Discussion Paper No. 154

STABILITY OF SOCIAL CHOICES IN INFINITELY LARGE SOCIETIES

by

Elisha A. Pazner and Eugene Wesley

June 1975

The work on this paper was supported in part by the
National Science Science Foundation.

STABILITY OF SOCIAL CHOICES IN INFINITELY LARGE SOCIETIES

by

Elisha A. Pazner and Eugene Wesley*

Northwestern University and Tel-Aviv University

I. INTRODUCTION

Arrow's conditions for social welfare functions have been shown by Fishburn [4] and by Kirman and Sondermann [6] to be consistent with the hypothesis that the set of individuals is infinite. Closely related to Arrow's impossibility theorem [1] are the recent results obtained by Gibbard [5] and Satterthwaite [9] according to which every social choice function that is immune to manipulation (misrepresentation of preferences) by a single individual is either imposed or dictatorial whenever the number of individuals is finite. It is natural to enquire whether the apparent duality in the finite case between Arrow's negative results and those of Gibbard and Satterthwaite carries over to the case of an infinite set of individuals. Not surprisingly the answer is affirmative. When the set of individuals is infinite it is not difficult to exhibit an example of a non-dictatorial social choice function that is immune to manipulation by individuals (such an example is provided in Section II). This fact is in agreement with the commonly held notion that in large societies a state of stability can be approached which is not likely to be disturbed by the acts of any individual.

---

* The research of both authors was partially supported by National Science Foundation grants GS-31276X to Leo Hurwicz and GS-31346X to Stanley Reiter. They are indebted to an anonymous referee for helpful editorial suggestions.

A greater threat to stability under these circumstances would stem from the emergence of concerted actions among sizable coalitions. It is therefore of interest to investigate situations in which stability remains intact when joint strategies of groups are also taken into account. In the present context, where we deal with voting, we are led to investigate the effects of misrepresentation of preferences by entire coalitions. We ask whether a coalitionally non-manipulable social choice function is possible when the number of voters is infinite. The existence of a non-imposed, non-dictatorial social choice function F with this property is the subject of the main theorem in Section II.

The social structure engendered by the function F produced below, though non-dictatorial, possesses (as is explained in Section III) some features which are characteristic of oligarchical societies. Suppose, for example, that the set of individuals (voters) V corresponds to the points in the unit interval $[0,1]$. Then under the social choice function F there are arbitrarily small sub-intervals of individuals whose block-votes determine the social choice regardless of the votes of outsiders. That is, for any $\varepsilon > 0$ there is a sub-interval (a,b) of length ε whose members decide the social choice whenever they vote identically. The "ruling" sub-intervals all intersect one another. At the same time, no individual ever has any noticeable influence in determining the final outcome. In this situation, the society V can be seen as being governed by a vanishingly small elite. On the other hand, an almost paradoxical equality prevails between any two voters in view of their insignificance in influencing the social choice. This feature also motivates the discussion of participatory voting in the next section. In particular, the difference between the notions of incentive compatibility and cheatproofness in the social choice context is examined in Section IV.

In the last section we consider the case where the set of alternatives is infinite. We then inquire whether a coalitionally non-manipulable, non-imposed, and non-dictatorial social choice function exists which is also Pareto. It appears that in order for a social choice function with these properties to be possible, the order of infinity (cardinality) of V would have to be extremely large.

II. EXISTENCE OF COALITIONALLY-CHEATPROOF SOCIAL CHOICE FUNCTIONS

Let $\mathcal{A}$ denote a finite set of alternatives and let Σ denote the class of strong orderings (complete, asymmetric and transitive binary relations) on $\mathcal{A}$. Let V denote an infinite set of individuals. A Social Choice Function (SCF) F is a function $F: \Sigma^V \to \mathcal{A}$. An element in the domain of F is called a preference profile and is denoted by $\{P_i\}_{i\in V}$ where $P_i \in \Sigma$ and $i \in V$. An SCF is individually-manipulable at $\{P_i\}_{i\in V} \in \Sigma^V$ if there exists a $\{P_i'\}_{i\in V} \in \Sigma^V$ such that for some i_o, $F(\{P_i'\}_{i\in V}) P_{i_o} F(\{P_i\}_{i\in V})$ where $P_i' = P_i$ for all $i \neq i_o$. An SCF F is individually cheat-proof if there is no preference profile at which it is individually manipulable; F is dictatorial if there exists a j in V such that for all profiles $\{P_i\}_{i\in V}$ and for all $x \neq F(\{P_i\}_{i\in V})$ in the range of F, $F(\{P_i\}_{i\in V}) P_j\, x$; F is imposed if its range is a singleton. Finally, F will be said to satisfy the unanimity criterion if whenever for all i, $x P_i\, a$ for all $a \neq x$ in $\mathcal{A}$, then $F(\{P_i\}_{i\in V}) = x$. As shown by Gibbard [5] and Satterthwaite [9] (a simple proof of which is provided by Schmeidler and Sonnenschein in [10]), when the number of individuals is finite there exists no individually cheat-proof and non-dictatorial SCF if the range of F contains at least three alternatives. As shown by the following example, this impossibility result does not carry over to the case of an infinite set of individuals.

Let $\mathcal{A} = \{a_1, a_2, \ldots a_m\}$ and let V be infinite.
Set $F(\{P_i\}_{i\in V}) = a_j$ $(j = 1, 2, \ldots, m)$ where j is the first index for which $\bar{D}(a_j, \{P_i\}_{i\in V}) \stackrel{\text{def}}{=} \{i\in V \mid a_j$ is maximal with respect to $P_i\}$ is infinite.
This F is clearly not dictatorial; it is not imposed since it satisfies the unanimity criterion. Since no single individual can by himself alter the finiteness or infiniteness of any $\bar{D}(a_j, \{P_i\}_{i\in V})$ $(j = 1, 2, \ldots m)$ this F is clearly individually cheat-proof. This completes our example.

We turn now to the definition of coalitional cheat-proofness. An SCF $F\colon \Sigma^V \to \mathcal{A}$ is said to be <u>coalitionally-manipulable</u> if for some profile $P = \{P_i\}_{i\in V}$ and some non-empty subset $E \subset V$ and some profile $P' = \{P_i'\}_{i\in V}$ where $P_i' = P_i$ for all $i \notin E$, $F(P')P_i\, F(P)$ for all $i \in E$. If F is not coalitionally-manipulable it is said to be <u>coalitionally-cheat-proof</u>.

The main object of the present paper is to establish that when the set of individuals is infinite there exists a non-imposed and non-dictatorial SCF that is coalitionally cheat-proof.

Before proceeding to the main theorem, we cite some definitions and results which will be used in the proof below. For proofs of these results, see Bourbaki [2].

<u>Definition 1</u>: Let V denote an infinite set. A non-empty set $\mathcal{J}$ of subsets of V is a <u>filter</u> on V, if each of the following conditions are met:

(a) Any subset of V which includes an element of $\mathcal{J}$ belongs to $\mathcal{J}$.

(b) Any finite intersection of elements of $\mathcal{J}$ belongs to $\mathcal{J}$.

(c) The empty set does not belong to $\mathcal{F}$.

Definition 2: A set of $\mathcal{U}$ of subsets of V is an ultra-filter on V if:

(a) $\mathcal{U}$ is a filter, and

(b) for every subset S of V, either $S \in \mathcal{U}$ or $S^{comp} \in \mathcal{U}$ (where $S^{comp} = V \setminus S$).

Definition 3: A filter $\mathcal{F}$ over V is a principal filter if there exists an $i \in V$ such that $\{i\} \in \mathcal{F}$. If no such i exists, $\mathcal{F}$ is called a non-principal filter.

Example: Let $\mathcal{F}$ be the set of subsets S of V such that S^{comp} is finite: $\mathcal{F}$ is then a non-principal filter.

Proposition 1: (See Bourbaki [2] for proof): Given any filter $\mathcal{F}$ over V , where V is an infinite set, there exists an ultra-filter $\mathcal{U}$ such that for any $S \in \mathcal{F}$, S is also in $\mathcal{U}$.

Proposition 2: For any finite partition $V_1, V_2, \ldots V_n$ of an infinite set V, ($\bigcup_{i=1}^{m} V_i = V$, $V_i \cap V_j = \emptyset$ for all $i \neq j$), if $\mathcal{U}$ is an ultra-filter on V then there exists an $i_o (i_o = 1, \ldots, n)$ such that $V_{i_o} \in \mathcal{U}$.

We now have the necessary equipment to prove the following:

THEOREM: If V is infinite and $\mathcal{A}$ is finite, then there exists a coalitionally cheat-proof SCF $F: \Sigma^V \to \mathcal{A}$ which is neither imposed nor dictatorial.

Proof: Let $\mathcal{U}$ be a non-principal ultra-filter on V. For all $a \in \mathcal{A}$ and all $P = \{P_i\}_{i \in V} \in \Sigma^V$, let $D(a, P) \overset{def}{=} \{v \in V \mid$ a is maximal with respect to $P_v\}$ and consider the sets $D(a_1, P), D(a_2, P), \ldots, D(a_m, P)$, where m is the number of elements in $\mathcal{A}$. Since $\bigcup_{i=1}^{m} D(a_i, P) = V$ and $D(a_i,P) \cap D(a_j,P) = \emptyset$ for any $i \neq j (i, j = 1, 2, \ldots, m)$ there exists an i_o such that $D(a_{i_o}, P)$ belongs to $\mathcal{U}$. Define the SCF F by $F(P) = a_{i_o}$. Since $\mathcal{U}$ is nonprincipal, F is clearly non-dictatorial; it is not imposed since it satisfies the unanimity condition.

We prove now that F is coalitionally-cheat-proof. Suppose to the contrary that F is coalitionally-manipulable. Then for some $P \in \Sigma^V$ there exists an $E \subset V$ and some $P' \in \Sigma^V$ where $P'_i = P_i$ for all $i \notin E$, such that $F(P') \; P_i \; F(P)$ for all $i \in E$. Let $a = F(P)$ and $a' = F(P')$. Since $E \neq \emptyset$, $F(P) \neq F(P')$ and thus $a \neq a'$. Let $C \overset{def}{=} D(a, P)$. Then $C \in \mathcal{U}$ and there are two possibilities: (a) $E \cap C \neq \emptyset$; (b) $E \cap C = \emptyset$. Consider possiblity (a). If $j \in E \cap C$ then since $j \in C$, $F(P) \; P_j \; F(P')$ implying $j \notin E$, a contradiction. Consider the possibility (b). By definition $F(P') = a'$ where a' satisfies $C' = D(a', P') \in \mathcal{U}$. But $C' \cap C \neq 0$, since both $C' \in \mathcal{U}$ and $C \in \mathcal{U}$. So let $j \in C' \cap C$. Then $a' \; P'_j \; b$ for all $b \neq a' (b \in \mathcal{A})$, since $j \in C'$. Now since $E \cap C = \emptyset$, $j \notin E$ (since $j \in C$) and thus $P'_j = P_j$ by the definition of E. Therefore

$a' \; P_j \; b$ for all $b \neq a'$ $(b \in \mathcal{A})$ and thus in particular $a' \; P_j \; a$, a contradiction to $j \in C$. This completes the proof. We remark that if the strong ordering assumption governing the preference ordering is relaxed and we permit the profiles to include weak orderings as well, then the theorem remains valid.

III. THE SIMULATION OF OLIGARCHICAL SOCIETIES

The SCF $F: \Sigma^V \to \mathcal{A}$ defined in the preceding theorem, though non-dictatorial, generates a social order over V which is strongly reminiscent of an institutional hierarchy. If V corresponds to the set of reals in [0, 1] and F is an SCF defined as above by means of an ultra-filter over V then the following is true: For any $\epsilon > 0$ there exists a subinterval $[a_1, a_2]$ such that $(a_2 - a_1) < \epsilon$ and such that if the individuals in $[a_1, a_2]$ vote in block then under the SCF F these voters get their way regardless of the actions of those outside the interval. The "decisive intervals" all intersect one another. At the same time, no individual has any influence over the final social outcome. For any voter $x \in [0, 1]$, if x's vote is counter to the wishes of all (other) members of some ruling interval, then x is ignored.

Is there an invisible dictator? Kirman and Sondermann [6], in dealing with the social welfare function of Fishburn's work [4], maintained that even in the infinite case an Arrovian social welfare function must be dictatorial in some sense. Although the arguments of Kirman and Sondermann can be carried over to the present situation as well, a more appropriate interpretation in this context would appear to be that a vanishingly small elite rather than an invisible dictator exists.

In other situations, coalitionally cheat-proof SCF's can produce a somewhat more democratic social structure. Consider the following example: Let V_1 be the set of reals in $(-\infty, \infty)$. Then an ultrafilter $\mathcal{U}_1$ over V_1 exists with the following property: Let $F_1 \colon \Sigma^V \to \mathcal{A}$ be defined by means of $\mathcal{U}_1$ as in the theorem above; then for any finite interval (c_1, c_2) if all voters outside the interval unite behind a single alternative $a \in \mathcal{A}$, then the SCF F_1 chooses that alternative, regardless of the voters inside (c_1, c_2).

IV. THE PROBLEM OF PARTICIPATION

A disturbing feature of the social choice function dealt with here is the salient lack of individual motivation to vote, since no individual is ever able to influence the social outcome. The recurrent question of "why bother to vote" stands out quite sharply in the present context. Actually the analysis here may help to shed some light on the difference between incentive compatibility and cheat-proofness in the social choice context.

In the standard Arrow-Debreu model of an economy with pure private goods only, the requirement of cheat-proofness is essentially equivalent to that of incentive compatibility. For, in such an economy even when any single agent has no influence on prices (as is the case for instance when there is a continuum of atomless agents) he clearly does have an incentive to "vote" (e.g. to state his excess-demand under the competitive mechanism) since this is the only way for him to get his most preferred bundle. However, if public goods are present in such an atomless economy, since no single agent can have any influence on the amounts of public goods provided, the "free rider" problem (i.e. lack of incentive compatibility) comes into play. Note that cheat-proofness is not an issue here since the lack of motivation to "vote" implies in particular a lack of motivation to cheat.

In the social choice context, the closest analogy to an economic model would be to a pure public goods economy in which no single commodity is individually appropriable. This follows from the observation that the set of alternatives is common to all and that unrestricted domain of preferences is assumed. In such a case, when the set of voters is infinite, cheat-proofness again results from the fundamental lack of individual incentives to participate in the voting game.

The problem of participatory motivation, though particularly acute in the infinite case is present in the finite case as well. As the existing cheat-proofness literature has essentially ignored the problem of incentive compatibility in this participatory sense, this problem arises there as well.

V. SOCIAL CHOICE FUNCTIONS WITH INFINITE ALTERNATIVES

Having shown the existence of a coalitionally cheat-proof non-dictatorial SCF $F : \Sigma^V \to \mathcal{A}$ when the set of alternative $\mathcal{A}$ is finite, we inquire whether the theorem carries over for the case where $\mathcal{A}$ is infinite. It is easily seen that it does. For if $\mathcal{A}$ is infinite, let $\mathcal{A}' \subset \mathcal{A}$ contain only three alternatives and let Σ_0 be the set of preferences on $\mathcal{A}'$. As before, let Σ be the set of preferences on $\mathcal{A}$. For each $p \in \Sigma$, let $\mathring{p} \in \Sigma_0$ be the preference on $\mathcal{A}'$ obtained by restricting p to $\mathcal{A}'$. For each $P = \{P_i\}_{i \in V} \in \Sigma^V$, let $\mathring{P} = \{\mathring{P}_i\}_{i \in V} \in \Sigma_\circ^V$. In accordance with the foregoing theorem there exists a non-imposed, non-dictatorial, coalitionally cheat-proof SCF $F_\circ : \Sigma_\circ^V \to \mathcal{A}'$. For each $P \in \Sigma^V$, let $F(P) = F_\circ(\mathring{P})$. It is then easily seen that $F : \Sigma^V \to \mathcal{A}$ is non-imposed, non-dictatorial, and coalitionally cheat-proof. However, F is not Pareto. (We say that an SCF $F : \Sigma^V \to \mathcal{A}$ is Pareto if there does not exist $a, b \in \mathcal{A}$ and a profile $P = \{P_i\}_{i \in V}$ such that $F(\{P_i\}_{i \in V}) = a$ and for all $i \in V$, $b \; P_i \; a$.) Clearly, unless restrictions are made on the profiles, no Pareto SCF exists. (Consider a profile $P = \{P_i\}_{i \in V}$ where $P_i = \bar{p}$ ($\bar{p}$ is some fixed preference order) for all $i \in V$, and $\bar{p}$ contains no maximal element.) We are therefore interested in whether a Pareto SCF, F, exists over the set of profiles $\{P = \{P_i\}_{i \in V} \mid$ each P_i contains a maximal element $\}$. More precisely, suppose $\mathcal{A}$ is countably infinite and let Σ_1 be the set of complete preference orders over $\mathcal{A}$ that have maximal elements. Let V be an infinte set of voters. We inquire as to whether there is a coalitionally cheat-proof, non-dictatorial Pareto SCF $F : \Sigma_1^V \to \mathcal{A}$.

Under the assumption that $\mathcal{A}$ is countably infinite it may then be shown (Proposition A1, see Appendix) that if V is countably infinite, no such SCF exists. (This and all that follows remains true if $\mathcal{A}$ is of the same cardinality as the continuum.) It is then natural to ask whether a coalitionally cheat-proof non-dictatorial Pareto SCF $F : \Sigma_1^V \to \mathcal{A}$ exists if V is larger, i.e. if V is uncountable. Using considerations similar to those of Ulam [12] one may in fact show that no such SCF is possible whenever V is of cardinality $\aleph_1, \aleph_2, \aleph_3, \ldots$ or even $\aleph_{\aleph_0}$. This will be seen as a consequence of the discussion which soon follows. The question then remains, does there exist any large enough V for which a coalitionally cheat-proof, non-dictatorial, Pareto SCF $F : \Sigma_1^V \to \mathcal{A}$ exists. The answer depends upon whether a set of measurable cardinality exists.* To see this connection, suppose that V is an infinite set of voters for which a coalitionally cheat-proof, non-dictatorial, Pareto SCF $F : \Sigma_1^V \to \mathcal{A}$ exists, where $\mathcal{A}$ is countably infinite. Given an SCF $F : \Sigma_1^V \to \mathcal{A}$ with these properties, let $\mathcal{U}_1$ be the following set of subsets of V:

(1) $V \in \mathcal{U}_1$

(2) $W \subset V$ is in $\mathcal{U}_1$ iff there exists a pair of preferences $p', p'' \in \Sigma$ with maximal elements $a', a'', a' \neq a''$ such that for the profile $P = \{P_i\}_{i \in V}$ where $P_i = p'$ for $i \in W$, $P_j = p''$ for $j \in W^{comp}$, $F(P) = a'$.

It may be shown that $\mathcal{U}_1$ is a non-principal ultra-filter (Proposition A2, see Appendix). Moreover, if $A_1, A_2, \ldots$ is any countable sequence of sets belonging to the ultra-filter, then $\bigcap_{i=1}^{\infty} A_i$ is also in $\mathcal{U}_1$ (Proposition A3). Thus V must be of measurable cardinality. Conversely, if V is of measurable cardinality then it may be shown that a Pareto, non-dictatorial, coalitionally cheat-proof SCF $F : \Sigma_1^V \to \mathcal{A}$ exists (Proposition A4). It is not known whether sets of measurable cardinality exist. It has been shown [12], however, that sets of measurable cardinality,

---

*A set C is said to be of measurable cardinality if there exists a non-principal ultra-filter $\mathcal{U}$ over C such that for every countable sequence $A_1, A_2, \ldots$ of subsets of C, where $A_i \in \mathcal{U}$ for all i, the set $\bigcap_{i=1}^{\infty} A_i$ also belongs to $\mathcal{U}$. (This definition of measurable cardinal conforms to the one appearing in [11].)

if they exist, are very large. Even if they do exist, their existence cannot be proven within Zermelo-Fraenkel set theory.

One may thus be inclined to believe that for all practical purposes, there is no set of voters V large enough for a coalitionally cheat-proof, non-dictatorial, Pareto SCF $F: \Sigma_1^V \to \mathcal{A}$ to exist. This assessment would doubtless be acceptable from the point of view of the social scientist. Mathematically, however, the question of whether or not a set V exists having this attribute (and which would thus be of measurable cardinality) has been of considerable interest in certain analytical problems. In [7], an affirmative solution to the Borel determinateness problem in game theory is produced under the assumption that measurable cardinals exist. Until recently, no alternative proof was known that did not make similar suppositions.

APPENDIX

Proposition A1: Let $\mathcal{A} = \{a_1, a_2, \ldots\}$ be a countably infinite set of alternatives, and $V = \{1,2,\ldots\}$ be a countably infinite set of voters. Let Σ be the set of strong preference orders on $\mathcal{A}$, let $\Sigma_1 \subset \Sigma$ be the set of preference orders on $\mathcal{A}$ that have maximal elements. If an SCF $F: \Sigma_1^V \to \mathcal{A}$ is Pareto and coalitionally cheatproof then it is dictatorial.

Proof: By method of contradiction, suppose that F is Pareto, coalitionally cheatproof, and non-dictatorial. We consider the following lemma.

Lemma: Let $\{p = p_1, p_2, \ldots, p_n, p_{n+1}, \ldots\}$ be any profile in Σ_1^V such that for some n, $p_{n+1}, = p_{n+2} = \ldots,$ and let $F: \Sigma_1^V \to \mathcal{A}$ be non-dictatorial, coalitionally cheatproof, and Pareto. Then $F(p) = a$, where $a \in \mathcal{A}$ is maximal with respect to p_{n+1}.

Proof of Lemma: Let Σ^{n+1} be the set of profiles $(p_1, \ldots, p_{n+1})$ where for each $1 \leq i \leq n+1$, p_i is a strong preference order over $\mathcal{A}$ with a maximal element. Let $F_{n+1}: \Sigma^{n+1} \to \mathcal{A}$ be defined by

$$F_{n+1}(p_1, \ldots, p_n, p_{n+1}) = F(p_1, p_2, \ldots, p_n, p_{n+1}, p_{n+1}, p_{n+1}, \ldots)$$

Then F_{n+1} is a non-imposed coalitionally cheat-proof SCF defined over profiles of $V_{n+1} \overset{\text{def}}{=} \{1,2,\ldots,n+1\}$. The Gibbard [5] and Satterthwaite [9] theorem (which assumes that V_{n+1} is finite but imposes no upper bound on the size of $\emptyset$) then asserts that F_{n+1} must be dictatorial. If $1 \leq i \leq n$ is a dictator for F_{n+1}, then this same i is a dictator for F, a contradiction. Hence $n+1$ must be a dictator for F_{n+1}. Thus F depends only on p_{n+1}. Hence $F(p) = F(p_1,p_2,\ldots,p_n,p_{n+1},p_{n+1},\ldots) =$ $= F(p') = F(p_1',p_2',\ldots,p_n',p_{n+1},p_{n+2},\ldots)$ for any $p_1',\ldots,p_n'$. In particular, $F(p) = F(p_{n+1},p_{n+1},\ldots,p_{n+1},p_{n+1},\ldots) = a$ by the Pareto condition.

To complete the proof of Proposition A1, consider the profile

$p = p_1,p_2,p_3,\ldots$

where

$$p_1 = a_1 >_{p_1} a_2 >_{p_1} a_3 >_{p_1} a_4 >_{p_1} a_5 >_{p_1} a_6 >_{p_1} a_7 >_{p_1} \ldots.$$

$$p_2 = a_2 >_{p_2} a_1 >_{p_2} a_3 >_{p_2} a_4 >_{p_2} a_5 >_{p_2} a_6 >_{p_2} a_7 >_{p_2} \ldots.$$

$$p_3 = a_3 >_{p_3} a_2 >_{p_3} a_1 >_{p_3} a_4 >_{p_3} a_5 >_{p_3} a_6 >_{p_3} a_7 >_{p_3}$$

$$p_4 = a_4 >_{p_4} a_3 >_{p_4} a_2 >_{p_4} a_1 >_{p_4} a_5 >_{p_4} a_6 >_{p_4} a_7 >_{p_4} \ldots.$$

$$\ldots$$

Then $F(p) = a_n$ for some n. Let $p_1' = p_1, p_2' = p_2,\ldots,p_n' = p_n$, and let

$$p_{n+1}' = a_{n+1} >_{p_{n+1}'} a_n >_{p_{n+1}'} \ldots >_{p_{n+1}'} a_1 >_{p_{n+1}'} a_{n+2} >_{p_{n+1}'} a_{n+3} >_{p_{n+1}'} \ldots$$

and let $p_{n+j}' = p_{n+1}'$ for all positive j. Consider $F(p') = F(p_1',p_2',\ldots)$. In accordance with the lemmas $F(p') = a_{n+1}$. Thus for all i in the coalition $E_{n+2} = \{n+2,n+3,\ldots,\}, F(p') p_i F(p)$, which contradicts coalitional cheat-proofness. The contradiction shows that F must be dictatorial.

Proposition A2: Let $\mathcal{A}$ be a countably infinite set of alternatives and V an infinite set of voters. Let Σ_1 be the set of preference orders over $\mathcal{A}$ that have maximal elements. Suppose there exists a non-dictatorial, Pareto, coalitionally cheatproof SCF $F:\Sigma_1^V \to \mathcal{A}$. Given such an F, let $\mathcal{U}$ be the following family of subsets $W \subset V$:

(1) $W = V \in \mathcal{U}$;

(2) $\emptyset \neq W \subset V$ is in $\mathcal{U}$ iff there exists a pair of preferences $P', P'' \in \Sigma_1$ with maximal elements $a', a'', a' \neq a''$ such that for the profile $p = \{p_i\}_{i \in V}$ with $p_i = P'$ for $i \in W$, $p_j = P''$ for $j \notin W$ $F(p = \{p_i\}_{i \in V}) = a'$.

Then $\mathcal{U}$ is a non-principal ultra-filter.

Proof: We must prove: (a) For each $W_1, W_2 \in \mathcal{U}$, $W_1 \cap W_2 \neq \emptyset$ and $W_1 \cap W_2 \in \mathcal{U}$, and (b) for every $W \subset V$, either $W \in \mathcal{U}$ or $W^{comp} \in \mathcal{U}$.

(a) Suppose $W_1, W_2 \in \mathcal{U}$ and $W_1 \cap W_2 = \emptyset$. Since $W_1 \in \mathcal{U}$, this implies the existence of $P', P'' \in \Sigma_1$ with maximal elements $a', a'', a' \neq a''$ such that for $p = \{p_i\}_{i \in W} \in \Sigma_1^V$ where $p_i = P'$ for $i \in W_1$, $p_j = P''$ for $i \notin W_1$, $F(p) = a'$. However, from Gibbard and Satterthwaite's result about finite sets of voters it follows directly that under F , if the coalitions W_1, W_1^{comp} each vote in a single block, one of them must dictate. The fact that W_1 gets its way for P' implies that W_1 dicates. We further assert that if members of W_1 vote as a block, they dicate even if members of W_2 do not vote as a block. Otherwise coalitional cheat-proofness would be violated. (To see this consider a P''' in Σ_1 which has a' as <u>minimal</u> element, where a' is at the same time the maximal element of P'. Let $\hat{p} = \{\hat{p}_i\}_{i \in V}$ be defined by $\hat{p}_i = P'$ for $i \in W_1$, $\hat{p}_j = P'''$ for $i \notin W_1$. If for some $\bar{p} = \{\bar{p}_i\}_{i \in V}$, where $\bar{p}_i = p_i$ for $i \in W_1$, $F(\bar{p}) \neq a'$, then F is manipulable at $\hat{p}$ for some subset $E' \subset W_1^{comp}$.). Similarly W_2 must get its way whenever its members vote in block. Since $W_1 \cap W_2 = \emptyset$, W_1 and W_2 can vote different ways. Let W_1 and W_2 each vote in block so that the preference of W_1 has a different maximum than that of W_2. Then W_1 and W_2 must each get their way, a contradiction.

Suppose $W_1 \cap W_2$ were not in $\mathcal{U}$. Consider the coalitions $W_1 - W_2$, $W_1 \cap W_2$, $W_2 - W_1$, each of whose members vote in block. The Gibbard-Satterthwaite theorem then implies that one of these three coalitions dictates under F. $W_1 \cap W_2$ does not dicatate; otherwise it would be in $\mathcal{U}$. If $W_1 - W_2$ dictates then W_2 does not dictate, which violates the assumption that $W_2 \in \mathcal{U}$. Similarly if $W_2 - W_1$ dictates we also get a contradiction. Thus $W_1 \cap W_2 \in \mathcal{U}$.

(b) Let W be any non-empty subset of V. If $W^{comp} = \emptyset$ then $W = V \in \mathcal{U}$ by definition of $\mathcal{U}$. Suppose $W^{comp} \neq \emptyset$. Let P', P'' be any two complete preference orders in Σ_1 having maximal elements $a', a'', a' \neq a''$. Let $p = \{p_i\}_{i \in V} \in \Sigma_1^V$ be the profile such that $p_i = P'$ for $i \in W$, $p_i = P''$ for $i \in W^{comp}$. Applying the Gibbard-Satterthwaite result to the present situation, it follows that either

W or W^{comp}, when voting in block, must dictate under F. If W dictates then $W \in \mathcal{U}$, if W^{comp} dictates the $W^{comp} \in \mathcal{U}$.

Proposition A3: Let V, F, and $\mathcal{U}$ be defined as in Proposition A2. Then for any countable sequence of subsets A_1, A_2, . . . of V, where $A_i \in \mathcal{U}$ for all i, the intersections $\bigcap_{i=1}^{\infty} A_i$ is in $\mathcal{U}$.

Proof: Let $B_0 = V$, $B_1 = A_1$, $B_2 = A_1 \cap A_2$, . . ., $B_n = A_1 \cap A_2 \cap \ldots \cap A_n$, . . ., and $C_1 = B_0 - B_1$, $C_2 = B_1 - B_2$, . . ., $C_n = B_{n-1} - B_n$, ... Let $C_0 = \bigcap_{i=1}^{\infty} A_i$. Clearly $C_i \neq C_j$ for all $0 \leq i, j$, and $C_i \notin \mathcal{U}$ for all $i \geq 1$. Moreover, $\bigcup_{i=0}^{\infty} C_i = V$, and $B_i \in \mathcal{U}$ for all i, $i = 0, 1, 2, \ldots$. Suppose that $C_0 \notin \mathcal{U}$. If the members of each coalition C_j vote in block, it then follows as a consequence of Proposition A1 that one coalition, C_i, must dictate. It then easily follows that $C_i \in \mathcal{U}$. Consequently $C_i \cap B_i = \emptyset$ must be in $\mathcal{U}$, a contradiction.

Proposition A4: If V is of measurable cardinality and $\mathcal{A}$ is countably infinite, then there exists a non-dictatorial, Pareto, coalitionally cheat-proof SCF $F: \Sigma_1^V \to \mathcal{A}$.

Proof: Since V is of measurable cardinality, there exists a non-principal ultra-filter, $\mathcal{U}$, over V such that for any countable sequence of sets $A_1, A_2, \ldots, \in \mathcal{U}$, the intersection $\bigcap_{i=1}^{\infty} A_i$ is also in $\mathcal{U}$. It is known (see [3], p. 182 Proposition 4.2.7) that this implies that for any set $B = \{B_t\}_{t \in R}$ of subsets B_t of V, where the index set R is of cardinality $2^{\aleph_0}$ and each $B_t \in \mathcal{U}$, the intersection $\bigcap_{t \in R} B_t$ is then also in $\mathcal{U}$. It therefore follows that for any set $\{C_t\}_{t \in R}$ of subsets of $\mathcal{U}$, if $\bigcup_{t \in R} C_t = V$ and $C_{t_1} \cap C_{t_2} = \emptyset$ for $t_1 \neq t_2$ then for some $t_o \in R$, $C_{t_o} \in \mathcal{U}$. Now given any $p = \{p_i\}_{i \in V} \in \Sigma_1^V$ and any $\overline{P} \in \Sigma_1$, let $Q_{\overline{P}} = \{i | i \in V$ and $p_i = \overline{P}\}$. Since there are only $2^{\aleph_0}$ elements in Σ_1, there are only $2^{\aleph_0}$ possible $Q_{\overline{P}}$'s. For each $P', P'' \in \Sigma_1$, if $P' \neq P''$ then $Q_{P'} \cap Q_{P''} = \emptyset$. Consequently for some $\hat{P} \in \Sigma_1$, $Q_{\hat{P}} \in \mathcal{U}$. We then set $F(p = \{p_i\}_{i \in V})$ equal to the maximal element of that $\hat{P}$. We do the same for any $p \in \Sigma_1^V$. Formally, $F(p = \{p_i\}_{i \in V}) \stackrel{def}{=}$ the maximal element of that $\hat{P} \in \Sigma_1$ such that $\{i | i \in V$ and $p_i = \hat{P}\} \in \mathcal{U}$. It is easily verified that the function $F: \Sigma_1^V \to \mathcal{A}$ constitutes a non-dictatorial, Pareto, coalitionally cheat-proof SCF.

REFERENCES

[1] ARROW, K. J., Social Choice and Individual Value, 2nd ed. Wiley, New York, 1963.

[2] BOURBAKI, N., Elements of Mathematics: General Topology, Part I, Addison-Wesley, Reading, Massachusetts, 1966, Chapter 1, Section 6.

[3] CHANG, C. C. and H. J. KEISLER, Model Theory, North Holland Publishing Company, Amsterdam, London; American Elsevier Publishing Company, Inc., New York, 1973.

[4] FISHBURN, P. C., "Arrow's Impossibility Theorem: Concise Proof and Infinite Voters," Journal of Economic Theory, 2 (1970), 103-106.

[5] GIBBARD, A., "Manipulation of Voting Schemes: A General Result," Econometrica, 41 (1973), 587-601.

[6] KIRMAN, A. P. and D. SONDERMANN, "Arrow's Theorem, Many Agents, and Invisible Dictators," Journal of Economic Theory, 5 (1972), 267-277.

[7] MARTIN, D. A. "Measurable Cardinals and Analytical Games," Fundamental Mathematica 66 (1970), 287-291.

[8] PAZNER, E. A. and E. WESLEY, "Cheatproofness Properties of the Plurality Rule in Large Societies," The Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Illinois, Discussion Paper No. 155, June 1975; accepted for publication in Review of Economic Studies.

[[9] SATTERTHWAITE, M., "Strategy Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions", Journal of Economic Theory, 10 (1975), 187-217.