CMS-EMS
Center for Mathematical Studies in Economics
And Management Science

Discussion Paper #1548

Auction Design with Fairness Concerns: Subsidies vs. Set-Asides

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April 8, 2012

JEL Classification: D44, H57

Keywords: Auctions, subsidies

NORTHWESTERN UNIVERSITY

Auction Design with Fairness Concerns: Subsidies vs. Set-Asides *

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Abstract

Government procurement and allocation programs often use subsidies and set-asides favoring small businesses and other target groups to address fairness concerns. These concerns are in addition to standard objectives such as efficiency and revenue. We study the design of the optimal mechanism for a seller concerned with efficiency, subject to a constraint to favor a target group. In our model, buyers' private values are determined by costly pre-auction investment. If the constraint is distributional, i.e. to guarantee that the target group wins 'sufficiently often', then the constrained efficient mechanism is a flat subsidy. This is consistent with findings in the empirical literature. In contrast, if the constraint is to ensure a certain investment level by the target group, the optimal mechanism is a type dependent subsidy. In this case a set aside may be better than a flat or percentage subsidy.

*The authors would like to thank seminar audiences at the University of Toronto, and University of Western Ontario.

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1 Introduction

Government programs, both procurement and allocation, are often required to favor target groups like small businesses, and businesses run by minorities and veterans.[1] To this end, the procedures used to assign licenses/contracts are 'distorted' relative to standard auctions. Subsidies and set asides are the two distortions regularly observed in practice. In a subsidy scheme, members of target groups receive a lump sum or percentage discount off their bid. In a set aside program, a portion of contracts/licenses are reserved exclusively for members of the target group to bid on.

In this paper, we investigate the design of the constrained efficient mechanism, taking as given that the seller is committed to favor a target group (i.e. some given subset of the buyers, e.g. small businesses).[2] We consider two possible formulations of what it means to favor the target group. For each of these formulations, we answer the following questions questions. i) What is the efficient mechanism? ii) Of the two observed distortions, i.e. subsidies and set asides, which is superior?

Our model considers a seller of a single unit selling to a population of buyers that come from two groups, a regular group and a target group. Each buyer's value is her private information but her identity (i.e. the group to which she belongs) is publicly observable. We assume that buyers make costly ex-ante investments that are unobserved and non-contractible by the seller. This is motivated by the observation that in many settings, participants can invest in cost reduction or quality enhancements. The choice of mechanism by the seller will influence the participants' investment levels.[3] Buyers have access to different investment technologies depending on whether they belong to the regular or target group. Each buyer's level of investment stochastically determines her private value. As a result, the distribution of values of both groups of buyers is endogenously determined by the seller's mechanism.[4]

We investigate two formulations for the seller's constraint to 'favor' the target group.

[1]Section 15(g)(1) of the Small Business Act in the US requires the government to direct 23% of all spend each year to small businesses that are owned by minorities, veterans and women. The EU, through its common agriculture policy, employs such methods to encourage and promote young farmers. Major construction projects that receive public funding like sports arenas are required to channel a certain amount of business to minority contractors. In Japan, roughly half of the annual Ministry of Land, Infrastructure and Transportation's expenditures made their way to small and medium enterprises.

[2]For now, we ignore revenue considerations, see discussions about related literature (Section 1.1) and extensions (Section 5) below.

[3]This is especially true with government procurement and allocation, since a large fraction of the participants' annual revenues and profits result from their business with the government.

[4]The case where the distribution of buyers' values in the two groups is fixed and exogenously given is thus a special case. We fully solve this as we build up to our main results.

First we consider a distributional requirement, modeled as requiring that the *expected* quantity allocated to the target group is at least some given threshold, α.[5] We find that the constrained efficient mechanism for the seller is always a flat subsidy for members of the target group, *independent* of their value. While the amount of subsidy depends on the details of the environment, the structure is independent of any assumptions on the distributions of buyer values. As a corollary, we show that a set aside can never be constrained efficient, again independent of distributional assumptions.

Our result follows from an approach similar to contract theory. First, we investigate the constrained efficient mechanism, given fixed investment levels by the regular and target groups (i.e. when the two groups have exogenously given distributions). We restrict attention to settings where the efficient (Vickrey) auction, coupled with buyers' investment choices, results in an allocation rule that does not satisfy the distributional constraint. In this case we show the optimal allocation rule gives buyers from the target group a (appropriately chosen) flat subsidy λ. In other words, the highest value buyer from the target group wins whenever her value is within λ of the highest overall.[6]

We then study an auxiliary setting where the seller can directly select buyers' investment choices. In this setting, we show that the seller will choose to have the target group buyers invest more than they would have under the Vickrey auction, and couple this with an appropriately chosen flat subsidy to buyers in the target group. Finally, we show that the equilibrium choices of buyers faced with this flat subsidy mechanism are the same as what the seller has chosen. Intuitively, the reason for this is the 'Vickrey pricing.' Each buyer gets to keep her contribution to the social surplus. Therefore, there is no agency problem– her incentives to invest are (surprisingly) perfectly aligned with those of an efficiency motivated seller/social-planner.

Our findings agree with those in [**?**] who consider a similar question in the context of US timber auctions. They develop and estimate a structural model of entry and bidding (in first and second price auctions) to assess the efficiency and revenue consequences of a shift from set asides to an appropriately chosen subsidy. Their counterfactual analysis shows that subsidies outperform set asides on both revenue and efficiency.

In our second setting, the seller wants the target group to reach a given investment level. In this setting we find that the constrained efficient mechanism involves a subsidy that depends on the type of the buyer. We show that the level of subsidy may be non-

[5]Implicitly, if the seller ran several such auctions in a year, as is the case with say timber auctions, the distributional goals would be met in aggregate by the law of large numbers.

[6]As is standard, the winning buyer pays the lowest amount she could have reported as her value and still won.

monotonic, ruling out both lump sum and percentage subsidies. In other words, the optimal subsidy is far from detail free. We exhibit an example where a set aside would be better than the optimal type-independent subsidy. In effect, faced with a (unmodeled) constraint to use a 'simple' or 'detail-free' mechanism, a seller may choose to use a set-aside.

Intuitively, buyers' incentives to invest given the auction depends on their profit in the auction as a function of their private value. From the revenue equivalence theorem, we know that the expected profit of a buyer in an auction depends on the allocation probability of buyers of lower value. In effect, the winning probability of a low value buyer increases the surplus of all types above it, and therefore has 'more impact' on the incentive to invest, relative to the winning probability of a high value buyer. This results in a tension with the efficiency objective- the mechanism needs to trade-off efficiency with incentives for the buyers to invest at the desired levels. This tension gives rise to a subsidy that is type dependent, and in general non-monotonic, ruling out both flat and percentage subsidies. A set aside may do better than a type-independent subsidy.[7]

Our results may be of independent interest from a technical standpoint. There has been little work on optimal auction design with ex-ante asymmetric buyers, because these are hard to handle via standard optimal control techniques.[8] We proceed by discretizing the space of possible valuations, and characterizing the optimal mechanism in this setting. Subsequently, we show how to pass to the limit by considering the limit of successively fine discretizations. The technique is not limited to the application considered here.

1.1 Related Literature

In the theoretical literature on auctions, there has been some work on settings where the buyers can make a pre-auction investment. Tan [?] studies a setting where ex-ante symmetric buyers can make pre-auction investment, and compares the equilibria in first and second price auctions. Piccione and Tan [?] characterize the optimal mechanism in such a setting, and compare it to one where the seller only announces the mechanism after buyers choose their investment levels. In both papers, the buyers are ex-ante symmetric and the seller has no additional 'fairness' concerns. Azoramena & Cantillion [?] consider a setting where only one buyer can make an investment (hence buyers are not ex-ante symmetric), and find that a first price auction may provide higher revenue than a second price auction. Li, Lovejoy and Gupta [?] consider a procurement setting where the procurer announces his mechanism after sellers have chosen their private investment levels, and show

[7] For further discussion and intuition, see Example 2 in Section 4.2.

[8] We refer the interested reader to the discussion in Appendix A.

that the equilibrium in investment levels depends on the outside option of the procurer.

There is a small related literature on information acquisition in auctions– i.e. buyers incur costs to learn their own private information, and the mechanism must incentivize the buyers to invest in this information acquisition. Bergemann & Välimaki [?] find that the VCG mechanism is efficient in this setting, while Shi [?] characterizes the expected revenue maximizing mechanism.

At a conceptual level, there is a large literature studying 'affirmative action', i.e. how to distribute a scarce resource across a population while favoring some target group, under various formulations of the planner's objectives. The closest paper to ours is that of Athey, Coey and Levin [?]. They develop and estimate a structural model of entry and bidding to compare quantitatively the effect of subsides vs. set asides, using data from timber auctions conducted by the US Forestry Services. Like our model they assume private values. Entry is also costly in their model similar to our assumption of costly ex-ante investment, however the distribution of values in their paper is fixed. They fix the format of the auction and (sealed bid first price or second price) and under that format they compare the impact of using a set aside vs. a subsidy, while we take a mechanism design approach. A recent paper by Hickman [?] studies a similar question to this paper's, both theoretically and empiricially, in a matching model in the context of college admissions. Loury and Fryer [?] study a model where agents have hetereogenous returns to investment in human capital, and the distribution of this effectiveness is different based on the group to which they belong. They compare the efficient allocation policy for a social planner when the principal observes agents' identity versus when the principal does not. Chung [?] observes that given any affirmative action policy, multiple equilibria may exist in the induced subgame among agents (e.g. choice of investment in human capital). He considers the redesign of affirmative policies to take into account this implementation problem.

2 Model

There is a seller with a single indivisible good for sale whose opportunity cost is 0. He is faced with potential buyers from two groups. There are $n_1 > 0$ buyers from the 'regular' group, and $n_2 > 0$ buyers from the 'target' group. Both seller and buyers are risk netural. The identity of a buyer (i.e. the group to which she belongs) is publicly observable. Each buyer has a privately known value for the good which we normalize to be between $[0, 1]$.

The seller's objective is to maximize efficiency, subject to a constraint to favor buyers from the target group. As stated earlier,we model this constraint in two ways. The first is that buyers from the target group win the good with an ex-ante probability of at least (some exogenously given) α.

Exogenously Given Distribution To develop intuition, consider a simple special case. Assume that buyers' values are drawn independently from two commonly known, exogenously given distributions- F_1 is the CDF of the distribution of values buyers in the 'regular' group, while F_2 is the corresponding CDF of buyers in the target group.

By the revelation principle, we can restrict attention to direct revelation mechanisms. Each buyer reports his value v. The seller commits to assign a buyer from group $g = 1, 2$ the good with interim probability $a_g(v)$ as a function of her report v, in return for an interim payment of $p_g(v)$.

Since buyers' values are private information, the seller is faced with the usual incentive compatility constraint:

$$\forall v \in V,\ g \in \{1, 2\}: \quad va_g(v) - p_g(v) \geq va_g(v') - p_g(v'). \tag{IC}$$

The usual voluntary participation constraint applies:

$$\forall v \in V,\ g \in \{1, 2\}: \quad va_g(v) - p_g(v) \geq 0. \tag{IR}$$

There must exist a feasible ex-post allocation rule (i.e. sells at most 1 unit at each realized profile) that corresponds to the interim allocation rule. :

$$(a_1(\cdot), a_2(\cdot)) \text{ is feasible.} \tag{F}$$

(IC), (IR) and (F) are standard. Regarding the latter— Mierendorff [**?**] characterizes exactly the set of feasible rules in this setting. The new constraint is the seller's equity or distributional constraint, which can be written as:

$$n_2 \int_V f_2(v) a_2(v) \mathrm{d}v \geq \alpha. \tag{E}$$

In other words, the expected quantity allocated to the target group, or alternately, the ex-ante probability that the good goes to a buyer in the target group, must be at least α.

The constrained efficient mechanism is the solution to:

$$\max_{(a_g(\cdot),p_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \int_V v a_g(v) f_g(v)\mathrm{d}v \tag{Opt-Exogenous}$$

$$\text{s.t.} \qquad (\text{IC}), (\text{IR}), (\text{E}), (\text{F}).$$

Pre-auction Investment We now introduce our full model. Instead of buyers' values being drawn from an exogenously given distribution, we suppose each buyer chooses a level of investment which stochastically determines her value. Each buyer may choose any level of investment in $\Re_+$. If a buyer chooses a level of investment t, she realizes a private value drawn independently from a distribution with CDF $F(\cdot, t)$. Investment is costly- a buyer in group g pays $c_g(t)$ if she chooses investment level t.[9]

Therefore, a buyer from group g faced with a mechansim with allocation rule a_g and payment rule p_g makes an expected profit of

$$\int_V (v a_g(v) - p_g(v)) f(v,t)\mathrm{d}v - c_g(t), \tag{Profit}$$

when she chooses investment level t. She will therefore choose t to maximize (Profit).

For tractability we impose standard regularity conditions on c and F. Together, these assumptions require that the returns to investment are uncertain, and decreasing.

Assumption 1. For $g = 1, 2$, the cost of investment $c_g(t)$, is increasing, convex, and continuously differentiable in t, i.e $c_g'(t)$ exists, is continuous, and $c_g'(t) \geq 0, c_g''(t) \geq 0$.

Assumption 2. $F(v, t)$ is a continuously differentiable function of t. Further:

1. Higher investment leads to stochastically better values (in the sense of first order stochastic dominance), i.e.

$$t > t' \implies F(v,t) \leq F(v,t'). \tag{FOSD}$$

2. Decreasing returns to investment:

$$\frac{\partial^2 F(v,t)}{\partial t^2} \leq 0. \tag{DR}$$

[9] In other words, buyers within the same group are homogenous. Similar to Fryer & Lowry, [?], one could extend this model to consider one where buyers within the same group have different cost functions, with the distribution of cost functions being different across the two groups. The results would be similar.

A 'nice' $F(v,t)$ that satisfies Assumption 2 from the literature on R& D (see e.g. [?]) is exhibited below.

EXAMPLE 1. For any distinguished distribution with c.d.f $F(v)$ over V, an investment technology that gives a buyer a value from the distribution

$$F(v,t) = F^t(v),$$

satisfies Assumption 2.

Under this technology, there is a distinguished distribution $F(v)$. An investment level t corresponds to t i.i.d. draws from this distribution, from which the highest is chosen.[10]

We make one more assumption.

ASSUMPTION 3. In (E),

$$\alpha = \frac{n_2}{n_1 + n_2}. \tag{1}$$

Further, costs are such that:

$$\forall t : c_1(t) < c_2(t). \tag{2}$$

This assumption is purely to aid interpretation. Assumption 3 (1) makes clear why we call it an 'equity' constraint: it requires that buyers from the target group have the same ex-ante probability of winning as buyers from the regular group. If both groups choose the same level of investment, any symmetric allocation rule (i.e. one which ignores buyers' group identities) satisfies the equity constraint. Assumption 3 (2) fits the applications we have in mind, i.e. the target group is disadvantaged relative to the regular group.

Some additional results will consider a simpler setting where the regular group has values drawn from a fixed distribution.

DEFINITION 1. *We say that only the target group invests if the regular group has values drawn from a fixed distribution* $F_1(\cdot)$*,*

$$F_1(v) \equiv F(v, t_1).$$

We are now in a position to describe the seller's problem. The seller still chooses interim allocation and pricing rules $(a_g, p_g)_{g=1,2}$. In additon the seller 'prescribes' an investment level t_g for each group g. This determines the distribution of the values of the buyers,

[10] Hence the connection to the R& D literature, where this would correspond to t independent R& D efforts.

$F(\cdot, t_g)$. The interim allocation and pricing rules are with respect to these distribution. The mechanism must satisfy (E) (IC), (IR), (F) as before. Further, the mechanism must be such that the a buyer in group g chooses the prescribed investment level t_g. From (Profit), we have:

$$t_g \in \arg\max_t \int_V (va_g(v) - p_g(v))f(v,t)\mathrm{d}v - c_g(t) \tag{SR}$$

The constrained efficient mechanism for the seller, therefore, must solve:

$$\max_{(t_g, a_g(\cdot), p_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \int_V va_g(v) f_g(v, t_g)\mathrm{d}v \tag{Opt-D}$$
$$\text{s.t.} \qquad \text{(IC), (IR), (E), (F), (SR).}$$

2.1 Target Investment Level

An alternate way to model a requirement that the seller 'favor' the target group is to choose a mechanism such that buyers in the target group are incetivized to achieve some given minimum investment level $\underline{t}$.

Under the maintained assumptions, the constraint for the seller can be written as:

$$\underline{t} \leq t^\star \in \arg\max_t \int_V (va_2(v) - p_2(v))f(v,t)\mathrm{d}v - c_2(t) \tag{Min Investment}$$

The constrained efficient mechanism for the seller, therefore, must solve

$$\max_{(t_g, a_g(\cdot), p_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \int_V va_g(v) f_g(v, t_g)\mathrm{d}v \tag{Opt-TI}$$
$$\text{s.t.} \qquad \text{(IC), (IR), (Min Investment), (F), (SR).}$$

3 Auction Design with Fixed Distributions

To develop intuition, we first work with the simple case when the regular and target group buyers' values are drawn from fixed, exogenously given distributions.

Recall that that seller's problem is to maximize efficiency, subject to (IC), (IR) and (F) constraints, plus the equity or distributional constraint (E). We begin by recalling some standard observations about incentive compatibility.

Lemma 1. *Given an allocation rule a_g, there exists a pricing rule p_g which satisfies* (IC) *if and only if $a_g(v)$ is nondecreasing in v, i.e.*

$$a_g(\cdot) \text{ nondecreasing.} \tag{M}$$

Lemma 2. *Given non-decreasing allocation rules $a_g(v)$, the pricing rule:*

$$p_g(v) = va_g(v) - \int_0^v a_g(v')dv', \tag{P}$$

satisfies both Incentive Compatibility (IC) *and Individual Rationality* (IR)*.*

The proofs are standard and ommitted. One aside— any pricing rule

$$p_g(v) = va_g(v) - \int_0^v a_g(v')\mathrm{d}v' + c,$$

$c \leq 0$ will satisfy (IC) and (IR). Indeed, these are the only pricing rules which satisfy both (IC) and (IR). Since the seller's objective is efficiency, the choice of c is irrelevant, we therefore use $c = 0$.

By Lemmas 1 and 2, the seller's problem reduces to:

$$\max_{(a_g(\cdot),p_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \int_V va_g(v)f_g(v)\mathrm{d}v$$
$$\text{s.t.} \qquad (\text{M}), (\text{E}), (\text{F}).$$

Our main result in this section is that the optimal mechanism for the seller involves a (appropriately chosen) flat subsidy for the target group.

Theorem 1. *Suppose the Vickrey auction violates the equity constraint* (E)*. Then, the optimal mechanism for the seller can be characterized thus:*

- *Allocation Rule: there exists $\lambda^\star > 0$, such that the optimal mechanism gives the good to the highest value buyer in the target whenever her value is within $\lambda^\star$ of the highest value overall, and to the highest value regular buyer otherwise.*
- *Pricing Rule: the winning buyer pays the Vickrey price, i.e. the lowest value she could have bid and still won the good.*

Note that in the absence of the (E) constraint, the Vickrey auction is optimal for the seller (which corresponds to a subsidy of $\lambda = 0$). The interesting case then is when the

Vickrey auction violates (E). In this case (E) must bind at the optimal solution. The proof involves a relaxation argument– i.e. relaxing the (E) constraint with the appropriate dual variable λ. Collecting terms, the result follows.

Corollary 1. *The optimal mechanism for the seller is a flat/ lump sum subsidy for buyers in the target group. It cannot be implemented as a set aside.*

This theorem follows from the structure of interim feasible allocation rules. We know that optimal mechanism for the seller must be a corner point of the feasible region. A set aside is strictly in the interior of the space of feasible allocation rules, and is thus ruled out.

4 Pre-Auction Investment

We now consider our 'full model' where buyers make costly pre-auction investments. First, we examine the design of the constrained efficient mechanism when the 'fairness' constraint for the seller takes the form of a distributional constraint. Then, we discuss the constrained efficient mechanism to achieve a minimum investment level for the target group.

4.1 Distributional Constraint

At a high level, we first consider a 'relaxed' problem where we relax the (SR) constraint. The seller is still faced with private information and the equity constraint. This corresponds to an environment where the seller chooses the investment level of the buyers directly. We show that the optimal mechanism in this setting will involve a lower effort choice by the seller for the target group than the regular group. Further, there is a flat subsidy to buyers in the target group. We conclude by observing that when faced with a mechanism with that level of subsidy, the equilibrium effort choices of buyers is the same as the effort choice by the seller in the relaxed problem. Therefore, a second price auction with a flat subsidy for buyers in the target group is the constrained efficient mechanism for a seller faced with a distributional constraint.

In this relaxed setting, a seller concerned with efficiency solves:

$$\max_{(t_g, a_g(\cdot), p_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \left(\int_V v a_g(v) f_g(v, t_g) \mathrm{d}v - c_g(t_g) \right) \tag{Opt-Aux}$$

s.t. (IC), (IR), (E), (F).

We first characterize the level of investment a buyer will choose given an interim allocation rule a_g and interim pricing rule p_g.

LEMMA 3. *Given Assumptions 1 and 2; a buyer from group g facing interim allocation rule a_g and pricing rule p_g will choose (the unique) level of investment t_g which solves:*

$$-\int_V a_g(v)\frac{\partial F(v,t)}{\partial t}dv - c_g'(t) = 0. \tag{FOC}$$

PROOF. Recall a buyer in group g, faced with an interim allocation rule a_g and an interim pricing rule p_g, will choose a level of investment t to solve:

$$\max_t \int_V \big(va_g(v) - p_g(v)\big) f(v,t)\mathrm{d}v - c_g(t).$$

Using the pricing equation (P), the buyer's problem reduces to:

$$\max_t \int_V \left(\int_0^v a_g(v')\mathrm{d}v'\right) f(v,t)\mathrm{d}v - c_g(t).$$

Reversing the order of integration, we get:

$$\max_t \int_V a_g(v)(1-F(v,t))\mathrm{d}v - c_g(t).$$

Note that under our maintained assumptions first order conditions are necessary and sufficient for optimality, and the result follows. □

Next, we study the outcome when the seller uses the efficient (Vickrey) mechanism.

LEMMA 4. *Suppose the seller uses the efficient (Vickrey) mechanism and lets buyers freely choose investment levels. Then, there is an equilibrium in which all buyers in group g choose t_g^e.*[11] *Further, (a_g^e, p_g^e), the interim allocation and pricing rules that result, solve the auxiliary problem in the absence of equity constraints. Formally:*

$$(t_g^e, a_g^e, p_g^e) \in \mathrm{argmax}_{(t_g, a_g(\cdot), p_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \left(\int_V va_g(v)f_g(v,t_g)dv - c_g(t_g)\right)$$
$$s.t. \text{ (IC), (IR), (F)}.$$

[11]There may be additional equilibria. If we are in a setting where only the target group invests (Definition 1), there is a unique optimal investment level for the target group.

In other words, in the absence of any equity concerns, running the efficient mechanism and letting buyers choose their investment levels has an equilibrium which results in the first best level of investment by the buyers.[12]

The interesting case is when the efficient mechanism violates the seller's equity constraint.

Observation 1. If we are in a setting where only the target group invests (Definition 1), and the seller's equity constraint is violated by the efficient mechanism, then

$$t_2^e < t_1$$

The equity constraint causes the solution to (Opt-Aux) to be distorted relative to efficiency. Intuitively, it is 'too expensive' to bridge the gap between the regular and target groups completely. The solution to (Opt-Aux) therefore reduces the investment gap betwen the two groups relative to the levels under the efficient mechanism. Additionally, an appropriately chosen flat subsidy is given to the target group to satisfy (E).

Lemma 5. *The solution* (t_g^a, a_g^a, p_g^a) *to* (Opt-Aux) *can be characterized thus— the selling mechanism is a flat subsidy with the standard pricing rule:*

1. *Allocation Rule: there exists* $\lambda^a > 0$*, such that the good goes to the highest value buyer in the target whenever her value is within* λ^a *of the highest value overall, and to the highest value regular buyer otherwise.*

2. *Pricing Rule: the winning buyer pays the Vickrey price, i.e. the lowest value she could have bid and still won the good.*

Observation 2. If we are in a setting where only the target group invests (Definition 1), the investment gap between the two groups is less in the solution to (Opt-Aux) than under the efficient mechanism, i.e.

$$t_2^e < t_2^a.$$

To relate the solution of this auxiliary problem to the seller's problem, we need to reinstate the sequential rationality constraint. Our main result in this section shows that the solution to this relaxed problem for the seller is feasible for the seller's original problem, and therefore optimal.

[12]Throughout, as is standard, we ignore the implementation problem of selecting the 'desired' equilibrium. For further details in this context, we refer the reader to [?].

Theorem 2. *Let* (t_g^a, a_g^a, p_g^a) *be the solution to* (Opt-Aux). *This mechanism also satisfies* (SR)*, so the solution is also optimal in the original seller's problem* (Opt-D)*.*

This implies that the optimal mechanism for the seller when faced with a distributional constraint is a flat subsidy for buyers in the target group.

4.2 Target Investment Level

We now consider the case when the 'fairness' concern of the seller is modeled as a minimum investment level for the target group: the seller wants to choose a mechanism subject to target group's investment choice being above a given threshold $\underline{t}$. To make the analysis interesting, we focus on the case when this threshold is larger than the investment level the target group would make when faced with the efficient mechanism, i.e. $\underline{t} > t_2^e$.

Our analysis uses the characterization of buyer investment levels in Lemma 3.

Theorem 3. *The constrained efficient mechanism to implement a level of investment* $\underline{t} > t_2^e$ *will involve a type dependent subsidy* $s(\cdot)$ *to buyers in the target group— buyers in the target group have their reported value* v *adjusted to* $v + s(v)$*. The good is given to the highest adjusted reported value over all buyers. The winning buyer pays the lowest bid he could have made and still won the good. Further, the subsidy takes the functional form:*

$$s(v) = \frac{\eta}{f(v,\underline{t})} \frac{\partial F(v,t)}{\partial t}\Big|_{t=\underline{t}}, \tag{3}$$

for an appropriately chosen $\eta < 0$*. If* $v + s(v)$ *is not non-decreasing in* v*, then further ironing is required.*

Intuitively, buyers' incentives to invest depend on their profit in the auction they get as a function of their private value. Recall that a buyer with value v from group g will make a profit of $\int_0^v a_g(v')\mathrm{d}v'$. In effect, the winning probability of a low value buyer increases the surplus of all types above it, and therefore has 'more impact' on the incentive to invest relative to a high value buyer. This results in a tension with the objective- the mechanism needs to trade-off efficiency with incentives for the buyers to invest at the desired levels t_1, t_2. Ironing is necessary if the first best tradeoff leaves the interim allocation probabilities non-monotone.

In general, $s(v)$ will be non-monotone in v— $s(0) = s(1) = 0$ because $\frac{\partial F(v,t)}{\partial t}$ vanishes when evaluated at $v = 0, 1$. Therefore, a flat subsidy does exactly the 'wrong' thing. High types of the target group are rewarded with a higher probability of winning than in the efficient mechanism. By contrast, the optimal mechanism increases less the winning probability of high types relative to the efficient mechanism, since $s(v)$ is small for high types. A set aside may do better than a subsidy in achieving this.

EXAMPLE 2. In this example, we numerically compare two suboptimal mechanisms that are seen in practice— 'percentage' subsidy and set asides. For simplicity, we will assume only the target group invests (Definition 1), and there is 1 buyer in each group. Investment will impact the distribution of buyer values as in Example 1, i.e.

$$F(v,t) = F^t(v),$$

for $F(v)$ uniform on $[0,1]$. Suppose $t_1 = 9$, and the seller's constraint being such that $\underline{t} = 9$.[13] A subsidy is as described above– a buyer from the target group has the subsidy added to his reported value, and wins if his adjusted value is the highest overall. In a set aside, with probability $p < 1$, the efficient mechanism is run, but only with buyers from the target group competing. With the complementary probability $1-p$, the efficient mechanism is run with all buyers competing.

The optimal set aside achieves an expected efficiency of .913, while a percentage subsidy achieve .911. The flat subsidy in this case outperforms both at .918. However, in the set aside, the target effort level can be achieved via less distortion- the target group wins only 86.1% of the time with the set aside, as opposed to 92.7% of the time under the optimal flat subsidy.[14] Relaxing the assumption that only the target group invests, therefore, a set aside will dominate a flat subsidy (since the regular group will invest less under the flat subsidy).

5 Extensions and Conclusions

Government procurement and allocation programs often use subsides and set asides to assist target groups. Our paper suggests when the use of one or the other would be appropriate. If a goal of the allocation program is to ensure that a target group wins sufficiently often, our analysis shows that a flat subsidy for the target group would impair efficiency less than a set aside. If the goal of the allocation program is to ensure a certain investment level by the target group, then a type dependent subsidy is constrained efficient. Recognizing that such subsidies may be impractical to implement one might restrict attention to flat or percentage subsides. In this case, we find that a set aside may be better than a flat or percentage subsidy.

[13] This is numerically equivalent to n_1, n_2 buyers, with $t_1 = 9/n_1$ and $\underline{t} = 9/n_2$.

[14] These calculations were done using the R package. The code is attached in Appendix C.

Our analysis assumed an environment where only a single unit was being allocated but this is not essential. The results hold for the case when an agent's utility for consumption depends on a one dimensional parameter of private information and an appropriate increasing differences property.

Our results hold even if the assumption of private values is generalized to interdependent values under the usual conditions. For example, each buyer receives an independent private signal of the value of the object for sale. Their value is increasing in both their own signal and the signal seen by others. An increase in investment increases the value the buyer would perceive, holding fixed the signals observed by all the buyers.

Finally, one might ask about the case of a seller concerned with revenue instead. Our techniques still hold. The counterpart to Theorems 1 and 2 would be that the optimal mechanism is a flat subsidy in the space of *virtual valuations*. For most distributions, this will not have a 'natural' implementation.

A Proofs from Section 3

First, we state a theorem due to Mierendorff [?] on the structure of feasible interim allocation rules.

Theorem 4 (Mierendorff [?]). *An interim allocation rule (a_1, a_2) is feasible if and only if for each $V_1, V_2 \subseteq V$:*

$$n_1 \int_{V_1} a_1(v) f_1(v)\, dv + n_2 \int_{V_2} a_2(v) f_2(v)\, dv \leq 1 - \left(1 - \int_{V_1} f_1(v)\, dv\right)^{n_1} \left(1 - \int_{V_2} f_2(v)\, dv\right)^{n_2}$$

If a_1, a_2 are nondecreasing, then they are feasible if and only if for each $v_1, v_2 \in V$:

$$n_1 \int_{v_1}^{1} a_1(v) f_1(v)\, dv + n_2 \int_{v_2}^{1} a_2(v) f_2(v)\, dv \leq 1 - (F_1(v_1))^{n_1} (F_2(v_2))^{n_2}$$

The latter form is easier to work with, and sufficient for our purposes since Lemma 1 guarantees that feasible allocation rules are monotone.

Proof of Theorem 1. The seller's problem is:

$$\max_{(a_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \int_V v a_g(v) f_g(v) \mathrm{d}v$$

$$\text{s.t.} \qquad a_1, a_2 \text{ non-decreasing in } v,$$

$$\int_0^1 a_2(v) f_2(v) \mathrm{d}v \geq \alpha,$$

$$\forall v_1, v_2: \quad n_1 \int_{v_1}^1 a_1(v) f_1(v) \mathrm{d}v + n_2 \int_{v_2}^1 a_2(v) f_2(v) \mathrm{d}v \leq 1 - (F_1(v_1))^{n_1} (F_2(v_2))^{n_2}$$

The Vickrey auction solves the seller's problem in the absence of the equity constraint. Therefore if it is infeasible in the seller's problem, the equity constraint (E) binds in the optimal solution.

The main technical issue in characterizing the optimal is that it is hard to write the seller's problem in a 'standard' form– the (F) is problematic. In the literature on optimal auctions with ex-ante symmetric buyers (e.g, [?], [?], [?]) the feasibility constraint is:

$$\forall v: \quad n \int_v^1 a(v) f(v) \mathrm{d}v \leq 1 - F(v)^n.$$

This can be written in standard form by creating a state variable $\beta(v)$, which evolves as:

$$\beta'(v) = (a(v) - F^{n-1}(v)) f(v).$$

The feasibility constraint now requires that $\beta(v) \geq 0$ for all v, with end point condition $\beta(1) = 0$. No such re-writing of (F) is possible, so a different approach is needed.[15]

We resolve this by discretizing the program seller's problem. At a high level, our approach is as follows:

1. Fix a positive integer n, let $\epsilon_n = \frac{1}{n}$. Consider the partition of the space of possible valuations $[0, 1]$ into n intervals of length ϵ_n, i.e. $\mathcal{P}_n = \{[0, \epsilon_n), [\epsilon_n, 2\epsilon_n), \ldots, [(n-1)\epsilon_n, 1)\}$. Consider the seller's problem with the additional constraint that the allocation rule a_g, for each g, be measurable with respect to this partition $\mathcal{P}_n$, i.e. be

[15] One approach seen in [?] in a different setting is to first characterize the optimal a_2 for any given feasible a_1, and then solve for the optimal a_1 using an optimal control approach. In both his setting and ours, the optimal a_1 may have jumps, and in this case the optimization problem cannot be written in a standard form. He resolves this by approximating the optimal allocation rule by taking the limit of the optimal of a sequence of constrained optimization programs. Each constrained program restricts the seller to choosing a Lipschitz continuous allocation rule with Lipschitz constant k, and the takes the limit as $k \uparrow \infty$.

constant on each interval $[k\epsilon_n, (k+1)\epsilon_n)$.

2. This constrained problem can be written as a standard linear program. We characterize the solution for any n, and show that it has the desired properties.

3. We take the limit of the optimal allocation rule for n as $n \uparrow \infty$ and show that it must be optimal in the unconstrained problem. The theorem follows.

STEP 1: A DISCRETIZATION Fix n. The constrained problem is:

$$\begin{aligned}
&\max_{(a_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \int_V v a_g(v) f_g(v) \mathrm{d}v \\
\text{s.t.} \quad & a_1, a_2 \text{ non-decreasing in } v, \\
& n_2 \int_0^1 a_2(v) f_2(v) \mathrm{d}v \geq \alpha, \\
\forall v_1, v_2: \quad & n_1 \int_{v_1}^1 a_1(v) f_1(v) \mathrm{d}v + n_2 \int_{v_2}^1 a_2(v) f_2(v) \mathrm{d}v \leq 1 - (F_1(v_1))^{n_1} (F_2(v_2))^{n_2} \\
\forall k < n, g = 1,2: \quad & a_g(v) \text{ constant on } \left[\frac{k}{n}, \frac{k+1}{n}\right).
\end{aligned}$$

We can rewrite this as a linear program, considering each interval on which a_g is constrained to be constant as a 'type'. Define, for each $i = 1, 2, \ldots, n$:

$$\begin{aligned}
f_g^n(i) &= \int_{\frac{i-1}{n}}^{\frac{i}{n}} f_g(v) \mathrm{d}v, \\
F_g^n(i) &= \sum_{j \leq i} f_g^n(i) \left(= F_g(\tfrac{i}{n})\right), \\
v_g^n(i) &= \mathbb{E}_g[v | v \in [\tfrac{i-1}{n}, \tfrac{i}{n})] = \frac{1}{f_g^n(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} v f_g(v) \mathrm{d}v, \\
V_g^n &= \{v_g^n(1), v_g^n(2), \ldots, v_g^n(n)\}.
\end{aligned}$$

With these definitions, the constrained problem can be rewritten as:

$$\max_{(a_g^n(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \sum_{i=1}^{n} v_g^n a_g^n(i) f_g^n(i) \tag{OPTn}$$

$$\text{s.t.} \qquad a_g^n(i) \text{ non-decreasing in } i,$$

$$n_2 \sum_{i=1}^{n} a_2^n(i) f_2^n(i) \geq \alpha, \tag{4}$$

$$\forall i_1, i_2 \leq n: \quad \sum_g n_g \sum_{i \geq i_g} a_g^n(i) f_g^n(i) \leq 1 - \left(F_1^n(i_1 - 1)\right)^{n_1} \left(F_2(i_2 - 1)\right)^{n_2}$$

We should note that the feasibility constraints for discrete types were first formulated in [?].

Step 2: A Linear Programming Approach The Vickrey auction solves the seller's problem in the absence of the equity constraint. Therefore if it is infeasible in program (OPTn), the equity constraint (4) binds in the optimal solution. Let λ_n be the dual variable corresponding to (4), and $\lambda_n^\star$ be the value of the this variable in the optimal dual solution.

The solution to seller's problem is therefore the solution to the Lagrangean relaxation of (En), with the optimal dual value of $\lambda_n^\star$. Collecting terms, the seller's problem can be written as:

$$\max_{(a_g^n(\cdot))_{g=1,2}} n_1 \sum_{i=1}^{n} v_1^n a_1^n(i) f_1^n(i) + n_2 \sum_{i=1}^{n} (v_2^n + \lambda_n^\star) a_2^n(i) f_2^n(i)$$

$$\text{s.t.} \qquad a_g^n(i) \text{ non-decreasing in } i,$$

$$\forall i_1, i_2 \leq n: \quad \sum_g n_g \sum_{i \geq i_g} a_g^n(i) f_g^n(i) \leq 1 - \left(F_1^n(i_1 - 1)\right)^{n_1} \left(F_2(i_2 - 1)\right)^{n_2}$$

Comparing with the seller's problem in the absence of the (E) constraint, it follows that the solution is a modified Vickrey auction where a buyer from the target group with value v is treated as if she had value $v + \lambda^\star$. If there are ties, the tie is resolved in favor of group 2 with some probability π such that the (E) constraint binds.[16]

Therefore, for each n, the solution to the discretized problem is a flat subsidy in favor of the target group 2. Denote the optimal interim allocation rule as $a_g^{n\star}$.

[16] Since each type has positive probability, ties happen with strictly positive probability if they happen at all.

Step 3: Limit as $n \to \infty$ This will involve two subparts. First we show the existence of a solution to the seller's original problem. Then we show that the limit of the solutions constrained problem (OPTn) as n goes to ∞ recovers a solution of the original problem.

Step 3.1: Existence (This step is based on the Proof of Theorem 8 in [?]) Let

$$\begin{aligned} E^\star = & \sup_{(a_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \int_V v a_g(v) f_g(v) \mathrm{d}v \\ \text{s.t.} \quad & a_1, a_2 \text{ non-decreasing in } v, \\ & n_2 \int_0^1 a_2(v) f_2(v) \mathrm{d}v \geq \alpha, \\ \forall v_1, v_2: \quad & n_1 \int_{v_1}^1 a_1(v) f_1(v) \mathrm{d}v + n_2 \int_{v_2}^1 a_2(v) f_2(v) \mathrm{d}v \leq 1 - (F_1(v_1))^{n_1} (F_2(v_2))^{n_2} \end{aligned}$$

Let $(a_1^k(\cdot), a_2^k(\cdot))$ be a sequence of allocation rules feasible in the seller's problem such that

$$\sum_{g=1,2} n_g \int_V v a_g^k(v) f_g(v) \to E^\star \text{ as } k \to \infty.$$

Since these functions are monotone, they are of bounded variation. By Helley's Theorem, there exists a subsequence, and non-decreasing functions a_g such that $a_g^k \to a_g^\star$ pointwise, almost everywhere. By Mierendorff [?] we know that the space of feasible interim allocation rules is weakly compact considered as elements of L_2. Therefore, taking a further subsequence $a_g^k \to a_g^\star$ weakly. Since $n_2 \int_0^1 a_2^k(v) f_2(v) \mathrm{d}v \geq \alpha$ for all k, weak convergence implies that $n_2 \int_0^1 a_2^\star(v) f_2(v) \mathrm{d}v \geq \alpha$. Weak convergence further implies that

$$\sum_{g=1,2} n_g \int_V v a_g^\star(v) f_g(v) \mathrm{d}v = E^\star,$$

concluding our proof of existence.

Step 3.2: Limit Consider the optimal solution $(a_1^\star, a_2^\star)$ identified above. We define a 'flattened' version of this solution:

$$\underline{a}_g^n(i) = \frac{1}{f_g^n(i)} \int_{\frac{i-1}{n}}^{\frac{i}{n}} a_g^\star(v) f_g(v) \mathrm{d}v$$

Note that by construction, $\underline{a}_g^n(i)$ is non-decreasing in i, feasible, and satisfies (4)— that is, it is feasible in (OPTn). Therefore,

$$\sum_{g=1,2} n_g \sum_{i=1}^{n} v_g^n a_g^{n\star}(i) f_g^n(i) \geq \sum_{g=1,2} n_g \sum_{i=1}^{n} v_g^n \underline{a}_g^n(i) f_g^n(i),$$

since $a_g^{n\star}$ is optimal in (OPTn). However, by construction:

$$\sum_{g=1,2} n_g \sum_{i=1}^{n} v_g^n \underline{a}_g^n(i) f_g^n(i) \to E^\star \text{ as } n \to \infty,$$

The solutions to (OPTn) can be viewed as feasible allocation rules in the original continuum type space as:

$$a_g^{n\star}(v) = a_g^{n\star}(i) \text{ for } \tfrac{i-1}{n} \leq v < \frac{i}{n}.$$

The space of feasible allocation rules is a weakly compact set considered as elements of L_2. Therefore of the sequence of allocation rules $a_g^{n\star}$, there must be a weakly convergent subsequence, $a_g^{n\star} \to \bar{a}_g$. Recall that:

$$\sum_{g=1,2} n_g \sum_{i=1}^{n} v_g^n a_g^{n\star}(i) f_g^n(i) \geq \sum_{g=1,2} n_g \sum_{i=1}^{n} v_g^n \underline{a}_g^n(i) f_g^n(i).$$

By construction, this implies:

$$\sum_{g=1,2} n_g \int_v v a_g^{n\star}(v) f_g(v) \mathrm{d}v \geq \sum_{g=1,2} n_g \sum_{i=1}^{n} v_g^n \underline{a}_g^n(i) f_g^n(i).$$

Taking limits as $n \to \infty$, we have by the defintion of weak convergence

$$\implies \sum_{g=1,2} n_g \int_V v \bar{a}_g(v) f_g(v) \mathrm{d}v \geq E^\star.$$

Since $\bar{a}_g$ is feasible in the original problem, it follows that:

$$\sum_{g=1,2} n_g \int_V v \bar{a}_g(v) f_g(v) \mathrm{d}v \leq E^\star.$$

Therefore,

$$\sum_{g=1,2} n_g \int_V v\bar{a}_g(v) f_g(v) \mathrm{d}v = E^\star.$$

Next, by Helly's theorem, $a_g^{n\star} \to \bar{a}_g$ pointwise almost everywhere. It follows that $\bar{a}_g$ can be implemented by a flat subsidy in favor of the target group. □

PROOF OF COROLLARY 1. Further, the corner points of this polytope characterized by the class of hierarchical allocation rules ([?]). Each corner point is a priority rule over types, such that in any given profile of types, the buyer with the type with the highest priority wins (ties broken randomly).

Now, any interim allocation rule generated from an auction with a set aside cannot be a corner point of the polytope- it corresponds to the convex combination of the standard priority rule (i.e. highest type from either group wins), and the priority rule that places all types from the target group above the priority rule from the regular group. We know that the optimal solution to a linear program must be a corner point of the feasible region. Therefore the optimal mechanism described in Theorem 1 cannot be implemented as a subsidy. □

B Proofs from Section 4

PROOF OF LEMMA 4. We know that the solution to:

$$\max_{(a_g(\cdot), p_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \left(\int v a_g(v) f_g(v, t_g) \mathrm{d}v - c_g(t_g) \right)$$

$$\text{s.t.} \qquad \text{(IC), (IR), (F).}$$

is the Vickrey auction for any given (t_1, t_2). The resulting interim allocation rule for a given (t_1, t_2) is the efficient allocation rule. The resulting interim allocation rules are:

$$a_1^e(v) = F^{n_1-1}(v, t_1) F^{n_2}(v, t_2) \tag{5a}$$

$$a_2^e(v) = F^{n_1-1}(v, t_1) F^{n_2-1}(v, t_2) \tag{5b}$$

Therefore the seller's problem (Opt-Aux) in the absence of the equity constraint can be written as:

$$\max_{(t_g)_{g=1,2}} \sum_{g=1,2} n_g \left(\int_V v a_g^e(v) f_g(v, t_g) \mathrm{d}v - c_g(t_g) \right).$$

The objective function can be rewritten as:

$$\sum_{g=1,2} n_g \int_V v a_g^e(v) f_g(v, t_g) \mathrm{d}v - n_1 c_1(t_1) - n_2 c_2(t_2).$$

Note the first term is equal to the expectation of a single draw from the distribution $F(v) = F^{n_1}(v, t_1) F^{n_2}(v, t_2)$.

$$\begin{aligned} \implies &= \mathbb{E}_F(v) - n_1 c_1(t_1) - n_2 c_2(t_2), \\ &= \int_V (1 - F^{n_1}(v, t_1) F^{n_2}(v, t_2)) \mathrm{d}v - n_1 c_1(t_1) - n_2 c_2(t_2). \end{aligned}$$

Taking first order conditions, we have that the optimal (t_1^e, t_2^e) jointly solve

$$-\int_v F^{n_1 - 1}(v, t_1) F^{n_2}(v, t_2) \frac{\partial F(v, t_1)}{\partial t_1} \mathrm{d}v - c_1'(t_1) = 0, \tag{6a}$$

$$-\int_v F^{n_1}(v, t_1) F^{n_2 - 1}(v, t_2) \frac{\partial F(v, t_2)}{\partial t_2} \mathrm{d}v - c_2'(t_2) = 0. \tag{6b}$$

But this implies that t_1^e, t_2^e also satisfy (SR)— (6a),(6b) are identical to (FOC) in Lemma 3, since substituting from the interim allocation alloation (5a), (5b), we have:

$$\begin{aligned} -\int_v a_1^e(v) \frac{\partial F(v, t_1)}{\partial t_1} \mathrm{d}v - c_1'(t_1) = 0, \\ -\int_v a_2^e(v) \frac{\partial F(v, t_2)}{\partial t_2} \mathrm{d}v - c_2'(t_2) = 0. \end{aligned}$$

□

Proof of Lemma 5. The auxiliary problem (Opt-Aux) can be trivially rewritten as:

$$\max_{(t_g)_{g=1,2}} \varphi(t_1, t_2) - n_1 c_1(t_1) - n_2 c_2(t_2),$$

where

$$\varphi(t_1, t_2) = \max_{(a_g(\cdot), p_g(\cdot))_{g=1,2}} \sum_{g=1,2} n_g \left(\int_V v a_g(v) f_g(v, t_g) \mathrm{d}v \right)$$
$$\text{s.t.} \qquad (\text{IC}), (\text{IR}), (\text{E}), (\text{F}).$$

By Theorem 1, for any given t_1, t_2, the solution involves a flat subsidy in favor of the target group. Therefore, this is also true for the optimal t_1, t_2. The first of the Lemma follows.

To see the second part, recall that if the seller used a Vickrey auction (with no subsidy), the effort choices would be $t_2^e < t_1^e$. □

Proof of Theorem 2. If E is slack in the solution to the optimal, then the efficient mechanism solves auxiliary problem (Opt-Aux), and we are done by Lemma 4. So let us suppose the efficient mechanism is infeasible. In that case, (E) must bind in the solution to the optimal.

By Lemma 5, the solution will be a flat subsidy in favor of the target group. Let $\varphi(\lambda, t_1, t_2)$ denote the expected value of the winning buyer when a flat subsidy of λ favors the target group, and the two groups have investment levels t_1 and t_2 respectively.

$$\begin{aligned}\varphi(\lambda, t_1, t_2) = \quad & n_1 \int_\lambda^1 v F(v, t_1)^{n_1 - 1} F(v - \lambda, t_2)^{n_2} f(v, t_1) \mathrm{d}v \\ & + n_2 \int_0^{1-\lambda} v F(v, t_2)^{n_2 - 1} F(v + \lambda, t_1)^{n_1} f(v, t_2) \mathrm{d}v \\ & + n_2 \int_{1-\lambda}^1 v F(v, t_2)^{n_2 - 1} f(v, t_2) \mathrm{d}v.\end{aligned}$$

We can therefore rewrite (Opt-Aux) as:

$$\max_{t_1,t_2} \varphi(\lambda, t_1, t_2) - n_1 c_1(t_1) - n_2 c_2(t_2)$$

$$\text{s.t. } n_2 \int_0^1 a_2(v) f(v, t_2) \mathrm{d}v \geq \alpha$$

$$a_2(v) = \begin{cases} F(v, t_2)^{n_2-1} F(v + \lambda, t_1)^{n_1} & v \in [0, 1 - \lambda] \\ F(v, t_2)^{n_2-1} & v \in [1 - \lambda, 1]. \end{cases}$$

$$a_1(v) = \begin{cases} 0 & v \in [0, \lambda] \\ F(v - \lambda, t_2)^{n_2} F(v, t_1)^{n_1} & v \in [\lambda, 1]. \end{cases}$$

We can rewrite as an unconstrained problem:

$$\max_{t_1,t_2} \varphi(\lambda^\star(t_1, t_2), t_1, t_2) - n_1 c_1(t_1) - n_2 c_2(t_2),$$

where $\lambda^\star$ implicitly solves:

$$n_2 \int_0^1 a_2(v) f(v, t_2) \mathrm{d}v \geq \alpha$$

$$a_2(v) = \begin{cases} F(v, t_2)^{n_2-1} F(v + \lambda, t_1)^{n_1} & v \in [0, 1 - \lambda] \\ F(v, t_2)^{n_2-1} & v \in [1 - \lambda, 1]. \end{cases}$$

The first order conditions are:

$$\frac{\partial \varphi(\lambda^\star(t_1, t_2), t_1, t_2)}{\partial \lambda} \frac{\partial \lambda^\star(t_1, t_2)}{\partial t_1} + \frac{\partial \varphi(\lambda^\star(t_1, t_2), t_1, t_2)}{\partial t_1} - n_1 c_1'(t_1) = 0$$

$$\frac{\partial \varphi(\lambda^\star(t_1, t_2), t_1, t_2)}{\partial \lambda} \frac{\partial \lambda^\star(t_1, t_2)}{\partial t_2} + \frac{\partial \varphi(\lambda^\star(t_1, t_2), t_1, t_2)}{\partial t_2} - n_2 c_2'(t_2) = 0$$

Taking derivatives and collecting terms,

$$\frac{\partial \varphi}{\partial \lambda} = -\lambda \int_\lambda^1 \frac{\partial F^{n_1}(v, t_1)}{\partial v} \frac{\partial F^{n_2}(v - \lambda, t_2)}{\partial v} \mathrm{d}v$$

Recall $\lambda^\star(t_1, t_2)$ solves:

$$n_2 \int_0^{1-\lambda} F^{n_2-1}(v, t_2) F^{n_1}(v + \lambda, t_1) f(v, t_2) \mathrm{d}v + n_2 \int_{1-\lambda}^1 F^{n_2-1}(v, t_2) f(v, t_2) \mathrm{d}v = \alpha$$

Differentiating with respect to t_1, we have:

$$n_1 n_2 \frac{\partial \lambda^\star}{\partial t_1} \int_0^{1-\lambda} F^{n_2-1}(v,t_2) F^{n_1-1}(v+\lambda,t_1) f(v,t_2) f(v+\lambda,t_1) \mathrm{d}v$$
$$+ n_1 n_2 \int_0^{1-\lambda} F^{n_2-1}(v,t_2) F^{n_1-1}(v+\lambda,t_1) \frac{\partial F(v+\lambda,t_1)}{\partial t_1} f(v,t_2) \mathrm{d}v = 0$$

Therefore

$$\frac{\partial \varphi}{\partial \lambda} \frac{\partial \lambda^\star}{\partial t_1} = \lambda \int_0^{1-\lambda} \frac{\partial F^{n_2}(v,t_2)}{\partial v} \frac{\partial F^{n_1}(v+\lambda,t_1)}{\partial t_1} \mathrm{d}v.$$

Recall that

$$\begin{aligned} \varphi(\lambda,t_1,t_2) = \quad & n_1 \int_\lambda^1 v F(v,t_1)^{n_1-1} F(v-\lambda,t_2)^{n_2} f(v,t_1) \mathrm{d}v \\ & + n_2 \int_0^{1-\lambda} v F(v,t_2)^{n_2-1} F(v+\lambda,t_1)^{n_1} f(v,t_2) \mathrm{d}v \\ & + n_2 \int_{1-\lambda}^1 v F(v,t_2)^{n_2-1} f(v,t_2) \mathrm{d}v. \end{aligned}$$

Rewriting,

$$\begin{aligned} \varphi(\lambda,t_1,t_2) = \quad & \int_\lambda^1 v \frac{\partial F(v,t_1)^{n_1}}{\partial v} F(v-\lambda,t_2)^{n_2} \mathrm{d}v + \int_0^{1-\lambda} v \frac{\partial F(v,t_2)^{n_2}}{\partial v} F(v+\lambda,t_1)^{n_1} \mathrm{d}v \\ & + \int_{1-\lambda}^1 v \frac{\partial F(v,t_2)^{n_2}}{\partial v} \mathrm{d}v. \end{aligned}$$

Via a change of variables in the first integrand,

$$\begin{aligned} = \quad & \int_0^{1-\lambda} (v+\lambda) \frac{\partial F(v+\lambda,t_1)^{n_1}}{\partial v} F(v,t_2)^{n_2} \mathrm{d}v + \int_0^{1-\lambda} v \frac{\partial F(v,t_2)^{n_2}}{\partial v} F(v+\lambda,t_1)^{n_1} \mathrm{d}v \\ & + \int_{1-\lambda}^1 v \frac{\partial F(v,t_2)^{n_2}}{\partial v} \mathrm{d}v. \end{aligned}$$

Gathering terms:

$$
\begin{aligned}
= \quad & \lambda \int_0^{1-\lambda} \frac{\partial F(v+\lambda, t_1)^{n_1}}{\partial v} F(v, t_2)^{n_2} \mathrm{d}v + \int_0^{1-\lambda} v \frac{\partial F(v, t_2)^{n_2} F(v+\lambda, t_1)^{n_1}}{\partial v} \mathrm{d}v \\
& + \int_{1-\lambda}^{1} v \frac{\partial F(v, t_2)^{n_2}}{\partial v} \mathrm{d}v.
\end{aligned}
$$

Doing integration by parts on each of the three integrands and gathering terms once again,

$$
\begin{aligned}
= \quad & 1 + \lambda - \int_0^{1-\lambda} F^{n_1}(v+\lambda, t_1) \frac{\partial F(v, t_2)^{n_2}}{\partial v} - \int_0^{1-\lambda} F(v, t_2)^{n_2} F(v+\lambda, t_1)^{n_1} \mathrm{d}v \\
& - \int_{1-\lambda}^{1} F(v, t_2)^{n_2} \mathrm{d}v
\end{aligned}
$$

Taking partial derivatives with respect to t_1, we have:

$$
\frac{\partial \varphi}{\partial t_1} = \quad - \int_0^{1-\lambda} \frac{\partial F^{n_1}(v+\lambda, t_1)}{\partial t_1} \frac{\partial F(v, t_2)^{n_2}}{\partial v} - \int_0^{1-\lambda} F(v, t_2)^{n_2} \frac{\partial F(v+\lambda, t_1)^{n_1}}{\partial t_1} \mathrm{d}v
$$

Therefore

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t_1} + \frac{\partial \varphi}{\partial \lambda} \frac{\partial \lambda^\star}{\partial t_1} &= - \int_0^{1-\lambda} F(v, t_2)^{n_2} \frac{\partial F(v+\lambda, t_1)^{n_1}}{\partial t_1} \mathrm{d}v \\
&= - \int_\lambda^1 F(v\lambda, t_2)^{n_2} \frac{\partial F(v, t_1)^{n_1}}{\partial t_1} \mathrm{d}v \\
&= -n_1 \int_\lambda^1 a_1(v) \frac{\partial F(v, t_1)}{\partial t_1} \mathrm{d}v \\
&= -n_1 \int_0^1 a_1(v) \frac{\partial F(v, t_1)}{\partial t_1} \mathrm{d}v.
\end{aligned}
$$

Therefore the first order condition with resepct to t_1 can be written as:

$$
- \int_0^1 a_1(v) \frac{\partial F(v, t_1)}{\partial t_1} \mathrm{d}v - c_1'(t_1) = 0,
$$

which by observation is the sequential rationality constraint for buyers in group 1, (SR) in the first order form via Lemma 3, (FOC). The argument for the target group/ the FOC w.r.t t_2 is completely symmetric and omitted. □

Proof of Theorem 3. Again, the proof follows by a discretization argument analogous to the one in the proof of Theorem 1. The details are omitted. Let $\underline{t}$ be the level of investment that the seller would like to implement. Since the efficient auction violates the seller's problem, this constraint will bind at the optimal solution.

We will continue with our relaxation approach. Let η_g be the dual variable corresponding to (FOC). The 'adjusted' value of a buyer in the target group is:

$$\varphi_2(v) = v + \frac{\eta}{f(v,\underline{t})}\frac{\partial F(v,t_1)}{\partial t},$$

If $\varphi_2(v)$ is non-decreasing in v, then the lemma follows by observation. If $\varphi_2(v)$ is not non-decreasing in v, then (M) is not satisfied 'for free', and additional ironing is required. □

C Code for Example 2

```
gamma2 <- 0.008 # this is the cost faced by our target group

b1d <- 9  # the distribution of the regular group is F(v) = v^b1d

vttl <- function (v,t,l) ((v+l)^(b1d)*v^(t)*log(v))
vtl <- function (v,t) (v^(t) *log(v))

term1 <- function (t,l) (integrate(vttl,0,1-l,t,l)$value)
term2 <- function (t,l) (integrate(vtl,1-l,1,t)$value)

evaluate <- function (l,t,g) (term1(t,l)+ term2(t,l) +g)  #value of the FOC for target buyer when exerting effort t,
#getting l in subsidy and g is the cost per unit effort.

#we now need to find the level of subsidy that will get us going to a particular target level of effort

targeteff <- 9

reqsubsidy <- uniroot(evaluate, c(0,0.75),targeteff, gamma2)$root #tells us how much subsidy we will have to put in.

#ex-ante probability good goes to buyer 2
t1quant <- function (v,t,l) ((v+l)^b1d*t*v^(t-1))
t2quant <- function (v,t) (t*v^(t-1))

quant <- function (t,l) (integrate(t1quant,0,1-l,t,l)$value + integrate(t2quant, 1-l, 1, t)$value)
subsidydistr <- quant(targeteff,reqsubsidy)

#we now similarly need to find the level of set-aside that will get us to that target level of effort

saincintegrand <- function(v,t,alpha) ((alpha+ (1-alpha)*v^b1d)*v^t*log(v))
b2safoc <- function(alpha,t,g) (integrate(saincintegrand,0,1,t,alpha)$value +g)

reqsetaside <- uniroot(b2safoc,c(0,1), targeteff, gamma2)$root

sadist <- reqsetaside + (1-reqsetaside)*quant(targeteff,0)

b1saeff<- b1d

#finally we need to compare implied efficiency.

saefficiency <- reqsetaside*(targeteff/(targeteff+1)) + (1-reqsetaside)*((b1saeff+targeteff)/(1+b1saeff+targeteff)) #that does it for set-aside

b1contrib <- function(v,t,l) (v^b1d*b1d*(v-l)^t)
efficient1 <- function(t,l) (integrate(b1contrib,l,1,t,l)$value)

b2contrib <- function(v,t,l) ((v+l)^b1d*t*v^(t))
```

```
b2contribh <- function(v,t,l) (t*v^t)
efficient2<- function(t,l) (integrate(b2contrib,0,1-l,t,l)$value + integrate(b2contribh,1-l,1,t,l)$value )

subefficiency<- efficient1(targeteff,reqsubsidy)+efficient2(targeteff,reqsubsidy)
#####

pvttl <- function (v,t,l) ((v*(1+l))^(b1d)*v^(t)*log(v))
pvtl <- function (v,t) (v^(t) *log(v))

pterm1 <- function (t,l) (integrate(pvttl,0,(1/(1+l)),t,l)$value)
pterm2 <- function (t,l) (integrate(pvtl,(1/(1+l)),1,t)$value)

pevaluate <- function (l,t,g) (pterm1(t,l)+ pterm2(t,l) +g)

preqsubsidy <- uniroot(pevaluate, c(0,0.75),targeteff, gamma2)$root #tells us how much subsidy we will have to put in.

pb1contrib <- function(v,t,l) (v^b1d*b1d*(v/(1+l))^t)
pefficient1 <- function(t,l) (integrate(pb1contrib,0,1,t,l)$value)

pb2contrib <- function(v,t,l) ((v*(1+l))^b1d*t*v^(t))
pb2contribh <- function(v,t,l) (t*v^t)
pefficient2<- function(t,l) (integrate(pb2contrib,0,1/(1+l),t,l)$value + integrate(pb2contribh,1/(1+l),1,t,l)$value )

psubefficiency<- efficient1(targeteff,preqsubsidy)+efficient2(targeteff,preqsubsidy)
```