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Preference for Randomization and Ambiguity Aversion

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Preference for Randomization and Ambiguity Aversion

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Abstract

Raiffa (1961) criticizes *ambiguity-averse* preferences by claiming that *hedging* is possible with randomization of choices. We argue that the timing of randomization is crucial for hedging. *Ex-ante randomizations*, which are randomizations before a state is realized, could provide only *ex-ante hedging* but not *ex-post hedging*, in contrast to *ex-post randomizations*, which are randomizations after a state is realized. However, these two randomizations have been assumed to be indifferent under the *reversal of order axiom* proposed by Anscombe and Aumann (1963). We, therefore, propose a weaker axiom, the *indifference axiom*, which allows heterogeneous attitudes toward the timing of randomization. By using this new axiom as well as standard axioms, we provide an extension of Gilboa and Schmeidler's (1989) *Maxmin preferences* that

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treats a preference for ex-ante randomizations separately from a preference for ex-post randomizations. In the representation, a single parameter characterizes a preference for ex-ante randomizations. By parsimoniously changing only the value of that single parameter, the representation can be consistent with Raiffa's (1961) normative argument as well as recent experimental evidence.

Keywords: Ambiguity; randomization; Ellsberg paradox; maxmin utility.

JEL Classification Numbers: D81, D03.

1 Introduction

Ellsberg (1961) proposed the following thought experiment: Consider an urn containing balls, each of which is either red or black. There is no further information about the contents of the urn. You bet on the color of the ball that you will draw. If your bet turns out to be correct, then you get a positive payoff (i.e., 1). Typically, subjects are indifferent between betting on either color. However, they strictly prefer the fifty-fifty objective lottery between 1 and 0 to the bets. This behavior is called *ambiguity aversion*.

Raiffa (1961) criticizes ambiguity-averse preferences with this argument: by flipping a fair coin to choose on which color to bet, you can *hedge* and obtain a constant expected payoff (i.e., the fifty-fifty lottery between 1 and 0) for each color of the ball you will draw. (See Figure 1.) Since this argument has such strong intuitive and normative appeal, this preference for randomization has received little theoretical consideration in the literature.

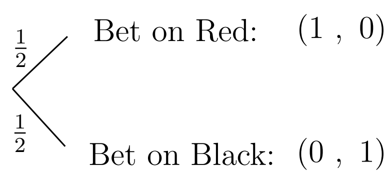

Figure 1: Raiffa's (1961) critique (The first and second coordinates in payoff profiles respectively show the payoffs when the color of the drawn ball is red and black. For each color of the drawn ball, payoffs 1 and 0 are equally likely to occur.)

After careful consideration, however, you would realize that even if you flip a coin, you have to face ambiguity again after either side of coin appears. That is, as the left tree in Figure 2 shows, the randomization performed by flipping a coin is *ex ante*, i.e., before a state (red or black) is realized. In contrast, the randomization with which you can remove all the ambiguity is *ex post*, i.e., after a state is realized.

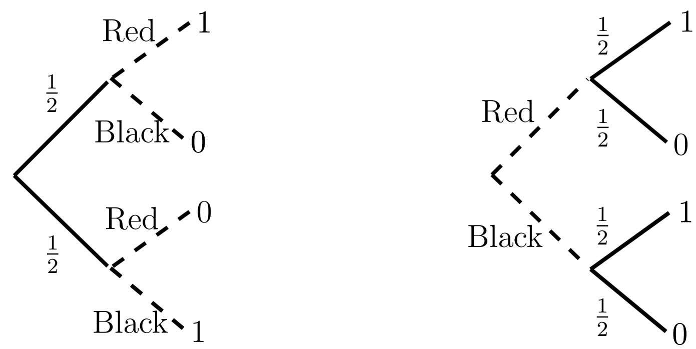

Figure 2: Ex-ante randomization (left) and ex-post randomization (right) in tree (The solid lines correspond to the risk introduced by flipping a coin, while the dotted lines corresponds to the ambiguity of the color of the drawn ball.)

In other words, ex-ante randomizations provide *ex-ante hedging*, i.e., hedging in ex-ante expected payoffs, but not *ex-post hedging*, i.e., hedging in ex-post payoffs. Admittedly, when a coin is flipped in the Ellsberg's (1961) example, the ex-ante expected payoff is constant (i.e., the fifty-fifty lottery between 1 and 0) and involves no ambiguity, as Raiffa (1961) argues. After either side of the coin appears, however, ex-post payoffs associated with each bet is ambiguous.

The above consideration shows that people would treat ex-ante randomizations and ex-post randomizations differently. Indeed, recent experiments found heterogeneous but systematic relationship between attitudes toward ex-ante randomizations and those toward ex-post randomizations.[1] The present paper proposes an axiomatic model of preferences for randomizations which describes the heterogeneous attitudes toward both types of randomizations as well as Raiffa's (1961) normative argument, in a parsimonious yet tractable way.

[1]For instance, see Dominiak and Schnedler (2010) and Spears (2009). These experiments are discussed in detail in Section 1.1.

In one sense, the seminal paper by Anscombe and Aumann (1963) address the issue of the timing of randomizations. In their domain, an *ex-ante randomization* is defined as a lottery on payoff profiles over states of the world, such as the lottery illustrated in Figure 1. Ex-ante randomizations are henceforth indicated by $\oplus$. For example, the ex-ante randomization obtained by flipping a coin in the Ellsberg's (1961) example is denoted as $\frac{1}{2}(1,0) \oplus \frac{1}{2}(0,1)$ and shown as P in Figure 3.

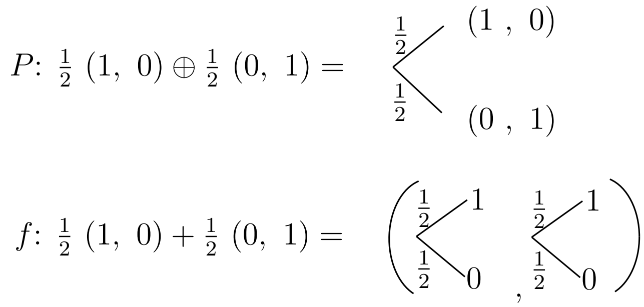

Figure 3: Ex-ante randomization P and ex-post randomization f

In Anscombe and Aumann's (1963) domain, an *ex-post randomization* is defined as a state-wise randomization of payoff profiles and indicated by $+$, as is conventional in the literature. For example, the fifty-fifty ex-post randomization in the Ellsberg's (1961) example, with which you can remove all the ambiguity, is denoted as $\frac{1}{2}(1,0) + \frac{1}{2}(0,1)$ and shown as a constant payoff profile f in Figure 3.

However, one axiom assumed by Anscombe and Aumann (1963), the *reversal of order axiom*, implies that an ex-ante randomization is indifferent to its ex-post randomization. For example, the reversal of order axiom implies that P and f are indifferent. Hence, this axiom precludes the study of a preference for ex-ante randomizations separately from a preference for ex-post randomizations.

For this reason, we do not assume the reversal of order axiom. Instead, we propose a new and weaker axiom, the *indifference axiom*. To explain this new axiom, first notice that one way to justify the reversal of order axiom is a *state-wise* comparison: if you look at P in Figure 3 state-wise (i.e., coordinate-wise), P offers 1 and 0 equally likely for each state in the same way as f. Indeed, this is the comparison which has been implicitly made by

Raiffa (1961): by flipping a coin, you can hedge and obtain the constant expected lottery payoff.

There is, however, another natural comparison to make between P and f. If you look at each payoff profile in the support of P, P offers nonconstant payoff profiles, namely, $(1,0)$ and $(0,1)$, which would be less attractive than the constant payoff profile f under ambiguity aversion. This way of evaluating is called *support-wise* comparison. The indifference axiom states that two ex-ante randomizations are indifferent if the two randomizations are indifferent *not only according to the state-wise comparison but also according to the support-wise comparison*, in contrast to the reversal of order axiom.

Using the indifference axiom together with the standard axioms used in Gilboa and Schmeidler (1989), we characterize the *Ex-ante/Ex-post (EAP) Maxmin preferences* that capture a preference for ex-ante randomizations and also, but separately a preference for ex-post randomizations as follows:

$$V(P) = \delta \min_{\mu \in C} \int_S \Big(\int_{\mathscr{F}} u(f_s) dP(f) \Big) d\mu(s) + (1-\delta) \int_{\mathscr{F}} \Big(\min_{\mu \in C} \int_S u(f_s) d\mu(s) \Big) dP(f), \quad (1)$$

where S is the set of states, C is a subset of the set of all finitely additive probabilities on S, and u is a von Neumann-Morgenstern utility function.

In representation (1), the set C of priors captures a preference for ex-post randomizations as in Gilboa and Schmeidler (1989). On the other hand, the relative weight δ between the first and the second terms captures a preference for ex-ante randomizations, as will be formally shown in Section 3.4. To see this, observe that in the first term, the minimum is taken outside of the integral not only with respect to ex-post randomizations but also with respect to ex-ante randomizations, in contrast to the second term. Therefore, in the first term, ex-ante randomizations provide hedging as much as ex-post randomizations, in contrast to the second term.

Indeed, representation (1) satisfies the reversal of order axiom if and only if $\delta = 1$. Moreover, this special case implies that by flipping a coin, the decision maker can remove all disutilities caused by the ambiguity in the Ellsberg's (1961) example, as Raiffa (1961) argues should be the case. Given that our purpose is to develop a model which does not satisfy the reversal of order axiom, one might wonder why it does not suffice to consider the other special case in which $\delta = 0$. However, this special case trivially implies the *independence axiom* on ex-ante randomizations so that there is *no* strict preference for ex-ante randomizations.

The remainder of Section 1 is organized as follows: Section 1.1 demonstrates how EAP Maxmin preferences can describe recent experimental evidence; in Section 1.2, the related literature is discussed. Section 2 then introduces the setup. Section 3 provides an axiomatization of EAP Maxmin preferences. Axioms are provided in Section 3.1 and the representation theorem and sketch of proof are provided in Section 3.2. Sections 3.3, 3.4, and 3.5 investigate properties of EAP Maxmin preferences. Finally, further relationships between our axioms and the axioms developed by Anscombe and Aumann (1963) are investigated in Section 4. All formal proofs are in the appendix.

1.1 Experiments

Dominiak and Schnedler (2010) have studied the relationship between attitudes toward ex-ante randomizations and those toward ex-post randomizations. EAP Maxmin preferences can parsimoniously describe their experimental evidence. Table 1 shows the numbers of subjects who exhibited a corresponding attitude toward ex-ante and ex-post randomizations.[2]

Dominiak and Schnedler's (2010) experimental result might be summarized by the following two points. First, subjects who prefer ex-post randomizations differ in their attitudes toward ex-ante randomizations. This result is inconsistent not only with the reversal of order axiom but also with Raiffa's (1961) critique. Second, almost all subjects who are ex-post randomization neutral are ex-ante randomization neutral as well. The result for this group therefore contrasts with the result for the previous group in that only this group displays the

[2]In this table, loving and aversion mean strict loving and strict aversion. The table excludes four subjects who exhibited strict ambiguity loving (i.e., strict ex-post randomization aversion) because ambiguity loving is outside of our focus in the present paper.

		Ex-post randomization		
		loving	neutral	
	loving	6	0	$\delta < 0$
Ex-ante randomization	neutral	17	12	$\delta = 0$
	averse	12	2	$\delta > 0$
		35	14	

Table 1: Attitudes toward ex-ante and ex-post randomizations observed by Dominiak and Schnedler (2010) (EAP Maxmin utility model can describe the date just depending on the sign of δ.)

overall consistency predicted by the reversal of order axiom and implied by Raiffa's (1961) critique.

The first observation is explained by the heterogeneity of parameter δ as follows: Suppose EAP Maxmin preferences exhibit ex-post randomization loving. Then, as will be shown in Section 3.5, the preferences exhibit ex-ante randomization loving, neutrality, and aversion, if and only if $\delta > 0$, $\delta = 0$, and $\delta < 0$, respectively, which is consistent with Table 1. The second observation is also consistent with EAP Maxmin preferences. As will be shown in Section 3.5, among EAP Maxmin preferences, ex-post randomization neutrality implies ex-ante randomization neutrality for any δ, which is also consistent with Table 1.

Spears (2009) independently conducted similar experiments to Dominiak and Schnedler (2010) and has obtained similar tendencies. On the other hand, in a field experiment, Dwenger, Kübler, and Weizsäcker (2010) have found a significant evidence for ex-ante randomization loving, which is consistent with $\delta > 0$.

1.2 Related Literature

To our knowledge, no other axiomatic papers have studied a preference for ex-ante randomizations and a preference ex-post randomizations separately.[3]

[3] In a different context of intertemporal decision making under risk, the recursive expected utility models entail non-neutral attitudes toward timing of randomization. For instance, see Kreps and Porteus (1978).

However, there are a few axiomatic papers which relax the reversal of order axiom in different contexts. Among them, to the best of our knowledge, the first is Drèze (1987), which identifies state-dependent utilities and subjective probability. A more recent paper is Seo (2009), which axiomatizes the second order subjective expected utility. Both papers are different from the present paper in the motivations. Indeed, Seo (2009) assumes the independence axiom on ex-ante randomizations, so that no strict preference for ex-ante randomizations.

In terms of applications, the present paper is related to a literature on game theory that studies ambiguity-averse players, in which mixed strategies correspond to ex-ante randomizations (i.e., lotteries on pure strategies). The special cases of EAP Maxmin preferences and EAP Choquet preferences (i.e., Choquet counterpart of EAP Maxmin), where $\delta = 0$ or 1, have been used in the literature as follows:[4] Klibanoff (1996) and Lo (1996) have applied EAP Maxmin preferences with $\delta = 1$; Eichberger and Kelsey (2000) have applied EAP Choquet preferences with $\delta = 0$; Mukerji and Shin (2002) have applied EAP Choquet preferences with $\delta = 0$ as well as with $\delta = 1$. As these authors note, both assumptions $\delta = 1$ and $\delta = 0$ could provide unintuitively extreme predictions in some games, respectively. In such games, it can be shown that, $\delta \in (0, 1)$ could predict more reasonable behavior of ambiguity-averse players than $\delta = 0$ and 1.

2 Setup

For any set X, let $\Delta(X)$ be the set of distributions over X with finite supports. An element in $\Delta(X)$ is called a *lottery* on X. Let $\delta_x \in \Delta(X)$ denote a point mass on x.

Let S be a set of states and let Σ be an algebra of subsets of S. Let Z denote a set of outcomes. A payoff profile f is called an *act* and defined to be a Σ-measurable function from S into $\Delta(Z)$ with finite range. Let $\mathscr{F}$ be the set of all acts.

[4]Since Choquet expected utilities with convex capacity have Maxmin representations, our axiomatization of EAP Maxmin preferences is also an axiomatize EAP Choquet preferences with convex capacities.

A preference relation $\succsim$ is defined on $\Delta(\mathscr{F})$.[5] As usual, $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric parts of $\succsim$. A *constant act* is an act f such that $f(s) = f(s')$ for all $s, s' \in S$.[6] For $f \in \mathscr{F}$, an element $l_f \in \Delta(Z)$ is a *certainty equivalent* for f if $f \sim l_f$.

Finally, ex-ante randomizations and ex-post randomizations are formally defined as follows:

Definition: For all $\alpha \in [0,1]$ and $P, Q \in \Delta(\mathscr{F})$, $\alpha P \oplus (1-\alpha)Q \in \Delta(\mathscr{F})$ is a lottery on acts such that $(\alpha P \oplus (1-\alpha)Q)(f) = \alpha P(f) + (1-\alpha)Q(f) \in [0,1]$ for each $f \in \mathscr{F}$. This operation is called an *ex-ante randomization*.[7]

Definition: For all $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$, $\alpha f + (1-\alpha)g \in \mathscr{F}$ is an act such that $(\alpha f + (1-\alpha)g)(s)(z) = \alpha f(s)(z) + (1-\alpha)g(s)(z) \in [0,1]$ for each $s \in S$ and $z \in Z$. This operation is called an *ex-post randomization*.

3 Axiomatization

To characterize EAP Maxmin preferences, instead of the reversal of order axiom, we assume the indifference axiom as well as the axioms used in Gilboa and Schmeidler (1989).

3.1 Axioms

The first six axioms are due to Gilboa and Schmeidler (1989). However, since the reversal of order axiom is not assumed, both the continuity axiom and the certainty independence axiom are assumed for ex-ante randomizations and also, but separately, for ex-post randomizations.

Axiom (Weak Order): $\succsim$ is complete and transitive.

Axiom (Continuity): $\succsim$ is von Neumann-Morgenstern continuous with respect to ex-ante

[5] Elements in $\Delta(\mathscr{F})$ are denoted by P, Q, and R. Elements in $\mathscr{F}$ are denoted by f, g , and h. Elements in $\Delta(Z)$ are denoted by l, q, and r.

[6] Elements in $\Delta(Z)$ are identified as constant acts.

[7] For degenerate lotteries on acts, we write $\alpha f \oplus (1-\alpha)g \in \Delta(\mathscr{F})$, instead of $\alpha\delta_f \oplus (1-\alpha)\delta_g$, for any $\alpha \in [0,1]$, and $f, g \in \mathscr{F}$.

randomizations as well as ex-post randomizations.[8]

Axiom (Nondegeneracy): There exist $z_+, z_- \in Z$ such that $z_+ \succ z_-$.

Axiom (Monotonicity): For all $f, g \in \mathscr{F}$,

$$f(s) \succsim g(s) \text{ for all } s \in S \Rightarrow f \succsim g.$$

If a preference relation $\succsim$ satisfies the axioms above, then each act $f \in \mathscr{F}$ admits a certainty equivalent $l_f \in \Delta(Z)$. The next axiom is called *uncertainty aversion* in Gilboa and Schmeidler (1989).

Axiom (Ex-post Randomization Loving): For all $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$,

$$f \sim g \Rightarrow \alpha f + (1-\alpha) g \succsim f.$$

Mixing constant acts, ex-ante as well as ex-post, does not provide any hedging. This suggest the next axiom.

Axiom (Ex-ante/Ex-post Certainty Independence):
(i) For all $\alpha \in (0,1]$, $P, Q \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$,

$$P \succsim Q \Leftrightarrow \alpha P \oplus (1-\alpha) l \succsim \alpha Q \oplus (1-\alpha) l.$$

(ii) For all $\alpha \in (0,1]$, $f, g \in \mathscr{F}$, and $l \in \Delta(Z)$,

$$f \succsim g \Leftrightarrow \alpha f + (1-\alpha) l \succsim \alpha g + (1-\alpha) l.$$

As noted in the introduction, the final axiom, the indifference axiom, states that two ex-ante randomizations are indifferent if the two randomizations are indifferent not only according to the state-wise comparison but also according to the support-wise comparison,

[8] Formally, (i) For all $P, Q, R \in \Delta(\mathscr{F})$, if $P \succ Q$ and $Q \succ R$, then there exist α and β in $(0,1)$ such that $\alpha P \oplus (1-\alpha) R \succ Q$ and $Q \succ \beta P \oplus (1-\beta) R$; (ii) For all $f, g, h \in \mathscr{F}$, if $f \succ g$ and $g \succ h$, then there exist α and β in $(0,1)$ such that $\alpha f + (1-\alpha) h \succ g$ and $g \succ \beta f + (1-\beta) h$.

in contrast to the reversal of order axiom. To formalize the state-wise comparison and the support-wise comparison, *state-wise reduction* and *support-wise reduction* of ex-ante randomizations are introduced as follows:

Definition: For all $P \in \Delta(\mathscr{F})$ such that $P = P(f^1)f^1 \oplus \cdots \oplus P(f^n)f^n$,
(i) an act

$$f_P = (P_s)_{s \in S}$$

is called the *state-wise reduction*, where $P_s = P(f^1)f^1(s) + \cdots + P(f^n)f^n(s)$ for all $s \in S$.
(ii) a lottery

$$l_P = P(f^1)l_{f^1} + \cdots + P(f^n)l_{f^n}$$

is called the *support-wise reduction*.[9]

In words, the state-wise reduction f_P offers the reduced marginal distribution P_s of P at each state s.[10] The supportwise reduction l_P offers the certainty equivalent l_f instead of each act f in the support of P. (See Figure 4, for examples of each comparison based on these reductions.) Given these definitions, the indifference axiom is defined as follows:[11]

Axiom (Indifference): For all $P, Q \in \Delta(\mathscr{F})$,

$$\left\{ \begin{array}{ll} \text{(i)} & f_P \sim f_Q \\ \text{(ii)} & l_P \sim l_Q \end{array} \right) \Rightarrow P \sim Q.$$

As suggested, a stronger axiom without condition (ii) is equivalent to the reversal of order axiom.[12] To demonstrate the implication of the indifference axiom, in comparison with the reversal of order axiom, consider two fifty-fifty ex-ante randomizations P and Q shown in Figure 4.

[9]For an act f, certainty equivalent l_f might not be unique. Hence, supportwise reduction l_P might not be unique in general. However, any supportwise reductions are indifferent regardless of various choices of certainty equivalents of each act, under the expected utility on $\Delta(Z)$ implied by the other axioms.

[10]Kreps (1988, p. 106) has also proposed this reduction.

[11]If the indifference axiom is strengthened to apply (i) and (ii) independently (that is, (i) *or* (ii) $\Rightarrow P \sim Q$), together with the other axioms in the theorem, Anscombe and Aumann's (1963) subjective expected utility is obtained.

[12]See Proposition 5 in Section 4, for a formal statement.

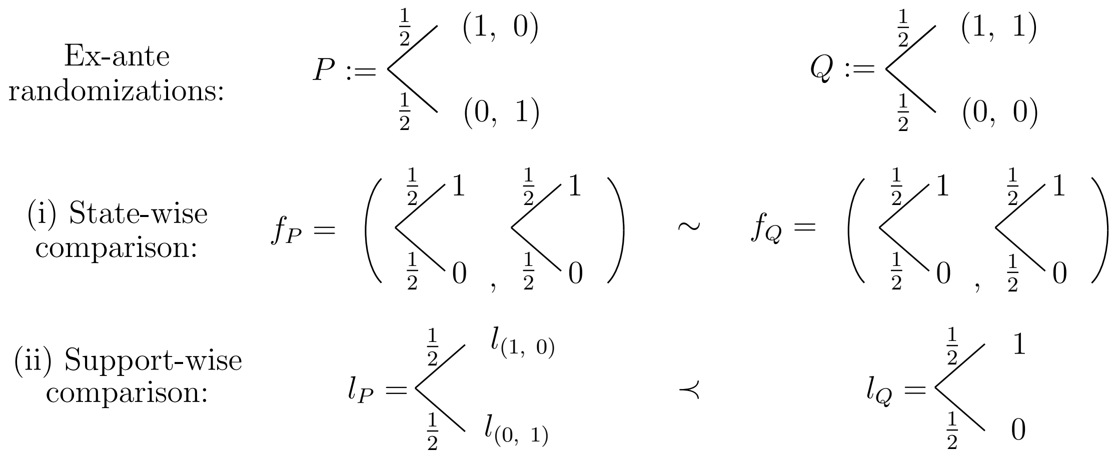

Figure 4: State-wise comparison and support-wise comparison between P and Q (Both are indifferent according to the state-wise comparison but are not indifferent according to the support-wise comparison.)

On the one hand, both P and Q offer the fifty-fifty lottery between 1 and 0 at each state. Therefore, f_P and f_Q become the same, so that condition (i) is satisfied. Hence, the state-wise comparison implies that P and Q are indifferent, as required by the reversal of order axiom.

On the other hand, since the acts in the support of P are ambiguous, l_P becomes a lottery on certainty equivalents of ambiguous acts. While, l_Q becomes the fifty-fifty lottery between 1 and 0 because the acts in the support of Q are constant, namely $(1,1)$ and $(0,0)$. Hence, according to this support-wise comparison, Q is better than P. This is because, under ambiguity aversion, $l_Q \succ (1,0) \sim (0,1) \sim l_P$.[13] Therefore, condition (ii) is not satisfied.

Based on the above two comparisons, therefore, the indifference axiom does not require that P and Q are indifferent, as opposed to the reversal of order axiom. This conclusion of the indifference axiom would intuitively make more sense than that of the reversal of order axiom, because Q is essentially the objective fifty-fifty lottery between 1 and 0 and, hence, involves no ambiguity, in contrast to P.

[13] Under ambiguity aversion, $l_Q \succ (1,0) \sim (0,1)$. Hence, the objective expected utility theory implies $l_P \sim (1,0) \sim (0,1)$.

3.2 Representation

Before stating the result, we mention that the topology to be used on the space of finitely additive set functions on Σ is the weak* topology.

Theorem: *For a preference relation $\succsim$ on $\Delta(\mathscr{F})$, the following statements are equivalent:*
(i) The preference relation satisfies Weak Order, Continuity, Nondegeneracy, Monotonicity, Ex-post Randomization Loving, Ex-ante/Ex-post Certainty Independence, and Indifference.
(ii) There exist a real number δ, a nonempty convex closed set C of finitely additive probability measures on Σ, and a nonconstant mixture linear function $u : \Delta(Z) \to \mathbb{R}$, such that $\succsim$ is represented by the function $V : \Delta(\mathscr{F}) \to \mathbb{R}$ of the form

$$V(P) = \delta \min_{\mu \in C} \int_S \Big(\int_{\mathscr{F}} u(f(s)) dP(f) \Big) d\mu(s) + (1-\delta) \int_{\mathscr{F}} \Big(\min_{\mu \in C} \int_S u(f(s)) d\mu(s) \Big) dP(f).$$

Definition: A preference relation $\succsim$ on $\Delta(\mathscr{F})$ is called an *Ex-ante/Ex-post (EAP) Maxmin* preference if it satisfies axioms in (i) of Theorem.

3.2.1 Sketch of Proof

In this section, we provide a sketch of proof. Formal proof is in the appendix.

By the standard argument as in Gilboa and Schmeidler (1989), there exists a function V representing $\succsim$ on $\Delta(\mathscr{F})$, which is unique up to positive affine transformation. The axioms of ex-post randomization loving, ex-ante/ex-post certainty independence, and indifference will show that V can be taken so that the restriction U of V on $\mathscr{F}$ has a Maxmin representation. That is, there exists a set C of priors and a mixture linear function u on $\Delta(Z)$ such that $U(f) = \min_{\mu \in C} \int_S u(f(s)) d\mu(s)$. To prove this formally, our new axiom, the indifference axiom, is necessary. Without the indifference axiom, U can be any monotonic function of the maximin utility.

Define $\mathscr{D} = \{ \big(U(f_P), U(l_P)\big) \in \mathbb{R}^2 \mid P \in \Delta(\mathscr{F})\}$. On the set $\mathscr{D}$, a binary relation $\hat{\succsim}$ is defined as follows: for all $(a,b),(a',b')\in\mathscr{D}$,

$$(a,b)\,\hat{\succsim}\,(a',b')\Leftrightarrow V(P)\geq V(Q),$$

where $P,Q\in\Delta(\mathscr{F})$, $\big(U(f_P),U(l_P)\big)=(a,b)$, and $\big(U(f_Q),U(l_Q)\big)=(a',b')$. The indifference axiom implies that $\hat{\succsim}$ is a well-defined binary relation.

The purpose of the proof is to show that there exists a real number δ such that for any $(a,b),(a',b')\in\mathscr{D}$, the following equivalence holds:

$$(a,b)\,\hat{\succsim}\,(a',b')\Leftrightarrow \delta a+(1-\delta)b\geq\delta a'+(1-\delta)b'. \tag{2}$$

Together with the definition of $\hat{\succsim}$, this implies that

$$V(P)\geq V(Q)\Leftrightarrow \delta U(f_P)+(1-\delta)U(l_P)\geq\delta U(f_Q)+(1-\delta)U(l_Q).$$

Since both V and U are unique up to positive affine transformation and $V=u=U$ on $\Delta(Z)$, then $V(P)=\delta U(f_P)+(1-\delta)U(l_P)$ for all $P\in\Delta(\mathscr{F})$. A straightforward argument shows that this equation proves that V has an EAP Maxmin representation.

To prove (2), it is convenient to define a subset $\mathscr{C}=\{(u(l),u(l))\in\mathbb{R}^2\mid l\in\Delta(Z)\}$ of $\mathscr{D}$. Then, it will be shown that if $(a,b)\in\mathscr{D}$, $(c,c)\in\mathscr{C}$, and $\alpha\in[0,1]$, then $\alpha(a,b)+(1-\alpha)(c,c)\in\mathscr{D}$. (See Figure 5.) In addition, $\hat{\succsim}$ satisfies completeness, transitivity, a weaker version of continuity, monotonicity on $\mathscr{C}$, and *certainty independence* defined as follows: for all $(a',b'),(a,b)\in\mathscr{D}$, $(c,c)\in\mathscr{C}$, and $\alpha\in[0,1]$, $(a,b)\,\hat{\succsim}\,(a',b')\Leftrightarrow\alpha(a,b)+(1-\alpha)(c,c)\,\hat{\succsim}\,\alpha(a',b')+(1-\alpha)(c,c)$.[14]

It is well-known, however, that in general in $\mathbb{R}^2$, an additive linear representation such as (2) requires more than the independence axiom.[15] In the rest of the proof, we overcome this difficulty by taking two steps. First, we show (2) on a subset $\mathscr{T}$ of $\mathscr{D}$. Then, we extend the result into $\mathscr{D}$.

[14] The certainty independence of $\hat{\succsim}$ from the certainty additivity of U.
[15] See Debreu (1960).

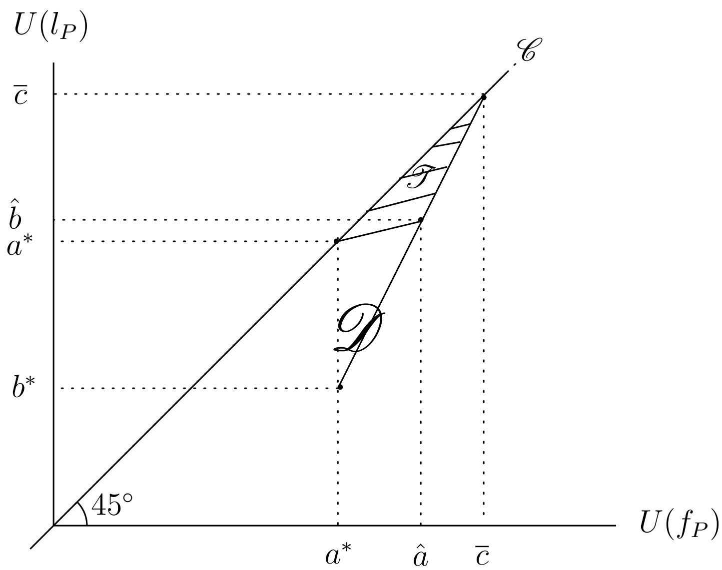

Figure 5: Indifference curves of $\hat{\succsim}$ on $\mathscr{T}$

Fix any point (a^*, b^*) in the interior of $\mathscr{D}$.[16] Consider the case in which $(a^*, a^*) \mathrel{\hat{\succsim}} (a^*, b^*)$.[17] Take $\bar{c}$ such that $\bar{c} \geq a^*$. Then, because of the monotonicity, $(\bar{c}, \bar{c}) \mathrel{\hat{\succsim}} (a^*, a^*)$. Hence, by a weaker version of continuity, we will show that there exists a point $(\hat{a}, \hat{b})$ on the line segment joining $(\bar{c}, \bar{c})$ and (a^*, b^*) such that $(\hat{a}, \hat{b}) \mathrel{\hat{\sim}} (a^*, a^*)$.[18] Since $(a^*, a^*) \in \mathscr{C}$, the certainty independence property shows that any points on the line segment joining (a^*, a^*) and $(\hat{a}, \hat{b})$ are indifferent.

Let $\mathscr{T}$ be the triangle which consists of vertices $(\bar{c}, \bar{c}), (a^*, a^*)$, and $(\hat{a}, \hat{b})$. By this definition, any point in $\mathscr{T}$ can be represented as a convex combination between $(\bar{c}, \bar{c})$ and a point on the line segment joining (a^*, a^*) and $(\hat{a}, \hat{b})$. Since any points on the line segment joining (a^*, a^*) and $(\hat{a}, \hat{b})$ are indifferent and $(\bar{c}, \bar{c})$ belongs to $\mathscr{C}$, therefore, the certainty independence property will show that the indifference curves on $\mathscr{T}$ are parallel to the line segment and, hence, linear, as shown in Figure 5. This is because, mixing $(\bar{c}, \bar{c})$ does not change preferences.

[16] If the set C of priors is nondegenerate, such a point exists. If C is degenerate, then $U(f_P) = U(l_P)$, so that the EAP Maxmin representation holds trivially.

[17] The other case can be proved in a symmetric way. For details, see footnote 24 in the appendix.

[18] In the appendix, the weaker version of continuity is called *certainty continuity* and defined as follows: for all $(a, b), (a', b') \in \mathscr{D}$, $(c, c) \in \mathscr{C}$, if $(a, b) \mathrel{\hat{\succsim}} (a', b') \mathrel{\hat{\succsim}} (c, c)$ or $(c, c) \mathrel{\hat{\succsim}} (a', b') \mathrel{\hat{\succsim}} (a, b)$, there exists $\alpha \in [0, 1]$ such that $(a', b') \mathrel{\hat{\sim}} \alpha(a, b) + (1 - \alpha)(c, c)$.

Finally, to extend this indifference curves on $\mathscr{T}$ into the whole domain $\mathscr{D}$ with linearity preserved, we study a particular property of $\mathscr{D}$. Choose $(a,b) \in \mathscr{D}$. There exists $P \in \Delta(\mathscr{F})$ such that $(a,b) = (U(f_P), U(l_P))$. Let $\mu^* \in \arg\min_{\mu \in C} \int_S (\int_{\mathscr{F}} u(f(s)) dP(f)) d\mu(s)$. Then

$$U(f_P) = \int_{\mathscr{F}} \int_S u(f(s)) d\mu^*(s) dP(f) \geq \int_{\mathscr{F}} \Big(\min_{\mu \in C} \int_S u(f(s)) d\mu(s) \Big) dP(f) = U(l_P).$$

Therefore, for all $(a,b) \in \mathscr{D}$,

$$a \geq b. \tag{3}$$

Let $(c^*, c^*) = \left(\frac{1}{2}\bar{c} + \frac{1}{2}a^*, \frac{1}{2}\bar{c} + \frac{1}{2}a^*\right)$. Then (c^*, c^*) belongs to $\mathscr{C}$ as well as $\mathscr{T}$. By (3), a convex combination of (c^*, c^*) and any point in $\mathscr{D}$ belongs to $\mathscr{T}$, if the relative weight on (c^*, c^*) is close enough to 1. (This statement is not true without (3), because for $(a,b) \in \mathscr{T}$, it must hold that $a \geq b$.) This observation, together with the certainty independence property shows that the indifference curves of $\hat{\succsim}$ on $\mathscr{T}$ can be extended into $\mathscr{D}$ with linearity preserved. Hence, there exists a desired number δ such that (2) holds.[19]

3.2.2 Uniqueness

By Theorem, EAP Maxmin preferences can be represented by a triple (δ, C, u). Next, we give the uniqueness property of this representation.

Proposition 1: *The following two statements are equivalent:*

(i) Two triples (δ, C, u) and (δ', C', u') represent the same EAP Maxmin preference.

(ii) (a) $C = C'$, and there exist real numbers α and β such that $\alpha > 0$ and $u = \alpha u' + \beta$; and

(b) If C is nondegenerate, then $\delta = \delta'$.

[19]In our proof, the certainty additivity and the concavity of V and U play an essential role. These two properties exactly characterize Maxmin preferences. Therefore, with our proof, EAP Maxmin is the most general EAP representation. In Saito (2008, 2010), a stronger EAP representation is obtained in the context of other-regarding preferences.

3.3 Characterizations of δ

The parameter δ has a direct behavioral characterization in terms of ex-ante randomization loving and *preference for late randomization*:

Axiom (Ex-ante Randomization Loving): For all $\alpha \in (0,1)$ and $f, g \in \mathscr{F}$,

$$f \sim g \Rightarrow \alpha f \oplus (1-\alpha) g \succsim f.$$

Ex-ante randomization neutrality and *ex-ante randomization aversion* are defined in the same way by changing the right-hand side of the definition to $\alpha f \oplus (1-\alpha)g \sim f$ and to $f \succsim \alpha f \oplus (1-\alpha)g$, respectively.

Axiom (Preference for Late Randomization): For all $\alpha \in (0,1)$ and $f, g \in \mathscr{F}$,

$$\alpha f + (1-\alpha) g \succsim \alpha f \oplus (1-\alpha) g.$$

Preference for late randomization means that an ex-post randomization is preferred over its ex-ante randomization. This is because an ex-post randomization provides hedging in ex-post utilities, whereas an ex-ante randomization provides hedging only in ex-ante expected utilities. In addition, *indifference for timing of randomization* is defined in the same way by changing $\succsim$ to $\sim$, which is nothing but the reversal of order axiom among two acts.

Proposition 2: *Suppose $\succsim$ is an EAP Maxmin preference with nondegenerate C.*
(i) $\succsim$ exhibits ex-ante randomization loving if and only if $\delta \geq 0$.
(ii) $\succsim$ exhibits a preference for late randomization if and only if $\delta \leq 1$.

Note that given the representation, it is easy to see that EAP Maxmin preferences with $\delta = 0$ and $\delta = 1$ exhibit ex-ante randomization neutrality and an indifference for timing of randomization, respectively.

3.4 Comparative Attitudes toward Ex-ante Randomization

We now study comparative attitudes toward ex-ante randomizations.

Definition: Given two preference relations $\succsim_1$ and $\succsim_2$, $\succsim_1$ is said to be *more ex-ante randomization loving* than $\succsim_2$ if, for every $P \in \Delta(\mathscr{F})$ and every $f \in \mathscr{F}$,

$$P \succsim_2 f \Rightarrow P \succsim_1 f.$$

The next proposition shows that δ captures the attitude toward ex-ante randomizations.[20]

Proposition 3: *Suppose two EAP Maxmin preferences* $\{\succsim_i\}_{i=1,2}$ *are represented by* $\{(\delta_i, C_i, u_i)\}_{i=1,2}$*, where* C_1 *and* C_2 *are nondegenerate. Then, the following statements are equivalent:*
(i) $\succsim_1$ *is more ex-ante randomization loving than* $\succsim_2$*.*
(ii) $\delta_1 \geq \delta_2$*,* $C_1 = C_2$*, and there exist real numbers* α *and* β *such that* $\alpha > 0$ *and* $u_1 = \alpha u_2 + \beta$*.*

Note that in (ii), both of the preferences coincide in C as well as in u under normalization. Therefore, Proposition 3 says that stronger ex-ante randomization loving is characterized only by a *larger* value of δ. Therefore, δ can be interpreted as an *index of ex-ante randomization loving.*

3.5 Relationship between Attitudes toward Ex-ante Randomizations and Attitudes toward Ex-post Randomizations

To conclude this section, implications of EAP Maxmin preferences on the relationship between attitudes toward ex-ante randomizations and those toward ex-post randomizations are characterized. In particular, it will be shown that the implications of EAP Maxmin preferences are consistent with Dominiak and Schnedler's (2010) experimental evidence, which was summarized by two points in Table 1 in Section 1.1 as follows.

Firstly, among strict ex-post randomization loving subjects, the attitude toward ex-ante randomizations is quite heterogeneous; but secondly most ex-post randomization neutral subjects are ex-ante randomization neutral as well. These results are formally described by

[20] Our notion of comparative attitude toward ex-ante randomizations is similar in spirit to the literature on comparative ambiguity aversion such as Ghirardato and Marinacci (2002).

EAP Maxmin preferences as follows:

Proposition 4: *Suppose $\succsim$ is an EAP Maxmin preference.*

(i) (a) Suppose $\delta > 0$. Then, $\succsim$ exhibits strict ex-post randomization loving if and only if $\succsim$ exhibits strict ex-ante randomization loving.[21]

(b) Suppose $\delta < 0$. Then, $\succsim$ exhibits strict ex-post randomization loving if and only if $\succsim$ exhibits strict ex-ante randomization aversion.

(c) Suppose $\delta = 0$. Then, $\succsim$ exhibits ex-ante randomization neutrality.

(ii) For any δ, if $\succsim$ exhibits ex-post randomization neutrality, then $\succsim$ exhibits ex-ante randomization neutrality.

Part (i) shows that the heterogeneity observed in the experiment can be described simply by the sign of δ. Part (ii) shows that among EAP Maxmin preferences, ex-post randomization neutrality implies ex-ante randomization neutrality, as observed in the experiment.

4 Concluding Remarks

To conclude the paper, the relationship between our axioms and axioms used in Anscombe and Aumann (1963) are discussed. As noted, we will show that *the state-wise indifference axiom*, which is a strengthening of the indifference axiom by dropping the support-wise comparison, is equivalent with the reversal of order axiom. Based on this result above, we also show that the reversal of order axiom implies the indifference axiom but not vice versa, and the indifference axiom, in turn, implies the reduction of compound lotteries axiom but not vice versa.

First, the reversal of order axiom proposed by Anscombe and Aumann (1963) is formally defined as follows:

Axiom (Reversal of Order): For all set $\{f^i\}_{i=1}^n$ of acts and set $\{\alpha_i\}_{i=1}^n$ of nonnegative

[21] Strict loving and strict aversion are defined by strict preferences.

numbers such that $\sum_{i=1}^{n} \alpha_i = 1$,

$$\alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n \sim \alpha_1 f^1 + \cdots + \alpha_n f^n.$$

As noted, the reversal of order axiom turns out to be equivalent to the following axiom:

Axiom (State-wise Indifference): For all $P, Q \in \Delta(\mathscr{F})$,

$$f_P \sim f_Q \Rightarrow P \sim Q.$$

Proposition 5: *The reversal of order axiom and the state-wise indifference axiom are equivalent.*

Note that the state-wise indifference axiom is a strengthening of the indifference axiom by dropping the requirement of the support-wise comparison. Hence,

Corollary : *The reversal of order axiom implies the indifference axiom.*

As the example illustrated by Figure 4 in Section 3 shows, the converse of Corollary is not true. However, the indifference axiom implies the reversal of order axiom *among constant acts.* Formally,

Axiom (Reduction of Compound Lotteries): For all set $\{l^i\}_{i=1}^{n}$ of lotteries and set $\{\alpha_i\}_{i=1}^{n}$ of nonnegative numbers such that $\sum_{i=1}^{n} \alpha_i = 1$,

$$\alpha_1 l^1 \oplus \cdots \oplus \alpha_n l^n \sim \alpha_1 l^1 + \cdots + \alpha_n l^n.$$

Proposition 6: *The indifference axiom implies the reduction of compound lotteries axiom.*

Appendix: Proofs

A Proof of Theorem

The necessity of axioms is easy to check. Note that Monotonicity is imposed only on the set $\mathscr{F}$ of acts. Thus, EAP Maxmin preferences immediately satisfy the axiom since the preferences reduce into Gilboa and Schmeidler's (1989) Maxmin preferences on $\mathscr{F}$. To show Continuity, note that the set of finitely additive probabilities measures is compact under the product topology. Therefore, the closed subset C is compact. Hence, the Berge's Maximum Theorem can be applied.

In the following, we will prove the sufficiency. The first lemma shows the existence of a utility function of $\succsim$ as follows:

Lemma 1: *Suppose that the preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem. Then, there exists a utility function V representing $\succsim$ such that*
(i) for all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $V(\alpha P \oplus (1-\alpha)l) = \alpha V(P) + (1-\alpha)V(l)$,
(ii) V is unique up to positive affine transformation,
(iii) moreover, there exists a nonempty convex closed set C of finitely additive probability measures on Σ, and a mixture linear function $u : \Delta(Z) \to \mathbb{R}$ such that $V(f) = \min_{\mu \in C} \int_S u(f(s)) d\mu(s)$.

Proof of Lemma 1: From the implication of the von Neumann-Morgenstern's Theorem, there exists a mixture linear function $u : \Delta(Z) \to \mathbb{R}$ representing $\succsim$ restricted to $\Delta(Z)$. In addition, u is unique up to positive affine transformation. Thus, choose u such that $u(z_+) = 1$ and $u(z_-) = -1$.

For an arbitrary $P \in \Delta(\mathscr{F})$, define

$$M_P = \{\alpha P \oplus (1-\alpha)l | l \in \Delta(Z) \text{ and } \alpha \in [0,1]\}.$$

Thus, M_P is the set of ex-ante randomizations of P and the constant acts. Using the von Neumann-Morgenstern's Theorem again, there is a function $V_P : M_P \to \mathbb{R}$ representing

$\succsim$ restricted to M_P, which is mixture linear with respect to the ex-ante randomizations. In addition, V_P is unique up to positive affine transformation. Thus, choose V_P such that $V_P(z_+) = 1$ and $V_P(z_-) = -1$.

For all $l, r \in \Delta(Z)$ $V_P(l) \geq V_P(r) \Leftrightarrow l \succsim r \Leftrightarrow u(l) \geq u(r)$. Hence, there exists an increasing function $v : u(\Delta(Z)) \to \mathbb{R}$ such that $V_P(l) = v(u(l))$ for all $l \in \Delta(Z)$. Note that by Proposition 6, $\succsim$ satisfies Reduction of Compound Lotteries. This property together with mixture linearities of V_p and u shows that v is also mixture linear.[22] In addition, by the normalization, $v(1) = 1$ and $v(-1) = -1$. Hence, we can conclude that v is the identity function, so that $V_P(l) = u(l)$.

Now, we define a real-valued function V on $\Delta(\mathscr{F})$ which represents $\succsim$ by $V(P) = V_P(P)$ for all $P \in \Delta(\mathscr{F})$. Note that V is well defined, because if $R \in M_P \cap M_Q$, then $V_P(R) = V_Q(R)$. In addition, for all $\alpha \in [0, 1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$,

$$V(\alpha P \oplus (1-\alpha)l) = V_P(\alpha P \oplus (1-\alpha)l) = \alpha V_P(P) + (1-\alpha)V_P(l) = \alpha V(P) + (1-\alpha)V(l).$$

Hence, parts (i) and (ii) hold.

Finally, to show (iii), fix $\alpha \in [0, 1]$, $f \in \mathscr{F}$, and $l \in \Delta(Z)$. Note Indifference shows that for all $\alpha \in [0, 1]$, $f \in \mathscr{F}$, and $l \in \Delta(Z)$, $\alpha f \oplus (1-\alpha)l \sim \alpha f + (1-\alpha)l$.[23] Hence, $V(\alpha f + (1-\alpha)l) = V(\alpha f \oplus (1-\alpha)l) = \alpha V(f) + (1-\alpha)V(l)$, where the second equality is by (i). Moreover, Ex-post Randomization Loving shows that if $V(f) = V(g)$, then $V(\alpha f + (1-\alpha)g) \geq V(f)$ for all $f, g \in \mathscr{F}$ and $\alpha \in [0, 1]$. Therefore, by Gilboa and Schmeidler (1989), these two properties (the certainty additivity and the concavity) and continuity with respect to ex-post randomizations shows that V has a Maxmin representation on $\mathscr{F}$. Hence, part (iii) holds. ■

[22] Choose $a, b \in u(\Delta(Z))$ and $\alpha \in [0, 1]$ to show $v(\alpha a + (1-\alpha)b) = \alpha v(a) + (1-\alpha)v(b)$. There exist $l, r \in \Delta(Z)$ such that $u(l) = a$ and $u(r) = b$. Then by Reduction of Compound Lotteries, $v(\alpha a + (1-\alpha)b) = v(u(\alpha l + (1-\alpha)r)) = V_P(\alpha l + (1-\alpha)r) = V_P(\alpha l \oplus (1-\alpha)r) = \alpha V_P(l) + (1-\alpha)V_P(r) = \alpha v(a) + (1-\alpha)v(b)$.

[23] To see this, let $P = \alpha f \oplus (1-\alpha)l$. Then, for all $s \in S$, $P_s = \alpha f(s) \oplus (1-\alpha)l$, note that condition (i) in Indifference is trivially satisfied. In addition, since $l_f \sim f$, Ex-post Certainty Independence shows $l_P = \alpha l_f + (1-\alpha)l \sim \alpha f + (1-\alpha)l$, so that condition (ii) in Indifference is satisfied as well. Hence, Indifference shows $P \sim \alpha f + (1-\alpha)l$.

Henceforth, we assume that the preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem. Then by Lemma 1, there exists a utility function V representing $\succsim$ and satisfying the properties (i), (ii), and (iii) in Lemma 1. Let U be the restriction of V on $\mathscr{F}$. Thus, U is a maxmin representation defined with a set C of priors and a von Neumann-Morgenstern utility function u. Hereafter, we fix this V, U, C, and u.

In the following, we use the following equations several times. For all $P \in \Delta(\mathscr{F})$,

$$\begin{aligned} U(f_P) &= \min_{\mu\in C}\int_S u(P_s)d\mu(s) &= \min_{\mu\in C}\int_S \Big(\int_{\mathscr{F}} u(f(s))dP(f)\Big)d\mu(s), \\ U(l_P) &= \int_{\mathscr{F}} U(l_f)dP(f) &= \int_{\mathscr{F}} \Big(\min_{\mu\in C}\int_S u(f(s))d\mu(s)\Big)dP(f). \end{aligned} \tag{4}$$

That is, $U(f_P)$ and $U(l_P)$ respectively coincide with the first term and the second term of EAP Maxmin utility.

By using (4), the next lemma proves Theorem in the case where C is degenerate.

Lemma 2: *Suppose that the preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem. If C is degenerate, there exists a real number δ such that (δ, C, u) is an EAP Maxmin representation.*

Proof of Lemma 2: Suppose $C = \{\mu\}$ for some $\mu \in \Delta(S)$. Then for all $P \in \Delta(\mathscr{F})$, $U(f_P) = \int_S \int_{\mathscr{F}} u(f(s))dP(f)d\mu(s) = \int_{\mathscr{F}} \int_S u(f(s))d\mu(s)dP(f) = U(l_P)$, where the second equality is by the Fubini's Theorem. Therefore, $f_P \sim l_P$. Hence, Indifference shows that for all $P \in \Delta(\mathscr{F})$, $P \sim f_P \sim l_P$. Therefore, $V(P) = U(f_P) = U(l_P)$. Thus, there exists a real number δ such that $V(P) = \delta U(f_P) + (1-\delta)U(l_P)$. ■

Hereafter, we consider the case where C is nondegenerate. We will prove three lemmas to complete the proof. First, we will show a representation result for a preference relation $\hat{\succsim}$ on a subset set $\mathscr{D}$ of $\mathbb{R}^2$. To show the statement, we define two properties of the preference as follows:

(i) $\hat{\succsim}$ is said to satisfy *Certainty Independence*, if the following condition holds: for all $(a,b),(a',b') \in \mathscr{D}$, $(c,c) \in \mathscr{C}$, $\alpha \in [0,1]$, $(a,b)\,\hat{\succsim}\,(a',b')$ if and only if $\alpha(a,b) + (1-\alpha)(c,c)\,\hat{\succsim}\,\alpha(a',b') + (1-\alpha)(c,c)$,

(ii) $\hat{\succsim}$ is said to satisfy *Certainty Continuity*, if the following condition holds: for all $(a,b),(a',b') \in \mathscr{D}$, $(c,c) \in \mathscr{C}$, if $(a,b) \mathrel{\hat{\succsim}} (a',b') \mathrel{\hat{\succsim}} (c,c)$ or $(c,c) \mathrel{\hat{\succsim}} (a',b') \mathrel{\hat{\succsim}} (a,b)$, there exists $\alpha \in [0,1]$ such that $(a',b') \mathrel{\hat{\sim}} \alpha(a,b) + (1-\alpha)(c,c)$.

Lemma 3: *Let $\mathscr{D} \subset \mathbb{R}^2$ and $\mathscr{C} = \mathscr{D} \cap \{(a,a)|a \in \mathbb{R}\}$. Suppose that*

(a) for any $(a,b) \in \mathscr{D}$, $a \geq b$,

(b) for all $(a,b) \in \mathscr{D}$, $(c,c) \in \mathscr{C}$, and $\alpha \in [0,1]$, $\alpha(a,b) + (1-\alpha)(c,c) \in \mathscr{D}$,

(c) there exists $(a^*,b^*) \in \mathscr{D}$ and $(\overline{c},\overline{c}),(\underline{c},\underline{c}) \in \mathscr{C}$ such that $a^* > b^*$ and $\overline{c} > a^* > \underline{c}$.

If a preference relation $\hat{\succsim}$ on $\mathscr{D}$ satisfies Completeness, Transitivity, Monotonicity on $\mathscr{C}$, Certainty Independence, and Certainty Continuity, then there exists a unique real number δ such that $(a,b) \mathrel{\hat{\succsim}} (a',b') \Leftrightarrow \delta a + (1-\delta)b \geq \delta a' + (1-\delta)b'$.

Proof of Lemma 3: Consider the case where $(a^*,a^*) \mathrel{\hat{\succsim}} (a^*,b^*)$. By assumption (c), $\overline{c} > a^*$. Monotonicity on $\mathscr{C}$ shows $(\overline{c},\overline{c}) \mathrel{\hat{\succ}} (a^*,a^*)$. Then by Certainty Continuity, there exist $\overline{\alpha} > 0$ such that $(a^*,a^*) \mathrel{\hat{\sim}} \overline{\alpha}(a^*,b^*) + (1-\overline{\alpha})(\overline{c},\overline{c})$. Denote $\big(\overline{\alpha}a^* + (1-\overline{\alpha})\overline{c}, \overline{\alpha}b^* + (1-\overline{\alpha})\overline{c}\big)$ by $(\hat{a},\hat{b})$. [24] Then, $(\hat{a},\hat{b}) \mathrel{\hat{\sim}} (a^*,a^*)$.

Let $\mathscr{T}$ be a triangle including the interior which consists of the vertices $(\overline{c},\overline{c})$, (a^*,a^*), and $(\hat{a},\hat{b})$ as shown in Figure 5 in Section 3.2.1.[25]

The first step shows that the preference $\hat{\succsim}$ is well defined on $\mathscr{T}$.

Step 1: $\mathscr{T}$ is a nondegenerate subset of $\mathscr{D}$.

Proof of Step 1: Since $a^* > b^*$ and, in addition, $\overline{\alpha} > 0$, then $\hat{a} > \hat{b}$. Therefore, $(a^*,a^*) \neq (\hat{a},\hat{b}) \neq (\overline{c},\overline{c})$. Hence, $\mathscr{T}$ is not degenerate. Choose any $(a,b) \in \mathscr{T}$ to show $(a,b) \in \mathscr{D}$. Since $\mathscr{T}$ is the triangle, the Carathéodory's Theorem (Hiriart-Urruty and Lemaréchal (1949, p.

[24] In the other case where $(a^*,b^*) \succsim (a^*,a^*)$, there exist $\underline{\alpha} > 0$ such that $(a^*,a^*) \mathrel{\hat{\sim}} \underline{\alpha}(a^*,b^*) + (1-\underline{\alpha})(\underline{c},\underline{c})$. Denote $\big(\underline{\alpha}\underline{c} + (1-\underline{\alpha})a^*, \underline{\alpha}\underline{c} + (1-\underline{\alpha})b^*\big)$ by $(\tilde{a},\tilde{b})$. Then, instead of the triangle $\mathscr{T}$ defined in the proof, consider a triangle, including the interior, which consists of the vertices $(\tilde{c},\tilde{c})$, $(\underline{c},\underline{c})$, and (a^*,a^*). Then, the rest of the proof goes through exactly in the same way.

[25] Formally, $\mathscr{T} = \big\{(a,b) \in \mathbb{R}^2 | a \geq b, \big\langle (a^*-\hat{a}, \hat{b}-a^*), (a,b)-(a^*,a^*)\big\rangle \geq 0, \big\langle (\overline{c}-\hat{a}, \hat{b}-\overline{c}),\ (a,b)-(\overline{c},\overline{c})\big\rangle \geq 0\big\}$, where $\langle\cdot,\cdot\rangle$ is a inner product.

29, Theorem 1.3.6)) shows that there exist $\alpha, \beta \in [0,1]$ such that

$$\begin{aligned}(a,b) &= \alpha(\overline{c},\overline{c}) + \beta(a^*,a^*) + (1-\alpha-\beta)(\hat{a},\hat{b}) \\ &= (\alpha+\beta)(c,c) + (1-\alpha-\beta)(\hat{a},\hat{b}),\end{aligned}$$

where $c \equiv \frac{\alpha}{\alpha+\beta}\overline{c} + \frac{\beta}{\alpha+\beta}a^*$. Therefore, since $(\hat{a},\hat{b}) \in \mathscr{D}$ and $(c,c) \in \mathscr{C}$, it follows from assumption (b) that $(a,b) \in \mathscr{D}$. $\square$

The next step shows the existence of the desired real number δ on the restricted domain $\mathscr{T}$ as follows:

Step 2: There exists a real number δ such that for any $(a,b),(a',b') \in \mathscr{T}$, $(a,b) \hat{\succsim} (a',b') \Leftrightarrow \delta a + (1-\delta)b \geq \delta a' + (1-\delta)b'$.

Proof of Step 2:

Substep 2.1: For all $(a,b) \in \mathscr{T}$, there exists a unique number $\alpha \in [0,1]$ such that $(a,b) \hat{\sim} \alpha(\overline{c},\overline{c}) + (1-\alpha)(a^*,a^*)$.

Proof of Substep 2.1: Choose any $(a,b) \in \mathscr{T}$. Since $\mathscr{T}$ is the triangle, the Carathéodory's Theorem, again, shows that there exist $\alpha,\beta \in [0,1]$ such that $(a,b) = \alpha(\overline{c},\overline{c}) + \beta(a^*,a^*) + (1-\alpha-\beta)(\hat{a},\hat{b})$. Since $(\hat{a},\hat{b}) \hat{\sim} (a^*,a^*)$, Transitivity and Certainty Independence show $(a,b) \hat{\sim} \alpha(\overline{c},\overline{c}) + (1-\alpha)(a^*,a^*)$. Since $\overline{c} > a^*$, Monotonicity on $\mathscr{C}$ shows that α is unique. Hence, Substep 2.1 is proved.

For all $(a,b) \in \mathscr{T}$, define

$$c(a,b) = \alpha\overline{c} + (1-\alpha)a^*,$$

where α is as in Substep 2.1.

Substep 2.2: For all $(a,b) \in \mathscr{T}$, $\dfrac{a-c(a,b)}{a-b} = \dfrac{\hat{a}-a^*}{\hat{a}-\hat{b}}$.

Proof of Substep 2.2: Choose any $(a,b) \in \mathscr{T}$. Then,

$$\frac{a-c(a,b)}{a-b} = \frac{a-\alpha\overline{c}-(1-\alpha)a^*}{a-b} = \frac{\hat{a}-a^*}{\hat{a}-\hat{b}},$$

The first equality holds because $c(a,b) = \alpha\bar{c} + (1-\alpha)a^*$. To see the second equality, note that by the Carathéodory's Theorem, $(a,b) = \alpha(\bar{c},\bar{c}) + \beta(a^*,a^*) + (1-\alpha-\beta)(\hat{a},\hat{b})$ for some $\alpha,\beta \in [0,1]$. Thus, $a - \alpha\bar{c} - (1-\alpha)a^* = (1-\alpha-\beta)(\hat{a}-a^*)$ and $a - b = (1-\alpha-\beta)(\hat{a}-\hat{b})$. Hence, Substep 2.2 is proved.

Define

$$\delta = 1 - \frac{\hat{a}-a^*}{\hat{a}-\hat{b}}.$$

By substituting this into the result of Substep 2.2, we conclude that for all $(a,b) \in \mathscr{T}$,

$$c(a,b) = \delta a + (1-\delta)b.$$

Substep 2.3. For any $(a,b),(a',b') \in \mathscr{T}$, $(a,b) \mathrel{\hat{\succsim}} (a',b') \Leftrightarrow \delta a + (1-\delta)b \geq \delta a' + (1-\delta)b'$.
Proof of Substep 2.3: Choose any $(a,b),(a',b') \in \mathscr{T}$. Then

$$\begin{aligned}
(a,b) \mathrel{\hat{\succsim}} (a',b') \quad &\Leftrightarrow \quad (c(a,b),c(a,b)) \mathrel{\hat{\succsim}} (c(a',b'),c(a',b')) && (\because \text{Substep 2.1})\\
&\Leftrightarrow \quad c(a,b) \geq c(a',b') && (\because \text{Monotonicity on } \mathscr{C})\\
&\Leftrightarrow \quad \delta a + (1-\delta)b \geq \delta a' + (1-\delta)b'. && (\because \text{Definition of } c) \qquad \square
\end{aligned}$$

The next step extends the result of Step 2 on the whole domain $\mathscr{D}$ as follows:

Step 3: For all $(a,b),(a',b') \in \mathscr{D}$, $(a,b) \mathrel{\hat{\succsim}} (a',b') \Leftrightarrow \delta a + (1-\delta)b \geq \delta a' + (1-\delta)b'$.
Proof of Step 3: Let $c^* = \frac{1}{2}\bar{c} + \frac{1}{2}a^*$. Choose any $(a,b),(a',b') \in \mathscr{D}$. Then, by assumption (a), $a \geq b$ and $a' \geq b'$. Thus, it follows that there exists $\alpha \in (0,1]$ such that $\alpha(a,b) + (1-\alpha)(c^*,c^*)$ and $\alpha(a',b') + (1-\alpha)(c^*,c^*)$ belong to $\mathscr{T}$. Therefore,

$$\begin{aligned}
(a,b) \mathrel{\hat{\succsim}} (a',b') \quad &\Leftrightarrow \quad \alpha(a,b) + (1-\alpha)(c^*,c^*) \mathrel{\hat{\succsim}} \alpha(a',b') + (1-\alpha)(c^*,c^*)\\
&\Leftrightarrow \quad \delta(\alpha a + (1-\alpha)c^*) + (1-\delta)(\alpha b + (1-\alpha)c^*) && (\because \text{Step 2})\\
&\qquad\quad \geq \delta(\alpha a' + (1-\alpha)c^*) + (1-\delta)(\alpha b' + (1-\alpha)c^*)\\
&\Leftrightarrow \quad \delta a + (1-\delta)b \geq \delta a' + (1-\delta)b'. && \square
\end{aligned}$$

This completes the proof of Lemma 3. ■

With using Lemma 3, we prove the sufficiency of Theorem as follows. Let

$$\mathscr{D} = \{(U(f_P), U(l_P)) \in \mathbb{R}^2 | P \in \Delta(\mathscr{F})\} \text{ and } \mathscr{C} = \{(u(l), u(l)) \in \mathbb{R}^2 | l \in \Delta(Z)\}.$$

To define a binary relation $\hat{\succsim}$ on $\mathscr{D}$, it is convenient to define a real-valued function v on $\mathscr{D}$ as follows: For all $(a, b) \in \mathscr{D}$, define $v(a, b) = V(P)$, where $P \in \Delta(\mathscr{F})$ such that $U(f_P) = a$ and $U(l_P) = b$. (It will be shown that v is well defined by Lemma 5.) With this function v, we define a binary relation $\hat{\succsim}$ on $\mathscr{D}$ as follows. For all $(a, b), (a', b') \in \mathscr{D}$, define

$$(a, b) \,\hat{\succsim}\, (a', b') \Leftrightarrow v(a, b) \geq v(a', b').$$

It will be shown by Lemma 4 that the set $\mathscr{D}$ satisfies the properties of (a), (b), and (c) in Lemma 3. In addition, it will be shown by Lemma 5 that the preference relation $\hat{\succsim}$ satisfies the properties in Lemma 3. Define $W(P) = \delta U(f_P) + (1 - \delta)U(l_P)$ for all $P \in \Delta(\mathscr{F})$. Then, by these lemmas, for all $P, Q \in \Delta(\mathscr{F})$,

$$\begin{aligned}
P \succsim Q \;&\Leftrightarrow\; V(P) \geq V(Q) && (\because \text{Lemma 1}) \\
&\Leftrightarrow\; v(U(f_P), U(l_P)) \geq v(U(f_Q), U(l_Q)) && (\because \text{Definition of } v) \\
&\Leftrightarrow\; (U(f_P), U(l_P)) \,\hat{\succsim}\, (U(f_Q), U(l_Q)) && (\because \text{Definition of } \hat{\succsim}) \\
&\Leftrightarrow\; \delta U(f_P) + (1 - \delta)U(l_P) \geq \delta U(f_Q) + (1 - \delta)U(l_Q) && (\because \text{Lemma 3})
\end{aligned}$$

This shows W as well as V represent $\succsim$ on $\Delta(\mathscr{F})$. In addition, for all $P \in \Delta(\mathscr{F}), l \in \Delta(Z)$, and $\alpha \in [0, 1]$,

$$\begin{aligned}
W(\alpha P \oplus (1 - \alpha)l) &= \delta U(f_{\alpha P \oplus (1-\alpha)l}) + (1 - \delta)U(l_{\alpha P \oplus (1-\alpha)l}) \\
&= \delta(\alpha U(f_P) + (1 - \alpha)U(f_l)) + (1 - \delta)(\alpha U(l_P) + (1 - \alpha)U(l_l)) \\
&= \alpha W(P) + (1 - \alpha)W(l).
\end{aligned}$$

Thus, W as well as V satisfy property (i) in Lemma 1. Furthermore, $W = U = V$ on $\mathscr{F}$. Since V is unique up to positive affine transformation, it follows that $V(P) = W(P) \equiv$

$\delta U(f_P)+(1-\delta)U(l_P)$ for all $P\in\Delta(\mathscr{F})$. Substituting (4) shows that V has a EAP Maxmin representation. This completes the sufficiency of Theorem, given Lemmas 4 and 5.

In the following, we will prove Lemma 4 and Lemma 5.

Lemma 4: *Suppose that the preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem. Then, the following results hold: (a) for any $(a,b)\in\mathscr{D}$, $a\geq b$,*
(b) for all $(a,b)\in\mathscr{D}$, $(c,c)\in\mathscr{C}$, and $\alpha\in[0,1]$, $\alpha(a,b)+(1-\alpha)(c,c)\in\mathscr{D}$,
(c) there exists $(a^*,b^*)\in\mathscr{D}$ and $(\overline{c},\overline{c}),(\underline{c},\underline{c})\in\mathscr{C}$ such that $a^*>b^*$ and $\overline{c}>a^*>\underline{c}$.

Proof of Lemma 4: To show (a), choose $(a,b)\in\mathscr{D}$. There exists $P\in\Delta(\mathscr{F})$ such that $(a,b)=(U(f_P),U(l_P))$. Let $\mu^*\in\arg\min_{\mu\in C}\int_S(\int_{\mathscr{F}}u(f(s))dP(f))d\mu(s)$. Then

$$a=U(f_P)=\int_{\mathscr{F}}\int_S u(f(s))d\mu^*(s)dP(f)\geq\int_{\mathscr{F}}\Big(\min_{\mu\in C}\int_S u(f(s))d\mu(s)\Big)dP(f)=U(l_P)=b.$$

To show (b), choose any $(a,b)\in\mathscr{D}$, $(c,c)\in\mathscr{C}$, and $\alpha\in[0,1]$. Then, there exist $P\in\Delta(\mathscr{F})$ and $l\in\Delta(Z)$ such that $(a,b)=(U(f_P),U(l_P))$ and $u(l)=c$. Hence, a direct calculation based on (4) shows $U(f_{\alpha P\oplus(1-\alpha)l})=\alpha U(f_P)+(1-\alpha)u(l)=\alpha a+(1-\alpha)c$ and $U(l_{\alpha P\oplus(1-\alpha)l})=\alpha U(l_P)+(1-\alpha)u(l)=\alpha b+(1-\alpha)c$. Therefore, $\alpha(a,b)+(1-\alpha)(c,c)\in\mathscr{D}$.

In the following, we will show (c). By the nondegeneracy of C, there exist $f,g\in\mathscr{F}$ such that $\frac{1}{2}f+\frac{1}{2}g\succ f\sim g$.[26] There exists $z_+,z_-\in Z$ such that $z_+\succ z_-$ because of Nondegeneracy of $\succsim$. Let $l_0=\frac{1}{2}\delta_{z_+}+\frac{1}{2}\delta_{z_-}$. By appropriately mixing l_0 with f and g, we respectively obtain f^* and g^* such that $\frac{1}{2}f^*+\frac{1}{2}g^*\succ f^*\sim g^*$ and $z_+\succ\frac{1}{2}f^*+\frac{1}{2}g^*\succ z_-$. Then, $U(f_{\frac{1}{2}f^*\oplus\frac{1}{2}g^*})=U(\frac{1}{2}f^*+\frac{1}{2}g^*)>U(f^*)=U(l_{\frac{1}{2}f^*\oplus\frac{1}{2}g^*})$. In addition, $U(z_+)>U(\frac{1}{2}f^*+\frac{1}{2}g^*)>U(z_-)$.

Define $(a^*,b^*)=(U(f_{\frac{1}{2}f^*\oplus\frac{1}{2}g^*}),U(l_{\frac{1}{2}f^*\oplus\frac{1}{2}g^*}))$, $\overline{c}=U(z_+)$, and $\underline{c}=U(z_-)$. Therefore, $a^*>b^*$ and $\overline{c}>a^*>\underline{c}$. ■

Lemma 5: *Suppose that the preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem. Then, the preference relation $\hat{\succsim}$ satisfies Completeness, Transitivity, and Monotonicity on*

[26] Otherwise, $f\sim g\Rightarrow\frac{1}{2}f+\frac{1}{2}g\sim f$ for all $f,g\in\mathscr{F}$. This implies the subjective expected utility, so that C becomes degenerate.

$\mathscr{C}$, Certainty Independence, and Certainty Continuity.

Proof of Lemma 5: To show Completeness and Transitivity, it suffices to show v is a well-defined function. That is, if there exist $P, Q \in \Delta(\mathscr{F})$ such that $(U(f_P), U(l_P)) = (U(f_Q), U(l_Q))$, then $V(P) = V(Q)$. The equality $(U(f_P), U(l_P)) = (U(f_Q), U(l_Q))$ shows that conditions (i) and (ii) in Indifference are satisfied. Therefore, Indifference shows $V(P) = V(Q)$.

To show Monotonicity in $\mathscr{C}$, choose any $(c, c), (c', c') \in \mathscr{C}$. Then there exist $l, l' \in \Delta(Z)$ such that $u(l) = c$ and $u(l') = c'$. Hence, $(c, c) \hat{\succsim} (c', c') \Leftrightarrow v(u(l), u(l)) \geq v(u(l'), u(l')) \Leftrightarrow V(l) \geq V(l') \Leftrightarrow u(l) \geq u(l') \Leftrightarrow c \geq c'$.

To show Certainty Independence, choose any $(a, b), (a', b'), (c, c) \in \mathscr{D}$ and $\alpha \in [0, 1]$. Then, there exist $P, Q \in \Delta(\mathscr{F})$ and $l \in \Delta(Z)$ such that $(a, b) = (U(f_P), U(l_P))$, $(a', b') = (U(f_Q), U(l_Q))$, and $(c, c) = (u(l), u(l))$. A direct calculation based on (4) shows

$$\begin{aligned}
\alpha(a, b) + (1 - \alpha)(c, c) &= (U(f_{\alpha P \oplus (1-\alpha)l}), U(l_{\alpha P \oplus (1-\alpha)l})), \\
\alpha(a', b') + (1 - \alpha)(c, c) &= (U(f_{\alpha Q \oplus (1-\alpha)l}), U(l_{\alpha Q \oplus (1-\alpha)l})).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(a, b) \hat{\succsim} (a', b') &\Leftrightarrow V(P) \geq V(Q) \\
&\Leftrightarrow \alpha V(P) + (1 - \alpha)V(l) \geq \alpha V(Q) + (1 - \alpha)V(l) \\
&\Leftrightarrow V(\alpha P \oplus (1 - \alpha)l) \geq V(\alpha Q \oplus (1 - \alpha)l) \qquad (\because \text{Lemma 1 (i)}) \\
&\Leftrightarrow \alpha(a, b) + (1 - \alpha)(c, c) \hat{\succsim} \alpha(a', b') + (1 - \alpha)(c, c).
\end{aligned}$$

To show Certainty Continuity, choose any $(a, b), (a', b') \in \mathscr{D}$, $(c, c) \in \mathscr{C}$ such that $(a, b) \hat{\succsim} (a', b') \hat{\succsim} (c, c)$. Then, there exist $P, Q \in \Delta(\mathscr{F})$ and $l \in \Delta(Z)$ such that $(U(f_P), U(l_P)) = (a, b)$, $(U(f_Q), U(l_Q)) = (a', b')$, and $u(l) = c$. Then $P \succsim Q \succsim l$. Then by Continuity, there exists $\alpha \in [0, 1]$ such that $Q \sim \alpha P \oplus (1 - \alpha)l$. Therefore,

$$(a', b') = (U(f_Q), U(l_Q)) \hat{\sim} (U(f_{\alpha P \oplus (1-\alpha)l}), U(l_{\alpha P \oplus (1-\alpha)l})) = \alpha(a, b) + (1 - \alpha)(c, c).$$

The other case where $(c,c)\hat{\succsim}(a',b')\hat{\succsim}(a,b)$ is proved in the same way.

■

B Proof of Propositions

To prove propositions, it is convenient to show the following claim:

Claim: *For EAP Maxmin preferences, if C is nondegenerate, then there exist $f^*, g^*, \hat{f}, \hat{g} \in \mathscr{F}$ such that $\frac{1}{2}f^* + \frac{1}{2}g^* \succ f^* \sim g^*$, $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ \hat{f} \sim \hat{g}$, and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim \frac{1}{2}f^* + \frac{1}{2}g^*$.*

Proof of Claim: By the proof of statement (c) in Lemma 4, if C is nondegenerate, then there exist $f^*, g^* \in \mathscr{F}$ such that $\frac{1}{2}f^* + \frac{1}{2}g^* \succ f^* \sim g^*$ and $z_+ \succ \frac{1}{2}f^* + \frac{1}{2}g^* \succ z_-$. We will show the existence of $\hat{f}$ and $\hat{g}$ as in the claim, in the following two exhaustive cases.

Case 1: $\frac{1}{2}f^* + \frac{1}{2}g^* \succsim \frac{1}{2}f^* \oplus \frac{1}{2}g^*$. By Continuity, there exists $\alpha \in [0,1]$ such that $\frac{1}{2}f^* + \frac{1}{2}g^* \sim \alpha\delta_{z_+} \oplus (1-\alpha)(\frac{1}{2}f^* \oplus \frac{1}{2}g^*)$. Let $\hat{f} = \alpha\delta_{z_+} + (1-\alpha)f^*$ and $\hat{g} = \alpha\delta_{z_+} + (1-\alpha)g^*$. Then, $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ \hat{f} \sim \hat{g}$ and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim \frac{1}{2}f^* + \frac{1}{2}g^*$.

Case 2: $\frac{1}{2}f^* \oplus \frac{1}{2}g^* \succsim \frac{1}{2}f^* + \frac{1}{2}g^*$. By mixing z_-, instead of z_+, the claim is proved in the same way as in Case 1. ■

B.1 Proof of Proposition 1

It is easy to see that (ii) implies (i). In the following, we will show that (i) implies (ii). Fix $\succsim$ on $\Delta(\mathscr{F})$. Let (δ, C, u) and (δ', C', u') represent $\succsim$, then u and u' are affine representations of $\succsim$ restricted on $\Delta(Z)$. Hence, by the standard uniqueness results, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u = \alpha u' + \beta$. The uniqueness of C follows from Gilboa and Schmeidler (1989), so that $C = C'$.

To show $\delta = \delta'$, let V and V' be EAP Maxmin representations defined by (δ, C, u) and (δ', C', u'), respectively. Let U and U' be the restrictions of V and V' on $\mathscr{F}$, respectively. Then, $U = \alpha U' + \beta$. Since C is nondegenerate, Claim shows that there exist $f^*, g^*, \hat{f}, \hat{g} \in \mathscr{F}$ such that $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ \hat{f} \sim \hat{g}$, and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim \frac{1}{2}f^* + \frac{1}{2}g^*$

Hence, $U(\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g}) > U(\hat{f})$ and $U(\frac{1}{2}f^*+\frac{1}{2}g^*) = \delta U(\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g}) + (1-\delta)U(\hat{f})$. Therefore,

$$\delta = \frac{U(\frac{1}{2}f^*+\frac{1}{2}g^*) - U(\hat{f})}{U(\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g}) - U(\hat{f})} = \frac{U'(\frac{1}{2}f^*+\frac{1}{2}g^*) - U'(\hat{f})}{U'(\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g}) - U'(\hat{f})} = \delta',$$

where the second equality holds because $U = \alpha U' + \beta$.

B.2 Proof of Proposition 2

Suppose $\succsim$ is an EAP Maxmin preference represented by V with nondegenerate C. Let U be the restriction of V on $\mathscr{F}$. By the nondegeneracy of C, there exist $f^*, g^* \in \mathscr{F}$ such that $U(\frac{1}{2}f^*+\frac{1}{2}g^*) > U(f^*) = U(g^*)$.

To show (i), fix $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$ such that $f \sim g$. Since $V(\alpha f \oplus (1-\alpha)g) = \delta U(\alpha f + (1-\alpha)g) + (1-\delta)U(f)$, then

$$\begin{aligned} V(\alpha f \oplus (1-\alpha)g) \geq U(f) \quad &\Leftrightarrow \quad \delta U(\alpha f + (1-\alpha)g) \geq \delta U(f) \\ &\Leftrightarrow \quad \delta(U(\alpha f + (1-\alpha)g) - U(f)) \geq 0. \end{aligned}$$

Since $U(\alpha f + (1-\alpha)g) \geq U(f)$ and $U(\frac{1}{2}f^*+\frac{1}{2}g^*) > U(f^*)$, therefore, $\succsim$ exhibits ex-ante randomization loving if and only if $\delta \geq 0$.

To show (ii), fix $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$. Since $V(\alpha f \oplus (1-\alpha)g) = \delta U(\alpha f + (1-\alpha)g) + (1-\delta)(\alpha U(f) + (1-\alpha)U(g))$, then

$$\begin{aligned} & U(\alpha f + (1-\alpha)g) \geq V(\alpha f \oplus (1-\alpha)g) \\ \Leftrightarrow \quad & (1-\delta)U(\alpha f + (1-\alpha)g) \geq (1-\delta)(\alpha U(f) + (1-\alpha)U(g)) \\ \Leftrightarrow \quad & (1-\delta)(U(\alpha f + (1-\alpha)g) - (\alpha U(f) + (1-\alpha)U(g))) \geq 0. \end{aligned}$$

Since $U(\alpha f + (1-\alpha)g) \geq \alpha U(f) + (1-\alpha)U(g)$ and $U(\frac{1}{2}f^*+\frac{1}{2}g^*) > \frac{1}{2}U(f^*) + \frac{1}{2}U(g^*)$, therefore, $\succsim$ exhibits a preference for late randomization if and only if $\delta \leq 1$.

B.3 Proof of Proposition 3

Fix two EAP Maxmin preferences $\{\succsim_i\}_{i=1,2}$ represented by $\{(\delta_i, C_i, u_i)\}_{i=1,2}$. For all $i \in \{1,2\}$, suppose that C_i is nondegenerate; let V_i be EAP Maxmin representation defined with (δ_i, C_i, u_i); and let U_i be the restriction of V_i on $\mathscr{F}$.

Step 1: (i) implies (ii).

Proof of Step 1: Suppose $\succsim_1$ is more ex-ante randomization loving than $\succsim_2$. A straightforward argument shows $U_1 = U_2$, so that $C_1 = C_2$.[27] In the following, we will show $\delta_1 \geq \delta_2$.

Since C_2 is nondegenerate, Claim shows that there exist $\hat{f}, \hat{g} \in \mathscr{F}$ such that $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ_2 \hat{f} \sim_2 \hat{g}$ and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim_2 \frac{1}{2}f^* + \frac{1}{2}g^*$. Since $U_1 = U_2$, $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ_1 \hat{f} \sim_1 \hat{g}$. Since $\succsim_1$ is more ex-ante randomization loving than $\succsim_2$, $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \succsim_1 \frac{1}{2}f^* + \frac{1}{2}g^*$. Since $U_i(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) - U_i(\hat{f}) > 0$ for all $i \in \{1,2\}$, therefore,

$$
\begin{aligned}
\delta_1 &\geq \frac{U_1(\frac{1}{2}f^* + \frac{1}{2}g^*) - U_1(\hat{f})}{U_1(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) - U_1(\hat{f})} && \Big(\because \delta_1 U_1(\tfrac{1}{2}\hat{f} + \tfrac{1}{2}\hat{g}) + (1-\delta_1)U_1(\hat{f}) = V_1(\tfrac{1}{2}\hat{f} \oplus \tfrac{1}{2}\hat{g}) \geq U_1(\tfrac{1}{2}f^* + \tfrac{1}{2}g^*)\Big) \\
&= \frac{U_2(\frac{1}{2}f^* + \frac{1}{2}g^*) - U_2(\hat{f})}{U_2(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) - U_2(\hat{f})} && (\because U_1 = U_2) \\
&= \delta_2. && \Big(\because \delta_2 U_2(\tfrac{1}{2}\hat{f} + \tfrac{1}{2}\hat{g}) + (1-\delta_2)U_2(\hat{f}) = V_2(\tfrac{1}{2}\hat{f} \oplus \tfrac{1}{2}\hat{g}) = U_2(\tfrac{1}{2}f^* + \tfrac{1}{2}g^*)\Big)
\end{aligned}
$$

$\square$

Step 2: (ii) implies (i).

Proof of Step 2: Suppose $\delta_1 \geq \delta_2$, $C_1 = C_2$, and there exist $\alpha > 0$, $\beta \in \mathbb{R}$ such that $u_1 = \alpha u_2 + \beta$. Then, $U_1 = \alpha U_2 + \beta$. Fix any $P \in \Delta(\mathscr{F})$ and $f \in \mathscr{F}$ such that $P \succsim_2 f$ to show $P \succsim_1 f$. We will show the result, in the following two exhaustive cases.

Case 1: $U_2(f_P) = \int_{\mathscr{F}} U_2(g)dP(g)$. Then $V_2(P) = U_2(f_P) \geq U_2(f)$. Since $U_1 = \alpha U_2 + \beta$, $V_1(P) = U_1(f_P) \geq U_1(f)$, as desired.

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[27] For all $i \in \{1,2\}$, since u_i is unique up to positive affine transformation, we can normalize u_i by $u_i(z_+) = 1$ and $u_i(z_-) = -1$, without loss of generality. Let, $l_0 = \frac{1}{2}\delta_{z_+} + \frac{1}{2}\delta_{z_-}$. Then, $u_i(l_0) = 0$ for all $i \in \{1,2\}$. Suppose to the contrary that $U_1 \neq U_2$. Then, without loss of generality, assume that there exists $f \in \mathscr{F}$ such that $U_1(f) > U_2(f)$. Moreover, by the constant linearity, without loss of generality, assume $1 > U_2(f) > 0$. Fix a positive number ε such that $\varepsilon < \min\{U_1(f) - U_2(f), 1 - U_2(f)\}$. Define $l = (U_2(f)+\varepsilon)\delta_{z_+} + (1 - U_2(f) - \varepsilon)l_0$. Then $U_i(l) = U_2(f) + \varepsilon < U_1(f)$ for all i. Let $P = \delta_l$. Then, $P \succ_2 f$ but $f \succ_1 P$. This is a contradiction. Hence, $U_1 = U_2$, so that $C_1 = C_2$.

Case 2: $U_2(f_P) \neq \int_{\mathscr{F}} U_2(g)dP(g)$. Since $U_1 = \alpha U_2 + \beta$, $U_1(f_P) \neq \int_{\mathscr{F}} U_1(g)dP(g)$. Therefore,

$$\begin{aligned}
\delta_1 &\geq \delta_2 \\
&\geq \tfrac{U_2(f) - \int_{\mathscr{F}} U_2(g)dP(g)}{U_2(f_P) - \int_{\mathscr{F}} U_2(g)dP(g)} \quad (\because \delta_2 U_2(f_P) + (1-\delta_2)\textstyle\int_{\mathscr{F}} U_2(g)dP(g) = V_2(P) \geq U_2(f)) \\
&= \tfrac{U_1(f) - \int_{\mathscr{F}} U_1(g)dP(g)}{U_1(f_P) - \int_{\mathscr{F}} U_1(g)dP(g)} \quad (\because U_1 = \alpha U_2 + \beta)
\end{aligned}$$

Hence, $V_1(P) = \delta_1 U_1(f_P) + (1-\delta_1)\int_{\mathscr{F}} U_1(g)dP(g) \geq U_1(f)$, as desired. □

■

B.4 Proof of Proposition 4

Suppose $\succsim$ is an EAP Maxmin preference represented by V. Let U be the restriction of V on $\mathscr{F}$.

We will show (i). To show (a), suppose $\delta > 0$. It is easy to see that strict ex-post randomization loving implies strict ex-ante randomization loving. To see the converse, suppose that $\succsim$ exhibits strict ex-ante randomization loving. Fix $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$ such that $f \sim g$ to show $\alpha f + (1-\alpha)g \succ f$. Since $\alpha f \oplus (1-\alpha)g \succ f$, then $U(f) < V(\alpha f \oplus (1-\alpha)g) = \delta U(\alpha f + (1-\alpha)g) + (1-\delta)U(f)$, so that $\delta U(f) < \delta U(\alpha f + (1-\alpha)g)$. Since $\delta > 0$, then $U(\alpha f + (1-\alpha)g) > U(f)$. Part (b) is proved in the same way.

Next, we will show (c). Suppose $\delta = 0$ to show ex-ante randomization neutrality. To show this fix $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$ such that $f \sim g$. It suffices to show $\alpha f \oplus (1-\alpha)g \sim f$. This indifference holds because $\delta = 0$ shows $V(\alpha f \oplus (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g) = U(f)$. Hence, (i) is proved.

In the following, we will show (ii). Assume ex-post randomization neutrality to show ex-ante randomization neutrality. To show this, fix $\alpha \in (0,1)$ and $f, g \in \mathscr{F}$ such that $f \sim g$. It suffices to show $\alpha f \oplus (1-\alpha)g \sim f$. Ex-post randomization neutrality implies that $U(\alpha f + (1-\alpha)g) = U(f)$. This implies $V(\alpha f \oplus (1-\alpha)g) = U(f)$, as desired.

B.5 Proof of Proposition 5

To see that Reversal of Order implies State-wise Indifference, fix $P, Q \in \Delta(\mathscr{F})$ such that $f_P \sim f_Q$ to show $P \sim Q$. Then, there exist sets $\{f^i\}_{i=1}^n$ and $\{g^j\}_{j=1}^m$ of acts and sets $\{\alpha_i\}_{i=1}^n$ and $\{\beta_j\}_{j=1}^m$ of nonnegative numbers such that $P = \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n$, and $Q = \beta_1 g^1 \oplus \cdots \oplus \beta_m g^m$. Then, Reversal of Order shows $P \sim \alpha_1 f^1 + \cdots + \alpha_n f^n = f_P \sim f_Q = \beta_1 g^1 + \cdots + \beta_m g^m \sim Q$.

To see that State-wise Indifference implies Reversal of Order, fix any set $\{f^i\}_{i=1}^n$ of acts and set $\{\alpha_i\}_{i=1}^n$ of nonnegative numbers such that $\sum_{i=1}^n \alpha_i = 1$. Let $P = \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n$ and $Q = \alpha_1 f^1 + \cdots + \alpha_n f^n$ to show $P \sim Q$. Then for all $s \in S$, $P_s = \alpha_1 f^1_s + \cdots + \alpha_n f^n_s = Q_s$, so that $f_P \sim f_Q$. Then, State-wise Indifference shows $P \sim Q$.

B.6 Proof of Proposition 6

To see that Indifference implies Reduction of Compound Lotteries, fix any set $\{l^i\}_{i=1}^n$ of lotteries and set $\{\alpha_i\}_{i=1}^n$ of nonnegative numbers such that $\sum_{i=1}^n \alpha_i = 1$. Let $P = \alpha_1 l^1 \oplus \cdots \oplus \alpha_n l^n$ and $Q = \alpha_1 l^1 + \cdots + \alpha_n l^n$ to show $P \sim Q$. Then, $P_s = \alpha_1 l^1 + \cdots + \alpha_n l^n = Q_s$ for all $s \in S$, so that condition (i) is satisfied. In addition, $l_P = \alpha_1 l^1 + \cdots + \alpha_n l^n = l_Q$, so that condition (ii) is also satisfied. Hence, Indifference shows $P \sim Q$.

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