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On a Fundamental Result in Stochastic Dominance Theory

by

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(Revised version)

ABSTRACT

This paper focuses on the characterization of the first degree stochastic dominance ordering in terms of expectations of monotonic nondecreasing utility functions.

Several difficulties arising in relation to the existence of expectations in the work of Hanoch and Levy [4] are pointed out and resolved through an extended and modified framework.

On a Fundamental Result in Stochastic Dominance Theory

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Several authors [1,3,4,6] have addressed the question of partial ordering of probability distributions in terms of utility functions. In this work we focus on two of the most common orderings.

Given two (cumulative) one-dimensional distribution functions F and G the binary relations first-degree stochastic dominance (FSD) and second-degree stochastic dominance (SSD) are denoted by $(\underset{\sim}{F})$ and $(\underset{\sim}{S})$, respectively, and defined as follows:

$$G(x)(\underset{\sim}{F})F(x) \quad \text{iff} \quad G(x) \leq F(x) \quad \text{for all } x \in R$$

and

$$G(x)(\underset{\sim}{S})F(x) \quad \text{iff} \quad \int_{-\infty}^{x} G(t)dt \leq \int_{-\infty}^{x} F(t)dt \quad \text{for all } x \in R.$$

The integrals throughout are Stieltjes-Lebesgues integrals.

Unifying and integrating previous results on characterizations of the above orderings Hanoch and Levi [4] presented one of the most general frameworks available to date. However, it seems that they have overlooked several existence aspects related to their treatment. In particular, in this paper we show that their results do not apply to several situations that are valid within their given framework. We then suggest an extended set-up and prove its validity for both finite and infinite situations.

Prior to Hanoch and Levi [4], Quirk and Saposnik [6] and Hadar and Russel [3] characterized the first degree stochastic dominance. Their results are summarized by the following theorem.

Theorem 1

Given two distributions F and G, $F(x)\underset{\sim}{(F)}G(x)$ if and only if

$$\int u(x)dF(x) \geq \int u(x)dG(x) \tag{1}$$

for any bounded nondecreasing and continuously differentiable function $u(x)$.

A desirable extension of the above theorem would be any characterization of the class of functions $u(x)$ for which (1) holds, provided $F(x)\underset{\sim}{(F)}G(x)$. An attempt in this direction was made by Hanoch and Levi [4] who suggested a framework where neither differentiability nor boundedness of the utility function u were required. Specifically, they claimed the following:

Theorem 2

Let F and G be two distributions and $u(x)$ a nondecreasing function, with finite values for any finite x; then $F(x)\underset{\sim}{(F)}G(x)$ if and only if

$$\Delta Eu = \int u(x)dF(x) - \int u(x)dG(x) \geq 0.$$

We note in passing that the statement of Theorem 2 generalized only the necessary part ot Theorem 1.

However, Hanoch and Levi did not explicitly address the existence ot the integrals defining ΔEu; even though this is a critical issue when unbounded functions are considered.

In fact, the existence of ΔEu imposes necessary restrictions on the class of functions to be considered.

The proof of Theorem 2 given in [4], is based on the following fundamental lemma. [4, Lemma 1].

Lemma 3

Let F and G be two distributions and u(x) a nondecreasing function with finite values for any finite scaler x; then

$$\Delta Eu = \int u(x)dF(x) - \int u(x)dG(x) = \int [G(x)-F(x)]\, du(x)$$

It can be easily verified that this result does not always hold when both $\int udF$ and $\int udG$ are defined on $[-\infty, \infty]$, since in these cases ΔEu is not well defined.* The key argument of the proof given in [4] is that $\int d[u(F-G)] = 0$. Setting $u(x) = x$ and letting

$$F(x) = \begin{cases} -\frac{1}{x} & x \le -1 \\ 1 & x \ge -1 \end{cases}, \quad G(x) = \begin{cases} -\frac{2}{x} & x \le -2 \\ 1 & x \ge -2 \end{cases}$$

to obtain $\int d[u(F-G)] = -1$, we note that this argument fails when infinite expectations are allowed. Consequently it follows that by introducing the operator ΔEu and using the relation $\int d[u(F-G)] = 0$, Hanoch and Levi have implicitly restricted their treatment to situations where both $\int udF$ and $\int udG$ are finite. Furthermore, a careful examination yields that their proof of Lemma 3 is incomplete and not rigorous even for the finite case, since they have deliberately interchanged $\lim_{x\to\infty} \lim_{n\to\infty} u_n(x)$ with $\lim_{n\to\infty} \lim_{x\to\infty} u_n(x)$.

*We will omit the arguments in the functions appearing in the integrals in those cases where no misunderstanding should arise.

In this work we modify and ratify the statement of Theorem 2 and then prove its validity for both finite and infinite situations.

We first recall that given a function $u(x)$, $u^+(x)$ and $u^-(x)$ are defined as follows:

$$u^+(x) = \begin{cases} u(x) & \text{if } u(x) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } u^-(x) = \begin{cases} -u(x) & u(x) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

If F is a distribution we say that u is F- integrable if both $\int u^+ dF$ and $\int u^- dF$ are finite. u is F- quasi-integrable if at least one of the two integrals $\int u^+ dF$ and $\int u^- dF$ is finite. (See [2,5].) Hence, $\int udF$ is well defined on $[-\infty, \infty]$ for F-quasi integrable functions. ($\int udF = \int u^+ dF - \int u^- dF$.)

The following lemma is the basis for our modification.

<u>Lemma 5</u>

Let F and G be two distributions and let $u(x)$ be a non-decreasing function of the real line to $[-\infty, \infty]$ which is bounded from below (or above) by zero. Suppose that $G(x) \geq F(x)$ for all x, then

$$\int u(x)dG(x) \leq \int u(x)dF(x).$$

<u>Proof</u>. The conditions on $u(x)$ ensure its quasi-integrability with respect to any distribution.

The integrals are defined by respresenting $u(x)$ as the pointwise limit of a sequence of step functions. Furthermore, the monotonicity of $u(x)$ implies that one can choose the step functions to be nondecreasing and their corresponding partitions of the line to be intervals. (See the representation given in [5,p.34]). Hence, we can assume without loss of generality that $u(x)$ itself is a step function.

Suppose that the partition of the line corresponding to $u(x)$ is determined by the points $x_o < x_1, \ldots, < x_n$, where $x_o = -\infty$, $x_n = \infty$. If $x_{i-1} < x < x_i$ denote $u(x) = C_i$, $i=1,\ldots,n$.

Let $R(x)$ be a distribution(hence, continuous on the right) then

$$\int u(x)dR(x) = \sum_{i=1}^{n} C_i \, (R^*(x_i) - R^*(x_{i-1}))$$

where

$$R^*(x_i) = \begin{cases} R(x_i) & \text{if } u(x_i) = C_i \\ R(x_i-) & \text{if } u(x_i) = C_{i+1} \end{cases}$$

Using the monotonicity we observe $C_i < C_{i+1}$ to obtain

$$\int u(x)dG(x) = \sum_{i=1}^{n-1} (C_i - C_{i+1}) \, G^*(x_i) + C_n \leq \sum_{i=1}^{n-1} (C_i - C_{i+1}) F^*(x_i)$$

$$+ C_n = \int u(x)dF(x)$$

and the proof is complete.

Note that if $u(x)$ was assumed to be non-increasing (instead of non-decreasing), we would have $\int udG \geq \int udF$.

Finally we state our modification of Theorem 2. Note that ΔEu is not introduced to avoid the ambiguity of $\infty - \infty$.

Theorem 6

Let F and G be two distributions on the real line. Then $F(x) \underset{\sim}{(F)} G(x)$ if and only if any non-decreasing function $u(x)$ from the real line to $[-\infty,\infty]$ satisfying $\int udG > -\infty$, is also F-quasi-integrable and $\int udF \geq \int udG$.

Proof: Sufficiency. Let y be a given real and consider the following step function

$$Uy(x)=\begin{cases} 0 & x \leq y \\ 1 & x > y \end{cases}$$

$Uy(x)$ is clearly quasi-integrable. If $R(x)$ is a distribution then $\int Uy(x)dR(x) = 1-R(y)$. Hence $\int Uy(x)dF(x) \geq \int Uy(x)dG(x)$ yields $F(y) \leq G(y)$.

Necessity. Suppose that $G(x) \geq F(x)$ for all x. From Lemma 5 we have

$$\int u^+dG \leq \int u^+dF \quad \text{and} \quad \int u^-dG \geq \int u^-dF.$$

$\int udG > -\infty$ implies that $\int u^-dF \leq \int u^-dG < \infty$ and thus $u(x)$ is F- quasi-integrable and

$$\int udF = \int u^+dF - \int u^-dF \geq \int u^+dG - \int u^-dG = \int udG.$$

To conclude our discussion (and motivate the assumption $\int udG > -\infty$ in Theorem 6) we show that if $F(x) \underset{\sim}{(F)} G(x)$ the quasi-integrability with respect to one distribution does not imply this property with respect to the second. First observe that if $F(x)\underset{\sim}{(F)}G(x)$ F - quasi-integrability does not imply G -

quasi-integrability. This is illustrated by the following example:

Let
$$G(x) = \begin{cases} \frac{-1}{2x} & x \leq -1 \\ \frac{1}{2} & -1 \leq x \leq 1 \\ 1 - \frac{1}{2x} & x \geq 1 \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 - \frac{1}{2x} & x \geq 1 \end{cases}$$

and $u(x) = x$.

$\int udF = \infty$ but $\int udG$ is not defined.

The next example shows that F - quasi-integrability is not implied by G - quasi-integrability when $F(x)(\underset{\sim}{F})G(x)$.

Let
$$G(x) = \begin{cases} -\frac{1}{2x} & x \leq -1 \\ \frac{7+x}{12} & -1 \leq x \leq 2 \\ 1 - \frac{1}{x^2} & x \geq 2 \end{cases}$$

$$F(x) = \begin{cases} -\frac{1}{2x} & x \leq -1 \\ \frac{1}{2} & -1 \leq x \leq 2 \\ 1 - \frac{1}{x} & x \geq 2 \end{cases}$$

and $u(x) = x$. $\int udG = -\infty$ but $\int udF$ is not defined.

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