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VIGILANT MEASURES OF RISK AND THE DEMAND FOR CONTINGENT CLAIMS

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Abstract. I examine a class of utility maximization problems with a not necessarily law-invariant utility, and with a not necessarily law-invariant risk measure constraint. The objective function is an integral of some function $\mathcal{U}$ with respect to some probability measure P, and the constraint set contains some risk measure constraint which is not necessarily P-law-invariant. This introduces some heterogeneity in the perception of uncertainty. The primitive $\mathcal{U}$ is a function of some given underlying random variable X and of a contingent claim Y on X. Many problems in economic theory and financial theory can be formulated in this manner, when a heterogeneity in the perception of uncertainty is introduced. Under a consistency requirement on the risk measure that will be called *Vigilance*, supermodularity of the primitive $\mathcal{U}$ is sufficient for the existence of optimal continent claims, and for these optimal claims to be comonotonic with the underlying random variable X. Vigilance is satisfied by a large class of risk measures, including all distortion risk measures. An explicit characterization of an optimal contingent claim is also provided in the case where the risk measure is a convex distortion risk measure.

1. Introduction

Problems of "utility" maximization in financial theory are not only some of the most fundamental issues, but they have also been recently re-examined in general abstract settings, hence allowing for various applications. See, for instance, Carlier and Dana [9, 10], Cvitanić, Schachermayer, and Wang [14], Dana [15], Föllmer and Schied [21], Hugonnier and Kramkov [31], Kramkov and Schachermayer [36, 37], Owen and Žitković [45], Schachermayer [51], Schied [52, 53, 54, 55], and Schied and Wu [56], to cite only a few. The generality of such problems

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allows for numerous interpretations, depending on the particular consumer demand problem at hand. By far the most common application of these problems is the – by now folkloric – problem of portfolio choice under uncertainty. The road paved by the seminal contributions of Merton [40, 41] has been often walked on, and the literature on portfolio choice is far too large for us to give a meaningful review here. Nevertheless, some of the recent contributions include He and Zhou [29], Jin and Zhou [33, 34, 35], and Zhang, Jin, and Zhou [62], for instance. These authors use a novel approach to utility maximization, based on a quantile reformulation of optimization problems, whereby problems involving a given choice variable are replaced by problems involving this variable's quantile function. Such quantile problems often turn out to be easier to handle and better behaved.

In this paper I use similar quantile techniques to show the existence of monotonic solutions to a general utility maximization problem, with a non-law-invariant indirect utility and a non-law-invariant risk measure constraint. This problem can be seen as a problem of demand for contingent claims under uncertainty, and it is an abstraction of many common problems in economic theory that were hitherto only considered in a framework of complete homogeneity of beliefs about the realizations of an underlying uncertainty. It can be formulated as

$$\sup_{Y \in \Theta} V\left(X, Y\right) := \int \mathcal{U}\left(X, Y\right)\ dP \tag{1.1}$$

where X is a given random variable on a probability space (S, Σ, P), $B(\Sigma)$ is the linear space of all bounded and Σ-measurable functions on S, $\Theta \subset B(\Sigma)$ is a given non-empty constraint set, and $\mathcal{U}(X, Y)$ is bounded and Σ-measurable for each $Y \in \Theta$. The objective function $V(X, Y)$ is understood to be a decision maker's (DM) indirect utility function, that is, $V(X, Y)$ represents the DM's expected utility of terminal wealth.

1.1. **Objective vs. Subjective Uncertainty.** When the constraint set Θ of Problem (1.1) contains, for example, another party's individual rationality constraint (participation constraint), one can distinguish between two types of problems, depending on how the underlying uncertainty is perceived by both parties: (i) either both parties agree on the distribution of this uncertainty (which will hence be induced by the probability measure P), or (ii) they have different perceptions of such randomness. The first type of problem is one where uncertainty can be called *homogeneous*, whereas the second type is a problem in which uncertainty can be referred to as being *heterogeneous*.

Surprisingly, the literature is mostly silent on problems of the form (1.1) where the uncertainty is heterogeneous, whereas problems of the form (1.1) with homogeneous uncertainty are abundant. For example, the vast majority of problems of optimal insurance design, or demand for insurance coverage are based on the classical formulation of Arrow [3], Borch [8], and Raviv [48], and are usually stated as a problem of the form (1.1) with homogeneous uncertainty. That is, both the insurer and the insured share the same beliefs about the realizations of some underlying insurable loss random variable X. In these problems, monotonicity of an optimal insurance contract Y is typically desired because such contracts can avoid *ex-post* moral hazard that might arise from a voluntary downward misrepresentation of the loss by the insured.

Problems of debt contracting between investors (lenders) and entrepreneurs (borrowers), such as the ones studied by Gale and Hellwig [22], Townsend [59], or Williamson [61], are also usually stated as a problem of the form (1.1) with homogeneous uncertainty. In this case, a contract specifies the repayment Y that the borrower makes to the lender as a function of the (uncertain) return X on the project being financed. The monotonicity of an optimal contract as a function of the return on investment is a coveted feature since such contracts will be *de facto* truthtelling, and will avoid any misrepresentation of the profitability of the project by the borrower.

Principal-agent problems have also been traditionally stated as problems of the form (1.1) with homogeneous uncertainty, as in the work of Grossman and Hart [27], Holmstrom [30], Mirrlees [44], Page [46], or Rogerson [49], for instance. In that setting, a contract specifies the wage Y that an agent receives from the principal, as a function of the (uncertain) outcome, or output X that occurs as a result of the agent's activity. Since the work of Rogerson [49], monotonicity of the optimal wage contract in the observed output is usually sought after[1].

Numerous other problems can be formulated as in (1.1), such as problems of demand for financial securities given a pricing or budgeting constraint. In such problems, monotonicity properties of optimal claims were studied first by Dybvig [18, 19]. Problem (1.1) can also be interpreted as a Neyman-Pearson type problem. In these problems, monotonicity properties of optimal solutions were discussed by Schied [52], for instance. Whatever the nature of the problem might be, it is interesting to examine under what conditions an optimal choice of the choice variable Y is monotone in the underlying variable X, and the theory of monotone comparative statics has usually been very fruitful in answering similar questions of monotonicity of solutions. However, it is of no help here, as the next section argues.

1.2. **The Theory of Monotone Comparative Statics and its Limitations.** The importance of monotone comparative statics analyses in economic theory is well-understood. One can even say that at the core of the motivation behind a sizeable collection of problems in economic theory, very often lies the question of whether or not a quantity is a monotone function of a parameter, or whether a variable output changes monotonically with a variable input. This is even more so significant if the monotonicity of an optimal such output as a function of an input parameter is desired, and indeed, monotone comparative statics techniques have proven to be very fruitful (see [58, 60]). Such techniques can be, and have been used in consumer theory, theory of production, portfolio choice theory, financial economics, and contract theory to answer some basic and intuitive questions.

The theory of monotone comparative statics is typically concerned with the behavior of a solution to a given optimization problem when a primitive of the problem changes. Specifically, let $(L, \geqslant_L)$ be a lattice, $B \subseteq L$ a choice set, $(T, \geqslant_T)$ a partially ordered set interpreted as a set of parameters, and $f : L \times T \to \mathbb{R}$ a given objective function. For the problem of choosing an $x \in B$ that maximizes the objective function given a value t of the parameter, the chief concern is the isotonicity of an optimal choice $x^*(t)$ of x as a function of t, that is,

$$t_1 \geqslant_T t_2 \implies x^*(t_1) \geqslant_L x^*(t_2) \tag{1.2}$$

The classical theory of monotone comparative statics [42, 43, 57, 58] seeks conditions on the function f that guarantee that eq. (1.2) holds.

Athey [5] examined a problem of monotone comparative statics in the presence of uncertainty, where the objective function is an integral of some function with respect to some measure. Specifically, let $(L, \geqslant_L)$ be a lattice, $B \subseteq L$ a choice set, $S = \prod_{i=1}^{m} S_i$ with $S_i \subseteq \mathbb{R}$ for $i = 1, \ldots, m$, $\Theta \subseteq \mathbb{R}$ a set of parameters, μ a finite nonnegative product measure on S, and u :

[1] For various reasons that are beyond the scope of this paper.

$L \times S \to \mathbb{R}$ and $\psi : S \times \Theta \to \mathbb{R}$ given bounded measurable functions. Define the objective function $\Phi : L \times \Theta \to \mathbb{R}$ by

$$\Phi(x, \theta) = \int_S u(x, s)\, \psi(s, \theta)\, d\mu(s)$$

For the problem of choosing an $x \in B$ that maximizes the objective function given a value θ of the parameter, the problem of monotone comparative statics in this situation of uncertainty is to find conditions on the primitives u and ψ so that an optimal choice $x^*(\theta)$ of x is a nondecreasing function of θ, that is,

$$\theta_1 \geqslant \theta_2 \implies x^*(\theta_1) \geqslant_L x^*(\theta_2)$$

In particular, in both situations of certainty and uncertainty, the interest is in the variation of the optimal solution with respect to the lattice order $\geqslant_L$ on L, given a variation of the parameter (t or θ, respectively) in the order on the parameter set ($\geqslant_T$ or the usual order on $\mathbb{R}$, respectively). Often, however, these notions of order are too strong for the problem under consideration. For instance, in problems of the form (1.1), conditions on the primitive $\mathcal{U}$ for the optimal choice Y^* of Y to be monotone in X are desired. Specifically, under what conditions on $\mathcal{U}$ does one have that for all $s, s' \in S$, $X(s) \geqslant X(s') \Rightarrow Y^*(s) \geqslant Y^*(s')$? The classical techniques of monotone comparative statics are of no help in these situations since the lattice order on $B(\Sigma)$ is not adequate here. This order $\geqslant_L$ on $L = B(\Sigma)$ is defined by

$$Y_1 \geqslant_L Y_2 \text{ if and only if } Y_1(s) \geqslant Y_2(s), \text{ for all } s \in S$$

1.3. **A Class of Demand Problems.** Since the seminal contribution of Artzner et al. [4] and Delbaen [16], the literature on risk measures has grown exponentially[2]. This motivated an active area of research where many classical problems arising in financial theory were re-examined using general risk measures beyond the classical dispersion measures such as the variance[3]. The problem that will be examined in this paper can be seen as an abstraction of problems of this sort, and it takes the form

$$\sup_{Y = I \circ X} \left\{ \int \mathcal{U}(X, Y)\ dP \,\middle|\, 0 \leqslant Y \leqslant X,\ \rho(Y) \leqslant R \right\} \tag{1.3}$$

where X is a given underlying uncertainty on the measurable space (S, Σ, P), $Y = I \circ X$ is a claim contingent on this uncertainty, $\int \mathcal{U}(X, I \circ X)\ dP$ is a DM's expected utility of wealth with respect to the probability measure P, $\rho : B^+(\Sigma) \to \mathbb{R}$ is a given risk measure, and $R \in \mathbb{R}$ is fixed.

The first constraint is standard in many problems in economic theory. In the insurance framework [3, 48], this constraint says that an indemnity is nonnegative and cannot exceed the loss itself. In a framework of debt contracting [22], this constraint is a limited liability constraint. The second constraint is simply a risk measure constraint, but it can be interpreted as a budget constraint, or a pricing constraint. In some situations, this constraint can also be seen as another party's individual rationality constraint (participation constraint). For instance, in problems of insurance demand, the constraint $\rho(Y) \leqslant R$ would be a "premium constraint" of the form $\int Y\ dP \leqslant \Pi$, which is nothing more than a rephrasing of the (risk-neutral) insurer's participation constraint.

[2] See, for instance, Föllmer and Schied [21] for a review.
[3] See, for instance, Balbás [6] for a brief review.

In Problem (1.3), the objective function (indirect utility function) is not necessarily law-invariant with respect to P (Definition 2.6). This is an important point of departure from the recent literature on utility maximization, such as in Carlier and Dana [9, 10] or Schied [52]. Moreover, in the constraint set of Problem (1.3), the mapping $\rho : B^+(\Sigma) \to \mathbb{R}$ need not be law-invariant with respect to P. When ρ is not law-invariant with respect to P, this creates some heterogeneity in the perception of the uncertainty X, and poses some important mathematical complications. For instance, in the insurance framework, it might be that the DM and the insurer assign different "distributions" to the underlying uncertainty. The latter problem has been examined by Ghossoub [23, 25, 26].

This paper's main result (Theorem 3.1) is that when the risk measure ρ satisfies a property that will be called *Vigilance* (Definition 2.7) and a continuity property that will be called the *Weak DC-Property* (Definition 2.5), supermodularity of the function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ (Definitions 2.3 and B.5) is sufficient for an optimal choice of $Y = I \circ X$ to be a nondecreasing function of the underlying uncertainty X. Roughly speaking, vigilance of the risk measure ρ can be understood as a (weak) preference for comonotonicity with X (Definition 2.2), on the collection of all functions that are identically distributed for the probability measure P. Given two elements Y_1 and Y_2 of $B^+(\Sigma)$ that have the same distribution with respect to the probability measure P, vigilance of a risk measure $\rho : B^+(\Sigma) \to \mathbb{R}$ means that if any one of Y_1 or Y_2 is a nondecreasing function of X, it will assigned a lower value by ρ than the other function, and it will hence be seen as less risky. The Weak DC-Property of an operator roughly means that the operator preserves dominated convergence. This property is satisfied by a large class of operators on $B^+(\Sigma)$, such as the Lebesgue integral or the Choquet integral (Definition A.3).

The idea of vigilance introduced here is an extension of the notion of *vigilant beliefs* introduced by Ghossoub [23, 25, 26] in the insurance framework under Bayesian uncertainty, and extended by Amarante, Ghossoub, and Phelps [2] to the case of Knightian uncertainty (Ambiguity). Ghossoub [23, 25, 26] re-examines the classical Arrow-Borch-Raviv model of insurance demand, but allowing for belief heterogeneity between the insurer and the insurance buyer. He shows that if the insurer's subjective probability measure on the state space satisfies a property called *vigilance*, then an optimal contract for the insurance buyer takes the form of what the author calls a *generalized deductible contract*. Moreover, he characterizes the collection of all optimal contracts in terms of their distribution for the insurance buyer. In Ghossoub's [23, 25, 26] setting, where X is the insurable loss random variable, P is the insurance buyer's (subjective) probability measure on the state space, and Q is the insurer's (subjective) probability measure on the state space, Q is said to be (P, X)-vigilant if for any two insurance contracts $I_1(X)$ and $I_2(X)$ that are identically distributed under P, and such that I_2 is a nondecreasing function[4], it follows that $I_2(X)$ is preferred by the insurer to $I_1(X)$. Ghossoub [23, 25, 26] argues that (P, X)-vigilance of Q indicates a kind of credibility that the insurer assigns to the insurance buyer's assessment of the riskiness of the insurance contracts being compared, and he shows that vigilance is a (strictly) weaker assumption than that of a *monotone likelihood ratio*, when the latter can be defined[5]. In this paper, the definition of vigilance is extended from the notion of vigilant beliefs to the concept of a vigilant real-valued mapping ρ on the collection of functions Y over which a decision maker (DM) has a given preference. When $\rho(Y) = \int Y \, dP$, one will

[4]So that $I_2(X)$ and X are comonotonic (Definition 2.2).

[5]That is, when the measure P and Q are such that the laws $P \circ X^{-1}$ and $Q \circ X^{-1}$ have densities (with respect to Lebesgue measure), so that a likelihood ratio can be defined as the ratio of these densities.

recover Ghossoub's [23, 25, 26] definition of vigilant beliefs as a special case of the definition of vigilance given here (Definition 2.7).

1.4. **Utility Maximization with a Convex Distortion Risk Measure.** As an illustration, I consider a utility maximization problem under a risk measure constraint, where the risk measure is a convex distortion risk measure. These risk measures are automatically vigilant and possess the Weak DC Property. Our Theorem 3.1 then yields the existence of optimal contingent claims that are nondecreasing functions of the underlying random variable X. Using a quantile reformulation procedure, I show that under mild regularity conditions on the DM's utility function, an optimal contingent claim has an explicit characterization, and a crisp and tractable form.

Specifically, when $\mathcal{U}(X,Y) := u(W_0 + Y - X)$, for some utility function u satisfying some mild regularity conditions, and when the risk measure ρ is a Choquet integral of the form $\rho(Y) = \widehat{\int} Y\,dT \circ P$, for some convex distortion function T, then an optimal contingent claim takes the form

$$\mathcal{Y}^* = \max\left[0, \min\left\{F_X^{-1}(U), \left(u'\right)^{-1}\left(\lambda^* T'(1-U)\right) + F_X^{-1}(U) - W_0\right\}\right] \tag{1.4}$$

where λ^* is chosen so that $\int T'(1-U)\,\mathcal{Y}^*\,dP = R$, U is a uniformly distributed random variable on $(0,1)$, and F_X^{-1} denotes the quantile function of X (defined as in equation (4.1)).

1.5. **Outline.** The rest of this paper is organized as follows. Section 2 introduces some preliminary notation and definitions. Section 3 presents this paper's main result, which gives sufficient conditions for the existence of comonotonic optimal continent claims. Section 4 gives a characterization of comonotonic optimal contingent claims in terms of quantile functions, when the risk measure is a distortion risk measure. Section 5 examines a special case of demand for contingent claims under a convex risk measure, and gives an explicit characterization of a comonotonic optimal continent claim in this case. Finally, Section 6 concludes. Some proofs and some related analysis can be found in the Appendices.

2. Preliminaries and Notation

Let S denote the set of states of the world, and suppose that $\mathcal{G}$ is a σ-algebra of subsets of S, called events. Denote by $B(\mathcal{G})$ the supnorm-normed Banach space of all bounded, $\mathbb{R}$-valued and $\mathcal{G}$-measurable functions on $(S,\mathcal{G})$, and denote by $B^+(\mathcal{G})$ the collection of all $\mathbb{R}^+$-valued elements of $B(\mathcal{G})$. For any $f \in B(\mathcal{G})$, the supnorm of f is given by $\|f\|_{sup} := \sup\{|f(s)| : s \in S\} < +\infty$. For $C \subseteq S$, denote by $\mathbf{1}_C$ the indicator function of C. For any $A \subseteq S$ and for any $B \subseteq A$, denote by $A \backslash B$ the complement of B in A.

For any $f \in B(\mathcal{G})$, denote by $\sigma\{f\}$ the σ-algebra of subsets of S generated by f, and denote by $B(\sigma\{f\})$ the linear space of all bounded, $\mathbb{R}$-valued and $\sigma\{f\}$-measurable functions on $(S,\mathcal{G})$. Then by Doob's measurability theorem [1, Theorem 4.41], for any $g \in B(\sigma\{f\})$ there exists a Borel-measurable map $\zeta : \mathbb{R} \to \mathbb{R}$ such that $g = \zeta \circ f$. Denote by $B^+(\sigma\{f\})$ the cone of nonnegative elements of $B(\sigma\{f\})$.

Definition 2.1. A finite nonnegative measure η on a measurable space $(\Omega, \mathcal{A})$ is said to be *nonatomic* if for any $A \in \mathcal{A}$ with $\eta(A) > 0$, there is some $B \in \mathcal{A}$ such that $B \subsetneq A$ and $0 < \eta(B) < \eta(A)$.

For any $f \in B(\mathcal{G})$, if $\mathcal{A}$ is any sub-σ-algebra of $\mathcal{G}$ such that $\sigma\{f\} \subseteq \mathcal{A}$, and if P is any probability measure on the measurable space $(S, \mathcal{A})$, it will be said that f is a continuous random variable for P when the law $P \circ f^{-1}$ of f is a nonatomic Borel probability measure.

Definition 2.2. Two functions $Y_1, Y_2 \in B(\mathcal{G})$ are said to be comonotonic if

$$\Big[Y_1(s) - Y_1(s')\Big]\Big[Y_2(s) - Y_2(s')\Big] \geqslant 0, \text{ for all } s, s' \in S$$

For instance any $Y \in B(\mathcal{G})$ is comonotonic with any $c \in \mathbb{R}$. Moreover, if $Y_1, Y_2 \in B(\mathcal{G})$, and if Y_2 is of the form $Y_2 = I \circ Y_1$, for some Borel-measurable function I, then Y_2 is comonotonic with Y_1 if and only if the function I is nondecreasing.

Definition 2.3. A function $L : \mathbb{R}^2 \to \mathbb{R}$ is *supermodular* if for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, one has

$$L(x_2, y_2) + L(x_1, y_1) \geqslant L(x_1, y_2) + L(x_2, y_1)$$

Appendix B contains more material about supermodular functions on lattices (Definition B.5). For instance, if $g : \mathbb{R} \to \mathbb{R}$ is concave, and $a \in \mathbb{R}$, then the function $L_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_1(x, y) = g(a - x + y)$ is supermodular (see Example B.7).

2.1. **Vigilant Risk Measures and the Weak DC-Property.** Let P be a given probability measure on the measurable space $(S, \mathcal{G})$. In many situations of choice under uncertainty, the elements of choice are the elements of $B^+(\mathcal{G})$, as in the problem that will be examined in this paper. Often, a problem of choice involving these elements is stated as an optimization problem subject to some constraints. In an abstract form, some of these constraints can be stated in terms of operators $\Psi : B^+(\mathcal{G}) \to \mathbb{R}$, and might be called "aggregation constraints", or "risk measure constraints". Here I will define two special kinds of these operators (or risk measures): vigilant risk measures, and risk measures having the Weak DC-Property.

Definition 2.4. A risk measure is any mapping $\Psi : B^+(\mathcal{G}) \to \mathbb{R}$.

Definition 2.5 (Weak DC-Property)**.** A risk measure $\Psi : B^+(\mathcal{G}) \to \mathbb{R}$ is said to have the *Weak DC-Property* if for any $Y^* \in B^+(\mathcal{G})$ and for any sequence $\{Y_n\}_{n \geqslant 1} \subset B^+(\mathcal{G})$ such that

(1) $\lim_{n \to +\infty} Y_n = Y^*$ (pointwise), and

(2) there is some $Z \in B^+(\mathcal{G})$ such that $Y_n \leqslant Z$, for each $n \geqslant 1$,

the following holds:

$$\lim_{n \to +\infty} \Psi(Y_n) = \Psi(Y^*)$$

When Ψ is defined as a Lebesgue integral with respect to P, i.e. $\Psi(Y) = \int Y \, dP$ for each $Y \in B^+(\mathcal{G})$, then Lebesgue's Dominated Convergence Theorem [13, Th. 2.4.4] implies that Ψ has the Weak DC-Property. More generally, if Ψ is a Choquet integral (Appendix A) with respect to some continuous capacity ν on $(S, \mathcal{G})$ (Definition A.2), i.e. $\Psi(Y) = \widehat{\int} Y \, d\nu$ for each $Y \in B^+(\mathcal{G})$, then when seen as an operator on $B^+(\mathcal{G})$, Ψ has the Weak DC-Property. This is a consequence of [47, Th. 7.16][6].

Definition 2.6. Recall that P is a probability measure on $(S, \mathcal{G})$. A mapping $\Psi : B(\Sigma) \to \mathbb{R}$ is said to be *P-law-invariant*, or *law-invariant with respect to* P, if for any $\phi_1, \phi_2 \in B(\Sigma)$, $\Psi(\phi_1) = \Psi(\phi_2)$ whenever ϕ_1 and ϕ_2 have the same distribution according to P.

Definition 2.7 (Vigilance)**.** Let X be a given element of $B^+(\mathcal{G})$, and recall that P is a probability measure on $(S, \mathcal{G})$. Denote by Σ the σ-algebra $\sigma\{X\}$ of subsets of S generated by X. A risk measure $\Psi : B^+(\Sigma) \to \mathbb{R}$ is said to be (P, X)-*vigilant* if for any $Y_1, Y_2 \in B^+(\Sigma)$ such that

(i) Y_1 and Y_2 have the same distribution under P, i.e. $P \circ Y_1^{-1} = P \circ Y_2^{-1}$, and,

(ii) Y_2 is a nondecreasing function of X, i.e. Y_2 and X are comonotonic,

the following holds:

$$\Psi(Y_2) \leqslant \Psi(Y_1)$$

Clearly, if Ψ is P-law invariant then it is (P, X)-vigilant. This covers a large collection of operators on $B^+(\Sigma)$ such that a Lebesgue integral with respect to P, a Choquet integral with respect to a distortion of P (Appendix A), and so on. When Ψ is not P-law invariant, the same intuition as that behind Ghossoub's [23] definition applies here. Namely, the elements of choice are those elements of $B^+(\Sigma)$. Any $Y \in B^+(\Sigma)$ can be written as a function of X, i.e. of the form $Y = I \circ X$. One can then think of such Y, rather informally, as possessing two sources of variability: one which results from the state space itself and the indeterminacy of the state of nature at the time of decision, and the other which is associated with the variability of Y with respect to X, i.e. with the variability of the function I with respect to the identity function (on the range of X). The first kind of variability depends on the distribution of X, and it is called *baseline randomness* by Ghossoub [23]. The second kind of variability does not depend on the distribution of X, and it is called *idiosyncratic randomness* by Ghossoub [23]. (P, X)-vigilance of the risk measure $\Psi : B^+(\Sigma) \to \mathbb{R}$ can then be understood as requiring that:

(1) For a given variability of the first kind, fixed according to the probability measure P (that is, when the distribution of X is considered with respect to the measure P), a comparison of two contingent claims through Ψ can be restricted to a comparison of their idiosyncratic randomness only; and,

(2) If a given function I's variability is similar to that of the identity function Id (that is, I is comonotonic with Id), then less idiosyncratic randomness is attributed to I than to another function J for which the variability is not similar to that of Id, and hence $I \circ X$ will be attributed a lower value than $J \circ X$ through Ψ (i.e. a lower measure of risk).

[6] Theorem 7.16 in Pap [47] is a Dominated Convergence Theorem for the *Šipoš integral*, or the *symmetric Choquet integral*. However, the latter coincides with the Choquet integral for nonnegative functions.

3. Existence of Comonotonic Optimal Contingent Claims

Consider the setting of Section 2, and let X be a given element of $B^+(\mathcal{G})$ with closed range $X(S) = [0, M]$, where $M := \|X\|_{sup} < +\infty$. Denote by Σ the σ-algebra $\sigma\{X\}$ of subsets of S generated by X, and let P be a probability measure on $(S, \mathcal{G})$. I also assume that X is a continuous random variable[7] for P.

Let $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ be a given function, and let $\rho : B^+(\Sigma) \to \mathbb{R}$ be a given risk measure. The random variable X is fixed, and the objects P, $\mathcal{U}$, and ρ are considered to be the primitives of the following problem:

$$\sup_{Y \in B(\Sigma)} \left\{ \int \mathcal{U}(X, Y)\ dP \;\middle|\; 0 \leqslant Y \leqslant X,\ \rho(Y) \leqslant R \right\} \tag{3.1}$$

where $R \in \mathbb{R}$ is fixed. The following theorem gives sufficient conditions for the optimal choice Y^* of the choice variable Y to be a nondecreasing function of X. The proof of the theorem is given in Appendix C.

Theorem 3.1. *If the following hold:*

1. *Problem (3.1) has a nonempty feasibility set,*
2. *The mapping $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ is supermodular,*
3. *The mapping $\mathcal{E}$ on $B^+(\Sigma)$ defined by $\mathcal{E}(Y) := \mathcal{U}(X, Y)$ is uniformly bounded and sequentially continuous in the topology of pointwise convergence*[8]*,*
4. *ρ is (P, X)-vigilant, and,*
5. *ρ has the Weak DC-Property,*

then Problem (3.1) admits a solution Y^* which is comonotonic with X. Moreover, any other solution Z^* which is comonotonic with X and identically distributed as Y^* under P is such that $Y^* = Z^*$, P-a.s.

A few comments on the assumptions in Theorem 3.1 are in order. First, the assumption of nonemptiness of the feasibility set of Problem (3.1) is made simply to rule out trivial cases where no solution can exist. The assumption of supermodularity of the mapping $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ is not a strong assumption by any means. It is usually given in many situations by the very nature of the problem considered. This happens for instance when $\mathcal{U}(X, Y) = u(a - X + Y)$, for a concave utility function u and some $a \in \mathbb{R}$. See Example B.7 (1). Assumption (4) in Theorem 3.1 is typically obtained whenever $\mathcal{U}(X, Y) = u(a - X + Y)$, for some continuous and bounded utility function u, and some $a \in \mathbb{R}$. Assumptions (5) and (6) were discussed in Section 2.1.

Theorem 3.1 is a general result that can be used in many situations. For instance, it can be applied to problems of demand for insurance indemnities as in the model of Arrow [3] and Raviv [48], but also allowing for heterogeneous beliefs unlike the classical approach to that problem.

[7] This is a standard assumption (e.g. Carlier and Dana [9, 10], Dana [15], and parts of Schied [52]), and it holds in many instances, such as when it is assumed that a probability density function for X exists.

[8] That is, (i) there exists some $N < +\infty$ such that $\| \mathcal{E}(Y) \|_{sup} \leqslant N$ for each $Y \in B^+(\Sigma)$; and, (ii) if $\{Y_n\}_n$ is a sequence in $B^+(\Sigma)$ that converges pointwise to some $Y \in B^+(\Sigma)$, then the sequence $\{\mathcal{E}(Y_n)\}_n$ converges pointwise to $\mathcal{E}(Y)$.

4. A Quantile Characterization of Optimal Contingent Claims under a Distortion Risk Measure

Let $\mathcal{H} := \left\{ Y \in B(\Sigma) \;\middle|\; 0 \leqslant Y \leqslant X \text{ and } \rho(Y) \leqslant R \right\}$ denote the feasibility set of Problem (3.1). For each $Y \in B^{+}(\Sigma)$, let $F_Y(t) := P\big(\{s \in S : Y(s) \leqslant t\}\big)$ denote the cumulative distribution function (cdf) of Y with respect to the probability measure P, and let $F_X(t) := P\big(\{s \in S : X(s) \leqslant t\}\big)$ denote the cdf of X with respect to the probability measure P. Let $F_Y^{-1}(t)$ be the left-continuous inverse of the cdf F_Y (i.e. the *quantile function* of Y), defined by

$$F_Y^{-1}(t) := \inf \left\{ z \in \mathbb{R}^{+} \;\middle|\; F_Y(z) \geqslant t \right\}, \ \forall t \in [0,1] \tag{4.1}$$

Lemma 4.1. *If X is a continuous random variable for the probability measure P, then*

(1) *$U := F_X(X)$ is a random variable on the probability space (S, Σ, P) with a uniform distribution on $(0,1)$,*

(2) *$X = F_X^{-1}(U)$, P-a.s.*

If, in addition, ρ is (P,X)-vigilant, and the mapping $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ is supermodular, then for each $Y \in \mathcal{H}$, the function Y^* defined by $Y^* := F_Y^{-1}(F_X(X))$ is such that:

(1) *$Y^* \in \mathcal{H}$,*

(2) *Y^* is comonotonic with X,*

(3) *$\int \mathcal{U}\Big(X, Y^*\Big)\, dP \geqslant \int \mathcal{U}\Big(X, Y\Big)\, dP$, and,*

(4) *$\rho(Y^*) \leqslant \rho(Y)$.*

The proof of Lemma 4.1 is given in Appendix D. Under the assumptions of Lemma 4.1, it follows that for each $Y \in \mathcal{H}$ the following holds:

(1) $\int \mathcal{E}\Big(F_Y^{-1}(U)\Big)\, dP = \int \mathcal{U}\Big(F_X^{-1}(U), F_Y^{-1}(U)\Big)\, dP \geqslant \int \mathcal{U}\Big(X, Y\Big)\, dP$

(2) $\rho\left(F_Y^{-1}(U)\right) \leqslant \rho(Y)$.

where the mapping $\mathcal{E}$ on $B^{+}(\Sigma)$ is defined by $\mathcal{E}(Y) = \mathcal{U}(X, Y)$. Hence, by Lemma 4.1, one can look for a solution to Problem (3.1) of the form $F^{-1}(U)$, where F is the cdf of a function $Z \in B^{+}(\Sigma)$ such that $0 \leqslant Z \leqslant X$ and $\rho(Z) \leqslant R$.

4.1. Distortion Risk Measure.

In the following, the risk measure ρ is assumed to be a distortion risk measure. Specifically, I make the following assumption.

Assumption 4.2. *There exists a function $T : [0,1] \to [0,1]$ such that:*

(1) *T is increasing and continuously differentiable,*

(2) *$T(0) = 0$ and $T(1) = 1$,*

(3) *$\rho(Y) = \widehat{\int} Y \, dT \circ P$, for each $Y \in B^{+}(\Sigma)$, where integration is in the sense of Choquet as in Appendix A.*

It follows immediately from the definition of vigilance that the distortion risk measure ρ is (P, X)-vigilant. Moreover, by a classical result [47, Th. 7.16], ρ has the Weak DC-Property. Hence, Theorem 3.1 yields the existence of a solution to the following problem:

$$\sup_{Y \in B(\Sigma)} \left\{ \int \mathcal{U}(X, Y)\ dP \;\middle|\; 0 \leqslant Y \leqslant X,\ \rho(Y) := \widehat{\int} Y\ dT \circ P \leqslant R \right\} \tag{4.2}$$

Now, for any given $Y \in B^+(\Sigma)$, if $Y^* = F_Y^{-1}(F_X(X)) = F_Y^{-1}(U)$, then one can write[9]

$$\rho(Y^*) = \widehat{\int} F_Y^{-1}(U)\ dT \circ P = \int_0^1 T'(1-t) F_Y^{-1}(t)\ dt = \int T'(1-U) F_Y^{-1}(U)\ dP$$

and

$$\int \mathcal{U}\Big(X, Y^*\Big)\ dP = \int \mathcal{U}\Big(F_X^{-1}(U), F_Y^{-1}(U)\Big) = \int \mathcal{E}\Big(F_Y^{-1}(U)\Big)\ dP = \int_0^1 \mathcal{E}\left(F_Y^{-1}(t)\right)\ dt$$

Definition 4.3. Let $\mathcal{Q}$ denote the collection of all quantile functions. That is,

$$\mathcal{Q} := \Big\{ f : (0,1) \to \mathbb{R} \;\Big|\; f \text{ is nondecreasing and left-continuous} \Big\}$$

Let $\mathcal{Q}^*$ denote the collection of all quantile functions $f \in \mathcal{Q}$ of the form F^{-1}, where F is the cdf of some function $Y \in B^+(\Sigma)$ such that $0 \leqslant Y \leqslant X$. That is,

$$\mathcal{Q}^* = \Big\{ f \in \mathcal{Q} \;\Big|\; 0 \leqslant f(z) \leqslant F_X^{-1}(z),\ \text{for each } 0 < z < 1 \Big\} \tag{4.3}$$

4.2. **A Quantile Problem.** Consider the following problem:

$$\sup_{f \in \mathcal{Q}^*} \left\{ \int \mathcal{E}(f(U))\ dP \;\middle|\; \int T'(1-U) f(U)\ dP \leqslant R \right\} \tag{4.4}$$

In light of Lemma 4.1, the following result is immediate.

Proposition 4.4. *Suppose that the assumptions of Lemma 4.1 hold. If f^* is optimal for Problem (4.4), then the function $f^*(U)$ is optimal for Problem (4.2).*

Proof. Suppose that $f^* \in \mathcal{Q}^*$ is optimal for Problem (4.4), and let $Z^* \in B^+(\Sigma)$ be a corresponding function. That is, f^* is the quantile function of Z^*. Hence, $0 \leqslant Z^* \leqslant X$, and $\rho(Z^*) = \widehat{\int} Z^*\ dT \circ P = \int T'(1-U) F_{Z^*}^{-1}(U)\ dP = \int T'(1-U) f^*(U)\ dP \leqslant R$. Therefore, Z^* is feasible for Problem (4.2). Let $\widetilde{Z}^* := f^*(U)$. Then, by Lemma 4.1, $\widetilde{Z}^* = f^*(U)$ is feasible for Problem (4.2). To show optimality, let Z be any feasible solution for Problem (4.2), and let F be the cdf of Z. Then, by Lemma 4.1, the function $\widetilde{Z} := F^{-1}(U)$ is feasible for Probem (4.2), comonotonic with X, and satisfies:

- $\int \mathcal{U}\Big(X, \widetilde{Z}\Big)\ dP \geqslant \int \mathcal{U}\Big(X, Z\Big)\ dP$, and,

[9] This is a standard exercise. See, for instance, [34, p. 418] or [29, p. 213].

- $\widehat{\int} \widetilde{Z} \, dT \circ P \leqslant \widehat{\int} Z \, dT \circ P \leqslant R$.

Moreover, $\widetilde{Z}$ has also F as a cdf. To show optimality of $\widetilde{Z}^* = f^*(U)$ for Problem (4.2) it remains to show that

$$\int \mathcal{U}\left(X, \widetilde{Z}^*\right) \, dP \geqslant \int \mathcal{U}\left(X, \widetilde{Z}\right) \, dP$$

Now, let $f := F^{-1}$, so that $\widetilde{Z} = f(U)$. Since $\widetilde{Z}$ is feasible for Probem (4.2), it follows that

$$R \geqslant \rho\left(\widetilde{Z}\right) = \int T'(1-U) \, f(U) \, dP$$

Hence, f is feasible for Problem (4.4). Since f^* is optimal for Problem (4.4), it follows that

$$\int \mathcal{E}\left(f^*(U)\right) \, dP \geqslant \int \mathcal{E}\left(f(U)\right) \, dP$$

That is

$$\int \mathcal{U}\left(X, \widetilde{Z}^*\right) \, dP \geqslant \int \mathcal{U}\left(X, \widetilde{Z}\right) \, dP$$

Therefore, $\widetilde{Z}^* = f^*(U)$ is optimal for Problem (4.2). □

5. An Example: Utility Maximization with a Convex Distortion Risk Measure

Consider the previous setting, and assume the following.

Assumption 5.1. *There exists a utility function $u : \mathbb{R} \to \mathbb{R}$ such that:*

(1) *u is bounded and satisfies Inada's [32] conditions. That is:*
   - *u is bounded,*
   - *$u(0) = 0$,*
   - *u is strictly increasing and strictly concave,*
   - *u is continuously differentiable, and,*
   - *$u'(0) = +\infty$ and $\lim_{x \to +\infty} u'(x) = 0$.*

(2) *For each $Y \in B^+(\Sigma)$, $\mathcal{E}(Y) = \mathcal{U}(X, Y) = u(W_0 + Y - X)$.*

The strict concavity and the continuous differentiability of u imply that u' is both continuous and strictly decreasing. The latter implies that $(u')^{-1}$ is continuous and strictly decreasing, by the Inverse Function Theorem [50, pp. 221-223].

Assumption 5.2. *The DM has initial wealth W_0 such that $X \leqslant W_0$, P-a.s. That is,*

$$P\left(\left\{s \in S : X(s) > W_0\right\}\right) = 0$$

Assumption 5.2 simply states that the DM is well-diversified so that the particular exposure to X is sufficiently small, with respect to the DM's total portfolio exposure.

5.1. **A Utility Maximization Problem.** The initial Problem (3.1) then becomes the following utility maximization problem.

$$\sup_{Y \in B(\Sigma)} \left\{ \int u\left(W_0 + Y - X\right) \, dP \;\middle|\; 0 \leqslant Y \leqslant X, \; \rho(Y) := \widehat{\int} Y \, dT \circ P \leqslant R \right\} \tag{5.1}$$

The concavity of the utility function u (Assumption 5.1) implies that the function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ defined by $\mathcal{U}(x, y) = u(y - x)$ is supermodular (Example B.7 (1)). Moreover, the risk measure ρ given by $\rho(Y) = \widehat{\int} Y \, dT \circ P$ is (P, X)-vigilant and has the Weak DC-Property. Hence, Theorem 3.1 yields the existence of a solution to Problem (5.1).

Remark 5.3. For any $Y \in B(\Sigma)$ which is feasible for Problem 5.1, one has $0 \leqslant Y \leqslant X$. Therefore, by monotonicity of the Choquet integral (Proposition A.4 (5)), it follows that

$$0 \leqslant \widehat{\int} Y \, dT \circ P \leqslant \widehat{\int} X \, dT \circ P$$

Hence, for the risk measure constraint $\rho(Y) \leqslant R$ to be meaningful in the present context, it will be assumed that $R \leqslant \widehat{\int} X \, dT \circ P$. This will then imply that any $Y \in B(\Sigma)$ which is feasible for Problem 5.1 satisfies:

$$0 \leqslant \widehat{\int} Y \, dT \circ P \leqslant R \leqslant \widehat{\int} X \, dT \circ P$$

Now, let $U = F_X(X)$, and consider the following problem.

$$\sup_{f \in \mathcal{Q}^*} \left\{ \int u\left(W_0 + f(U) - F_X^{-1}(U)\right) \, dP \;\middle|\; \int T'(1 - U) f(U) \, dP \leqslant R \right\} \tag{5.2}$$

where $\mathcal{Q}^*$ is given by Equation (4.3).

Proposition 5.4. *Suppose that the assumptions of Lemma 4.1 hold. If f^* is optimal for Problem (5.2), then the function $f^*(U)$ is optimal for Problem (5.1).*

Proof. By Lemma 4.1, $X = F_X^{-1}(U)$, P-a.s. Therefore, for any quantile function $f \in \mathcal{Q}$,

$$\int u\left(W_0 + f(U) - F_X^{-1}(U)\right) \, dP = \int u\left(W_0 + f(U) - X\right) \, dP = \int \mathcal{E}\left(f(U)\right) \, dP$$

The rest follows from Proposition 4.4. □

Note that Proposition 5.4 holds for any distortion function T, and the convexity of T has not been assumed yet.

5.2. **Characterizing a Solution.**

Lemma 5.5. *Suppose that X is a continuous random variable for the probability measure P. If $f^* \in \mathcal{Q}^*$ satisfies the following:*

(1) $\int_0^1 T'(1-t) f^*(t)\ dt = R$,
(2) *There exists $\lambda \geqslant 0$ such that for all $t \in (0,1)$,*

$$f^*(t) = \underset{0 \leqslant y \leqslant F_X^{-1}(t)}{\arg\max} \left[u\Big(W_0 + y - F_X^{-1}(t)\Big) - \lambda T'(1-t)\, y \right],$$

then f^* solves Problem (5.2).

If, in addition, the rest of the assumptions of Lemma 4.1 hold, then the function $f^*(U)$ is optimal for Problem (5.1), where $U = F_X(X)$.

Proof. Suppose that $f^* \in \mathcal{Q}^*$ satisfies conditions (1) and (2) above. Then, in particular, f^* is feasible for Problem (5.2). To show optimality of f^* for Problem (5.2), let f by any other feasible solution for Problem (5.2). Then $\int_0^1 T'(1-t) f(t)\ dt \leqslant R$ and, for all $t \in (0,1)$,

$$u\Big(W_0 + f^*(t) - F_X^{-1}(t)\Big) - \lambda T'(1-t) f^*(t) \geqslant u\Big(W_0 + f(t) - F_X^{-1}(t)\Big) - \lambda T'(1-t) f(t),$$

that is,

$$\Big[u\Big(W_0 + f^*(t) - F_X^{-1}(t)\Big) - u\Big(W_0 + f(t) - F_X^{-1}(t)\Big)\Big] \geqslant \lambda T'(1-t)\Big[f^*(t) - f(t)\Big]$$

Integrating yields

$$\int_0^1 u\Big(W_0 + f^*(t) - F_X^{-1}(t)\Big)\ dt - \int_0^1 u\Big(W_0 + f(t) - F_X^{-1}(t)\Big)\ dt$$
$$\geqslant \lambda \left[R - \int_0^1 T'(1-t) f(t)\ dt\right] \geqslant 0$$

or,

$$\int u\Big(W_0 + f^*(U) - F_X^{-1}(U)\Big)\ dP = \int_0^1 u\Big(W_0 + f^*(t) - F_X^{-1}(t)\Big)\ dt$$
$$\geqslant \int_0^1 u\Big(W_0 + f(t) - F_X^{-1}(t)\Big)\ dt = \int u\Big(W_0 + f(U) - F_X^{-1}(U)\Big)\ dP$$

as required. The rest follows from Proposition 5.4. □

Lemma 5.5 suggests that in order to find a solution for Problem (5.2), one can start by solving the problem

$$\max_{0 \leqslant f_\lambda(t) \leqslant F_X^{-1}(t)} \left[u\Big(W_0 + f_\lambda(t) - F_X^{-1}(t)\Big) - \lambda T'(1-t) f_\lambda(t)\right] \tag{5.3}$$

for a given $\lambda \geqslant 0$, and for a fixed $t \in (0,1)$.

Assumption 5.6. *T is convex and $T'(1) < +\infty$.*

Assumption 5.6 yields an explicit characterization of an optimal solution to Problem (5.1), as given in the following Corollary.

Corollary 5.7. *Suppose that the assumptions of Lemma 4.1 hold, and let $U = F_X(X)$. If Assumption 4.2, Assumption 5.1, Assumption 5.2, and Assumption 5.6 also hold, then an optimal solution for Problem (5.1) takes the form:*

$$\mathcal{Y}^* = \max\left[0, \min\left\{F_X^{-1}(U), (u')^{-1}\left(\lambda^* T'(1-U)\right) + F_X^{-1}(U) - W_0\right\}\right] \tag{5.4}$$

where λ^* is chosen so that $\int T'(1-U)\,\mathcal{Y}^*\,dP = R$.

Proof. For a given $\lambda \geqslant 0$, and for a fixed $t \in (0,1)$, consider the problem:

$$\max_{f_\lambda(t)}\left[u\Big(W_0 + f_\lambda(t) - F_X^{-1}(t)\Big) - \lambda T'(1-t)\, f_\lambda(t)\right] \tag{5.5}$$

By Assumption 5.1, the first-order conditions are sufficient for an optimum for Problem (5.5), and they imply that the function

$$f_\lambda^*(t) := (u')^{-1}\left(\lambda T'(1-t)\right) + F_X^{-1}(t) - W_0$$

solves Problem (5.5). Concavity of u and convexity of T imply that the function $f_\lambda^* : (0,1) \to \mathbb{R}$ is nondecreasing, since F_X^{-1} is a nondecreasing function. Assumption 4.2 and Assumption 5.1 also yield left-continuity of f_λ^*. Consequently, $f_\lambda^* \in \mathcal{Q}$, the set of all quantile functions.

Now, define the function f_λ^{**} by

$$f_\lambda^{**}(t) := \max\left[0, \min\left\{F_X^{-1}(t), f_\lambda^*(t)\right\}\right] \tag{5.6}$$

It is then easy to check that $f_\lambda^{**} \in \mathcal{Q}$, since $f_\lambda^* \in \mathcal{Q}$ and since F_X^{-1} is a nondecreasing function. Moreover, $0 \leqslant f_\lambda^{**}(z) \leqslant F_X^{-1}(z)$, for each $z \in (0,1)$. Therefore, $f_\lambda^{**} \in \mathcal{Q}^*$. Finally, it is easily seen that $f_\lambda^{**}(t)$ solves Problem (5.3) for the given λ and t, since the concavity of u yields the concavity of the function $z \mapsto u\Big(W_0 + z - F_X^{-1}(t)\Big) - \lambda T'(1-t)\, z$, for each $t \in (0,1)$. Hence, in view of Lemma 5.5, it remains to show that there exists a $\lambda^* \geqslant 0$ such that $\int_0^1 T'(1-t)\, f_{\lambda^*}^{**}(t)\, dt = R$. This is given by Lemma E.1 in Appendix E. □

6. Conclusion

In this paper I examined a general utility maximization problem, with a non-law-invariant indirect utility and a non-law-invariant risk measure constraint. This problem of demand for contingent claims is an abstraction of many common problems in economic theory that were hitherto only considered in a framework of complete homogeneity of beliefs about the realizations of an underlying uncertainty. The problem examined in this paper takes the following form:

$$\sup_{Y \in B(\Sigma)}\left\{\int \mathcal{U}(X,Y)\, dP \;\middle|\; 0 \leqslant Y \leqslant X,\ \rho(Y) \leqslant R\right\} \tag{6.1}$$

where X is a given underlying uncertainty on some state space S, $\Sigma = \sigma\{X\}$, $Y = I \circ X$ is a claim contingent on this uncertainty, $\int \mathcal{U}(X, I \circ X)\ dP$ is a DM's expected utility of wealth with respect to the probability measure P, $\rho : B^+(\Sigma) \to \mathbb{R}$ is a given risk measure, and $R \in \mathbb{R}$ is fixed. The set $B(\Sigma)$ is the linear space of all bounded and Σ-measurable functions on S.

Conditions on the primitives $\mathcal{U}$ and ρ for the optimal choice Y^* of Y to be monotone in X are desired. Using some quantile reformulation techniques inspired by the theory of monotone equimeasurable rearrangements, I showed that a set of sufficient conditions on $\mathcal{U}$ and ρ for this to hold is that the risk measure ρ be *Vigilant* and satisfy the *Weak DC-Property*, and that the function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ be *supermodular*. Roughly speaking, vigilance of the risk measure ρ can be understood as a (weak) preference for comonotonicity, on the collection of all functions that are identically distributed for the probability measure P. The Weak DC-Property is verified by a large class of operators on $B^+(\Sigma)$, such as the Lebesgue integral or the Choquet integral (Appendix A). For instance, any distortion risk measure of the form $\rho(Y) = \widehat{\int} Y\ dP$ is (P, X)-vigilant. In this case, I gave a characterization of an optimal contingent claim in terms of a quantile problem, in which the uncertainty may be seen as being homogeneous. In the special case of a convex distortion risk measure, a full explicit characterization of an optimal contingent claim was given.

Appendix A. Related Analysis

A.1. **A Useful Result.**

Lemma A.1. *If $(f_n)_n$ is a uniformly bounded sequence of nondecreasing real-valued functions on some closed interval $\mathcal{I}$ in $\mathbb{R}$, with bound N (i.e. $|f_n(x)| \leqslant N,\ \forall x \in \mathcal{I},\ \forall n \geqslant 1$), then there exists a nondecreasing real-valued bounded function f^* on $\mathcal{I}$, also with bound N, and a subsequence of $(f_n)_n$ that converges pointwise to f^* on $\mathcal{I}$.*

Proof. [11, Lemma 13.15]. □

A.2. **Capacities and the Choquet Integral.**

Definition A.2. A (normalized) *capacity* on a measurable space (S, Σ) is a set function $\nu : \Sigma \to [0,1]$ such that

(1) $\nu(\varnothing) = 0$;

(2) $\nu(S) = 1$; and,

(3) ν is monotone: for any $A, B \in \Sigma,\ A \subseteq B \Rightarrow \nu(A) \leqslant \nu(B)$.

The capacity ν is said to be

(1) *Continuous from above* if for any sequence $\{A_n\}_n$ in $\mathcal{G}$ such that $A_{n+1} \subseteq A_n$ for each $n \geqslant 1$, one has $\lim\limits_{n \to +\infty} \nu(A_n) = \nu\left(\bigcap_{n=1}^{+\infty} A_n\right)$.

(2) *Continuous from below* if for any sequence $\{A_n\}_n$ in $\mathcal{G}$ such that $A_n \subseteq A_{n+1}$ for each $n \geqslant 1$, one has $\lim\limits_{n \to +\infty} \nu(A_n) = \nu\left(\bigcup_{n=1}^{+\infty} A_n\right)$.

(3) *Continuous* if it is both continuous from above and continuous from below.

For instance, if P is a probability measure on (S, Σ) and $T : [0,1] \to [0,1]$ is increasing, with $T(0) = 0$ and $T(1) = 1$, then the set function $\nu := T \circ P$ is a capacity on (S, Σ). Such a function T is usually called a *probability distortion*, and the capacity $T \circ P$ is usually called a *distorted probability measure*. If, moreover, the function T is continuous, then the set function $\nu := T \circ P$ is a capacity on (S, Σ) which is continuous. This is an immediate consequence of the continuity of the measure P for monotone sequences [13, Prop. 1.2.3] and the continuity of T. In particular, any probability measure is continuous.

Definition A.3. For a given capacity ν on (S, Σ) and a given $\phi \in B(\Sigma)$, the *Choquet integral* of ϕ with respect to ν is defined by

$$\widehat{\int} \phi \, d\nu := \int_0^{+\infty} \nu\left(\{s \in S : \phi(s) \geqslant t\}\right) \, dt + \int_{-\infty}^{0} \left[\nu\left(\{s \in S : \phi(s) \geqslant t\}\right) - 1\right] \, dt$$

where the integrals are taken in the sense of Riemann.

The Choquet integral with respect to a measure is simply the usual Lebesgue integral with respect to that measure [39, p. 59]. Unlike the Lebesgue integral, however, the Choquet integral is not an additive operator on $B(\Sigma)$. However, the Choquet integral is additive on comonotonic functions (Definition 2.2).

Proposition A.4. *Let ν be a capacity on (S, Σ).*

(1) *If $\phi_1, \phi_2 \in B(\Sigma)$ are comonotonic, then $\widehat{\int} (\phi_1 + \phi_2) \, d\nu = \widehat{\int} \phi_1 \, d\nu + \widehat{\int} \phi_2 \, d\nu$.*

(2) *If $\phi \in B(\Sigma)$ and $c \in \mathbb{R}$, then $\widehat{\int} (\phi + c) \, d\nu = \widehat{\int} \phi \, d\nu + c$.*

(3) *If $A \in \Sigma$ then $\widehat{\int} \mathbf{1}_A \, d\nu = \nu(A)$.*

(4) *If $\phi \in B(\Sigma)$ and $a \geqslant 0$, then $\widehat{\int} a \, \phi \, d\nu = a \, \widehat{\int} \phi \, d\nu$.*

(5) *If $\phi_1, \phi_2 \in B(\Sigma)$ are such that $\phi_1 \leqslant \phi_2$, then $\widehat{\int} \phi_1 \, d\nu \leqslant \widehat{\int} \phi_2 \, d\nu$.*

For more about capacities and Choquet integrals, I refer to Marinacci and Montrucchio [39].

Appendix B. Equimeasurable Monotone Rearrangements and Supermodularity

The concept of an equimeasurable rearrangement of a Borel-measurable function on $\mathbb{R}$ with respect to a finite Borel measure, and the notion of an equimeasurable rearrangement of a measurable function f from a measurable space into $\mathbb{R}$ with respect to a finite Borel measure on the range of f is by now part of the classical literature [7, 12, 28, 38]. It is the basic tool that will be used to show the existence of a monotone solution to the DM's problem. Here, the idea of an equimeasurable rearrangement of a random variable with respect to another random variable is discussed. The nomenclature used has been chosen with the present context in mind, whereby the same measurable space may be endowed with different measures. In this section I introduce a specific formulation of the nondecreasing rearrangement of any element Y of $B^+(\Sigma)$ with respect to the fixed underlying uncertainty X. Although some of the results presented here are not new, the approach is novel, to the best of my knowledge. All proofs and some additional results may be found in Ghossoub [26] or Ghossoub [24]. The latter introduces the idea of a rearrangement in a context of *non-additive* probability measures (capacities).

B.1. **The Nondecreasing Rearrangement.** Let $(S, \mathcal{G}, P)$ be a probability space, and let $X \in B^+(\mathcal{G})$ be a continuous random variable (i.e. $P \circ X^{-1}$ is nonatomic) with range $[0, M] := X(S)$, where $M := \sup\{X(s) : s \in S\} < +\infty$, i.e. X is a mapping of S onto the closed interval $[0, M]$. Denote by Σ the σ-algebra $\sigma\{X\}$, and denote by ϕ the law of X defined by

$$\phi(B) := P\Big(\Big\{s \in S : X(s) \in B\Big\}\Big) = P \circ X^{-1}(B)$$

for any Borel subset B of $\mathbb{R}$.

If $I, I_n : [0, M] \to [0, M]$, for each $n \geqslant 1$, I will write $I_n \downarrow I$, ϕ-a.s., to signify that the sequence $\{I_n\}_n$ is a nonincreasing sequence of functions and that $\lim\limits_{n \to +\infty} I_n(t) = I(t)$, for ϕ-a.a. $t \in [0, M]$. Similarly, I will write $I_n \uparrow I$, ϕ-a.s., to signify that the sequence $\{I_n\}_n$ is a nondecreasing sequence of functions and that $\lim\limits_{n \to +\infty} I_n(t) = I(t)$, for ϕ-a.a. $t \in [0, M]$.

Definition B.1. For any Borel-measurable map $I : [0, M] \to \mathbb{R}$, define the *distribution function* of I as the map $\phi_I : \mathbb{R} \to [0, 1]$ defined by

$$\phi_I(t) := \phi\Big(\Big\{x \in [0, M] : I(x) \leqslant t\Big\}\Big) \tag{B.1}$$

Then ϕ_I is a nondecreasing right-continuous function, and the function $t \mapsto 1 - \phi_I(t)$ is called the *survival function* of I.

Definition B.2. Let $I : [0, M] \to [0, M]$ be any Borel-measurable map, and define the function $\widetilde{I} : [0, M] \to \mathbb{R}$ by

$$\widetilde{I}(t) := \inf\Big\{z \in \mathbb{R}^+ \;\Big|\; \phi_I(z) \geqslant \phi\big([0, t]\big)\Big\} \tag{B.2}$$

The following proposition gives some useful properties of the map $\widetilde{I}$ defined above.

Proposition B.3. *Let $I : [0, M] \to [0, M]$ be any Borel-measurable map and let $\widetilde{I} : [0, M] \to \mathbb{R}$ be defined as in equation (B.2). Then the following hold:*

1. *$\widetilde{I}$ is left-continuous, nondecreasing, and Borel-measurable;*
2. *For each $t \in [0, M]$, $\phi_I\left(\widetilde{I}(t)\right) \geqslant \phi([0, t])$;*
3. *$\widetilde{I}(t) \geqslant 0$, for each $t \in [0, M]$, $\widetilde{I}(0) = 0$, and $\widetilde{I}(M) \leqslant M$;*
4. *If $I_1, I_2 : [0, M] \to [0, M]$ are such that $I_1 \leqslant I_2$, ϕ-a.s., then $\widetilde{I}_1 \leqslant \widetilde{I}_2$;*
5. *If $Id : [0, M] \to [0, M]$ denotes the identity function, then $\widetilde{Id} \leqslant Id$;*
6. *$\widetilde{I}$ is ϕ-equimeasurable with I, in the sense that for any Borel set B,*

$$\phi\Big(\Big\{t \in [0, M] : I(t) \in B\Big\}\Big) = \phi\Big(\Big\{t \in [0, M] : \widetilde{I}(t) \in B\Big\}\Big) \tag{B.3}$$

7. *If $\overline{I} : [0, M] \to \mathbb{R}^+$ is another nondecreasing, Borel-measurable map which is ϕ-equimeasurable with I, then $\overline{I} = \widetilde{I}$, ϕ-a.s.;*

(8) *If* $I, I_n : [0, M] \to [0, M]$, *for each* $n \geqslant 1$, *and* $I_n \downarrow I$, ϕ*-a.s., then* $\widetilde{I}_n \downarrow \widetilde{I}$, ϕ*-a.s.*

$\widetilde{I}$ will be called the nondecreasing ϕ-rearrangement of I (see also [20, pp. 224-225]). Now, define $Y := I \circ X$ and $\widetilde{Y} := \widetilde{I} \circ X$. Since both I and $\widetilde{I}$ are Borel-measurable mappings of $[0, M]$ into itself, it follows that $Y, \widetilde{Y} \in B^+(\Sigma)$. Note also that $\widetilde{Y}$ is nondecreasing in X, in the sense that if $s_1, s_2 \in S$ are such that $X(s_1) \leqslant X(s_2)$ then $\widetilde{Y}(s_1) \leqslant \widetilde{Y}(s_2)$, and that Y and $\widetilde{Y}$ are P-equimeasurable, that is, for any $\alpha \in [0, M]$, $P\left(\left\{s \in S : Y(s) \leqslant \alpha\right\}\right) = P\left(\left\{s \in S : \widetilde{Y}(s) \leqslant \alpha\right\}\right)$.

Call $\widetilde{Y}$ a **nondecreasing P-rearrangement of Y with respect to** X, and denote it by $\widetilde{Y}_P$ to avoid confusion in case a different measure on $(S, \mathcal{G})$ is also considered. For example, in case both P_1 and P_2 are probability measures on the measurable space $(S, \mathcal{G})$, denote by $\widetilde{Y}_{P_1}$ (resp. $\widetilde{Y}_{P_2}$) a nondecreasing P_1-rearrangement (resp. P_2-rearrangement) of Y with respect to X. In the general case, nothing can be said *a priori* about the relationship between $\widetilde{Y}_{P_1}$ and $\widetilde{Y}_{P_2}$. What can be asserted, however, is that:

(1) Both $\widetilde{Y}_{P_1}$ and $\widetilde{Y}_{P_2}$ are nondecreasing in X, and hence $\widetilde{Y}_{P_1}$ and $\widetilde{Y}_{P_2}$ are comonotonic, i.e. $\left[\widetilde{Y}_{P_2}(s) - \widetilde{Y}_{P_2}(s')\right]\left[\widetilde{Y}_{P_1}(s) - \widetilde{Y}_{P_1}(s')\right] \geqslant 0$, for all $s, s' \in S$;

(2) Y and $\widetilde{Y}_{P_1}$ are P_1-equimeasurable; and,

(3) Y and $\widetilde{Y}_{P_2}$ are P_2-equimeasurable.

Note that $\widetilde{Y}_P$ is P-a.s. unique. Note also that if Y_1 and Y_2 are P-equimeasurable and if $Y_1 \in L_1(S, \mathcal{G}, P)$, then $Y_2 \in L_1(S, \mathcal{G}, P)$ and $\int \psi(Y_1)\ dP = \int \psi(Y_2)\ dP$, for any measurable function ψ such that the integrals exist.

Similarly to the previous construction, for a given a Borel-measurable $B \subseteq [0, M]$ with $\phi(B) > 0$, there exists a ϕ-a.s. unique (on B) nondecreasing, Borel-measurable mapping $\widetilde{I}_B : B \to [0, M]$ which is ϕ-equimeasurable with I on B, in the sense that for any $\alpha \in [0, M]$,

$$\phi\Big(\Big\{t \in B : I(t) \leqslant \alpha\Big\}\Big) = \phi\Big(\Big\{t \in B : \widetilde{I}_B(t) \leqslant \alpha\Big\}\Big)$$

$\widetilde{I}_B$ is called the nondecreasing ϕ-rearrangement of I on B. Since X is $\mathcal{G}$-measurable, there exists $A \in \mathcal{G}$ such that $A = X^{-1}(B)$, and hence $P(A) > 0$. Now, define $\widetilde{Y}_A := \widetilde{I}_B \circ X$. Since both I and $\widetilde{I}_B$ are bounded Borel-measurable mappings, it follows that $Y, \widetilde{Y}_A \in B^+(\Sigma)$. Note also that $\widetilde{Y}_A$ is nondecreasing in X on A, in the sense that if $s_1, s_2 \in A$ are such that $X(s_1) \leqslant X(s_2)$ then $\widetilde{Y}(s_1) \leqslant \widetilde{Y}(s_2)$, and that Y and $\widetilde{Y}_A$ are P-equimeasurable on A, that is, for any $\alpha \in [0, M]$, $P\left(\left\{s \in S : Y(s) \leqslant \alpha\right\} \cap A\right) = P\left(\left\{s \in S : \widetilde{Y}_A(s) \leqslant \alpha\right\} \cap A\right)$.

Call $\widetilde{Y}_A$ a **nondecreasing P-rearrangement of Y with respect to X on** A, and denote it by $\widetilde{Y}_{A,P}$ to avoid confusion in case a different measure on $(S, \mathcal{G})$ is also considered. For example, in case both P_1 and P_2 are probability measures on the measurable space $(S, \mathcal{G})$, denote by $\widetilde{Y}_{A,P_1}$ (resp. $\widetilde{Y}_{A,P_2}$) a nondecreasing P_1-rearrangement (resp. P_2-rearrangement) of Y with respect to X on A. In the general case, nothing can be said *a priori* about the relationship between $\widetilde{Y}_{A,P_1}$ and $\widetilde{Y}_{A,P_2}$. What can be asserted, however, is that:

(1) Both $\widetilde{Y}_{A,P_1}$ and $\widetilde{Y}_{A,P_2}$ are nondecreasing in X on A, and hence $\widetilde{Y}_{P_1}$ and $\widetilde{Y}_{P_2}$ are comonotonic on A, i.e. $\left[\widetilde{Y}_{A,P_2}(s) - \widetilde{Y}_{A,P_2}(s')\right]\left[\widetilde{Y}_{A,P_1}(s) - \widetilde{Y}_{A,P_1}(s')\right] \geqslant 0$, for all $s, s' \in A$;

(2) Y and $\widetilde{Y}_{A,P_1}$ are P_1-equimeasurable on A; and,

(3) Y and $\widetilde{Y}_{A,P_2}$ are P_2-equimeasurable on A.

Note that $\widetilde{Y}_{A,P}$ is P-a.s. unique. Note also that if $Y_{1,A}$ and $Y_{2,A}$ are P-equimeasurable on A and if $\int_A Y_{1,A}\, dP < +\infty$, then $\int_A Y_{2,A}\, dP < +\infty$ and $\int_A \psi(Y_{1,A})\, dP = \int_A \psi(Y_{2,A})\, dP$, for any measurable function ψ such that the integrals exist.

Lemma B.4. *Let $Y \in B^+(\Sigma)$ and let $A \in \mathcal{G}$ be such that $P(A) = 1$ and $X(A)$ is a Borel set*[10]*. Let $\widetilde{Y}_P$ be the nondecreasing P-rearrangement of Y with respect to X, and let $\widetilde{Y}_{A,P}$ be the nondecreasing P-rearrangement of Y with respect to X on A. Then $\widetilde{Y}_P = \widetilde{Y}_{A,P}$, P-a.s.*

B.2. **Supermodularity and Hardy-Littlewood-Pólya Inequalities.** A *partially ordered set* (*poset*) is a pair $(T, \succcurlyeq)$ where $\succcurlyeq$ is a reflexive, transitive and antisymmetric binary relation on T. A point $t \in T$ is called an *upper bound* (resp. *lower bound*) for a subset S of T if $t \succcurlyeq x$ (resp. $x \succcurlyeq t$) for each $x \in S$. A point $t^* \in T$ is called a *least upper bound* (resp. *greatest lower bound*) for S if it is an upper bound (resp. lower bound) for S and for any other upper bound (resp. lower bound) t of S one has $t \succcurlyeq t^*$ (resp. $t^* \succcurlyeq t$). It is easily seen that the least upper bound and the greatest lower bound are unique. For any $x, y \in S$ denote by $x \vee y$ (resp. $x \wedge y$) the least upper bound (resp. greatest lower bound) of the set $\{x, y\}$. A poset $(T, \succcurlyeq)$ is called a *lattice* when $x \vee y, x \wedge y \in T$, for each $x, y \in T$.

For instance, the Euclidian space $\mathbb{R}^n$ is a lattice for the partial order $\succcurlyeq$ defined as follows: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, write $x \succcurlyeq y$ when $x_i \geqslant y_i$, for each $i = 1, \ldots, n$. It is then easy to see that $x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$ and $x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$.

Definition B.5. Let $(T, \succcurlyeq)$ be a lattice. A function $L : T \to \mathbb{R}$ is said to be *supermodular* if for each $x, y \in T$,

$$L(x \vee y) + L(x \wedge y) \geqslant L(x) + L(y) \tag{B.4}$$

In particular, a function $L : \mathbb{R}^2 \to \mathbb{R}$ is *supermodular* if for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, one has

$$L(x_2, y_2) + L(x_1, y_1) \geqslant L(x_1, y_2) + L(x_2, y_1) \tag{B.5}$$

A function $L : \mathbb{R}^2 \to \mathbb{R}$ is called *strictly supermodular* if for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$, one has

$$L(x_2, y_2) + L(x_1, y_1) > L(x_1, y_2) + L(x_2, y_1) \tag{B.6}$$

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[10] Note that if $A \in \Sigma = \sigma\{X\}$ then $X(A)$ is automatically a Borel set, by definition of $\sigma\{X\}$. Indeed, for any $A \in \sigma\{X\}$, there is some Borel set B such that $A = X^{-1}(B)$. Then $X(A) = B \cap X(S)$ [17, p. 7]. Thus $X(A) = B \cap [0, M]$ is a Borel subset of $[0, M]$.

Lemma B.6. *A function $L : \mathbb{R}^2 \to \mathbb{R}$ is supermodular (resp. strictly supermodular) if and only if the function $\eta(y) := L(x+h, y) - L(x, y)$ is nondecreasing (resp. increasing) on $\mathbb{R}$, for any $x \in \mathbb{R}$ and $h \geqslant 0$ (resp. $h > 0$).*

Example B.7. The following are useful examples of supermodular functions:

(1) If $g : \mathbb{R} \to \mathbb{R}$ is concave, and $a \in \mathbb{R}$, then the function $L_1 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_1(x, y) = g(a - x + y)$ is supermodular. If, moreover, g is strictly concave then L_1 is strictly supermodular.

(2) If $f : \mathbb{R} \to \mathbb{R}$ is concave, and $a \in \mathbb{R}$, then the function $L_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_2(x, y) = f(a + x - y)$ is supermodular. If, moreover, f is strictly concave then L_2 is strictly supermodular.

(3) The function $L_3 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_3(x, y) = -(y - x)^+$ is supermodular.

(4) If $\psi, \phi : \mathbb{R} \to \mathbb{R}$ are both nonincreasing or both nondecreasing functions, then the function $L_4 : \mathbb{R}^2 \to \mathbb{R}$ defined by $L_4(x, y) = \phi(x)\ \psi(y)$ is supermodular.

Lemma B.8 (Hardy-Littlewood-Pólya Inequalities)**.** *Let $Y \in B^+(\Sigma)$ and let $A \in \mathcal{G}$ be such that $P(A) > 0$ and $X(A)$ is a Borel set. Let $\widetilde{Y}_P$ be the nondecreasing P-rearrangement of Y with respect to X, and let $\widetilde{Y}_{A,P}$ be the nondecreasing P-rearrangement of Y with respect to X on A. If L is supermodular, then:*

(1) $\int L\Big(X, Y\Big)\ dP \leqslant \int L\left(X, \widetilde{Y}_P\right)\ dP$, *and if L is strictly supermodular then equality holds if and only if $Y = \widetilde{Y}_P$, P-a.s., and,*

(2) $\int_A L\Big(X, Y\Big)\ dP \leqslant \int_A L\left(X, \widetilde{Y}_{A,P}\right)\ dP$,

provided the integrals exist (i.e. they are not of the form $\infty - \infty$).

Lemma B.9. *Let $Y \in B^+(\Sigma)$ and let $A \in \mathcal{G}$ be such that $P(A) > 0$ and $X(A)$ is a Borel set. Let $\widetilde{Y}_P$ be the nondecreasing P-rearrangement of Y with respect to X, and let $\widetilde{Y}_{A,P}$ be the nondecreasing P-rearrangement of Y with respect to X on A. Then the following hold:*

(1) *If $0 \leqslant Y \leqslant X$, P-a.s., then $0 \leqslant \widetilde{Y}_P \leqslant X$; and,*

(2) *If $0 \leqslant Y \leqslant X$, P-a.s. on A, then $0 \leqslant \widetilde{Y}_{A,P} \leqslant X$, P-a.s. on A.*

Appendix C. Proof of Theorem 3.1

First note that by the assumption that X is a continuous random variable for the probability measure P, it follows that the Borel probability measure $P \circ X^{-1}$ is nonatomic. Now, suppose that $\mathcal{H} = \Big\{Y \in B(\Sigma)\ \Big|\ 0 \leqslant Y \leqslant X \text{ and } \rho(Y) \leqslant R\Big\} \neq \varnothing$, ρ is (P, X)-vigilant and has the Weak DC-Property, the mapping $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}$ is supermodular, and the mapping $\mathcal{E} : B^+(\Sigma) \to B(\Sigma)$ defined by $\mathcal{E}(Y) = \mathcal{U}(X, Y)$ is uniformly bounded and sequentially continuous in the topology of pointwise convergence.

Lemma C.1. *For each $Y \in \mathcal{H}$ there is $Y^* \in \mathcal{H}$ such that:*

(1) *Y^* is comonotonic with X, i.e. Y^* is of the form $I^* \circ X$ where $I^* : [0, M] \to [0, M]$ is nondecreasing;*

(2) *$\int \mathcal{U}\Big(X, Y^*\Big)\ dP \geqslant \int \mathcal{U}\Big(X, Y\Big)\ dP$; and,*

(3) *$\rho(Y^*) \leqslant \rho(Y)$.*

Proof. By Assumption, $\mathcal{H} \neq \varnothing$. Choose any $Y = I \circ X \in \mathcal{H}$, and let $Y^* := \widetilde{Y}_P$, where $\widetilde{Y}_P$ denotes the nondecreasing P-rearrangement of Y with respect to X. Then (i) $Y^* = \widetilde{I} \circ X$ where $\widetilde{I} : [0, M] \to [0, M]$ is nondecreasing, bounded, and Borel-measurable; and, (ii) $0 \leqslant Y^* \leqslant X$, by Lemma B.9. Furthermore, since ρ is (P, X)-vigilant, it follows that $\rho(Y^*) \leqslant \rho(Y)$, by definition of (P, X)-vigilance. But $\rho(Y) \leqslant R$ since $Y \in \mathcal{H}$. Therefore, $\rho(Y^*) \leqslant R$. Thus, $Y^* \in \mathcal{H}$ is comonotonic with X. Moreover, since the function $\mathcal{U}$ is supermodular, it follows from Lemma B.8 that $\int \mathcal{U}\Big(X, Y^*\Big)\ dP \geqslant \int \mathcal{U}\Big(X, Y\Big)\ dP$. □

Now, let $\mathcal{H}^{\uparrow}$ denote the collection of all elements of $\mathcal{H}$ that are comonotonic with X. Then $\mathcal{H}^{\uparrow} \neq \varnothing$, by Lemma C.1. Also, by Lemma C.1, one can choose a maximizing sequence $\{Y_n\}_n$ in $\mathcal{H}^{\uparrow}$ for Problem (3.1). That is, $\lim\limits_{n \to +\infty} \int \mathcal{U}\left(X, Y_n\right)\ dP = N := \sup_{Y \in B^+(\Sigma)} \left\{\int \mathcal{U}\left(X, Y\right)\ dP\right\} < +\infty$. Since $0 \leqslant Y_n \leqslant X \leqslant M := \|X\|_{sup}$, the sequence $\{Y_n\}_n$ is uniformly bounded. Moreover, for each $n \geqslant 1$ one has $Y_n = I_n \circ X$, with $I_n : [0, M] \to [0, M]$. Consequently, the sequence $\{I_n\}_n$ is a uniformly bounded sequence of nondecreasing Borel-measurable functions. Thus, by Lemma A.1, there is a nondecreasing function $I^* : [0, M] \to [0, M]$ and a subsequence $\{I_m\}_m$ of $\{I_n\}_n$ such that $\{I_m\}_m$ converges pointwise on $[0, M]$ to I^*. Hence, I^* is also Borel-measurable, and so $Y^* := I^* \circ X \in B^+(\Sigma)$ is such that $0 \leqslant Y^* \leqslant X$. Moreover, the sequence $\{Y_m\}_m$, defined by $Y_m = I_m \circ X$, converges pointwise to Y^*. Thus, by the assumption that ρ has the *Weak DC-Property*, $Y^* \in \mathcal{H}^{\uparrow}$. Now, by the assumption of uniform boundedness and sequentially continuity of the map $\mathcal{U}(X, .)$ in the topology of pointwise convergence, and by Lebesgue's Dominated Convergence Theorem [1, Theorem 11.21] one has

$$\int \mathcal{U}\left(X, Y^*\right)\ dP = \lim_{m \to +\infty} \int \mathcal{U}\left(X, Y_m\right)\ dP = \lim_{n \to +\infty} \int \mathcal{U}\left(X, Y_n\right)\ dP = N$$

Hence Y^* solves Problem (3.1). The P-a.s. uniqueness of Y^* is a consequence of Proposition B.3. This concludes the proof of Theorem 3.1. □

Appendix D. Proof of Lemma 4.1

Let $\zeta := P \circ X^{-1}$ be the image measure of P under X. By the assumption that X is a continuous random variable for the probability measure P, the Borel probability measure ζ is nonatomic. One can then define the P-a.s. unique nondecreasing P-rearrangement of X with respect to X, as in Appendix B. Since X is a nondecreasing function of X and P-equimeasurable with X, it follows from the P-a.s. uniqueness of the equimeasurable nondecreasing P-rearrangement (see Proposition B.3) that $X = F_X^{-1}\left(F_X\left(X\right)\right)$, P-a.s. (see also [21, Lemma A.21]). Moreover, since $\zeta = P \circ X^{-1}$ is nonatomic, it follows that $U := F_X\left(X\right)$ has a uniform distribution over $(0, 1)$ [21, Lemma A.21], that is, $P\Big(\{s \in S : F_X\left(X\right)(s) \leqslant t\}\Big) = t$ for each $t \in (0, 1)$.

Now, let $Y \in \mathcal{H}$ be given, and let $\widetilde{Y}$ denote the nondecreasing P-rearrangement of Y with respect to X. Since $Y \in B^+(\Sigma)$, it can be written as $\phi \circ X$ for some nonnegative Borel-measurable and bounded map ϕ on $X(S)$. Moreover, since $0 \leqslant Y \leqslant X$, ϕ is a mapping of $[0, M]$ into $[0, M]$. Since, ζ is nonatomic (by assumption), one can define the mapping $\widetilde{\phi} : [0, M] \to [0, M]$ as in Appendix B (see equation (B.2) on p. 18) to be the nondecreasing ζ-rearrangement of ϕ, that is,

$$\widetilde{\phi}(t) := \inf \left\{ z \in \mathbb{R}^+ \;\middle|\; \zeta\big(\{x \in [0, M] : \phi(x) \leqslant z\}\big) \geqslant \zeta\big(\,[0, t]\,\big) \right\}$$

Then, as in Appendix B, $\widetilde{Y} = \widetilde{\phi} \circ X$. Therefore, for each $s_0 \in S$,

$$\widetilde{Y}(s_0) = \widetilde{\phi}(X(s_0)) = \inf \left\{ z \in \mathbb{R}^+ \;\middle|\; \zeta\big(\{x \in [0, M] : \phi(x) \leqslant z\}\big) \geqslant \zeta\big(\,[0, X(s_0)]\,\big) \right\}$$

However, for each $s_0 \in S$,

$$\zeta\big(\,[0, X(s_0)]\,\big) = P \circ X^{-1}\big(\,[0, X(s_0)]\,\big) = F_X(X(s_0)) := F_X(X)(s_0)$$

Moreover,

$$\begin{aligned} \zeta\big(\{x \in [0, M] : \phi(x) \leqslant z\}\big) &= P \circ X^{-1}\big(\{x \in [0, M] : \phi(x) \leqslant z\}\big) \\ &= P\big(\{s \in S : \phi(X(s)) \leqslant z\}\big) = F_Y(z) \end{aligned}$$

Consequently, for each $s_0 \in S$,

$$\widetilde{Y}(s_0) = \inf \left\{ z \in \mathbb{R}^+ \;\middle|\; F_Y(z) \geqslant F_X(X)(s_0) \right\} = F_Y^{-1}(F_X(X(s_0))) := F_Y^{-1}(F_X(X))(s_0)$$

That is,

$$\widetilde{Y} = F_Y^{-1}(F_X(X))$$

where F_Y^{-1} is the left-continuous inverse of F_Y, as defined in equation (4.1). Therefore, the function $Y^* = F_Y^{-1}(F_X(X))$ coincides with the function $\widetilde{Y}$ (he nondecreasing P-rearrangement of Y with respect to X).

Consequently, it follows that Y^* is comonotonic with X. Moreover, by Lemma B.9, it follows that $0 \leqslant Y^* \leqslant X$. Also, as in the proof of Lemma C.1, one has $\int \mathcal{U}\Big(X, Y^*\Big)\, dP \geqslant \int \mathcal{U}\Big(X, Y\Big)\, dP$ (by supermodularity of $\mathcal{U}$), $\rho(Y^*) \leqslant \rho(Y)$ (by vigilance of ρ), and $Y^* \in \mathcal{H}$. □

Appendix E. A Useful Lemma

Lemma E.1. *Suppose that Assumption 4.2, Assumption 5.1, Assumption 5.2, and Assumption 5.6 hold. Let ψ be the function of the parameter $\lambda \geqslant 0$ defined by*

$$\psi(\lambda) := \int_0^1 T'(1-t) \max\left[0, \min\left\{F_X^{-1}(t), (u')^{-1}\big(\lambda T'(1-t)\big) + F_X^{-1}(t) - W_0\right\}\right] dt$$

Then there exists a $\lambda^* \geqslant 0$ such that $\psi(\lambda^*) = R$.

Proof. Since X is bounded and since F_X^{-1} is nondecreasing, it follows that for each $t \in [0,1]$,

$$\min\left\{F_X^{-1}(t), (u')^{-1}\left(\lambda T'(1-t)\right) + F_X^{-1}(t) - W_0\right\} \leqslant F_X^{-1}(t) \leqslant F_X^{-1}(1) \leqslant \|X\|_{sup} < +\infty.$$

Moreover, since T is convex and increasing, T' is nondecreasing and nonnegative, and so for each $t \in [0,1]$, $0 \leqslant T'(1-t) \leqslant T'(1)$. But $T'(1) < +\infty$, by Assumption 5.6. Hence, for each $t \in [0,1]$,

$$\min\left\{F_X^{-1}(t), (u')^{-1}\left(\lambda T'(1-t)\right) + F_X^{-1}(t) - W_0\right\} T'(1-t) \leqslant F_X^{-1}(1)\, T'(1) < +\infty$$

Hence, Lebesgue's Dominated Convergence Theorem [1, Theorem 11.21] implies that ψ is a continuous function of λ. Moreover, ψ is a noincreasing function of λ (by concavity of u, convexity of T, and monotonicity of the Lebesgue integral).

Now, Assumption 5.1 implies that

$$\psi(0) = \widehat{\int} X \; dT \circ P = \int_0^1 T'(1-t)\, F_X^{-1}(t) \; dt$$

and that

$$\lim_{\lambda \to +\infty} \psi(\lambda) = \int_0^1 T'(1-t) \max\left[0, \min\left\{F_X^{-1}(t), F_X^{-1}(t) - W_0\right\}\right] dt$$
$$= \int_0^1 T'(1-t) \max\left[0, F_X^{-1}(t) - W_0\right] dt$$

Furthermore, by Assumption 5.2, $F_X(W_0) = 1$. Therefore, for all $t \in (0,1)$, $F_X^{-1}(t) \leqslant W_0$, and so

$$\lim_{\lambda \to +\infty} \psi(\lambda) = 0$$

Consequently (recall Remark 5.3),

$$0 = \lim_{\lambda \to +\infty} \psi(\lambda) \leqslant R \leqslant \widehat{\int} X \; dT \circ P = \int_0^1 T'(1-t)\, F_X^{-1}(t) \; dt = \psi(0)$$

Hence, by the Intermediate Value Theorem [50, Theorem 4.23], there exists a $\lambda^* \geqslant 0$ such that $\psi(\lambda^*) = R$. ☐

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