DISCUSSION PAPER NO. 151

THE GLOBAL ASYMPTOTIC STABILITY OF OPTIMAL CONTROL WITH APPLICATIONS TO DYNAMIC ECONOMIC THEORY**

by

William A. Brock*

and

José A. Scheinkman*

Revised: May, 1975

Department of Economics
University of Chicago

*We thank M. Magill for helpful comments on this work. Financial support of the National Science Foundation is gratefully acknowledged.

**The origin of this paper is as follows. It began as an attempt by the first author to relate the results of the recent burst of activity in stability analysis of optimal control to well known theorems in control engineering. Viz. results on the optimal regulator problem [29, Chapter 3]. The first author wrote the paper. The second author's name appears because the work contained herein is very strongly influenced by the joint Brock-Scheinkman work on optimal control, and the second author was a constant source of idea input during the work.

Section 1: Introduction

In this paper we develop some results on the global asymptotic stability, G.A.S., of optimal control and show how these results may be used in dynamic economic theory.

In dynamic economic theory the distinction between short-run, long-run, flow equilibrium, and stock equilibrium is made precise. In a lot of such theorizing, optimal control models are posited to describe the objectives of agents in the economy. The theories of optimal economic growth, optimal accumulation of capital by a profit maximizing firm, optimal accumulation of financial assets and education or skills (called "human" capital) by a utility maximizing individual over his life cycle, optimal holding of money, capital and bonds by a consumer over his life time are just a few examples of optimal control models in modern economics, both neo-classical and anti-neo-classical.

In all of these models, from models of economics optimally controlled by central planning boards to models populated by neo-classical consumers who act as if they are well versed in Pontriagin's Maximal Principle, the following question turns up: If the model is infinite horizon, does the optimal solution converge to a steady state as time tends to infinity. And if the model is finite horizon, does the associated infinite horizon model display the convergence property so that this information may be used to generate information about the solution of the finite horizon problem.

The convergence question is as important for dynamic economics as the stability analysis is for static economics: A question which has occupied some of the best minds in economic theory for a number of years.

Results on the convergence question are virtually non-existent for problems with more than one state variable except for the case where future utilities or payoffs are not discounted. In this paper we develop a new set of results on the convergence question, relate them to recent work in this area, and demonstrate their usefulness for dynamic economic theory by applying them to models of optimal accumulation of capital by profit maximizing firms in the face of adjustment cost.

Section 2: Basic Stability Results: Lyapounov Functions of Type $\dot{k}^T W'' \dot{k}$

Consider the problem[1/]

$$W(k_0) \equiv \text{Max} \int_0^\infty e^{-\rho t}\, U(k,x)dt$$

$$\text{s.t.} \quad \dot{k} = x, \; k(0) = k_0 \tag{1}$$

Let

$$q(k_0) = \frac{\partial W}{\partial k}(k_0)$$

Then

$$\dot{q} = \rho q - H_2(q(k),k) \tag{2}$$

$$\dot{k} = H_1(q(k),k), \; k(0) = k_0 \tag{3}$$

where $H = \max_x \{U(k,x) + qx\}$ is called the present value Hamiltonian and

$H_1 = \frac{\partial H}{\partial q}$, $H_2 = \frac{\partial H}{\partial k}$. "Standard" (viz. Lee and Markus [13, p. 396] methods of optimal control stability analysis examine the "reduced form"

$$\dot{k} = H_1(q(k),k) \equiv F_2(k), \; k(0) = k_0 \tag{4}$$

and look at Lyapounov functions of the form $V = k^T G k$ or $V = \dot{k}^T G \dot{k}$ where G is some n x n symmetric matrix (usually positive definite). Assumptions are made on (4) and on G so trajectory derivative of V is negative. See Lee and Markus [13, p.396] for example.

The recent work (Cass-Shell, Rockafellar, Brock-Scheinkman) looks at Lyapounov functions

$$(q - \bar{q})^T G(k - \bar{k}), \; G = I, \text{ Cass-Shell, and Rockafellar, Brock-Scheinkman} \tag{5}$$

$$\dot{q}^T G \dot{k}, \text{ Brock-Scheinkman, Rockafellar} \tag{6}$$

In the recent work sufficient conditions are placed on the system (2), (3) so that the trajectory derivative of one of the above Lyapounov functions is negative.

Brock and Scheinkman [3] extend Hartman and Olech [9] type methods (using very complicated arguments) to the case (2), (3).

In this section, we show that there is a correspondence between the standard methods embodied in the reduced form (4) and the new methods. As we shall see this will make the new results intuitively clear and will simplify proofs. All this comes at a price, however. Our rendition of standard methods requires the value function $W(\cdot)$ be C^2 for some results and C^3 for others, Anyone who has tried to verify that the value is C^2 knows that that particular task is a royal pain in the rear. But we follow here a well established tradition in optimal control theory and avoid this issue. Let us get into the results.

First, we must state a basic result of Hartman and review the standard Lyapunov stability analysis that is needed for our purposes. Since, in most economic problems, the state variable k will be bounded (diminishing returns, depreciation of capital goods, etc. prevent the accumulation of an infinite amount of capital goods by an optimizing system), therefore, we will <u>assume that optimum k is bounded independently of time in all that follows</u>.

<u>Hartman's Lemma</u> ([8, p.539]). Let $F(z)$ be continuous on an open set $E \subseteq R^m$, and such that the solutions of

$$\dot{z} = F(z) \qquad (*)$$

are uniquely determined by initial conditions. Let $V(z)$ be a real valued function on E with the following properties:

(a) V is continuously differentiable on E.

(b) $\dot{V}(z) \leqq 0$ on E (where $\dot{V}(z)$ is the trajectory derivative of $V(z)$ for any z in E).

Let $z(t)$ be a solution of (*) for $t \geqq 0$. Then the limit points of $z(t)$ for $t \geqq 0$, in E, if any, are contained in the set $E_0 = \{z | \dot{V}(z) = 0\}$.

The proof of this is in [8 , p.539]. We point out here that if the trajectory $z(t)$ is bounded independently of time, then $z(t)$ possesses limit points. We also remind the reader that

$$\dot{V}(z) \equiv \frac{\partial V}{\partial z}(z) \quad F(z).$$

Also, Hartman's restriction, $V(z) \geqq 0$, in his statement ([8 , p.539]) is not needed for our purposes since we will be assuming that trajectories are bounded independently of time.

The standard way that limiting behavior of $z(t)$ is studied is to find a V such that $\dot{V}(z) \leqq 0$ for all z. Then we know by Hartman's Lemma that all limit points are in the set $E_0 = \{z | \dot{V}(z) = 0\}$. The structure of E_0 is then studied to see what kind of limiting behavior is possible. In our problems, E_0 will be assumed to be a set containing a finite number of points. Since for any initial condition, z_0, the set of limit points of the trajectory $z(t|z_0)$ starting from z_0 is connected, therefore, $z(t|z_0)$ must converge to an element of E_0. See Hartman's [8] chapter on "Miscellany on Monotony" for the basic methods outlined above. Let us continue.

Look at the Lyapounov function (5) written in terms of the value function for $G = I$; i.e.,

$$V = (q - \bar{q})^T (k - \bar{k}) = \left[\frac{\partial W}{\partial k}(k) - \frac{\partial W}{\partial k}(\bar{k})\right]^T (k - \bar{k}). \tag{7}$$

If W is concave (it will be if U is), then obviously

$$V \leqq 0 \quad \text{for } k \neq \bar{k} \tag{8}$$

Inequality (8) follows because

$$\left[\frac{\partial W}{\partial k}(k) - \frac{\partial W}{\partial k}(\bar{k})\right]^T (k - \bar{k}) \leqq 0 \tag{9}$$

for any concave function W, for any k, $\bar{k}$. Now if W is strictly concave (9) will hold with strict inequality for $k \neq \bar{k}$, and thus the absolute value of (9) forms a natural measure of "distance" of k from $\bar{k}$. Therefore, it is natural to require directly that the absolute value of (9) decreases along trajectories as do Cass and Shell or place sufficient conditions on the Jacobian matrix of $H(q,k)$ that imply the absolute value of (9) decreases along trajectories as do Brock and Scheinkman.

Look at the Lyapunov function (6) for $G = I$ in terms of the value. We get

$$\dot{q} = \frac{\partial^2 W}{\partial k^2} \dot{k}, \quad \dot{q}^T \dot{k} = \dot{k}^T \frac{\partial^2 W}{\partial k^2} \dot{k} \leqq 0 \tag{10}$$

if W is concave. Thus, $\dot{q}^T \dot{k}$ is a type of distance of k from zero. Let us now turn to derive an identity that will be useful to show a relationship between different sufficient conditions for negative trajectory derivative of Lyapounov functions for (4).

First, we need

Lemma 1: Let $v_2 = \dot{k}$, $\dot{v}_2 = J_{F_2} v_2$, $W'(k) = \frac{\partial W}{\partial k}$, $W''(k) = \frac{\partial^2 W}{\partial k^2}$, $\dot{W}''(k)_{ij}$

$$= \sum_r \frac{\partial}{\partial k_r} \frac{\partial^2 W}{\partial k_i \partial k_j} \cdot \dot{k}_r = \sum_r \frac{\partial}{\partial k_r} \frac{\partial^2 W}{\partial k_i \partial k_j} H_{q_r}$$

Then

$$W''(k) J_{F_2}(k) = [\rho W''(k) - H_{21} W''(k) - H_{22} - \dot{W}'']$$

$$W''(k) J_{F_2}(k) = W'' H_{11} W'' + W'' H_{12} \tag{11}$$

for all k.

Proof: Since

$$q = W'(k), \ \dot{q} = W''(k)\dot{k}, \ \dot{k} = H_1(W'(k),k) \equiv F_2(k) \tag{12}$$

and

$$\dot{q} = \rho q - H_2, \tag{13}$$

therefore,

$$W''(k)H_1 = \rho q - H_2 = \rho W'(k) - H_2(W'(k),k) \tag{14}$$

Now totally differentiate equation (14) w.r.t. to k to obtain

$$\dot{W}'' + W'' J_{F_2} = \rho W'' - H_{21}W'' - H_{22} \tag{15}$$

which is the first part of (11). The second part just follows by definition of J_{F_2}.

Q.E.D.

Remark 1: Notice that the equation

$$W'' H_1 = \rho W' - H_2 \tag{16}$$

is the total derivative w.r.t. to k of the equation

$$\rho W(k) = H(W'(k),k) \tag{17}$$

which is, as we shall see in more detail later, the Hamilton Jacobi equation for the optimal control problem

$$W(k_0) = \text{maximum} \int_0^\infty e^{-\rho t} U(k,x)dt$$

$$\text{s.t.} \quad \dot{k} = x, \; k(0) = k_0 \tag{18}$$

We now move on to use identity (11) to show a useful identity between the quadratic forms of the trajectory derivatives of two apparently unrelated Lyapounov functions.

It is easy to see that

$$\frac{d}{dt}(\dot{q}^T\dot{k}) = \frac{d}{dt}(F_1^T F_2) = (F_1, F_2)^T \begin{bmatrix} H_{11} & \rho/2\ I \\ \rho/2\ I & -H_{22} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{19}$$

But

$$\frac{d}{dt}(\dot{q}^T\dot{k}) = \frac{d}{dt}[\dot{k}^T W''(k)\dot{k}] = \frac{d}{dt}[v_2{}^T W''(k) v_2]$$

$$= \dot{v}_2{}^T W'' v_2 + v_2{}^T W'' \dot{v}_2 + v_2{}^T \dot{W}'' v_2$$

$$= v_2{}^T \{J_{F_2}{}^T W'' + W'' J_{F_2} + \dot{W}''\} v_2 \qquad (20)$$

Let Q denote the matrix of the quadratic form of R.H.S. (19). Notice that we have discovered two different ways of looking at the same Lyapunov function:

Way 1: $\frac{d}{dt}(\dot{q}^T\dot{k}) = (F_1,F_2)^T Q(F_1,F_2)$ - a "new" way,

Way 2: $\frac{d}{dt}(\dot{q}^T\dot{k}) = \frac{d}{dt}[v_2{}^T W''(k)\ v_2]$ - an "old" way

for the latter is just a Hartman [8 , p. 542] type of Lyapounov function with variable matrix $G(k) \equiv - W''(k)$ for the "reduced form"

$$\dot{k} = F_2(k).$$

Two theorems will state identities that will be useful in the sequel.

Theorem 1:

$$(F_1,F_2)^T Q(F_1,F_2) = (W'' F_2,F_2)^T Q\ (W'' F_2,F_2) = (W'' v_2,v_2)^T Q\ (W'' v_2,v_2)$$

$$= v_2{}^T \{J_{F_2}{}^T W'' + W'' J_{F_2} + \dot{W}''\} v_2 \qquad (21)$$

Proof: Follows by definitions and by Lemma 1.

Q.E.D.

Theorem 2: Assume that for each k (11) holds, then $\forall$ k, $\forall$x

$$(W'' x, x)^T Q\ (W'' x, x) \geqq 0 \qquad (22)$$

iff $\forall$ k, $\forall$x

$$x^T [J_{F_2}{}^T W'' + W'' J_{F_2} + \dot{W}'']x \geqq 0 \qquad (23)$$

Proof: Obvious by the identity (11) of Lemma 1. Compute as follows

$$(W''\, x,\ x)^T \begin{bmatrix} H_{11} & \rho/2 \\ \rho/2 & -H_{22} \end{bmatrix} \begin{bmatrix} W''x \\ x \end{bmatrix} = (W''x)^T\, H_{11}(W''x) + x^T(-H_{22})x$$

$$+ \rho\ (W''x)^T x = x^T\ W''\ H_{11}W''x$$

$$+ x^T(-H_{22})x + x^T(\rho W'')x. \tag{24}$$

By (11), we have

$$W''\ H_{11}W'' + W''\ H_{12} = \rho\ W'' - H_{21}W'' - H_{22} - \dot{W}'' \tag{25}$$

Therefore reshuffling (25), we get

$$\rho W'' - H_{22} = W''\ H_{11}W'' + W''\ H_{12} + H_{21}W'' + \dot{W}'' \tag{26}$$

Now substitute (26) into the matrix of (24)

$$W''\ H_{11}W'' - H_{22} + \rho W'' = 2\ W''\ H_{11}W'' + W''\ H_{12} + H_{21}W'' + \dot{W}'' \tag{27}$$

But, now calculate the matrix of (23) using

$$J_{F_2} = H_{11}W'' + H_{12}, \tag{28}$$

we get

$$J_{F_2}{}^T\ W'' + W''\ J_{F_2} + \dot{W}''$$

$$= \ [H_{11}\ W'' + H_{12}]^T\ W'' + W''[H_{11}W'' + H_{12}] + \dot{W}''$$

$$= W''\ H_{11}W'' + H_{21}W'' + W''\ H_{11}W'' + W''\ H_{12} \tag{29}$$

which is exactly R.H.S. (27).

Q.E.D.

Record (11) for convenience.

$$W''(k_0)J_{F_2}(k_0) = \rho\, W''(k_0) - H_{21}W''(k_0) - H_{22} - \dot{W}''(k_0). \tag{30}$$

This identity is relevant to the Brock and Scheinkman [3] adaptation of Hartman and Olech [9] to the case of optimal control. As we shall see if (30) holds, then an alternative proof of the Brock-Scheinkman Theorem [3 , p. 16] is available. And the proof has considerable intuitive appeal. Some preliminary results are needed.

Theorem 3: $\forall\ (q,k) \neq (\bar{q},\bar{k}), \forall\ (w_1,\ w_2) \neq 0$

$$w_1^T\, F_2 + w_2^T\, F_1 = 0 \Rightarrow\ w^T\, Q(q,k)w > 0 \tag{31}$$

implies $\exists$ $G(k)$ such that

$$\forall\ 0 \neq \alpha \in R^n,\ \forall\ k \neq \bar{k}$$

$$\alpha^T\, GF_2 = 0 \Rightarrow \alpha^T\, [GJ_{F_2} + (GJ_{F_2})^T + \dot{G}]\alpha > 0 \tag{32}$$

Proof: Put

$$w_1 = W''\, w_2 \tag{33}$$

in (31). We get by realizing that since

$$F_1 = W''F_2,$$

therefore,

$$w_2^T\, W''\, F_2 + w_2^T\, W''\, F_2 = 0 \Rightarrow (W''\, w_2,w_2)^T\, Q\, (W''\, w_2,w_2) > 0$$

But the above boils down to

$$w_2^T\, W''\, F_2 = 0 \Rightarrow (W''\, w_2,w_2)^T\, Q(W''\, w_2,w_2)$$

$$= w_2^T\, [J_{F_2}{}^T\, W'' + W''\, J_{F_2} + \dot{W}'']w_2 > 0$$

where the last follows from Theorem 2. Put $G = W''$.

Q.E.D.

Remark 2: Note that the condition

$$\alpha^T S F_2 = 0 \Rightarrow \alpha^T [\dot{S} + (S J_{F_2}) + (S J_{F_2})^T]\alpha < 0 \tag{34}$$

where

$$S = -W'' = \text{positive semi-definite} \tag{35}$$

for W concave is a standard, GAS assumption of Hartman-Olech type, Hartman [8, p. 548] for the system

$$\dot{k} = F_2(k) \tag{36}$$

Hence (34) implies GAS by the Hartman-Olech theorem.

At this point, for the record, it is worthwhile to insert an alternative proof of (30) by use of the Hamilton Jacobi equation. We will rewrite (30) in an equivalent form that is useful to have down.

Theorem 4: The following holds

$$W'' J_{F_2} + (W'' J_{F_2})^T + \dot{W}'' = W'' H_{11} W'' + - H_{22} + \rho W''. \tag{37}$$

Proof: Consider the Hamilton Jacobi equation [2/]

$$\rho W = H(W', k) \tag{38}$$

Differentiate both sides of this w.r.t. k twice to obtain

$$\rho W_i = \sum_j H_{q_j} W_{ji} + H_{k_i} \tag{39}$$

$$\rho W_{is} = \sum_j \frac{\partial H_{q_j}}{\partial k_s} W_{ji} + \sum_j H_{q_j} \frac{\partial W_{ji}}{\partial k_s} + \sum_j H_{k_i q_j} \frac{\partial q_j}{\partial k_s}$$

$$i,s = 1,2, \ldots, n \tag{40}$$

Now

$$J_{F_2} = H_{11} W'' + H_{12} \tag{41}$$

Therefore, L.H.S. (37) =

$$W'' \ H_{11}W'' + W'' \ H_{12} + (W''(H_{11}W'' + H_{12}))^T + \dot{W}''$$

Thus to show (37), we need only prove

$$W'' \ H_{12} + W'' \ H_{11}W'' + H_{12}^{\ T} \ W'' + \dot{W}'' = - H_{22} + \rho W'' \tag{42}$$

We will use (40) to establish (42). Calculate

$$\frac{\partial H_{q_j}}{\partial k_s} = \sum_r H_{q_j q_r} \frac{\partial q_r}{\partial k_s} + H_{q_j k_s} = \sum_r H_{q_j q_r} \frac{\partial^2 W}{\partial k_r \partial k_s} + H_{q_j k_s} \tag{43}$$

Thus

$$\rho \ W_{is} = \sum_j \left[\sum_r H_{q_j q_r} \frac{\partial^2 W}{\partial k_r \ \partial k_s} \right] \frac{\partial^2 W}{\partial k_i \partial k_j} + \sum_j H_{q_j k_s} \frac{\partial^2 W}{\partial k_i \partial k_j}$$

$$+ \sum_j H_{q_j} \frac{\partial^3 W}{\partial k_i \partial k_j \partial k_s} + \frac{\partial^2 H}{\partial k_i \partial k_s} + \sum_j H_{k_i q_j} \frac{\partial^2 W}{\partial k_i \partial k_j} \tag{44}$$

Now examine the (i,s) term of (42)

$$\frac{-\partial^2 H}{\partial k_i \partial k_s}$$

cancels against $\frac{\partial^2 H}{\partial k_i \partial k_s}$ in L.H.S. (42).

Now

$$[\dot{W}'']_{is} = \frac{d}{dt} \ W_{is} = \sum_r \frac{\partial W_{is}}{\partial k_r} \ H_{q_r} = \sum_r \frac{\partial^3 W}{\partial k_i \partial k_s \partial k_r} \ H_{q_r} \tag{45}$$

and this cancels out the expression

$$\sum_j H_{q_j} \frac{\partial^3 W}{\partial k_i \partial k_j \partial k_s}$$

in the RHS (44). Thus to prove (42), we need only check

$$[W'' H_{12}]_{is} + [W'' H_{11} W'']_{is} + [H_{12}{}^T W'']_{is} = \sum_j H_{k_i q_j} \frac{\partial^2 W}{\partial k_s \partial k_j}$$

$$+ \sum_j H_{q_j k_s} \frac{\partial^2 W}{\partial k_i \partial k_j} + \sum_j [\sum_r H_{q_j q_r} \frac{\partial^2 W}{\partial k_r \partial k_j}] \frac{\partial^2 W}{\partial k_i \partial k_j} \tag{46}$$

To check it write out the formulas for the (i,s)th element

$$[W'' H_{12}]_{is} = \sum_j \frac{\partial^2 W}{\partial k_i \partial k_j} H_{q_j k_s} \tag{47}$$

$$[H_{12}{}^T W'']_{is} = \sum_j H^T_{q_i k_j} \frac{\partial^2 W}{\partial k_j \partial k_s} = \sum_j H_{q_j k_i} \frac{\partial^2 W}{\partial k_j \partial k_s} \tag{48}$$

$$[W'' H_{11} W'']_{is} = \sum_j \sum_r \frac{\partial^2 W}{\partial k_i \partial k_j} H_{q_j q_r} \frac{\partial^2 W}{\partial k_r \partial k_s} \tag{49}$$

Obviously (47), (48), (49) imply (46).

Q.E.D.

At this point in time it is appropriate to take stock of what we have accomplished, and relate it to the sufficient conditions for the global asymptotic stability of optimal control derived by Cass-Shell, Rockafellar, and Brock-Scheinkman.

3/ We have shown that there is an equivalence between negative trajectory derivative of the Lyapounov function

$$V = v_2^T(-W''(k))v_2 \tag{50}$$

and positive definiteness in the directions (F_1,F_2) of the matrix

$$Q = \begin{bmatrix} H_{11} & \rho/2\ I_n \\ \rho/2\ I_n & -H_{22} \end{bmatrix} \tag{51}$$

which plays a central role in the Brock-Scheinkman analysis. Now the trajectory derivative of (50) is given by

$$\dot{V} = -v_2^T\ [J_{F_2}{}^T\ W'' + W''\ J_{F_2} + \dot{W}'']v_2 \tag{52}$$

But by (37)

$$J_{F_2}{}^T\ W'' + W''\ J_{F_2} + \dot{W}''$$

$$= W''\ H_{11}\ W'' - H_{22} + \rho W'' \tag{53}$$

Thus, the analyst has the option of checking positive definiteness of the matrix (53) in the direction

$$v_2 = F_2$$

or checking positive definiteness of Q in the direction (F_1,F_2) in order to test for GAS. Two tests are better than one since the Q test shows the analyst that

GAS holds for small ρ, but the L.H.S. of equation (53) does not reveal this although the R.H.S. suggests it.

The Q test is useful for uncovering the result: If the smallest eigenvalue, λ, of the matrices H_{11}, $- H_{22}$ is greater than $\rho/2$, then GAS holds. For $\lambda > \rho/2$ is sufficient for Q to be positive definite. 4/ Equation (53) offers no guide to such a result.

Equation (53), however, opens up a new line of research viz. the generalizations reported in Hartman's book [8 , p. 549] e.g., may now be carried out for the case of optimal control.

Let us now look at the Brock-Scheinkman generalization of the Hartman-Olech in Hartman's book [8 , p. 549] to the case of optimal control. An important special case of the Brock-Scheinkman result is: Assume that there is just one steady state, k, and it is L.A.S. for the "reduced form"

$$\dot{k} = F_2(k)$$

Also assume that for all (q,k), for all $w = (w_1, w_2) \neq 0$

$$w_1^T \, [H_1(q,k)] + w_2^T \, [\rho \, q - H_2(q,k)] = 0 \tag{54}$$

$$\Rightarrow (w_1, w_2)^T \, Q(q,k) \, (w_1, w_2) > 0$$

But we saw from (37) and Theorem 3 that (54) implies: For all k, for all $\alpha \neq 0$

$$\alpha^T (S(k)) \, H_1(W'(k), k)) = 0 \Rightarrow \alpha^T [(S(k) J_{F_2}) + (S(k) J_{F_2})^T + \dot{S}(k)]\alpha < 0 \tag{55}$$

where

$$S(k) = -W''(k) \tag{56}$$

and the (i,j)th element of $\dot{S}$ is defined by

$$[\dot{S}(k)]_{ij} = \sum_{r=1}^{n} \frac{\partial S(k)}{\partial k_r} H_{q_r}(W'(k),k) \tag{57}$$

And (55) together with the L.A.S. of $\bar{k}$ is just a special case of the Hartman-Olech sufficient condition for G.A.S. of $\bar{k}$. (Hartman [8 , p.549]).

The Brock-Scheinkman [3a] theorem does not follow immediately, however, because W" is not necessarily negative semi definite for them. In [3b] the Hartman-Olech method is extended to cases where G is not positive definite, and that allows the methods of this paper to be applied to prove the Brock-Scheinkman ([3a]) theorem.

Thus, we have succeeded in building a bridge between both methods for obtaining G.A.S. results for control systems: (1) Methods that rely on analysis of the "reduced form"

$$\dot{k} = H_1(W'(k),k)$$

and (2) Methods that rely on analysis of the system

$$\dot{q} = \rho\, q - H_2(q,k)$$

$$\dot{k} = H_1(q,k)$$

This bridge that we have established will aid in generalizing recieved results and will aid the analyst in finding useful sufficient conditions for the GAS of optimal control.

We turn now to section three where we establish another class of G.A.S. results derived from the reduced form and based on the Lyapounov function

$$V = v_2^T H_{11}^{-1} v_2.$$

Section 3: Basic Stability Results: Lyapounov Functions of The Type $\dot{k}^T H_{11}^{-1} \dot{k}$

Consider

$$\text{Max} \int_0^\infty e^{-\rho t} U(k,x)dt \equiv W(k_0)$$

$$\text{s.t.} \quad \dot{k} = x, \ k(0) = k_0$$

Necessary conditions of Pontriagin are:

$$\dot{q} = \rho q - H_k$$

$$\dot{k} = H_q, \ k(0) = k_0$$

Let

$$V = v_2^T H_{11}^{-1} v_2, \quad v_2 = \dot{k}$$

calculate the trajectory derivative of V along the "reduced form"

$$\dot{k} = H_1(q(k), k) \equiv F_2(k)$$

where the Jacobian matrix of $F_2(k)$ is given by

$$J_{F_2} = H_{11} W'' + H_{12}.$$

Doing this calculation, we get

$$\dot{V}(k) = \dot{v}_2^T H_{11}^{-1} v_2 + v_2^T H_{11}^{-1} \dot{v}_2 + v_2^T \left(\dot{H}_{11}^{-1}\right) v_2$$

$$= [(H_{11} W'' + H_{12})v_2]^T H_{11}^{-1} v_2 + v_2^T H_{11}^{-1} [(H_{11} W'' + H_{12})v_2]$$

$$+ v_2^T (\dot{H}_{11}^{-1}) v_2$$

$$= v_2^T [2 W'' + H_{12}^T H_{11}^{-1} + H_{11}^{-1} H_{12} + (\dot{H}_{11}^{-1})]v_2.$$

This calculation suggests

<u>Theorem 5</u>: Assume

$$H_{12}^T H_{11}^{-1} + H_{11}^{-1} H_{12} + (\dot{H}_{11}^{-1})$$

is negative definite in the direction

$$v_2 = F_2(k)$$

for each k. Then given any k_0, the optimum solution $k(t|k_0)$ converges to

$$E_0 = \{k|\dot{V}(k) = 0\}$$

as $t \to \infty$.

Proof: Since U(k,x) is concave in (k,x), therefore, the matrix W" is negative semi definite everywhere.

Thus,

$$\dot{V}(k) \leqq 0$$

and the conclusion follows by Hartman's Lemma.

Remark 3: If the objective U(k,x) is quadratic in (k,x), then H_{11}^{-1} is independent of time. Thus, $(\dot{H}_{11}^{-1}) = 0$ and a useful sufficient condition for global asymptotic stabiltiy is immediate from Theorem 5:

$$(H_{11}^{-1}\ H_{12})^T + (H_{11}^{-1}\ H_{12})$$

is negative definite. The proof of this assertion follows trivially from Hartman's Lemma because the reduced form

$$\dot{k} \equiv F_2(k)$$

is linear for U(k,x) quadratic.

Remark 4: The same argument as that used in Remark 3 allows us to prove that

$$(H_{11}^{-1}\ H_{12})^T + (H_{11}^{-1}\ H_{12}) \qquad (58)$$

negative definite at a steady state $\bar{k}$ is sufficient for $\bar{k}$ to be LAS. The proof is carried out by linearizing the necessary conditions of optimality around $\bar{k}$, and noticing that $(\dot{H}_{11}^{-1}) = 0$ at a steady state.

Notice also that the sufficient condition for stability (58) is independent of the discount rate ρ in contrast to the Cass-Shell, Brock-Scheinkman, Rockafellar conditions.

Later on we will show how (58) leads to a powerful sufficient condition for stability of a large class of adjustment cost models. Before we move into that, let us sketch an argument that LAS of all steady states implies there is only one steady state.

A rigorous uniqueness argument would take us far afield, but we can give heuristics here. Look at the reduced form

$$\dot{k} = H_1(q(k),k) \equiv F_2(k) \tag{59}$$

Linearize (59) around any rest point $\bar{k}$ to get

$$\dot{z}_2 = J_{F_2}(\bar{k})z_2, \ z_2(0) = k_0 - \bar{k} \tag{60}$$

Since each $\bar{k}$ is L.A.S. the determinant of the matrix

$$J_{F_2}(\bar{k})$$

does not change sign across the set of rest points $\{\bar{k}\}$. It will be nonzero except for "hairline" cases. Therefore, we follow the tradition of modern global analysis (see Dierker [7] for a discussion of modern global analysis and applications of it to economics), and assume away the hairline cases. Thus, the topological index theory reported in Milnor [15, p. 36] may be applied. To do this assume that a large enough homeomorph of an n dimensional solid ball can be found so that the vector field described by (59) "points inward" on the surface of this ball and all rest points of (59) are contained within it. Then it follows directly from a theorem in differential topology (Milnor [15, p. 36]) that if the determinant of $J_{F_2}(\bar{k})$ is nonzero and does not change sign across the set of rest points $\bar{k}$ (which by hypothesis are all L.A.S.), then there is only one rest point $\bar{k}$. The reader is referred to Dierker for the detailed development of this type of argument.

We might add that for the optimal growth problem the "inward pointing" condition amounts to little more than the economics that if a capital good, k_i, is near enough to zero then the optimizing system sets $\dot{k}_i > 0$, and if k_i is very large then $\dot{k}_i < 0$. The actual checking of the inward pointing hypothesis is a nontrivial task however. It is most useful as a guide of what to expect: viz. if all rest points are L.A.S. the analyst should not expect to find multiple rest points.

We have built up enough abstract technique. Now it is time to turn to applications. Our methods lead us to a very strong set of G.A.S. results for the important subclass of models where there are only two state variables. This case is important for (a) trade models, (b) human capital theory, (c) population models, and many more. Two goods models are the simplest models that allow for substitution possibilities in economics, and therefore, they play an important role in the theoretical literature.

Section 4: The Two State Case

Consider the system

$$\begin{aligned} \dot{q}_i &= \rho\, q_i - H_{k_i}(q,k) \\ \dot{k}_i &= H_{q_i}(q,k) \qquad\qquad i = 1,2 \\ k(0) &= k_0 \end{aligned} \tag{61}$$

Let

$$W(k_0) = \max \int_0^\infty e^{-\rho t}\, U(k,\dot{k})\,dt$$
$$\text{s.t. } k(0) = k_0$$

Lemma 2: If U is concave in $(k,\dot{k})$, then $W(\cdot)$ is concave.

Proof: Easy exercise in the definition of concavity. Let $(\bar{q},\bar{k})$ be a steady state of (61). Assume it is unique. Brock [2] provides sufficient conditions for unique steady state. It's finding sufficient conditions for stability that's hard. To that task we now turn.

Lemma 3: The shadow price q can be written as $q(k)$ and the matrix

$$\frac{\partial q}{\partial k}$$

is negative semi-definite and symmetric at each k where it exists. It exists almost everywhere; (i.e., except for a set of k of Lebesque measure zero).

Proof: By Arrow and Kurz [1, Chapter 2], $q(k) = \frac{\partial W}{\partial k}$, therefore,

$$\frac{\partial q}{\partial k} = \frac{\partial^2 W}{\partial k^2}$$

and W is concave so negative semi-definiteness and symmetry follows. Existence of W'' almost everywhere follows by Karlin [12, p. 405].

Consider the system of ordinary differential equations in the plane:

$$\left.\begin{aligned} \dot{k}_1 &= H_{q_1}(q_1(k),\ q_2(k),\ k_1,\ k_2) = H_q(q,k) \\ \dot{k}_2 &= H_{q_2}(q,k) \end{aligned}\right\} \equiv F_2(k)$$

$$k(0) = k_0 \tag{62}$$

The Jacobian matrix of F_2 is

$$J_{F_2} = H_{11} \quad \frac{\partial^2 W}{\partial k^2} + H_{12} \tag{63}$$

We state

Lemma 4: Hsu and Meyer's Bendixon Theorem [11, p. 164]. If Trace $J_{F_2}(k)$ doesn't change sign for all k, then there are no limit cycles for (62).

Proof: See Hsu and Meyer [11, p. 164] for the argument which follows immediately from Green's Theorem in the plane.

Lemma 5:[5/] Let A, B be two 2 x 2 real matrices that are symmetric; A positive definite and B negative definite, then

(i) Trace (AB) < 0.

If only semi-positive definiteness and semi-negative definiteness holds, then

(ii) Trace (AB) $\leqq$ 0.

If neither A nor B is 0 and

Trace AB = 0,

then

(iii) A and B are both singular.

Proof:

$$A\ B = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}\,b_{11} + a_{12}\,b_{12}, & - \\ -, & a_{12}\,b_{12} + a_{22}\,b_{22} \end{bmatrix}$$

So Trace (AB) = $a_{11}\,b_{11} + a_{22}\,b_{22} + 2\,a_{12}\,b_{12}$. Suppose Trace (AB) ≥ 0 then

$$0 > \underset{(-)}{a_{11}\,b_{11}} + \underset{(-)}{a_{22}\,b_{22}} \geq -2\ a_{12}\,b_{12}$$

therefore,

$$a_{11}|b_{11}| + a_{22}|b_{22}| \leq 2\ |a_{12}|\ |b_{12}| \tag{64}$$

Now positive definiteness of A implies

$$\underset{(+)}{a_{11}\ a_{22}} > a_{12}^2 \tag{65}$$

and negative definiteness of B implies

$$\underset{(+)}{b_{11}\ b_{22}} > b_{12}^2 \tag{66}$$

Using (65), (66) to get an upper bound for the R.H.S. of (64), we get

$$a_{11}|b_{11}| + a_{22}|b_{22}| < 2\,(a_{11}\ a_{22})^{1/2}\,(b_{11}\ b_{22})^{1/2} \tag{67}$$

Put

$$\alpha = |a_{11}\ b_{11}| \qquad \beta = |a_{22}\ b_{22}|$$

(67) is

$$\alpha + \beta < 2\ \alpha^{1/2}\ \beta^{1/2} \tag{68}$$

Therefore

$$\alpha^2 + \beta^2 + 2\,\alpha\beta < 4\,\alpha\beta$$

I.e.,

$$\alpha^2 + \beta^2 - 2\,\alpha\beta < 0$$

So

$$(\alpha - \beta)^2 < 0$$

Immediate contradiction. This proves (i). (ii) follows by a similar argument.

In order to prove (iii), we notice first that Tr (AB) = 0 implies that A or B is singular. This is so because letting λ_i denote the eigenvalues of AB, we have

$$\lambda_1 + \lambda_2 = \mathrm{Tr}\ (AB) = 0$$

Thus,

$$\lambda \equiv \lambda_1 = -\lambda_2,$$

and because $|A| \geqq 0$, $|B| \geqq 0$ by positive semi definiteness of A and negative semi definiteness of B, therefore,

$$|AB| = \lambda_1\ \lambda_2 = -\lambda^2 = \underset{(+)}{|A|}\ \underset{(+)}{|B|}$$

gives an immediate contradiction if both A, B non-singular. Thus, one of them, say A, is singular.

Suppose B is <u>not</u> singular. To get a contradiction follow through the implications of the hypothesis Tr (AB) = 0 in inequalities (64) - (68). If (67) is a strong inequality, we have the contradiction

$$(\alpha - \beta)^2 < 0$$

as before. Well, (67) is an equality when $a_{12} = 0$ because B is assumed non-singular. But

$$0 = |A| = a_{11}\ a_{22} - a_{12}^2 = a_{11}\ a_{22}$$

implies one of

$$a_{11},\ a_{22}$$

is zero. Suppose W.L.O.G. $a_{11} = 0$. Then

$$0 = \mathrm{Tr}\ (AB) = a_{11}\ b_{11} + a_{22}\ b_{22} + 2a_{12}\ b_{12} = a_{22}\ b_{22}.$$

But

$$0 < |B| = b_{11}\ b_{22} - b_{12}^2$$

implies

$$b_{22} \neq 0$$

so that

$$a_{22} = 0.$$

Thus,

$$A = 0$$

contradiction to

$$A \neq 0.$$

A similar argument works if B is singular and A is assumed non-singular. This ends the proof.

Remark 5: If $W'' \neq 0$ and if Trace $H_{qk} \leqq 0$ and H_{11} is positive definite, then no limit cycles exist for (62). This follows directly from Lemma 5 because $\mathrm{Tr}\,(H_{11}^{-1} W'') < 0$. Notice that $W''(k) \neq 0$ means that the stock demand "curve" for capital is not "perfectly elastic" at k.

Theorem 6: Assume that there is a bound $M(k_0)$ such that

$$||k(t|k_0)|| \leqq M(k_0)$$

for all $t \geq 0$. Also assume $W''(k) \neq 0$, H_{11} positive definite, and

$$\mathrm{Tr}\ H_{12}(k) \leqq 0$$

then

$$k(t|k_0) \to \bar{k},\ t \to \infty$$

when $\bar{k}$ is a rest point of (62).

Proof: By Lemma 5

$$\mathrm{Trace}\ (H_{11}\ \frac{\partial^2 W}{\partial k^2}) < 0$$

therefore, no limit cycle exists. By the Poincaré Bendixon Theorem (see section 5.8 of Hsu and Meyer [11]) $k(t|k_0)$ must become unbounded or converge to a limit cycle, or converge to a point. Limit cycles are ruled out by the Trace conditions and boundedness is assumed. Therefore,

$$k(t|k_0) \to \bar{k},\ t \to \infty.$$

Section 5: Applications

Application 1: Accumulation of two capital goods in the face of adjustment cost: Treadway's model [24]. Consider the problem of a firm maximizing profit in the face of adjustment costs

$$W(k_0) \equiv \text{Max} \int_0^{\infty} e^{-\rho t} \pi(t)dt$$

$$\text{s.t.} \quad \dot{k} = I - \eta k$$

$$k(0) = k_0$$

Define

$$H^0(q,k) = \max_I \{\pi + q (I - \eta k)\} = \max_I H(q,I,k) \equiv \max_I H$$

$$\eta = \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix}, \quad \pi = \max_L (F(k_1,k_2,L) - wL) - c(I_1,I_2)$$

Here $k \in R_+^2$, $I \in R^2$, $\eta_i \geqq 0$, $L \geqq 0$, π, F, c, q, ρ, w denote capital stocks, investment flow, depreciation factor for capital good i, labor, profits, production function for the one good that is sold at unit price, adjustment cost function, shadow price of capital, rate of interest, wage rate.

Solving

$$\frac{\partial H}{\partial I} = 0, \text{ yields} \quad q = \frac{\partial c}{\partial I}$$

Invert to solve for I as a function of q

$$I = \ell(q)$$

Let it be understood from this point on that we are talking about the maximized Hamiltonian, and drop the superscript "0".

H^0 becomes

$$H = F(k_1,k_2,L(k)) - wL(k) - c[\ell(q)] + q[\ell(q) - \eta k]$$

Therefore,

$$H_1 = H_q = -\frac{\partial c}{\partial I}\frac{\partial I}{\partial \ell}\frac{\partial \ell}{\partial q} + q\frac{\partial \ell}{\partial q} + \ell(q) - \eta k = \ell(q) - \eta k.$$

Hence,

$$H_{12} = -\eta.$$

Consider the "reduced form"

$$\dot{k} = H_1(q(k), k)$$

$$k(0) = k_0$$

Since it can be shown that the optimal solution starting from k_0 denoted by $k(t|k_0)$ is bounded (diminishing returns and depreciation bound capital stock), and

$$H_{12} = -\eta,$$

therefore by our negative Trace theorem (Theorem 6), we have

$$k(t|k_0) \to \bar{k},\ t \to \infty$$

This shows that a large class of two goods adjustment cost models converge.

The Treadway model shows that Theorem 6 yields a very strong stability theorem for the case of two capital goods. Theorem 6 may be useful in obtaining stability results for growth models with past consumption levels in the utility function (Ryder and Heal [21]), growth models with population (Pitchford [17]), human capital models (Heckman [10]), and any other case where the problem with two state variables has economic content.

We suspect that Theorem 6 will be useful in the main for problems where the trace of H_{qk} is non-positive. For problems where this trace restriction is unnatural, it may be better to use stability conditions like the Q condition, or to invert

$$q = W'(k)$$

to obtain

$$k = K(q)$$

and write

$$\dot{q} = \rho q - H_2(q,k)$$

as a reduced form, but we have not done this.

<u>Application 2</u>: Treadway's model with n capital goods.

Use the same notation as in Application 1 and introduce n capital goods.

Note that

$$\dot{k} = H_1 = \ell(q(k)) - \eta k \tag{69}$$

and that

$$H_{12} = -\eta \tag{70}$$

$$H_{11} = \frac{\partial \ell}{\partial q} \tag{71}$$

Write down the critical quantity to test for GAS from Theorem 5:

$$S \equiv H_{11}^{-1} H_{12} + (H_{11}^{-1} H_{12})^T + (\dot{H}_{11}^{-1}) \tag{72}$$

We know that S negative definite implies GAS via Theorem 5. When is S negative definite? Well, let's look first at the case

$$c(I)$$

convex and quadratic. In this case, $\frac{\partial \ell}{\partial q}$ is a constant positive semi definite matrix. Assume away the "hair line" case of

$$\frac{\partial^2 c}{\partial I^2}$$

singular. Thus,

$$H_{11} = \frac{\partial \ell}{\partial q}$$

is positive definite. Also since $\frac{\partial \ell}{\partial q}$ is independent of q, therefore,

$$(\dot{H}_{11}^{-1}) = 0.$$

Thus,

$$S = -[(H_{11}^{-1}\eta) + (H_{11}^{-1}\eta)^T]. \tag{73}$$

Since H_{11}^{-1} is positive definite and η is a positive diagonal matrix, therefore, S "looks" negative definite.[6/] The trouble is, though, that if the diagonal elements of η vary wildly enough in size, then S may not be negative definite. However, it certainly is if all elements of η are identical. Thus, we see from (73) that a very large class of adjustment cost models with quadratic adjustment costs are GAS. Notice that GAS is not lost when more variable factors L are introduced. Also, the production function may be any concave function of (k,L).

Furthermore, the same arguments above give LAS of all steady states for all adjustment cost models provided that the derivative of the cost function

$$\frac{\partial c}{\partial I}$$

does not depend on (k,L).

Note that S does not involve the interest rate, whereas the other stability tests in this paper are critically dependent upon the interest rate, and are likely to hold only when it is small.

Application 3: The Burmeister-Graham model.

Burmeister-Graham formulate a one consumption good n capital goods model of optimal economic growth, where it turns out that W" = 0 on the set of paths that they test for stability. Let us apply our methods to their problem. Write

$$\dot{k} = H_1(W'(k)\ k) \tag{74}$$

$$\ddot{k} = (H_{11}W'' + H_{12})\dot{k} = H_{12}\dot{k} \tag{75}$$

We get a large class of Burmeister-Graham type stability results immediately from (75). For they study cases where H_{12} is independent of time along optimal paths (that lie in their "nonspecialization region"). Our methods expose the assumption needed to get stability for them. Viz. H_{12} has all eigenvalues with negative real parts. We refer the reader to [26] for the economic interpretation of this assumption.

<u>Economic Interpretations of the Stability Condition</u>

Rough economic interpretations of the stability conditions contained in this paper are easy to come by. For

q = stock demand price vector for capital goods,

$q(k)$ = stock demand "curve" for capital goods,

$H_2 = H_k$ = flow demand price vector for capital goods services,

$H_k(q(k), k)$ = flow demand "curve" for capital goods services,

$H_1 = H_q$ = internal supply curve of investment goods,

$H_{22} = H_{kk}$ = "slope of flow demand curve,"

$H_{11} = H_{qq}$ = "slope of internal supply curve,"

$\frac{\partial q}{\partial k} = W''$ = "slope of stock demand curve,"

$H_{12} = H_{qk}$ = "shift in internal supply of investment goods when capital stock increases."

Thus, positive definiteness of

$$Q = \begin{bmatrix} H_{11} & \rho/2 \; I_n \\ \rho/2 \; I_n & -H_{22} \end{bmatrix}$$

is more likely to hold when the product of the slope of the internal supply curve and the slope of the flow supply curve is large relative to the interest rate ρ.

In order to interpret the stability test arising from the Lyapunov function

$$V = \dot{k}^T H_{11}^{-1} \dot{k}$$

look at

$$T = [2 W'' - (H_{11}^{-1} \eta) - (H_{11}^{-1} \eta)^T + (\dot{H}_{11}^{-1})]$$

for the case

$$(\dot{H}_{11}^{-1}) = 0.$$

Stability holds when T is negative definite. Now W" is negative semi definite. We see that T will be negative semi definite when

$$\eta = 0.$$

It will be negative semi definite when H_{11} is diagonal or when all diagonal elements of η are equal. We also see that the "larger" is the "slope" of the stock demand curve for capital; i.e., the more negative definite W" is the more likely is T to be negative definite.

It might be worth while to close this section with a few remarks by way of comparison with independently derived results of Swapan Dasgupta [28] and Araujo-Scheinkman [27]. Equation (73) indicates that S is more likely to be negative definite the faster capital depreciates. Dasgupta finds that large depreciation rates are stabilizing in the Treadway model. He formulates a notion of dominant diagonal for control systems and shows that it will be satisfied if depreciation rates are large enough. He also finds that high elasticities of substitution between inputs in the production function and insensitivity of optimal investment to the price of capital goods is stabilizing.

Araujo-Scheinkman use calculus-in-Banach-space techniques to show that a domina diagonal assumption of I.V. Pearce type (that is independent of the discount rate) leads to stability in a discrete time model. They are then able to show in a discre time version of our "Treadway" model that LAS holds under hypotheses closely related to our assumption that S is negative definite. It is beyond the scope of this paper to explore this relationship in detail here. We refer the reader to Araujo-Scheinkman for details.

Application 4: The Ryder-Heal [21] model.

The Ryder-Heal model is given by

$$\text{maximize} \int_0^\infty e^{-\delta t} u(c,k_1)dt \tag{76}$$

$$\text{s.t.} \quad \dot{k}_1 = \rho(c - k_1) \tag{77}$$

$$\dot{k}_2 = f(k_2) - \lambda k_2 - c \tag{78}$$

$$k(0) = k_0 \text{ given}$$

The differential equation

$$\dot{k}_1 = \rho(c - k_1) \tag{79}$$

is the derivative w.r.t. t of the integral equation

$$k_1(t) = \rho\, e^{-\rho t} \int_{-\infty}^{t} e^{\rho s} c(s)ds \tag{80}$$

Equation (80) defines the stock of consumption experience as an integral of past consumptions with exponentially declining weights put on consumption flows further back in time.

Equation (78) is the usual feasibility constraint in the Cass-Koopmans one good model of economic growth: consumption, c, plus investment, $\dot{k}_2$, plus depreciation, λk_2, equals flow output $f(k_2)$.

We shall show how the methods developed in this paper are useful in analysing the stability properties of optimal paths in this model. Basically, we shall show that if

$$\delta - \rho < 0 \tag{81}$$

then the rest point is likely to be LAS, and limit cycles will be less likely the more negative is $\delta - \rho$. We remark parenthetically that this steady state is unique in the Ryder-Heal model. 7/ This will be obvious upon inspection of the necessary conditions of optimality. The LAS will follow very quickly without computing any eigenvalues of the linearized system. Of course, our methods will not yield the complete characterization of the eigenvalues as do Ryder-Heal. Ryder-Heal compute the eigenvalues of the linearized system explicitly and draw up the restrictions necessary for half of the eigenvalues to have negative real parts, and half of the eigenvalues to have positive real parts. But the formulas for the roots of a fourth order polynomial as functions of the coefficients are so complicated that it is difficult to obtain "clean" results. The fact that $\delta - \rho < 0$ implies LAS of the steady state is likely seems to have escaped Ryder-Heal's notice. Thus, our methods contribute to the understanding of the stability properties of this well known model. Let us get into the analysis.

The Hamiltonian for the Ryder-Heal model is

$$H(q,k) \equiv \underset{c}{\text{maximum}} \{u(c,k_1) + q_1 [\rho(c - k_1)] + q_2 (f(k_2) - \lambda k_2 - c)\} \tag{82}$$

where (q_1, q_2, k_1, k_2) is determined by the differential equations

$$\dot{q}_1 = \delta q_1 - H_{k_1} = \delta q_1 - u_{k_1} + \rho q_1$$

$$\dot{q}_2 = \delta q_2 - H_{k_2} = \delta q_2 + \lambda q_2 - q_2 f'(k_2)$$

$$\dot{k}_1 = H_{q_1} = \rho(c - k_1)$$

$$\dot{k}_2 = H_{q_2} = f(k_2) - \lambda k_2 - c \tag{83}$$

where subscripts k_i, q_j on a variable denote the obvious partial differentiation.

Now the "reduced form"

$$\dot{k} = H_q\ (q(k),\ k) \tag{84}$$

has Jacobian matrix

$$J(k) = H_{qq}\ W'' + H_{qk} \tag{85}$$

Let us use information on the Trace of J to get restrictions on the eigenvalues of the linearization of (84) at the steady state $\bar{k}$. By Lemma 5 of Section 4

$$\operatorname{Tr} J(k) = \operatorname{Tr}(H_{qq}\ W'') + \operatorname{Tr}(H_{qk}) \leqq \operatorname{Tr}(H_{qk}) \tag{86}$$

since H_{qq} and W'' are positive semi definite and negative semi definite, respectively. Compute H_{qk}.

$$H_{qk} = \begin{bmatrix} H_{q_1k_1} & H_{q_1k_2} \\ H_{q_2k_1} & H_{q_2k_2} \end{bmatrix} = \begin{bmatrix} \rho(c_{k_1} - 1) & \rho c_{k_2} \\ -c_{k_1} & f'(k_2)-\lambda-c_{k_2} \end{bmatrix} \tag{87}$$

To complete the computation, we must compute

$$c_{k_1},\ c_{k_2}.$$

Now,

$$u_c(c,\ k_1) + \rho\ q_1 - q_2 = 0 \tag{88}$$

Using (88) we have

$$c_{k_2} = 0,\ c_{k_1} = -u_{cc}^{-1}\ u_{ck_1} \tag{89}$$

At the steady state

$$f'(\bar{k}_2) = \lambda + \delta$$

so that (at the steady state)

$$\mathrm{Tr}\ H_{qk} = -\rho\, u_{cc}^{-1}\ u_{ck_1} - \rho + \lambda + \delta - \lambda = \delta - \rho + \rho(-u_{cc}^{-1})u_{ck_1} \tag{90}$$

We are now ready for the main theorem of this section. One of the hypotheses will need clarification, but first let us state and prove

Theorem If (a): at the steady state $\bar{k}$

$$\mathrm{Tr}\ H_{qk} \equiv \delta - \rho + \rho(-u_{cc}^{-1})u_{ck_1} < 0 \tag{91}$$

and (b): there is a closed curve Γ containing the steady state such that

$$\dot{k} = H_q(q(k),\ k)$$

"points inward" on Γ. Then, the steady state is LAS (except for hairline cases).

Proof: The sum of the eigenvalues of J is Tr J < 0. If the eigenvalues are complex their real parts must, therefore, be negative. Thus, we need check real eigenvalues only. There is only one possibility that must be ruled out (one eigenvalue could be zero, of course, but this is a hairline case), and that is, letting λ_1, λ_2 denote the eigenvalues of J, the following:

$$\lambda_1 < 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 < 0. \tag{92}$$

We will now exploit topological index theory (Milnor [15, p. 36-37]).[8/] The fact that the rest point is unique and condition (b) holds is enough to rule out (92). Condition (b) means that the phase diagram of (84) looks like Figure 1 below

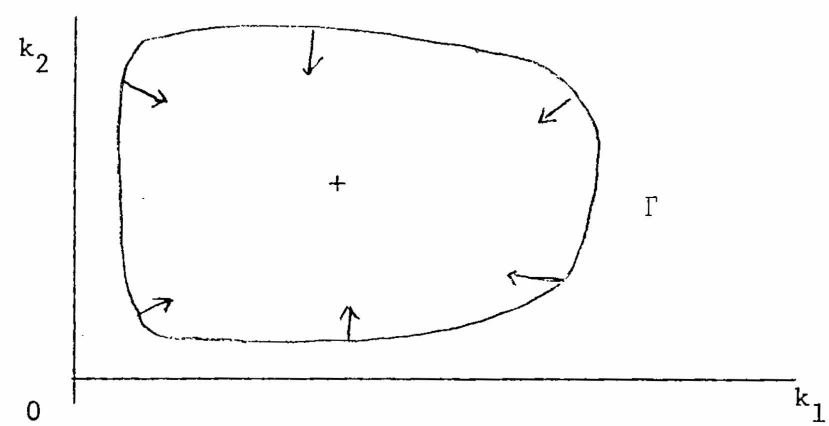

FIGURE 1

along Γ. All condition (b) is saying is that $\dot{k}_i > 0$ if k_i is small enough and $\dot{k}_i < 0$ if k_i is large enough - a rather mild restriction to place on a capital accumulation problem under the usual economic hypotheses.

Topological index theory (Milnor [15]) says that the index of Γ which is + 1 by the inward pointing condition equals the sum of the indices across steady states "inside" Γ. But by hypothesis there is only one steady state inside Γ. Thus, its index is + 1. But the index of a saddle point (e.g., (92)) is - 1. Therefore, saddle points are impossible. The theorem is proved.

A few remarks are in order.

Remark 1: A condition such as (b) while intuitively obvious and economically reasonable is not all that easy to verify.

Remark 2: Topological index theory fails to give such a nice result if zero eigenvalues are allowed. Ryder and Heal's formulae may be checked directly for the conditions that yield a zero eigenvalue. Notice that if one of them is zero the trace condition assumes that the other is negative so we have a "weak" form of LAS anyway.

Let us examine (91) for economic content. It says that LAS is more likely to hold when (a) $\delta - \rho$ decreases, (b) $u_{ck_1} < 0$, (c) $(-u_{cc}^{-1})$ increases (provided $u_{ck_1} < 0$). Now (a) says that LAS is likely to hold when the discount or future utility is small and when the effects of past consumption decay rapidly. This is reasonable. Assertion (b) states that the marginal utility of current consumption flow declines as the stock of past consumption experience increases - a presumption that most economists would agree upon. Statement (c) asserts that under the most likely presumption on the sign of u_{ck_1}, a decrease in the "concavity" of u in c $(-u_{cc}^{-1}$ increases) is stabilizing. This last result is the only one that is counter-intuitive. At any rate the trace analysis allows economically meaningful stability conditions to be derived rapidly for the Ryder-Heal model. This is important because the positive definiteness of the Q matrix (recall equation (19))is not a "natural" hypothesis to place upon the Ryder-Heal model. We close with a few remarks on limit cycles.

From (90)

$$\mathrm{Tr}\, H_{qk} = -\rho\, u_{cc}^{-1}\, u_{ck_1} - \rho + f'(k_2) - \lambda$$

It follows immediately from Lemma 4 of Section 4 that Tr H_{qk} must change sign on a limit cycle. The value of k_2 must oscillate above and below $\bar{k}_2$ along a limit cycle. Furthermore, roughly writing, the more negative is $\delta - \rho$, then the variation of k_2 about $\bar{k}_2$ must increase along the cycle as ρ increases. To see this just note that $\bar{k}_2$ is independent of ρ, and note that k_2 must be made close to zero in order that $f'(k_2)$ be made large enough to swamp the ρ-effect in order that

$$\mathrm{Tr}\, H_{qk}$$

change sign along the cycle.

This exercise shows how the methods developed here give insights into the determinants of size of limit cycles and necessary conditions for their existence.

Section 6: A Suggestion for Further Development: The Comparative Dynamics of Optimal Paths

Here we are concerned with the impact of a shift in a parameter, α, of a problem upon its optimal solution. As in static economics, in many cases, the hypothesis of L.A.S. of a steady state, $\bar{k}$, gives sign information on $\frac{\partial \bar{k}}{\partial \alpha}$: the "correspondence" principle of Samuelson. More importantly for dynamic analysis, the G.A.S. hypothesis tells us that the entire optimal path will be "pulled" in the direction of the steady state $\bar{k}(\alpha)$. I.e., G.A.S. of $\bar{k}$ for each α allows us to make "comparative dynamics" inferences from "static" shifts in $\bar{k}(\alpha)$. This rather vague inchoate observation also holds for finite horizon problems. For a long finite horizon problem's optimal path spends a large amount of time near the unique G.A.S. steady state of the corresponding infinite horizon problem, the well-known "turnpike" theorem of optimal economic growth (Takayama [23]). Therefore, a shift in the infinite horizon problem's G.A.S. steady state w.r.t. α exerts a corresponding "pull" on the solution of the corresponding finite horizon problem.

In the two state variable case G.A.S. should be useful to develop results on the shift of an optimal path when the initial conditions are changed. For here, one can use the policy function approach of dynamic programming to replace the four dimensional system

$$\dot{q} = \rho\, q - H_2$$

$$\dot{k} = H_1, \qquad k(0) = k_0$$

by the two dimensional

$$\dot{k} = H_1(q(k),k), \;\; k(0) = K_0$$

which is the familiar reduced form that we have been working with.

One can now exploit the fact that solutions of a pair of differential equations in the plane cannot "cross" together with G.A.S. to obtain results on the shift of the optimum path $k(t|k_0)$ when k_0 is changed.

The precise development of "comparative dynamics" and the role of G.A.S. in generating dynamic correspondence principles seems to me to be critically important for dynamic economics.

The G.A.S. results reported here should help in this task.

FOOTNOTES

1/ More general problems where

$$\dot{k} = T(k,x)$$

may be treated easily with the methods exposited in this paper. We have chosen to exposit the ideas contained in this paper for the case

$$\dot{k} = x$$

in order to minimize notational burden. Since the theorems to be presented have their hypotheses placed directly upon the Hamiltonian, therefore, no generality is lost by restricting our attention to the special case treated here.

2/ See Arrow and Kurz [1, p. 35] for an intuitive derivation of the Hamilton Jacobi equation and a list of references. We will say a few words about it here for the purpose of self containment.

Define the return function $R(k_0,t)$ by

$$R(k_0,t) \equiv \text{maximum} \int_t^\infty e^{-\rho t} U(k,x)dt$$

$$\text{s.t.} \quad \dot{k} = x$$

$$k(t) = k_0$$

Put

$$H = e^{-\rho t} U(k,x) + p\,x,$$

$$H^0 = \underset{x}{\text{maximum}} \; H.$$

Then Arrow and Kurz [1, p. 35] derive and define respectively

$$-\frac{\partial R}{\partial t} = H^0(p,k) \qquad \text{(a)}$$

$$p = \frac{\partial R}{\partial k} \qquad \text{(b)}$$

In our case,

$$R(k_0,t) = e^{-\rho t} R(k_0,0) \qquad \text{(c)}$$

Put

$$W(k_0) \equiv R(k_0,0),$$

and

$$q = e^{\rho t} p$$

$$H = e^{\rho t} H^0(p,k) = G(q,k)$$

Then it follows immediately from (a),(b),(c) that

$$\rho W(k_0) = G(q,k)$$

and

$$q(k) = W'(k)$$

from (b),(c).

Hence for all k_0

$$\rho W(k_0) = G(W'(k_0),k_0)$$

must hold.

3/ The Lyapunov function (50) is used in the study of the optimal linear regulator problem [29, Chapter 3]. At this point, we would like to thank M. Magill for stressing to us the close relation between (50) and (51).

Magill [30] has obtained stability results for optimal growth under uncertainty using methods stimulated by developments on the stochastic optimal linear regulator problem ([29, Chapter 3]). His paper was an important source of stimulation to us in generalizing the methods of [29, Chapter 3] to the discounted optimal growth problem in order to obtain new global asymptotic stability results.

4/ The following is true. Let the minimum eigenvalue of $H_{11}(q,k)$ be $\alpha(q,k)$ and the minimum eigenvalue of $-H_{22}(q,k)$ be $\beta(q,k)$. Then if

$$\alpha\ \beta > \rho^2/4 ,$$

then

$$Q(q,k) = \begin{bmatrix} H_{11}(q,k), & \rho/2\ I_n \\ \rho/2\ I_n, & -H_{22}(q,k) \end{bmatrix}$$

is positive definite.

To prove this, examine

$$x^T Qx = x_1^T H_{11} x_1 + x_2^T(- H_{22})x_2 + \rho x_1 x_2$$

$$\geqq \alpha x_1^T x_1 + \beta x_2^T x_2 + \rho x_1 x_2$$

$$\geqq \alpha x_1^T x_1 + \beta x_2^T x_2 - \rho |x_1 x_2|$$

$$\geqq \alpha ||x_1||^2 + \beta ||x_2||^2 - \rho ||x_1|| ||x_2||$$

where $||z|| \equiv (z^T z)^{1/2}$ for any vector z.

Now

$$\alpha ||x_1||^2 + \beta ||x_2||^2 - \rho ||x_1|| ||x_2||$$

$$> \alpha ||x_1||^2 + \beta ||x_2||^2 - (4\alpha\beta)^{1/2} ||x_1|| ||x_2||$$

So our problem reduces to showing that

$$\alpha ||x_1||^2 + \beta ||x_2||^2 - (4\alpha\beta)^{1/2} ||x_1|| ||x_2|| \geqq 0.$$

But this last quantity is just

$$(\alpha^{1/2} ||x_1|| - \beta^{1/2} ||x_2||)^2$$

which is non-negative. This ends the proof.

The importance of this observation is that it gives a nice geometric interpretation of the positive definiteness of Q. The hypothesis $\alpha\beta > \rho^2/4$ is an important sufficient condition for L.A.S. and G.A.S. in Rockafellar's analysis [20, p. 8] e.g. We have shown that it is sufficient for positive definiteness of Q, and hence, for stability by our analysis. Equation (53) is useless for discovering such a result. Therefore, the usefulness of Q is demonstrated once again.

5/ I thank Mr. Rau of the University of Rochester for correcting the original statement of this lemma and simplifying the proof.

6/ The reader should not be lulled into thinking that if A is positive definite, symmetric, and η is diagonal and positive definite that

$$E = A\eta + (A\eta)^T$$

is positive definite. It is easy to construct counterexamples in the 2 x 2 case.

It would be worthwhile to explore sufficient conditions on positive definite symmetric matrices A, and diagonal matrices η so that E is positive definite. Perhaps the methods of Arrow [25] will be useful in this regard. It is beyond the scope of this paper, however, to carry out this investigation.

7/ We are detailing with the Ryder-Heal nonsatiated case. They show that multiple steady states may exist if utility is satiated.

8/ Milnor has his phase diagram pointing "outward" along Γ rather than inward along Γ. By reversing the flow of time in our case, we are back to Milnor's case.

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