_{i,h^2+1}, s_{i,h^2}\} > \max\{b^2_{i,h^2+1}, s_{i,h^2}\} = \underline{p}^2$. This implies that $b^{1\vee 2}_{i,h^2+1} > b^2_{i,h^2+1}$ and $b^1_{i,h^2+1} = b^{1\vee 2}_{i,h^2+1} > s_{i,h^2}$, but this implies $b^1_{i,h^1+1} \geqslant b^{1\vee 2}_{i,h^2+1} = b^1_{i,h^2+1} > s_{i,h^2} \geqslant s_{i,h^1+1}$, contradicting the assumption that b^1_i receives only h^1 and not h^1+1. Thus, $\underline{p}^{1\vee 2} = \underline{p}^2$.

Now, assume that $h^1 = h^2 = h$. Let $b^{1\vee 2}_{ih} = b^k_{ih}$, $b^{1\wedge 2}_{ih} = b^{k'}_{ih}$ for $(k,k') = (1,2)$ or $(k,k') = (2,1)$. Thus, $\overline{p}^{1\vee 2} = \min\{b^{1\vee 2}_{ih}, s_{i,h+1}\} = \min\{b^k_{ih}, s_{i,h+1}\} = \overline{p}^k$ and $\overline{p}^{1\wedge 2} = \min\{b^{1\wedge 2}_{ih}, s_{i,h+1}\} = \min\{b^{k'}_{ih}, s_{i,h+1}\} = \overline{p}^{k'}$. Analogously, let $b^{1\vee 2}_{i,h+1} = b^k_{i,h+1}$, $b^{1\wedge 2}_{i,h+1} = b^{k'}_{i,h+1}$.[64] Thus, $\underline{p}^{1\vee 2} = \max\{b^{1\vee 2}_{i,h+1}, s_{i,h}\} = \min\{b^k_{i,h+1}, s_{i,h}\} = \underline{p}^k$ and $\underline{p}^{1\wedge 2} = \max\{b^{1\wedge 2}_{i,h+1}, s_{i,h}\} = \max\{b^{k'}_{i,h+1}, s_{i,h}\} = \underline{p}^{k'}$. ∎

Remark 8.18 *As the statement of this lemma should suggest, (30) is not necessarily true when $h^1 = h^2$. To see this, consider the following example. There are two objects, the two highest bids by opponents are $(6,2)$, $b^1_i = (7,3)$ and $b^2_i = (5,4)$. Then $h^1 = h^2 = 1$, $b^{1\wedge 2}_i = (5,3)$, $b^{1\vee 2}_i = (7,4)$, $\overline{p}^{1\wedge 2} = \overline{p}^2 = 5$; $\overline{p}^{1\vee 2} = \overline{p}^1 = 6$, $\underline{p}^{1\wedge 2} = \underline{p}^1 = 3$ and $\underline{p}^{1\vee 2} = \underline{p}^2 = 4$, which shows that (30) is not true, even relabeling the bids. Note also that this example does not depend on the specification of the tie-breaking rule, because there are no ties.*

Corollary 8.19 *Let Assumption 5.1 hold. Then,*

$$p_i(h, e, b^1_i, b_{-i}) - p_i(h, e, b^{1\wedge 2}_i, b_{-i}) = p_i(h, e, b^{1\vee 2}_i, b_{-i}) - p_i(h, e, b^2_i, b_{-i}).$$

Proof. Using Lemma 8.17, a straightforward inspection of (12) leads to the conclusion. ∎

8.3.3 Proof of Theorem 5.1

The proof of Theorem 5.1 requires two lemmas. The first one generalizes an argument first given by McAdams (2003, p. 1210). For the discussion below, let u_i be given by (10).

Lemma 8.20 *Under assumptions 5.1 and 5.2, u_i is modular.*

Proof. Fix b_{-i} and let $h^1, h^2, h^{1\wedge 2}$ and $h^{1\vee 2}$ be the final allocation given by b^1_i, b^2_i, $b^{1\wedge 2}_i$ and $b^{1\vee 2}_i$. Without loss of generality, assume $h^1 \leqslant h^2$. By Proposition 5.4,

[64] These k and k' could be different from the first part.

$h = h^1 = h^{1\wedge 2}$ and $h' = h^2 = h^{1\vee 2}$. For simplicity, define $p^1 \equiv p_i(h, e, \cdot, b_i^1)$; $p^{1\wedge 2} \equiv p_i(h, e, \cdot, b_i^{1\wedge 2})$; $p^2 \equiv p_i(h', e, \cdot, b_i^2)$ and $p^{1\vee 2} \equiv p_i(h', e, \cdot, b_i^{1\vee 2})$.

We want to show that

$$u_i(t_i, b_i^1, \cdot) - u_i(t_i, b_i^{1\wedge 2}, \cdot) = u_i(t_i, b_i^{1\vee 2}, \cdot) - u_i(t_i, b_i^2 . \cdot), \tag{31}$$

that is,

$$\begin{aligned}\left[\sum_{j=1}^{h} v_{ij} - p^1\right] - \left[\sum_{j=1}^{h} v_{ij} - p^{1\wedge 2}\right] &= \left[\sum_{j=1}^{h'} v_{ij} - p^{1\vee 2}\right] - \left[\sum_{j=1}^{h'} v_{ij} - p^2\right] \\ \iff p^{1\wedge 2} - p^1 &= p^2 - p^{1\vee 2},\end{aligned}$$

but the last expression is true by Corollary 8.19. ∎

Lemma 8.21 *(i) u_i has nondecreasing differences.*

(ii) If $h^1 = h_i^*(b_i^1, b_{-i}) < h_i^*(b_i^2, b_{-i}) = h^2$, then the inequality defining nondecreasing differences is actually strict.

Proof. Let $t_i^1 < t_i^2$ and assume that $b_i^1 <_i b_i^2$ and let $h = h_i^*(b_i^1, b_{-i}) = h_i^*(b_i^2, b_{-i})$. For simplicity, define $p^1 \equiv p_i(h, e^1, \cdot, b_i^1)$ and $p^2 \equiv p_i(h, e^2, \cdot, b_i^2)$. Then,

$$\begin{aligned}&\left(\sum_{j=1}^{h} v_{ij}^1 - p^2\right) - \left(\sum_{j=1}^{h} v_{ij}^1 - p^1\right) \\ &= \left(\sum_{j=1}^{h} v_{ij}^2 - p^2\right) - \left(\sum_{j=1}^{h} v_{ij}^2 - p^1\right).\end{aligned}$$

Given the second claim, this will be enough for the first part.

Now, let $h^1 = h_i^*(b_i^1, b_{-i}) < h_i^*(b_i^2, b_{-i}) = h^2$. For simplicity, define $p^1 \equiv p_i(h^1, e^1, \cdot, b_i^1)$ and $p^2 \equiv p_i(h^2, e^2, \cdot, b_i^2)$. We want to show that

$$\begin{aligned}&\left(\sum_{j=1}^{h^2} v_{ij}^1 - p^2\right) - \left(\sum_{j=1}^{h^1} v_{ij}^1 - p^1\right) \\ &< \left(\sum_{j=1}^{h^2} v_{ij}^2 - p^2\right) - \left(\sum_{j=1}^{h^1} v_{ij}^2 - p^1\right),\end{aligned}$$

which is equivalent to:

$$\sum_{j=h^1+1}^{h^2} v_{ij}^1 < \sum_{j=h^1+1}^{h^2} v_{ij}^2.$$

Since $t_i^1 < t_i^2$, the last inequality is obviously true by (9). ∎

Proof of Theorem 5.1: Define a modified n-auction exactly as the original, except for two differences:

- with probability $1/n$, there is a player playing uniformly in all bids; and
- the tie-breaking is modified so that it satisfies Assumption 5.2.

By JS' Theorems 6 and 9, each n-modified auction has an equilibrium in undominated* strategies with a zero probability of competitive ties, which is an equilibrium under any omniscient and effectively trade-maximizing tie-breaking rule, including standard tie-breaking rule. We want to argue that this equilibrium is actually in pure strategies.

Since the tie-breaking rule of the modified auction satisfies Assumption 5.2, by Lemma 8.20, u_i is modular, hence quasi-supermodular. By Lemma 8.21, u_i has non-decreasing differences. Moreover, since in the n-modified auction there is a player playing uniformly in all bids, then $h_i^*(b_i^1, b_{-i}) < h_i^*(b_i^2, b_{-i})$ occurs with positive probability for any pair of bids b_i^1 and b_i^2 satisfying $b_i^1 < b_i^2$. Again by Lemma 8.21, the inequality defining non-decreasing differences in u_i is actually strict with positive probability. Therefore, the assumptions of Lemma 8.10 are satisfied, which implies that $\Pi_i : T_{i\delta} \times A_i \times S_{-i} \to \mathbb{R}$ has the ID-monotonicity property in $T_{i\delta} \times A_i$.

Therefore, each n-auction satisfies the assumptions of Theorem 4.3 and all of its equilibria are actually in pure strategies that are monotonic when conditioned to each $T_{i\delta}$. Let b^n be the pure strategy equilibrium of the n-auction. By Jackson and Swinkels (2005, Theorem 9), this equilibrium is competitive tie-free, which means that it implies an allocation that does not depend on the tie-breaking rule. This means that we could erase the second bullet point in the definition of the n-modified auction and consider it with the original tie-breaking rule.[65]

An easy adaptation of Reny (2011, Lemma A.10) shows that for each i, $\{b_i^n\}$ has a subsequence that converges pointwise to some b_i, since each b_i^n is monotonic when restricted to $T_{i\delta}$. Naturally, $b = (b_i, b_{-i})$ can involve ties with positive probability. Let us define a specific tie-breaking rule for this case. For each $t \in T$, let the allocation determined by b^n be denoted by $a^n(t)$, which is defined up to a set of zero measure on T. Since the allocation is discrete, we can pass to subsequences, if necessary, and assume that $a^n(t)$ converge to some $a(t)$, for almost all t. Fixing this sub-sub-sequence (but maintaining the notation on n, for simplicity), $a(t)$ defines an omniscient tie-breaking rule. We will show below that this a does not really matter. Therefore we can conceive of the hypothetical game in which the knowledge to implement a is given. For this a, we have by construction that $U_i(b^n) \to U_i(b)$.

[65] The only impact of using the original tie-breaking rule in the n-modified auction is that, besides all pure strategies equilibria, we may also have equilibria in mixed strategies.

Now we claim that b is free of competitive ties for every agent. Indeed, suppose that b induces a tie for player i. By Jackson and Swinkels (2005, Lemma 8), there exists a strategy b_i' that is tie-free for player i and satisfies

$$U_i(b_i', b_{-i}) > U_i(b_i, b_{-i}). \tag{32}$$

Since b_i' is tie-free for player i, $u_i(t, b_i'(t_i), b_{-i}^n(t_{-i})) \to u_i(t, b_i'(t_i), b_{-i}(t_{-i}))$ for almost all t_{-i}. By Lebesgue dominated convergence theorem, $U_i(b_i', b_{-i}^n) \to U_i(b_i', b_{-i})$ and this, together with $U_i(b^n) \to U_i(b)$ and (32) would imply that

$$U_i(b_i', b_{-i}^n) > U_i(b_i^n, b_{-i}^n)$$

for all sufficiently high n. But since the modified n-auction has payoffs U_i^n very close to the original auction payoff U_i, we would obtain, for sufficiently high n,

$$U_i^n(b_i', b_{-i}^n) > U_i^n(b_i^n, b_{-i}^n).$$

However, this contradicts the assumption that b^n is an equilibrium of the modified auction. Therefore, the claim is established.

Now, we argue that b is actually an equilibrium of the original auction. For this, we can assume that a player has a profitable deviation b_i' so that (32) holds. In this case, we can repeat the same arguments above and again obtain a contradiction. This implies that b is actually an equilibrium of the game with the omniscient tie-breaking rule a. However, we have argued above that b is tie free for every player i. Therefore, a does not matter for the equilibrium condition and b is also an equilibrium of the original game, as we wanted to show. ∎

9 Appendix B: Grid Distributions

In this section, we define grid distributions for multidimensional signals, that is, we consider a usual setting, with $T_i \equiv \times_{j=1}^{l_i} \left[\underline{t}_{i,j}, \bar{t}_{i,j}\right]$.

9.1 Formal definition and Basic Properties

Let $\lceil x \rceil$ denote the minimum integer at least as large as x, for instance, $\lceil 2.7 \rceil$=3. For each $k \in \mathbb{N}$ and $i \in I$, define the functions $\mathbb{I}_i^k : T_i \to \{1, ..., k\}^{l_i}$ by:

$$\mathbb{I}_i^k(t_i) = \left(\left\lceil k\, \frac{t_{i,1} - \underline{t}_{i,1}}{\bar{t}_{i,1} - \underline{t}_{i,1}} \right\rceil, ..., \left\lceil k\, \frac{t_{i,j} - \underline{t}_{i,j}}{\bar{t}_{i,j} - \underline{t}_{i,j}} \right\rceil ..., \left\lceil k\, \frac{t_{i,l_i} - \underline{t}_{i,l_i}}{\bar{t}_{i,l_i} - \underline{t}_{i,l_i}} \right\rceil \right)$$

and $\mathbb{I}^k : T \to \{1, ..., k\}^L$, where $L = \sum_{i=1}^N l_i$, by $\mathbb{I}^k(t) = (\mathbb{I}_1^k(t_1), ..., \mathbb{I}_N^k(t_N))$. Note that $\mathbb{I}_i^k(t_i)$ gives the interval that contains each coordinate of t_i. We already know from the introduction that grid distributions have a constant density in each of these intervals. The following is a formal definition of grid distributions in general.

Definition 9.1 (Grid distribution) *The distribution τ is a grid distribution if it is absolutely continuous with respect to the Lebesgue measure (in $\mathbb{R}^L$), with correspondent Radon-Nikodym $f : T \to \mathbb{R}_+$ satisfying the following: there exists $k \in \mathbb{N}$ such that for almost all $t, t' \in T$,*

$$\mathbb{I}^k(t) = \mathbb{I}^k(t') \implies f(t) = f(t'). \tag{33}$$

In this case, we say that $f \in \mathcal{D}^k$. The set of densities associated to grid distributions is then $\mathcal{D}^\infty \equiv \cup_{k=1}^{\infty} \mathcal{D}^k$.

We say that t and t' are in the same *grid-cube* if $\mathbb{I}^k(t) = \mathbb{I}^k(t')$. More precisely, a *grid-cube* is a set $\{t \in T : \mathbb{I}^k(t) = c\}$ for some $c \in \{1, ..., k\}^L$. Also, we can define a *grid-interval* as a set $\{t_i \in T_i : \mathbb{I}_i^k(t_i) = c\}$ for some $c \in \{1, ..., k\}^{l_i}$ and some $i \in I$.[66] In this way, we can concisely define grid distributions as those absolutely continuous distributions with density functions that are constant in grid-cubes.

Another useful way of describing the set of grid distributions is through the image of a transformation $\mathbb{T}^k : \mathcal{D} \to \mathcal{D}$, where $\mathcal{D}$ is the set of densities in T. For simplicity, let us describe this transformation in the particular case in which $T = [0,1]^2$, that is, there are two players, each with types in $[0,1]$. In this case,

$$\mathbb{T}^k (f)(x, y) \equiv k^2 \int_{\frac{p-1}{k}}^{\frac{p}{k}} \int_{\frac{m-1}{k}}^{\frac{m}{k}} f(\alpha, \beta) \, d\alpha d\beta,$$

whenever $(x, y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for $m, p \in \{1, 2, ..., k\}$. Observe that $\mathbb{T}^k(f)$ is constant over each square $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Then $\mathcal{D}^k$ be the image of $\mathcal{D}$ by $\mathbb{T}^k$, that is, $\mathcal{D}^k \equiv \mathbb{T}^k(\mathcal{D})$. Thus, $\mathbb{T}^k$ is a projection from the infinite dimensional space $\mathcal{D}$ over the finite dimensional space $\mathcal{D}^k$. Indeed, $\mathcal{D}^k$ is finite dimensional set because any density function $f \in \mathcal{D}^k$ can be described by a matrix $A = (a_{ij})_{k \times k}$.

The transformation $\mathbb{T}^k$ is interesting because allows us to approximate any density $f \in \mathcal{D}$ by grid distributions in a convenient way. See section 6.5 below.

9.1.1 Basic Properties

The following is an easy application of the mean value theorem to two dimensions.

Lemma 9.2 *Let $g : (a, b] \times (c, d] \to \mathbb{R}$ be continuous. Then, there is $(x, y) \in (a, b) \times (c, d)$ such that*

$$\int_c^d \int_a^b g(\alpha, \beta) \, d\alpha d\beta = (c - d)(b - a) \, g(x, y).$$

[66] Of course a *grid-interval* is also a "(hyper)cube" in multidimensional settings ($l_i > 1$), but it is just an interval if each player's type is unidimensional ($l_i = 1$). Also, note that a *grid-cube* will be just a square if there are just two players with unidimensional types. We adopt this terminology to simplify later references.

Proof. This is a trivial application of the mean value theorem of integration. Define the function $f:(c,d] \to \mathbb{R}$ by

$$f(y) = \int_a^b g(\alpha, y)\, d\alpha.$$

It is clear that f is continuous. By the mean value theorem of integration, there exists $y \in (c,d)$ such that

$$\int_c^d \int_a^b g(\alpha,\beta)\, d\alpha d\beta = \int_c^d f(\beta)\, d\beta = (c-d) f(y).$$

Fixing y, the function $x \mapsto h(x) = g(x,y)$ is continuous. Thus, there exists $x \in (a,b)$ such that

$$f(y) = \int_a^b g(\alpha,y)\, d\alpha = \int_a^b h(\alpha)\, d\alpha = (b-a) h(x),$$

which concludes the proof. ∎

It is clear that the above proof easily extends to arbitrary dimensions. Therefore, we have the following:

Corollary 9.3 *Given $c \in \{1, ..., k\}^L$ and $f \in \mathcal{C}$, there exists $t \in T$ such that $\mathbb{I}^k(t) = c$ and $\mathbb{T}^k(f)(t) = f(t)$.*

9.2 Approximation results

Proof of Theorem 6.8. We first prove (iii). Given $f \in \mathcal{C}$, since T is compact, f is uniformly continuous. Thus, for any $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $\|t - t'\| < \frac{2}{k}\sqrt{L}$ implies that $|f(t) - f(t')| < \epsilon$. Now, define $f^k \equiv \mathbb{T}^k(f)$. We claim that $\|f - f^k\| < \epsilon$, that is, $|f(t) - f^k(t)| < \epsilon$, $\forall t \in T$. Fix $t \in T$ and let $c \equiv \mathbb{I}^k(t)$. By Corollary 9.3, there exists t' such that $\mathbb{I}^k(t') = c$ and $f(t') = f^k(t') = f^k(t)$. Since $\mathbb{I}^k(t) = c$, then $\|t - t'\| < \frac{2}{k}\sqrt{L}$ and we have $|f^k(t) - f(t)| = |f(t') - f(t)| < \epsilon$. This establishes (iii).[67]

For (ii), recall that continuous functions are dense in L^p (see, for instance, Aliprantis and Border (1999, Theorem 12.9)). Given $f \in L^p$ and $\epsilon > 0$, let $g \in \mathcal{C}$ be such that $\|f - g\| < \frac{\epsilon}{2}$ and pick k such that $\|\mathbb{T}^k(g) - g\| < \frac{\epsilon}{2}$. Then, $\|f - \mathbb{T}^k(g)\| < \epsilon$.

Now (i) is immediate, since $\mathbb{T}^k(f) \to f$ in the sup-norm implies that it also converges pointwise. It remains to show that the set of grid distributions is dense in Δ. Given $\tau \in \Delta$, let $f \in \mathcal{D}$ be (a version of) its Radon-Nikodym derivative with respect to the Lebesgue measure λ on T. Let $f^k \equiv \mathbb{T}^k(f)$ and τ^k the distribution generated by $f^k \in \mathcal{D}^k$. Then, for any continuous function $g: T \to \mathbb{R}$, $\int g d\tau^k = \int g f^k d\lambda \to \int g f d\lambda = \int g d\tau$, by the Lebesgue Dominated Convergence Theorem (it

[67] This also shows that $\mathbb{T}^k(f) \to f$ in the sup-norm.

is not difficult to see that the sequence gf^k is bounded). Since g is an arbitrary continuous function, $\tau^k \to \tau$ in the weak* topology, which establishes the first claim in the theorem. ∎

As observed in footnote 67, we also have the following:

Corollary 9.4 *If $f \in \mathcal{C}$, $\mathbb{T}^k(f) \to f$ in the sup-norm.*

Proof of Proposition 6.9. For notational simplicity, we will make the proof for the case in which $T = [0,1]^2$. The same argument can be easily generalized. We want to prove that if $x > x'$ and $y > y'$ then

$$\mathbb{T}^k(f)(x,y)\,\mathbb{T}^k(f)(x',y') \geq \mathbb{T}^k(f)(x,y')\,\mathbb{T}^k(f)(x',y).$$

Let $x \in \left(\frac{i-1}{k}, \frac{i}{k}\right]$, $x' \in \left(\frac{j-1}{k}, \frac{j}{k}\right]$, $y \in \left(\frac{q-1}{k}, \frac{q}{k}\right]$ and $y' \in \left(\frac{p-1}{k}, \frac{p}{k}\right]$, where $i > j$ and $q > p$. Thus, the above inequality is equivalent to:

$$\begin{aligned} & \int_0^{\frac{1}{k}}\int_0^{\frac{1}{k}} f\left(\alpha + \frac{i-1}{k}, \beta + \frac{q-1}{k}\right) d\alpha d\beta \cdot \int_0^{\frac{1}{k}}\int_0^{\frac{1}{k}} f\left(z + \frac{j-1}{k}, w + \frac{p-1}{k}\right) dz dw \\ \geq & \int_0^{\frac{1}{k}}\int_0^{\frac{1}{k}} f\left(\alpha + \frac{i-1}{k}, w + \frac{p-1}{k}\right) d\alpha dw \cdot \int_0^{\frac{1}{k}}\int_0^{\frac{1}{k}} f\left(z + \frac{j-1}{k}, \beta + \frac{q-1}{k}\right) dz d\beta. \end{aligned}$$

This can be rewritten as:

$$\begin{aligned} & \int_0^{\frac{1}{k}}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{k}}\int_0^{\frac{1}{k}} \left[f\left(\alpha + \frac{i-1}{k}, \beta + \frac{q-1}{k}\right) f\left(z + \frac{j-1}{k}, w + \frac{p-1}{k}\right) \right. \\ & \left. - f\left(\alpha + \frac{i-1}{k}, w + \frac{p-1}{k}\right) f\left(z + \frac{j-1}{k}, \beta + \frac{q-1}{k}\right) \right] d\alpha d\beta dz dw \\ \geq & \; 0. \end{aligned}$$

Now it is easy to see that affiliation implies that the integrand is non-negative for all $\alpha, \beta, z, w \in \left(0, \frac{1}{k}\right)$.

For the converse, suppose that $\mathbb{T}^k(f) \in \mathcal{A}\ \forall k \in \mathbb{N}$ but $f \notin \mathcal{A}$. This means that there exist x, x', y and y' such that $x > x'$, $y > y'$ and

$$f(x,y)\,f(x',y') < f(x,y')\,f(x',y).$$

Since $\mathbb{T}^k(f) \to f$ in the sup-norm (Corollary 9.4), there exists $k \in \mathbb{N}$ such that

$$\mathbb{T}^k(f)(x,y)\,\mathbb{T}^k(f)(x',y') < \mathbb{T}^k(f)(x,y')\,\mathbb{T}^k(f)(x',y),$$

which contradicts $\mathbb{T}^k(f) \in \mathcal{A}$. ∎

Proof of Proposition 6.10 By the Lebesgue Dominated Convergence Theorem,

$$U_i(s^{k_n}) = \int u_i(t, s^{k_n}(t)) f^{k_n}(t)\lambda(dt) \to \int u_i(t, s(t)) f(t)\lambda(dt) = U_i(s).$$

Thus, $U_i(s_i, s_{-i}) \geqslant U_i(s_i', s_{-i}) = \lim_{n\to\infty} U_i(s_i', s_{-i}^{k_n})$ for all $s_i' \in \mathcal{F}_i$. ∎

Proof of Proposition 6.11 By the previous results, there exists k_ε such that for all $k \geqslant k_\varepsilon$,

$$\begin{aligned}
\int u_i(t, s(t)) f^k(t)\lambda(dt) &\geqslant \int u_i(t, s(t)) f(t)\lambda(dt) - \frac{\varepsilon}{2} \\
&\geqslant \int u_i(t, s_i'(t_i), s_{-i}(t_{-i})) f(t)\lambda(dt) - \frac{\varepsilon}{2} \\
&\geqslant \int u_i(t, s_i'(t), s_{-i}(t_{-i}) f^k(t)\lambda(dt) - \varepsilon,
\end{aligned}$$

where the second inequality comes from the fact that s is an equilibrium for f. ∎

9.3 Steps in the proof of Theorem 6.2

In this section we describe the main steps in proving Theorem 6.2. For a detailed proof of this Theorem, see the supplement to this paper.

We assume that $f \in \mathcal{D}^k$ is symmetric and can be described by the matrix $(a_{ij})_{k\times k}$ as follows:

$$f(y,x) = a_{mp} \text{ if } (y,x) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right], \tag{34}$$

for $m, p \in \{1, 2, ..., k\}$. The definition of f at the zero measure set of points $\{(y,x) = \left(\frac{m}{k}, \frac{p}{k}\right) : m = 0 \text{ or } p = 0\}$ is arbitrary.

For the description of the steps below, assume that $x \in \left(\frac{p-1}{k}, \frac{p}{k}\right]$ and $z \in \left(\frac{m-1}{k}, \frac{m}{k}\right]$, for $m, p \in \{1, ..., k\}$.

1. We obtain the expressions:

$$\begin{aligned}
f(z|x) &= \frac{k a_{mp}}{\sum_{i=1}^k a_{ip}}; \\
F(z|x) &= \int_0^z f(\alpha|x)\, d\alpha = \frac{\sum_{i=1}^{m-1} a_{ip} + a_{mp}(kz - m + 1)}{\sum_{i=1}^k a_{ip}}
\end{aligned}$$

   and

$$\frac{f(z|x)}{F(z|x)} = \frac{k a_{mp}}{\sum_{i=1}^{m-1} a_{ip} + a_{mp}(kz - m + 1)}. \tag{35}$$

2. Using (35), we integrate $b(z) = z - \int_0^z \exp\left[-\int_u^z \frac{g(s|s)}{G(s|s)} ds\right] du$ to obtain:

$$b\left(\frac{m-1+\zeta}{k}\right) = \frac{m-1+\zeta}{k} - \frac{(r_m+\zeta)}{2k} + \frac{(r_m^2 - D_m)}{2k(r_m+\zeta)}. \tag{36}$$

where $\zeta = kz - m + 1$ and, for $m \geqslant 2$,

$$r_m \equiv \frac{\sum_{i=1}^{m-1} a_{im}}{a_{mm}}, \tag{37}$$

and

$$D_m \equiv \sum_{j=1}^{m-1} \left[\prod_{l=j}^{m-1} \left(\frac{r_{l+1}}{1 + r_l} \right) \right] \cdot \left[(1 + r_j)^2 - r_j^2 \right]. \tag{38}$$

3. Define $\Delta(x, z) \equiv \Pi(x, b(z)) - \Pi(x, b(x))$, where $\Pi(x, b(z))$ is given by $\Pi(x, b(z)) = [x - b(z)] F(z|x)$. Note that $b(\cdot)$ is SMPSE iff $\Delta(x, z) \leqslant 0$ for all $(x, z) \in [0, 1]^2$. Using $\chi \equiv kx - p + 1$ or $x = \frac{p-1+\chi}{k}$ and $\zeta \equiv kz - m + 1$ or $z = \frac{m-1+\zeta}{k}$, denote $2k \sum_{i=1}^{k} a_{ip} \Delta(x, z)$ by $\Delta_{pm}(\chi, \zeta)$ and obtain:

$$\begin{aligned} \Delta_{pm}(\chi, \zeta) \quad \equiv \quad & \left[2(\chi - \zeta + p - m) + \frac{\zeta^2 + 2\zeta r_m + D_m}{r_m + \zeta} \right] \\ & \cdot \left(\sum_{i=1}^{m-1} a_{ip} + a_{mp}\zeta \right) - \left(\chi^2 + 2\chi r_p + D_p \right) a_{pp}. \end{aligned} \tag{39}$$

4. We show that (39) is a quadratic function of $\chi \in (0, 1)$ that is: non-positive if $m = p$; decreasing in $(0, 1)$ if $m < p$ and increasing in $(0, 1)$ if $m > p$. Thus, $b(\cdot)$ is SMPSE iff for all $m, p \in \{1, ..., k\}, m \neq p$:

$$\begin{cases} \Delta_{pm}(0, \zeta) \leqslant 0, \forall \zeta \in [0, 1], \text{ if } m < p; \\ \Delta_{pm}(1, \zeta) \leqslant 0, \forall \zeta \in [0, 1], \text{ if } m > p. \end{cases} \tag{40}$$

Since $r_m + \zeta > 0$, the signal of $\Delta_{pm}(\chi, \zeta)$ is the same as the signal of

$$\tilde{\Delta}_{pm}(\chi, \zeta) \equiv (r_m + \zeta) \Delta_{pm}(\chi, \zeta). \tag{41}$$

From (40), $b(\cdot)$ is SMPSE iff, for all $m < p$ we have $\tilde{\Delta}_{pm}(0, \zeta) \leqslant 0$ and for $m > p$, $\tilde{\Delta}_{pm}(1, \zeta) \leqslant 0$.

5. From (39), it is easy to see that both $\tilde{\Delta}_{pm}(0, \zeta)$ and $\tilde{\Delta}_{pm}(1, \zeta)$ are polynomials of third degree in ζ which do not depend on χ. We carry χ below only to consider both cases in just one expression. We then show that $\partial_\zeta \tilde{\Delta}_{pm}(\chi, \zeta) = 0$ can be written as

$$c_2 \zeta^2 + c_1 \zeta + c_0 = 0, \tag{42}$$

where

$$\begin{aligned} c_2 &= -3a_{mp}; \\ c_1 &= 4a_{mp}(\chi + p - m) - 2\sum_{i=1}^{m-1} a_{ip}; \\ c_0 &= (\chi + p - m) \left[2a_{mp} r_m + 2\sum_{i=1}^{m-1} a_{ip} \right] - \left(\chi^2 + 2\chi r_p + D_p \right) a_{pp} + a_{mp} D_m. \end{aligned}$$

Let $\zeta_{mp}(\chi)$ denote the solution(s) to the quadratic equation (42). Now, condition (40) requires us to test:

- if $m < p$, whether $\zeta_{mp}(0) \in (0,1)$ and if this happens, test whether $\tilde{\Delta}_{pm}\left(0, \zeta_{mp}(0)\right) \leqslant 0$;
- if $m > p$, whether whether $\zeta_{mp}(1) \in (0,1)$ and if this happens, test whether $\tilde{\Delta}_{pm}\left(1, \zeta_{mp}(1)\right) \leqslant 0$.

This concludes the method.

The above definitions allow us to a more explicit statement of Theorem 6.2.

Theorem 9.5 *Suppose that there are two risk neutral players and $f \in \mathcal{D}^k$ is described by a matrix $(a_{ij})_{k\times k}$ as in (34). Let $\tilde{\Delta}_{pm}(\chi,\zeta)$ be defined by (41). Then f has a SMPSE if and only if all of the following inequalities are satisfied:*[68]

Case	Conditions to verify
$2 \leqslant m < p \leqslant k$	$\tilde{\Delta}_{pm}(0,0) \leqslant 0$
$2 \leqslant m < p \leqslant k$	$\tilde{\Delta}_{pm}\left(0,\zeta_{mp}(0)\right) \leqslant 0$
$1 \leqslant p < m \leqslant k$	$\tilde{\Delta}_{pm}(1,1) \leqslant 0;$
$1 \leqslant p < m \leqslant k$	$\tilde{\Delta}_{pm}\left(1,\zeta_{mp}(1)\right) \leqslant 0$

Table 1 - Necessary and sufficient conditions for equilibrium existence.

It is easy to see that in each square, we need to check less than six points. Thus, the number of inequalities to test is less than $6 \cdot \frac{k(k+1)}{2} = 3k^2 + 3k$.

9.4 Proof of Theorem 6.7.

The dominant strategy for each bidder in the second price auction is to bid his value: $b^2(t) = t$. Then, the expected payment by a bidder in the second price auction, P^2, is given by:

$$\begin{aligned} P^2 &= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} y f(y|x)\,dy \cdot f(x)\,dx = \\ &= \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} [y - b(y)]\, f(y|x)\,dy \cdot f(x)\,dx + \int_{[\underline{t},\bar{t}]}\int_{[\underline{t},x]} b(y)\, f(y|x)\,dy \cdot f(x)\,dx, \end{aligned}$$

[68]The conditions in Table 1 already incorporate some further simplifications for the case $m = 1$, not explained above.

where $b(\cdot)$ gives the equilibrium strategy for symmetric first price auctions. Thus, the first integral can be substituted by $\int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \frac{F(y|y)}{f(y|y)} f(y|x)\,dy \cdot f(x)\,dx$, from the first order condition: $b^{'}(y) = [y - b(y)] \frac{f(y|y)}{F(y|y)}$. The last integral can be integrated by parts, to:

$$\begin{aligned}
& \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y) f(y|x)\,dy \cdot f(x)\,dx \\
= & \int_{[\underline{t},\bar{t}]} \left[b(x) F(x|x) - \int_{[\underline{t},x]} b^{'}(y) F(y|x)\,dy \right] \cdot f(x)\,dx \\
= & \int_{[\underline{t},\bar{t}]} b(x) F(x|x) \cdot f(x)\,dx - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) F(y|x)\,dy \cdot f(x)\,dx
\end{aligned}$$

In the last line, the first integral is just the expected payment for the first price auction, P^1. Thus, we have

$$\begin{aligned}
D &= P^2 - P^1 \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \frac{F(y|y)}{f(y|y)} f(y|x)\,dy \cdot f(x)\,dx \\
&\quad - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) F(y|x)\,dy \cdot f(x)\,dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} f(y|x) - F(y|x) \right] dy \cdot f(x)\,dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\,dy \cdot f(x)\,dx
\end{aligned}$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\,d\alpha$, where $L(\alpha|t) = \exp\left[-\int_{\alpha}^{t} \frac{f(s|s)}{F(s|s)} ds \right]$. So, we have

$$\begin{aligned}
b^{'}(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y)\,d\alpha \\
&= \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\,d\alpha.
\end{aligned}$$

We conclude that

$$\begin{aligned}
D &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\,d\alpha \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\,dy \cdot f(x)\,dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \left[\int_{[\underline{t},y]} L(\alpha|y)\,d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\,dy \cdot f(x)\,dx
\end{aligned}$$

This is the desired expression. ∎

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