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CHEATPROOFNESS PROPERTIES OF THE PLURALITY RULE IN LARGE SOCIETIES

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I. INTRODUCTION

Vickrey [11] and Dummet and Farquharson [2] conjectured and Gibbard [5], Satterthwaite [9] and Schmeidler and Sonnenschein [10] proved that when the number of social alternatives is at least three, any non-imposed and non-dictatorial voting scheme is manipulable (in the sense of it being profitable for some voter at some profile to misrepresent his preferences in order to secure a social outcome preferred by him to that resulting in the event his vote reflects his true preferences). This manipulability (or noncheatproofness) result was obtained by the above mentioned authors under (the implicit) assumption that the number of individuals (voters) is finite. In the case of an infinite set of individuals the impossibility of a cheatproof social choice function no longer holds. This is shown in [8].

The concept of individual cheatproofness can be extended quite naturally to the notion of coalitional cheatproofness: Let $\mathcal{A}$ be a finite set of at least three distinct objects, called alternatives, and let Σ be the set of total, transitive, asymmetric (preference) orderings over $\mathcal{A}$. Let V be a non-empty set of individuals. The elements in Σ^V are referred to as preference profiles. A social choice function (SCF) is a function $f\colon \Sigma^V \to \mathcal{A}$. An SCF f is coalitionally manipulable if for some preference profile $p = \{p_i\}_{i \in V}$ in Σ^V and some (non-empty) coalition $A \subset V$ there exists a preference profile

$p' = \{p_i'\}_{i \in V}$ in Σ^V such that $p_j' = p_j$ for $j \notin A$ and $f(p')p_i f(p)$ for every $i \in A$. If f is not coalitionally manipulable, f is said to be coalitionally cheatproof.

In the case where V, the set of voters, is infinite it is shown in Pazner and Wesley [8] that there exists a coalitionally cheatproof non-imposed and non-dictatorial SCF Yet, while the existence of a coalitionally cheatproof social choice function is rigorously proven in [8], the essentially non-constructive method of proof used there does not make it possible to actually present any concrete example of such a cheatproof method of social choice.

In this paper, we turn to the constructive aspects of the problem of designing a coalitionally cheatproof social choice function for large societies. One would like, if possible, to exhibit a coalitionally cheatproof, non-imposed, non-dictatorial SCF in some explicit fashion.[1/] It appears, however, that this cannot be done. If $V = \{1, 2, \ldots\}$ and $\mathcal{A}$ has at least three alternatives, then the existence of an SCF $f\colon \Sigma^V \to \mathcal{A}$ having these properties cannot be proven unless one uses some form of the axiom of choice.[2/] Since every explicitly describable function is, in all likelihood, "constructible" within ZF (the Zermelo-Fraenkel axiom of set theory excluding the axiom of choice), a plausible conclusion is that no "definable" SCF exists bearing these properties.

[1/] This, problem does not arise if only individual cheatproofness is required; for an explicit example, see [8].

[2/] If $F\colon \Sigma^V \to \mathcal{A}$ is a coalitionally cheatproof, non-dictatorial, non-imposed SCF, then using F one can construct a non-principal ultrafilter over V (see Appendix). It is shown in [3], however, that the existence of a non-principal ultra-filter over V cannot be proven in ZF alone. (ZF $\equiv$ the Zermelo-Fraenkel axioms of set theory without the axiom of choice.)

The object of this work is to consider constructibly definable SCF's which, in some sense, very nearly satisfy the condition (coalitionally cheatproof, non-imposed, non-dictatorial). The social choice functions we consider are essentially variations of the plurality rule. The paper is divided into two parts. Section II deals with individual cheatproofness. We produce a limit theorem which asserts that when the number of voters is large (but finite) the plurality rule is individually cheatproof most of the time. In Section III, coalitional cheatproofness properties of a variant of the plurality rule is studied in the case where V, the set of voters, is countably infinite. An Appendix in which the nonconstructive character of the results in [8] are elaborated upon concludes the paper.

II. A LIMIT THEOREM ON THE PLURALITY RULE

Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ be a set of alternatives and let $V_n = \{v_1, \ldots, v_n\}$ be a set of individuals. Let Σ denote the set of all strong orderings (i.e., the set of total, asymmetric and transitive binary relations) on $\mathcal{A}$. Each element $p = (p_1, \ldots, p_n)$ in Σ^{V_n} (the set of functions from V_n to Σ) where $p_i \in \Sigma$ for all i, $1 \leqq i \leqq n$, is called a preference profile. A function f from Σ^{V_n} to $\mathcal{A}$ is called a social choice function (SCF). An SCF f is said to be individually manipulable at $p = (p_1, \ldots, p_n) \in \Sigma^{V_n}$ if there exist $p_i' \in \Sigma$ and an i_o, $1 \leq i_o \leq n$ such that $f(p_1', p_2', \ldots, p_n') p_{i_o} f(p_1, p_2, \ldots, p_n)$ where $p'_i = p_i$ for all $i \neq i_o$. f is individually cheatproof at p if it is not individually manipulable at that profile.

For every $p = (p_1, \ldots, p_n)$ in Σ^{V_n} and every i, $1 \leq i \leq k$, let $C(p,i) = \{v_j | v_j \in V_n$ and $a_i p_j a_\ell$ for all a_ℓ in $\mathcal{A}$ such that $a_\ell \neq a_i\}$, i.e. $C(p,i)$ is the set of individuals who most prefer a_i under the profile p. Let $|C(p,i)|$ be the number of individuals in $C(p,i)$. In conformance with this notation, let $|V_n|$ be the number of individuals in V_n, i.e. $|V_n| = n$. We define the plurality rule $F: \Sigma^{V_n} \to \mathcal{A}$ as follows:

Let $p = (p_1, \ldots, p_n)$ be a preference profile in Σ^{V_n}. If for some i, $1 \leq i \leq k$, $|C(p,i)| > |C(p,\ell)|$ for all ℓ, $1 \leq \ell \leq k$, $\ell \neq i$, then let $F(p) = a_i$. If not, let $F(p) = a_{j_1}$, where j_1 is the smallest index such that $|C(p,j_1)| \geq |C(p,\ell)|$ for all ℓ, $1 \leq \ell \leq k$.

Let $\mathcal{D}_n$ be the set of preference profiles in Σ^{V_n} for which F is individually cheatproof. Let $|\mathcal{D}_n|$, $|\Sigma^{V_n}|$ be the number of elements in $\mathcal{D}_n$ and Σ^{V_n}, respectively.

THEOREM 1:

$$\lim_{n\to\infty} \frac{|\mathcal{D}_n|}{|\Sigma^{V_n}|} = 1$$

To prove the theorem, two auxiliary lemmas will be utilized. In order to formulate them, we introduce some additional notation.

For any $\epsilon > 0$ and any natural number n, and any i,j $1 \leq i \leq k$, $1 \leq j \leq k$, $i \neq j$, let

$$T(n,\epsilon,i,j) = \{p \mid p \in \Sigma^{V_n}, \frac{|C(p,i) - C(p,j)|}{\sqrt{\frac{2n}{k}}} < \epsilon\}.$$

Let $|T(n,\epsilon,i,j)|$ be the number of preference profiles in $T(n,\epsilon,i,j)$. Then

LEMMA 1:

$$\lim_{n\to\infty} \frac{|T(n,\epsilon,i,j)|}{|\Sigma^{V_n}|} \leq \frac{2\epsilon}{\sqrt{2\pi}}$$

for any i,j, $1 \leq i \leq k$, $1 \leq j \leq k$, $i \neq j$.

PROOF: In proving the Lemma we make use of the central limit theorem in probability theory ([6], p. 290). We assume that a probability measure P is defined over Σ so that all preference orders in Σ are equally likely to occur when random choices are made[3/] (i.e. we assume that all possible profiles are equally probable for the society under consideration).

Suppose that each $v_m \in V_n$ randomly chooses a preference ordering p_m in accordance with the probability measure P. A randomly selected preference profile $p = (p_1, \ldots, p_n)$ is thereby obtained. The probabilities are then such that given any $a_{i_1}, a_{i_2} \in \mathcal{A}$, where $a_{i_1} \neq a_{i_2}$, we have $P(a_{i_1} p_m a_{i_2}) = P(a_{i_2} p_m a_{i_1})$. Let i, j, $i \neq j$, be any fixed natural number between 1 and k, inclusives. Define in the following manner the random variables $\xi_1, \xi_2, \ldots$ over the set of infinite sequences of preference orders:

$$\xi_\ell \overset{\text{def}}{=} \xi_\ell(p_1, p_2, \ldots) = \begin{cases} 1 & \text{if under } p_\ell,\ a_i \text{ is preferred over all other elements in } \mathcal{A} \\ -1 & \text{if under } p_\ell,\ a_j \text{ is preferred over all other elements in } \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(II.1)}$$

Each ξ_ℓ then depends only on the ℓ-th element p_ℓ of the infinite sequence, i.e. $\xi_\ell(p) = \bar{\xi}_\ell(p_\ell)$ where $\bar{\xi}_\ell$ is defined in obvious fashion. It is thus easily seen that

$$\{p = (p_1, \ldots, p_n) \mid p \in \Sigma^{V_n},\ \frac{\left|\sum_{\ell=1}^{n} \bar{\xi}_\ell(p_\ell)\right|}{\sqrt{\frac{2n}{k}}} < \epsilon\} = T(n, \epsilon, i, j)$$

and that consequently

[3/] Note that throughout, the lower-case p and p_i stand for preference profiles and orderings respectively while capital P denotes the probability measure.

$$P\left(\frac{|\sum_{\ell=1}^{n}\xi_\ell|}{\sqrt{\frac{2n}{k}}} < \epsilon\right) = \frac{|T(n,\epsilon,i,j)|}{|\Sigma^{V_n}|}$$

Therefore, it is sufficient to prove that

$$\lim_{n\to\infty} P\left(\frac{|\sum_{\ell=1}^{n}\xi_\ell|}{\sqrt{\frac{2n}{k}}} < \epsilon\right) \leq \frac{2\epsilon}{\sqrt{2\pi}} \tag{II.2}$$

For each random variable ξ_ℓ, let $\bar{a}_\ell = E(\xi_\ell)$, the expectation of ξ_ℓ. Then $\bar{a}_\ell = 0$ for all ℓ. Let $\bar{b}_\ell^2 = E(\xi_\ell - E(\xi_\ell))^2 = \frac{2}{k}$. Let $B_n^2 = \sum_{\ell=1}^{n} \bar{b}_\ell^2 = \frac{2n}{k}$.

Let F_ℓ be the distribution function of the random variable ξ_ℓ. Then for any $\tau > 0$

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{\ell=1}^{n} \int_{|x - \bar{a}_\ell| > \tau B_n} (x - \bar{a})^2 dF_\ell(x) = \lim_{n\to\infty} \frac{k}{2n} \sum_{\ell=1}^{n} \int_{x > \tau B_n} x^2 dF_\ell(x) = 0$$

Hence, the Lindeberg condition (Gnedenko [6], p. 289) is satisfied. Then by the central limit theorem (Gnedenko [6], p. 290), as $n \to \infty$

$$P\left\{\frac{1}{B_n} \sum_{k=1}^{n} (\xi_k - \bar{a}_k) < x\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$$

uniformly in x. Thus

$$P\left(-\epsilon < \frac{1}{B_n} \sum_{\ell=1}^{n} \xi_\ell < \epsilon\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\epsilon}^{\epsilon} e^{\frac{-z^2}{2}} dz < \frac{2\epsilon}{\sqrt{2\pi}}$$

or $\lim_{n\to\infty} P\left(\frac{|\Sigma\xi_\ell|}{\sqrt{\frac{2n}{k}}} < \epsilon\right) \leq \frac{2\epsilon}{\sqrt{2\pi}}$, from which the truth of the lemma follows.

LEMMA 2 Given any $\epsilon > 0$ and any positive integer n, let $T(n,\epsilon) = \bigcup_{\substack{1\leq i,j\leq k \\ i\neq j}} T(n,\epsilon,i,j)$. Then $\lim_{n\to\infty} \frac{|T(n,\epsilon)|}{|\Sigma^{V_n}|} \leq k^2 \frac{2\epsilon}{\sqrt{2\pi}}$

PROOF: Let $P(T(n,\varepsilon))$ be the probability that a randomly chosen preference profile in Σ^{V_n} occur in $T(n,\varepsilon)$. Then $P(T(n,\varepsilon)) = \frac{|T(n,\varepsilon)|}{|\Sigma^{V_n}|}$. However,

$$P(T(n,\varepsilon)) = P(\bigcup_{\substack{1\leq i,j\leq k \\ i\neq j}} T(n,\varepsilon,i,j)) \leq \sum_{\substack{1\leq i,j\leq k \\ i\neq j}} P(T(n,\varepsilon,i,j))$$

which for sufficiently large n is equal or less than $k^2 \frac{2\varepsilon}{\sqrt{2\pi}}$.

Then $\lim_{n\to\infty} \frac{|T(n,\varepsilon)|}{|\Sigma^{V_n}|} = \lim_{n\to\infty} P(T(n,\varepsilon)) \leq k^2 \frac{2\varepsilon}{\sqrt{2\pi}}$,

which completes the proof of the lemma.

Now, consider $\Sigma^{V_n} \setminus T(n,\varepsilon)$. It follows from Lemma 2 that

$$\lim_{n\to\infty} \frac{|\Sigma^{V_n} \setminus T(n,\varepsilon)|}{|\Sigma^{V_n}|} \geq 1 - \frac{k^2 2\varepsilon}{\sqrt{2\pi}} .$$

However, given any fixed $\varepsilon > 0$, for all sufficiently large n

$$\Sigma^{V_n} \setminus T(n,\varepsilon) \subset \mathcal{D}_n .$$

Thus, given any fixed $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\mathcal{D}_n|}{|\Sigma^{V_n}|} \geq 1 - \frac{k^2 2\varepsilon}{\sqrt{2\pi}} ,$$ from which the theorem follows.

Theorem 1 proved here indicates that the plurality rule (cum a reasonable tie-breaking device) is approximately cheatproof in large finite societies. More exactly, we have shown that as the number of individuals tends to infinity the proportion of profiles at which the plurality rule is individually cheatproof tends to one. This means that the issue of preference misrepresentation by any single voter in a large society can be ignored for all practical purposes when social choices are made according to the plurality rule. This of course should have been expected on purely intuitive grounds as any isolated individual does not really count when society is sizeable. Somewhat less intuitive is Theorem 2 in the next section in which it is stated that a variant of the plurality rule is almost coalitionally cheatproof in societies in which there is a countable infinity of voters.

III. THE PLURALITY RULE IN THE INFINITE CASE

Let $V = \{1, 2, \ldots\}$ be a countably infinite set of voters, $\mathcal{A} = \{a_1, \ldots, a_m\}$ a finite set of alternatives, Σ the set of total, transitive, asymmetric (preference) orders over $\mathcal{A}$. The elements $p = (p_1, p_2, \ldots) \in \Sigma^V$ are referred to as preference profiles. We would like to define a social choice function $f\colon \Sigma^V \to \mathcal{A}$ which chooses the alternative favored by a "plurality" of the voters. Since there are likely to be an infinite set of voters favoring each alternative we cannot determine plurality simply be counting. One approach we could try is to consider limits of some normalized quantity. For any integer $1 \leq n \leq \infty$ and any natural i, $1 \leq i \leq m$ and any $p \in \Sigma^V$, let

$$C_n(p,i) = \{j \mid 1 \leq j \leq n \text{ and } a_i p_j a_k \text{ for all } a_k \neq a_i,\ a_k \in \mathcal{A}\}.$$

Let $|C_n(p,i)|$ be the number of j's in $C_n(p,i)$.

A possible method for defining a plurality rule might be to consider the normalized limits

$$\lim_{n\to\infty} \frac{|C_n(p,1)|}{n},\ \lim_{n\to\infty} \frac{|C_n(p,2)|}{n},\ \ldots,\ \lim_{n\to\infty} \frac{|C_n(p,m)|}{n}$$

and to choose the k, $1 \leq k \leq m$ for which $\lim_{n\to\infty} \dfrac{|C_n(p,k)|}{n}$

is largest, whereupon the plurality is ascribed to a_k. However, this rule is usually not practicable.If one assumes that each voter is just as likely to prefer a_{j_1} to a_{j_2} as a_{j_2} to a_{j_1} and that these probabilities are independent of the choices of the remaining voters, then for each $1 \leq j \leq m$, the probability that voter i prefers a_j over all other alternatives is equal to $\frac{1}{m}$. Thus the limits

$$\lim_{n\to\infty} \frac{|C_n(p,k)|}{n} \qquad 1 \leq k \leq m$$

are with probability 1 all equal to $\frac{1}{m}$. There is thus virtual certainity of a tie. One might therefore try this type of approach but with a different normalization.

For each n, j, where $1 \leq n \leq \infty$, $1 \leq j \leq m$, and any $p \in \Sigma^V$ let

$$X_j^{(n)} = X_j^{(n)}(p) = \frac{|C_n(p,j)| - \frac{n}{m}}{\sqrt{n(\frac{1}{m})(\frac{m-1}{m})}} .$$

$X_j^{(n)}$ is thus a random-variable dependent on p. Under the foregoing probability assumption one can prove (using the de Moivre -Laplace integral theorem ([6], p. 93)) that

$$P(\{a \leq X_j^{(n)}(p) \leq b\}) \sim \int_a^b e^{\frac{-z^2}{2}} dz$$

i.e. $X_j^{(n)}$ is very nearly normally distributed. Similarly, one may also prove that $P(\{X_{j_1}^{(n)}(p) = X_{j_2}^{(n)}(p)\}) \sim 0$ for any j_1, j_2, where $0 \leq j_1 < j_2 \leq m$. One might then define $X_j(p)$ as

$$X_j(p) = \lim_{n\to\infty} X_j^{(n)}$$

if this limit exists; the plurality would then be ascribed to that a_k for which $X_k(p)$ is largest. However, it is not difficult to show that for almost all $p \in \Sigma^V$, the limit $\lim_{n\to\infty} X_j^{(n)} = \lim_{n\to\infty} \frac{|C_n(p,j)| - \frac{n}{m}}{\sqrt{n(\frac{1}{m})(\frac{m-1}{m})}}$

does not exist. Thus the problem of defining a suitable plurality rule when V is infinite is apparently insurmountable.

We are therefore led to settle for a social choice function which falls short of being a plurality rule in the strict sense. We consider the following SCF, $F: \Sigma^V \to \mathcal{A}$: For any $p \in \Sigma^V$, if $\lim_{n\to\infty} \frac{|C_n(p,j)|}{n}$ exists for all j, $1 \leq j \leq m$ and if for some k, $1 \leq k \leq m$

$$\lim_{n\to\infty} \frac{|C_n(p,k)|}{n} > \lim_{n\to\infty} \frac{|C_n(p,\ell)|}{n}$$

for all $\ell \neq k$, $1 \leq \ell \leq m$, then let $F(p) = a_k$. Otherwise, let $F(p) = a_1$. We note that for almost all p in Σ^V, $F(p) = a_1$. Thus F is very nearly an imposed regime. In those rare instances where a significant proportion of the population is unified against the regime's policy, the regime yields (the alternative a_1 may be thought to be the status-quo).

In presenting this SCF, our purpose is not to assert that it is an ideal one. Our contention, rather, is that in spite of the fact that the SCF, F, is less than ideal, it nevertheless illustrates the possibility of a workable democracy since it is in some sense coalitionally cheatproof and hence conducive to social stability (without being fully imposed or dictatorial).

It is not difficult to show that F is not coalitionally cheatproof in the strict sense. We shall claim, however, that for almost every profile $p \in \Sigma^V$, $F(p)$ is coalitionally cheatproof for practically all of the important coalitions. By important coalitions we mean the following:

In dealing with large populations, the coalitions of significance are usually definable by some simple phrase in the English language, eg. upper middle class, New Yorkers, coal miners, etc. Coalitions that are not easily definable are likely to be too complicated to be formed. If we carry this principle over to the case of an infinite population

$V = \{1, 2, \ldots\}$, we would conclude that the coalitions of significance are those that may be defined by a phrase in the English language, eg. the set of even numbers, the set of primes less than 1500, etc. Although there are $2^{\aleph_0}$ subsets of V, only $\aleph_0$ are definable by sentences of finite length. Thus a substantial restriction is imposed on the coalitions of interest.

We would like to assert the following: If $\hat{\mathcal{A}}$ is the set of coalitions definable in the English language, then for almost all p in Σ, F is coalitionally cheatproof for all $A \in \hat{\mathcal{A}}$. Although there seems to be no reason why such a result cannot be obtained, it is somewhat difficult to formulate this mathematically; we know of no adequate mathematical description of what constitutes a set which is definable in English. (What is and and is not English?) We would prefer instead to consider a more mathematically precise language -- the language of set theory. This language possesses a very small vocabulary -- the symbols $\in$, $\equiv$, $\forall$, $\exists$, $\sim$(not), $\wedge$ (and), $\vee$ (or), and some rigidly defined syntactical rules for forming sentences. In spite of its meagerness, the language is nevertheless sufficiently powerful to formulate all of the familiar concepts in classical analysis -- limits, irrationals, π, e, Bessel functions, etc. If we assume that classical concepts are adequate tools for forming models of social behavior, we would conclude that the language of set theory is also satisfactory for this purpose. Thus in many situations the assumption that the admissible coalitions are those that can be described in the language of set theory seems to be quite acceptable. Let $\mathcal{A}$ be the set of coalitions (subsets of V) that are describable in the language of set theory. We then have the following result:

THEOREM 2: For almost all $p \in \Sigma^V$, the SCF $F:\Sigma^V \to \mathcal{Q}$ described above is cheat-proof for all coalitions in $\mathcal{A}$[4/].

Proof: We make use of the fact that $\mathcal{A}$ is countably infinite.

Let $\bar{\mathcal{A}} = \{A_1, A_2, \ldots,\}$ be a sequence (an enumeration) of infinite coalitions in $\mathcal{A}$, whereby every infinite subset of V is indexed in the sequence. For every natural k let

$Q_k = \{p | p = \{p_1, p_2, \ldots,\} \in \Sigma^V$ and for every natural j between 1 and m (where m is the number of alternatives in $\mathcal{O}$ there are an infinite number of i's in A_k who favor a_j (under p) above all other alternatives.$\}$

By virtue of the Borel - Cantelli lemma ([6], p. 247) we find that the probability that a randomly selected profile $p \in \Sigma^V$ will belong to Q_k is 1, given any natural number k. Thus if we let $Q = \bigcap_{k=1}^{\infty} Q_k$, it follows that Prob $(\{p | p \in Q\})$, the probability that a randomly selected $p \in \Sigma^V$ will belong to Q, is also equal to 1. Let

$R \overset{def}{\equiv} \{p | p = (p_1, p_2, \ldots) \in \Sigma^V$ and for every natural j between 1 and m, $\lim_{n \to \infty} \frac{|C_n(p,j)|}{n}$ exists and is equal to $\frac{1}{m}\}$.

As a result of the strong law of large numbers, Prob $(\{p | p \in R\}) = 1$. Letting $S = R \cap Q$, we receive that Prob $(\{p | p \in S\}) = 1$. Clearly $F(p) = a_1$ for every $p \in S$. Moreover, given any $p \in S$, F(p) is coalitionally cheatproof with respect to every coalition $A \in \mathcal{A}$. This is because for any $A \in \mathcal{A}$ and all $p \in S$, an infinite number of voters in A will prefer a_1 to all other alternatives.

4/ By this we mean: For almost every $p \in \Sigma^V$, no $A \in \mathcal{A}$ exists such that for some $p' = (p'_1, p'_2, \ldots,) \in \Sigma^V$, where $p'_i = p_i$ for $i \notin A$, $F(p') p_j F(p)$ for all $j \in A$.

It should be noted that there are coalitions of interest which are not included in $\mathcal{A}$. For example, consider an arbitrary preference profile $p = (p_1, p_2, \ldots) \in \Sigma^V$ and let $A_p^{a_2}$ be the set of voters who most prefer a_2 under the profile p, i.e. $A_p^{a_2} = \{j | a_2 p_j a_i$ for all $a_i \neq a_2$, $a_i \in A\}$. For almost all p in Σ^V, $A_p^{a_2} \notin \mathcal{A}$. We can argue, however, that in many practical cases, coalitions like $A_p^{a_2}$ cannot actually arise. Completely free communication among the voters with regard to their true preferences would be required in order for this group to form. When existing conditions do not permit free communication, these coalitions are not likely to arise. In such situations, the formable coalitions are only those belonging to $\mathcal{A}$.

APPENDIX

Let $V = \{1,2,\ldots,\}$ be the set of natural numbers. A family $\mathcal{J}$ of subsets of V is called a filter if (1) $V \in \mathcal{J}$, $\emptyset \notin A$, (2) for every $A_1, A_2 \in \mathcal{J}$, $A_1 \cap A_2 \in A$, and (3) for any $A, B \subset V$, if $A \in \mathcal{J}$ and $B \supset A$ then $B \in \mathcal{J}$. A filter $\mathcal{J}$ is called on ultra-filter if for any $A \subset V$, either A or its complement belong to $\mathcal{J}$. $\mathcal{J}$ is a non-principal filter if for every $i \in V$, the set $\{i\}$ does not belong to $\mathcal{J}$.

The existence of a non-principal ultra-filter over V may be proven by means of Zorn's lemma ([1] , p. 61). It is also known that any existence proof for non-principal ultra-filter over V must use some form of the axiom of choice [5/]. A reasonable conclusion is that, although non-principal ultra-filters over V exist, no non-principal ultra-filter over V can be defined in any explicit manner.

Let $\mathcal{A}$ be a finite set of at least three distinct alternatives and let Σ be the set of complete, transitive, assymetric orderings over A. It has been proven by Pazner and Wesley in [8] that there exists a coalitionally cheat-proof, non-dictatorial and non-imposed SCF $f:\Sigma^V \to \mathcal{A}$ Like Fishburn's [4] result (in which the possibility of an Arrovian social welfare function is shown in the case of an infinite set of individuals) the existence proof depicted by Pazner and Wesley utilitzes ultra-filters and is non-constructive in character. A natural question is whether the social choice function, whose existence is proven in [8], can be exhibited in some specific fashion. In view of the equivalence (which

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5/ It is shown in [3] that ZF alone (the axioms of Zermelo-Fraenkel set theory excluding the axiom of choice) is not sufficient to prove the existence of a non-principal ultra-filter over V.

we now demonstrate) between the existence of ultra-filters and the existence of a coalitionally cheat-proof non-imposed, non-dictatorial SCF, the answer is apparently no.

It is shown in [8] that the existence of ultra-filters implies the existence of a coalitionally cheat-proof, non-imposed, non-dictatorial SCF $f:\Sigma^V \to \mathcal{A}$. To establish equivalence, the converse must also be proven. We do so in the following theorem:

<u>Theorem</u>[6/]: Let $f:\Sigma^V \to \mathcal{A}$ be a coalitionally cheat-proof, non-dictatorial, non-imposed social choice function. Using f, one can construct a non-principal ultra-filter $\mathcal{U}$ over V.

<u>Proof</u>: Let A be any subset of V and suppose that the members of A and A^{comp} (the complement of A) vote in block. The effect is the same as if there were a population of two voters only. Thus the Gibbard-Satterthwaite- result (which asserts that no cheat-proof non-dictatorial, non-imposed SCF exists when the population is finite) can be applied. Since f is non-imposed and cheat-proof, one of the two coalitions must dictate. Without loss in generality assume that A dictates whenever A and A^{comp} each vote in block. We then assert that even if the members of A^{comp} do not vote in block, A dictates. Otherwise coalitional cheat proofness would be violated. To see this, suppose by contradiction that there is a profile $\overline{p} = \{\overline{p}_i\}_{i \in V}$ and a preference order $p \in \Sigma$ with maximal element $a \in A$ such that for all $i \in A$ $\overline{p}_i = p$ while $f(\overline{p}) = b \neq a$. Suppose further that the "true" preferences of the members of A^{comp} are identical and that they all least prefer a. If the members of A^{comp} vote according to their

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6/ This theorem was adapted from a similar result appearing in [7].

true preferences while the members of A vote according to the preference p, then under the resulting profile $p' = \{p_i'\}_{i \in V}$, $f(p') = a$. If on the other hand, members of A^{comp} vote according to $\{\overline{p}_i\}_{i \in A^{comp}}$, then under the resulting profile, $\overline{p}$, $f(\overline{p}) = b$. Thus $f(\overline{p})\ p_i'\ f(p')$ for all $i \in A^{comp}$, in violation of coalitional cheatproofness.

Let $\mathcal{U}$ be the set of coalitions which dictate. Then it is easily seen that $V \in \mathcal{U}$, and for any $A_1, A_2 \in \mathcal{U}$, $A_1 \cap A_2$ is also in $\mathcal{U}$. Moreover, for any A, either A or A^{comp} is in $\mathcal{U}$. The non-dictatorial property of f insures that no coalition A consisting of a singleton belongs to $\mathcal{U}$. Thus $\mathcal{U}$ is a non-principal ultra-filter constructible from f.

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