BARGAINING AND REPUTATION IN SEARCH MARKETS

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Abstract.

In a two-sided search market agents are paired to bargain over a unit surplus. The matching market serves as an endogenous outside option for a bargaining agent. Behavioral agents are commitment types that demand a constant portion of the surplus. The frequency of behavioral types is determined in equilibrium. Even if the frequency of behavioral types is negligible, they affect the terms of trade and efficiency. In an unbalanced market where the entering flow of one side is short, there is one-sided reputation building in bargaining, and commitment types on the short side determine the terms of trade. In a balanced market where the entering flows are equal, there is two-sided reputation building in bargaining, and reputation concerns lead to inefficiency. An equilibrium with persistent delays is constructed. The magnitude of inefficiency is determined by the demands of the commitment types and is independent of their frequency. Access to the market exacerbates bargaining inefficiencies caused by behavioral types, even at the frictionless limit of complete rationality.

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Contents

1. Introduction and related literature

Outcomes of bilateral dynamic interactions, where agents are rational (i.e., not behavioral), can be drastically different from outcomes of bilateral interactions, where there is even a small amount of incomplete information concerning the rationality (or the "type") of the agents.[1] Bilateral bargaining outcomes are also highly sensitive to the outside options of the bargaining agents.[2] However, outside options and the distribution of types available for trade are typically determined by aggregate market forces. In turn, some prominent large markets are agglomerations of many bilateral bargaining relationships.[3] For large markets economic intuition suggests that the impact of a small number of "behavioral" agents on aggregate equilibrium variables should be small. But this intuition suggests a tension between the impact of behavioral agents in bilateral relationships and the effect of behavioral agents in a large market. To highlight this tension we analyze a bilateral bargaining relationship within the context of a larger market comprised of rational agents and a small number of behavioral agents. We explore how incentives driven by aggregate market forces and incentives generated during bilateral bargaining interact to determine equilibrium outcomes.

Consider a two player alternating offers bargaining game over a unit surplus where the time between offers is arbitrarily small. Under complete information the unique perfect equilibrium of the bargaining game is the Rubinstein (1982) outcome (see also Shaked and Sutton (1984), Sutton (1986) and Perry and Reny (1993)).

Suppose instead that there is incomplete information about the type of player 1. In particular, if agent 1 is potentially a (strategically inflexible) "commitment" type that insists on portion θ_1 of the bargaining surplus, and player 2 is a fully rational normal type with certainty, then player 1 obtains θ_1 and player 2 receives $1-\theta_1$ in any perfect equilibrium, even if the probability that player 1 is a commitment type is arbitrarily small (the one-sided reputation result of Myerson (1991)).

In addition to player 1, suppose that there is also incomplete information about the type of player 2. In particular, if both players are potentially commitment types that demand θ_1 and θ_2, then a war of attrition ensues, and the unique equilibrium payoff

[1] See, Milgrom and Roberts (1982), Kreps and Wilson (1982), Fudenberg and Levine (1989) and Fudenberg and Levine (1992) for demonstrations of this phenomenon in repeated games; or Myerson (1991), Kambe (1999), Abreu and Gul (2000), Compte and Jehiel (2002), or Abreu and Pearce (2007) for examples in bilateral bargaining.

[2] See, for example, Compte and Jehiel (2002).

[3] The labor market and the housing market are particular examples of such markets. For economic models of such markets see Rubinstein and Wolinsky (1990, 1985) and Serrano and Yosha (1993), or see Osborne and Rubinstein (1990) for a more complete overview. Related work on bargaining and matching with incomplete information includes Sobel (1991) and Samuelson (1992).

profile is inefficient with the "weak" agent (agent i) receiving $1-\theta_j$ and the "strong" agent receiving strictly less than θ_j (the two-sided reputation result of Abreu and Gul (2000)). However, now suppose that both players have access to an outside option. If agent i's outside option exceeds $1-\theta_j$ and j's outside option is less than $1-\theta_i$, then player i never yields to j, eliminating the incentive for j to build a reputation, and the outcome is identical to the one-sided incomplete information case where i receives θ_i and j receives $1-\theta_i$ (Lemma 1). Moreover, if both agents' outside options dominate yielding to the commitment type, then the incentive to build a reputation is entirely eliminated, and the unique equilibrium is again the Rubinstein outcome (Compte and Jehiel (2002)).

As outlined above, the outcome of bilateral bargaining depends heavily on the distribution over agent types while the distribution over agent types is an endogenous variable determined in a market equilibrium. In turn, the market equilibrium may depend on bargaining outcomes: agents that are traded infrequently, but that nevertheless obtain high values, are plentiful; while agents that are traded very frequently, or agents that have very low values, are scarce. Also, the bargaining outcome depends crucially on the outside options of agents, and again, outside options are endogenous variables determined in equilibrium. An agent that enjoys favorable bargaining outcomes will typically be more optimistic about his outside option than an agent who is less successful in bargaining.

To address the aforementioned issues of endogeneity, we present a two-sided search model where agents are paired to bargain over a unit surplus. The two sides of the market can be thought of as buyers and sellers of a homogeneous good. The matching market serves as the endogenous outside option for agents in a bargaining relationship. In each period, a constant measure of agents enters the market. Agents exit the market through successfully making a trade, or they leave voluntarily because there are no profitable trading opportunities in the market. A fraction of the entering population on each side is comprised of commitment types. The steady state frequency of behavioral types in the market is determined in equilibrium, and if the entering fraction of behavioral types is small, then so is the equilibrium frequency of behavioral types.

A central finding of this paper is that even a negligible number of behavioral agents significantly affect equilibrium outcomes, that is, equilibrium bargaining behavior, equilibrium terms of trade and outside options.[4] Compte and Jehiel (2002) demonstrated that if the outside options of normal types are sufficiently high, then commitment types have no effect on bargaining outcomes. In the market analyzed here, however, the endogenous outside options of the normal types are never large enough to deter the commitment types. In equilibrium, some normal types always trade with commitment types. This,

[4]In all the results described, the time between offers in the bargaining stage is assumed arbitrarily small.

in turn, makes rational agents in the market excessively greedy in bargaining. Consequently, even if behavioral types are negligible, they substantially affect the terms of trade and the efficiency of the bilateral bargaining outcome.

Although behavioral agents always have an impact on the bilateral outcome, the nature of their effect depends on aggregate forces. We focus on two cases: an unbalanced market where the entering flow of one side is short, and a balanced market where the entering flows of the two sides are equal. Unbalanced markets entail one-sided reputation building; and balanced markets entail two-sided reputation building, in equilibrium. Note that commitment types are present on both sides regardless of whether the market is balanced or unbalanced. Nevertheless, in an unbalanced market, only the short-side chooses to imitate the commitment types, whereas, in a balanced market, both sides imitate the commitment types.

In an unbalanced market, a fraction of the agents in the long side of the market must be leaving the market without trading in any steady state. Consequently, aggregate flows ensure that the outside option of the long-side is compatible with the demands of the commitment types while the outside option of the short-side is incompatible. However, if the short-side's outside option is incompatible and the long-side's outside option is compatible with the commitment type demands, then equilibrium play in the bargaining stage involves one-sided reputation building by agents on the short-side.

In balanced markets, the effects of the commitment types are most pronounced. In equilibrium, aggregate forces ensure that the outside options of both sides are compatible with the inflexible demands of the commitment types. So, the magnitude of inefficiency is determined by the inflexible demands of the commitment types. In order to further investigate the impact of the commitment types on the market and provide a direct comparison with previous literature, we construct an analytically tractable equilibrium that exhibits dynamics similar to previous models of two-sided reputation (Abreu and Gul (2000) and Compte and Jehiel (2002)). In this equilibrium the normal types play a war of attrition and always trade. A normal type of side 2 always trades with a commitment type of side 1 but a normal type of side 1 opts out against the commitment type with positive probability. Bargaining is inefficient and the inefficiency is caused by delays in reaching an agreement. As the fraction of commitment types entering the market approaches zero, the steady-state frequency of commitment types present in the market and the probability that a normal type of side 1 opts out against the commitment type also converge to zero. However, in sharp contrast to existing literature (Abreu and Gul (2000) and Compte and Jehiel (2002)), the inefficiency (manifested as delay) persists, even at the limit without commitment types. Consequently, the mere availability of outside market is sufficient to magnify the impact of commitment types and leads to substantial inter-bargaining inefficiency.

1.1. **Outline of the paper.** The paper is organized as follows: Section 2 describes the basic model. Section 3 presents a baseline for comparison by characterizing equilibria in a model without any commitment types. Section 4 analyzes the bargaining stage game and presents the required interim results. Section 5 presents the main one-sided reputation result for unbalanced markets. Section 6 presents the main two-sided reputation result for balanced markets. Section 6.1 constructs an inefficient equilibrium with two sided reputation building for balanced market and section 6.2 provides the comparative statics for the constructed equilibrium. All proofs are in the appendix.

2. The model

In each period agents belonging to two classes $i \in \{1, 2\}$ (for example, buyers and sellers) enter a matching market. Mass L_i of agents enter from each class. *We assume, without loss of generality, that* $L_2 \geq L_1$. Of the class i agents entering the market a fraction z_i are commitment types and the remaining $1 - z_i$ are normal types. We refer to an agent from class 1 as **agent 1** or **him** and to an agent from class 2 as **agent 2** or **her**. We refer to normal type agents from class 1 and class 2 as **player 1** and **player 2**, respectively. Also, we refer to commitment type agents from class 1 and class 2 as **commitment type 1** and **commitment type 2**, respectively.

In each period a portion of the unmatched agents in the market are randomly paired with a potential trading partner from the opposite class to play a bilateral bargaining game. In each bilateral bargaining game a unit surplus is available for division between the paired agents. Agents only receive utility if they can agree on the division of the unit surplus. If two matched agents agree on the division of the unit surplus, then they trade. Agents that trade leave the market permanently. The division of the unit surplus is determined in an alternating offers bargaining game with the possibility of opting out.

In the alternating offers bargaining game agent 1 is the proposer in the odd periods and agent 2 is the proposer in the even periods. The proposer can make an offer or can opt-out and terminate the bargaining relationship. If the proposer chooses to make an offer, then he/she proposes a division of the unit surplus. The responder can accept the offer, reject the offer or can opt-out and terminate the bargaining relationship. If agent i rejects agent j's offer, then agent i becomes the proposer after $\Delta > 0$ units of time. The extensive form of the bargaining game is given in Figure 1.

The bargaining game can terminate without an agreement because an agent voluntarily opts out (i.e., due to an endogenous break-up). If the bargaining game terminates without

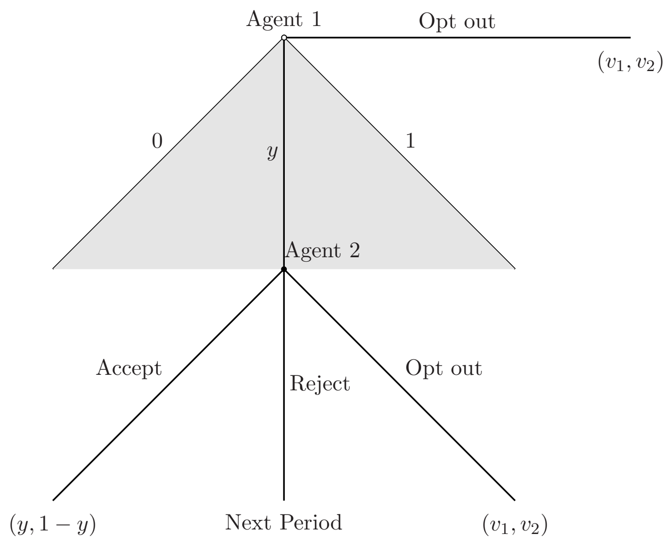

FIGURE 1. This depicts the bargaining game in any odd period where agent 1 speaks first. He can offer $y \in [0, 1]$ or opt-out. Agent 2 speaks second and can accept the offer, reject the offer or opt-out. The option value for agent i of opting out and returning to the unmatched matched population is equal to v_i. The option value v_i is determined endogenously in equilibrium. In equilibrium, v_i equals the maximum of the agent's equilibrium payoff from remaining unmatched in the market and the agent's payoff from exiting the market and receiving $x(\Delta)$.

an agreement, then both agents return to the unmatched population. An agent in the unmatched population can choose to wait for $t(\Delta)$ units of time (or $t(\Delta)/\Delta$ periods) for a new bargaining partner,[5] or can choose to leave the market and receive an exogenous outside option worth $x(\Delta)$. We assume that $t : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous function,* $t(\Delta) \geq \Delta$ *for all* Δ *and* $\lim_{\Delta\to 0} t(\Delta) = 0$. Also, we assume that $x : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous function and* $\lim_{\Delta\to 0} x(\Delta)/\Delta = 0$.

The period length Δ measures the amount of time it takes to formulate a counter-offer. So, it is a measure of with-in bargaining frictions. If $\Delta \approx 0$, then agents are able to make offers almost instantaneously. The parameter $t(\Delta)$ measures the amount of time it takes to generate a new bargaining opportunity and is a proxy for the magnitude of search frictions. If $t(\Delta) \approx 0$, then agents are able to generate new bargaining partners almost instantaneously.[6] The parameter $x(\Delta)$ is the exogenous outside option. The only role of the outside option is to ensure that agents who never trade leave the market instead

[5]Alternatively, one could assume, without introducing any additional complications, that matches are determined by a Poisson process with an expected time to matching equal to $t(\Delta)$.

[6]We parametrize $t(\Delta)$ and $x(\Delta)$ by the period length Δ for a more unified exposition. All our results would continue to hold even if we assume that these parameters are independent of the period length Δ.

of accumulating and clogging up the market. The assumption that $\lim_{\Delta\to 0} x(\Delta)/\Delta = 0$ focuses our analysis away from $x(\Delta)$ and on endogenous outside options, even at the limit.[7]

2.1. **Agent types.** Player i (a normal type belonging to class i) is impatient with instantaneous rate of time preference r_i. Consequently, if player i reaches an agreement that gives him y units of surplus in period s, then his utility is equal to $ye^{-r_i\Delta s}$. Alternatively, if he takes the exogenous outside option in period s, then his utility is equal to $x(\Delta)e^{-r_i\Delta s}$.

Commitment type i (a commitment type belonging to class i) is assumed to insist on share θ_i of the unit surplus and reject any offer that gives him less than θ_i. The demands of commitment type 1 and commitment type 2 are incompatible, that is $\theta_1 + \theta_2 > 1$. The commitment types never opt-out as long as there is positive probability that their opponent is a normal type, and they immediately opt-out otherwise. Consequently, the probability that two commitment types remain in a bargaining relationship forever is zero. The commitment types decide whether to leave or remain in the market using the same payoff calculation as the normal types.[8]

2.2. **The pool of unmatched agents and matching.** Let N_i^s and C_i^s denote the measure of unmatched player is and unmatched commitment type is in the matching market in period s, respectively. Also, let $n_i^s = N_i^s/(N_i^s+C_i^s)$ and $c_i^s = C_i^s/(N_i^s+C_i^s)$, that is, n_i^s and c_i^s are the proportions of player is and commitment type is among unmatched agent is in period s, respectively. Also, let $m_i^s = \min\{1, (N_j^s + C_j^s)/(N_i^s + C_i^s)\}$. If the measure of agent 1s in the pool of unmatched agents is equal to the measure of agent 2s, then the market tightness parameter $m_1^s = m_2^s = 1$. Otherwise, since one side of the market is larger, these agents are rationed and the market tightness for this side is less than one. The market tightness is the inverse of the "queue length".

The pool of unmatched agents available to be matched in period s is comprised of agents that entered the market in period s; agents whose bargaining arrangement dissolved as a result of an opt-out in period $s-t(\Delta)/\Delta$; and agents who in period $s-t(\Delta)/\Delta$ were not paired with a bargaining partner and who chose to remain in the market. An

[7]The assumption $\lim_{\Delta\to 0} x(\Delta)/\Delta = 0$ is immaterial for all of our results except Theorem 2 which requires the weaker assumption that $\lim_{\Delta\to 0} x(\Delta)/1 - e^{-t(\Delta)} = 0$. However, we make the stronger assumption in order to stress that the only role we wish $x(\Delta)$ to play is to ensure that the measure of active agents remains bounded in equilibrium.

[8]This is a much stronger assumption than we need on the preferences of the commitment types. All the results go through under the following behavioral assumption: if the probability of being traded is strictly positive and if the expected time at which a trade occurs is finite, then there exists an x^* such that the commitment types strictly prefer to remain in the market for all $x(\Delta) < x^*$.

agent i is matched with a player j with probability n_j^s and with a commitment type j with probability c_j^s, in period s. Consequently, measure $m_1^s N_1^s n_2^s = m_2^s n_1^s N_2^s$ of player 1, player 2 pairs; measure $m_1^s C_1^s n_2^s = m_2^s c_1^s N_2^s$ of commitment type 1, player 2 pairs; measure $m_1^s N_1^s c_2^s = m_2^s n_1^s C_2^s$ of player 1, commitment type 2 pairs; and measure $m_1^s C_1^s n_2^s = m_2^s c_1^s C_2^s$ of commitment type 1, commitment type 2 pairs are created, in each period s.

2.3. **Histories and strategies.** Agents have perfect recall. Agents do not observe the actions chosen by another agent unless they are in a bargaining relationship. Agents observe the actions of their bargaining partner while they bargain. Let h^t denote a period t history for agent i which contains all the information that agent i has observed up to period t and let H denote the set of all histories for player i.

A strategy for player i, $\sigma_i : H \rightarrow [0,1] \cup \{accept,\ reject,\ opt-out,\ leave,\ stay\}$. The strategy for player i, $\sigma_i(h^t) \in [0,1] \cup \{opt-out\}$, if player i is making an offer in any period t history h^t and $\sigma_i(h^t) \in \{accept,\ reject,\ opt-out\}$, if player i is responding in any period t history h^t. Also, at the end of each period, agents in the pool of unmatched agents can leave the market and take their exogenous outside option, or choose to stay in the market until the next period. Consequently, at a history h^t where player i needs to choose whether to leave or stay, $\sigma_i(h^t) \in \{leave,\ stay\}$. A behavior strategy is similarly defined but has the player randomizing over the action choices. We assume that agents of the same type and class use the same strategy, i.e., we focus on symmetric strategies. A belief for agent i is a function $\mu_j : H \rightarrow [0,1]$ that gives the probability that agent i places on his bargaining partner j being the commitment type, when player i is bargaining with agent j.[9]

2.4. **Steady state.** Our analysis focuses on the steady state of the system, i.e., we assume $(N_1^s, C_1^s, N_2^s, C_2^s) = (N_1, C_1, N_2, C_2)$ for all periods s. A steady state implies that the measure of agents leaving the market in each period (through successfully consummating a trade or through voluntary exit) equals the inflow of new agents into the market. Consequently, the steady state equations for the market are as follows:

$$\underbrace{(1-z_1)L_1}_{\text{Flow entry by player 1}} = \underbrace{N_1 m_1 n_2 p_{nn}}_{\text{Trade with player 2}} + \underbrace{N_1 m_1 c_2 p_{nc}}_{\text{Trade with commitment type 2}} + \underbrace{E_1^n,}_{\text{Voluntary exit}} \tag{1}$$

[9]At histories where player i is not bargaining with another agent, we set the belief function to equal the steady state frequency of commitment type j, i.e., $\mu_j = c_j$.

and similarly

$$z_1 L_1 = C_1 m_1 (n_2 p_{cn} + c_2 p_{cc}) + E_1^c, \tag{2}$$

$$(1 - z_2) L_2 = n_2 m_2 (N_1 p_{nn} + C_1 p_{cn}) + E_2^n, \tag{3}$$

$$z_2 L_2 = c_2 m_2 (N_1 p_{nc} + C_1 p_{cc}) + E_2^c, \tag{4}$$

where p_{nn} is the probability that player 1 and player 2 trade if matched, p_{cn} is the probability that commitment type 1 and player 2 trade if matched and p_{nc} and p_{cc} are similarly defined. E_i^n and E_i^c are the measures of player is and commitment type is leaving the market without trading at the end of each period, respectively. The vector of match probabilities p, as well as, the vector of outflows E are obtained from the strategy profiles.

2.5. **Equilibrium.** Let $v = (v_1, v_2)$, $c = (c_1, c_2)$, $L = (L_1, L_2)$ and $z = (z_1, z_2)$. $\Gamma(\Delta, c, v)$ denotes the bargaining stage game where the time between offers is Δ, opting out is worth v_i to player i, and the initial belief that player i's opponent is a commitment type, $\mu_i(h^0)$ is equal to c_i. In the bargaining stage game $\Gamma(\Delta, c, v)$, let $U_i(\sigma|n)$ and $U_i(\sigma|c)$ denote the payoff for player i conditional on facing a normal type or a commitment type respectively and let $U_i(\sigma) = (1 - c_j) U_i(\sigma|n) + c_j U_i(\sigma|c)$ denote the expected payoff for player i, if the agents use strategy profile σ.

$E(\Delta, L, z)$ denotes the search market. A *search equilibrium* σ for $E(\Delta, L, z)$ is comprised of a strategy σ_k for each agent type; a belief function μ_k for each agent type; and steady state measures (N_1, C_1, N_2, C_2), that are mutually compatible. More precisely, an equilibrium σ satisfies the following conditions:

- (i) The strategy profile σ and the belief profile μ comprises a *perfect Bayesian equilibrium* (PBE) in the bargaining stage-game $\Gamma(\Delta, c, v(\sigma))$, where $c_i = C_i/(N_i + C_i)$ is the equilibrium frequency of commitment type i and $v_i(\sigma)$ is the equilibrium value for player i.
- (ii) The equilibrium values for each player i satisfies the following recursive equation

  $$v_i(\sigma) = \max\{x(\Delta), e^{-r_i t(\Delta)} m_i U_i(\sigma) + (1 - m_i) e^{-r_i t(\Delta)} v_i(\sigma).\}$$

  In words, player i can take the exogenous outside option, or alternatively wait $t(\Delta)$ units of time and start a bargaining game $\Gamma(\Delta, c, v(\sigma))$ with probability $m_i = \min\{1, (N_j + C_j)/(N_i + C_i)\}$.
- (iii) The market remains in steady state, i.e., equations (1) - (4) are satisfied, given the equilibrium σ.

2.6. **Balanced and unbalanced markets.** Recall we assume, without loss of generality, that $L_2 \geq L_1$. First, as a baseline, assume that $z_1 = z_2 = 0$, i.e., assume that there are no commitment types in the market. We say that the market is *unbalanced* if $L_1 < L_2$ and we say that the market is *balanced* if $L_1 = L_2$.

The market with commitment types ($z_1 > 0$ and $z_2 > 0$) is a perturbation of the baseline without commitment types. If we start from an unbalanced market without any commitment types and perturb the model by assuming that a positive but small percentage of the entering population is comprised of commitment types, then $L_1 < L_2(1 - z_2)$. Consequently, in the general model with commitment types we say that a market is *unbalanced* if $L_1 < L_2(1 - z_2)$. Similarly, if we start from a balanced market without any commitment types and perturb the model by assuming that a positive but small percentage of the entering population is comprised of commitment types, then $L_1 \geq L_2(1-z_2)$. Consequently, in the general model with commitment types we say that a market is *balanced* if $L_1 \geq L_2(1 - z_2)$.

3. Baseline without commitment types

First, before turning to the model with commitment types, we study a model with only rational agents ($z_1 = z_2 = 0$). In the bargaining stage game equilibrium play unfolds according to the complete information alternating offers bargaining model of Rubinstein (1982). Recall that $\Delta \leq t(\Delta)$, that is, once in a bargaining relationship it takes less time to make a counter offer than to opt-out and search for a new bargaining partner. This implies, with only rational agents, players never opt-out after any history. So, play is identical to an alternating offers bargaining game without opt-outs which has the Rubinstein outcome as its unique equilibrium. Define, the Rubinstein payoffs, $u_1^*(\Delta) \equiv \frac{1-e^{-r_2\Delta}}{1-e^{-(r_1+r_2)\Delta}}$ and $u_2^*(\Delta) \equiv \frac{e^{-r_2\Delta}(1-e^{-r_1\Delta})}{1-e^{-(r_1+r_2)\Delta}}$.

In each period, an equal number of agents from class i and j leave the market as a result of successful trades. This is because all trade occurs in pairs. If the market is unbalanced ($L_2 > L_1$), then there are more agent 2s entering the market than agent 1s in each period and some agent 2s must leave the market voluntarily without trading, for the market to remain in steady state. So, in order to incentivize agents on the long side to voluntarily exit the market, equilibrium values for agent 2 must equal the exogenous outside option $x(\Delta)$, i.e., $v_2 = x(\Delta)$. However, since each agent 2 receives a substantial portion of the unit pie in the bargaining stage game, the market tightness m_2 for side 2 must be sufficiently smaller than 1, (or in alternative terminology, the queue length, $1/m_2$, must be sufficiently long) in order for agent i's value to equal $x(\Delta)$. Alternatively, if the markets are balanced ($L_2 = L_1$), then there is an equilibrium, with no queues on either side, in which case each side receives their Rubinstein payoff. The following summarizes these results.

The Complete Information Benchmark. *If* $z_1 = z_2 = 0$ *and* $x(\Delta) < \min_i e^{-r_i t(\Delta)} u_i^*(\Delta)$, *then*

- (i) *The agents receive their Rubinstein payoffs in the bargaining stage game, i.e.,* $U_i(\sigma) = u_i^*(\Delta)$,
- (ii) *If* $L_2 > L_1$, *then* $v_2(\sigma) = x(\Delta)$ *and* $v_1(\sigma) = e^{-r_1 t(\Delta)} u_1^*(\Delta)$,
- (iii) *If* $L_2 = L_1$, *then* $v_i(\sigma) = e^{-r_i t(\Delta)} u_i^*(\Delta)$ *and* $x(\Delta) \leq v_j(\sigma) \leq e^{-r_j t(\Delta)} u_j^*(\Delta)$,

in any search equilibrium σ *of* $E(\Delta, L, z)$. *Also, if* $L_1 = L_2$, *then there is a search equilibrium* σ *of* $E(\Delta, L, z)$ *such that* $v_1(\sigma) = e^{-r_1 t(\Delta)} u_1^*(\Delta)$ *and* $v_2(\sigma) = e^{-r_2 t(\Delta)} u_2^*(\Delta)$.

Proof. Follows from Rubinstein (1982), Shaked and Sutton (1984) or Osborne and Rubinstein (1990). For an argument see Appendix A. □

4. The bargaining stage game

This section presents results for the bargaining stage game used in the analysis of the full economy. Lemmata 1, 2 and 3 take as given the vector of outside options, v, and the vector of commitment type probabilities c and characterize PBE for the bargaining stage-game $\Gamma(\Delta, c, v)$.

We say *player* i *reveals rationality* if i accepts any offer less than θ_i in a period she responds, or proposes something other than θ_i in a period that she proposes. Let $r = \max\{r_1, r_2\}$. For any $y > 0$ and $m > 0$ we say that player i's payoff U_i *is approximately equal to* y, $U_i \approx y$, *and the error is* $o(\Delta^m)$, if there exists a constant $\kappa > 0$, that is independent of Δ, such that

$$y - \kappa(1 - e^{-r\Delta})^m \leq U_i \leq y + \kappa(1 - e^{-r\Delta})^m.$$

Also, if the error associated with a particular approximation is omitted, then this should be understood to mean that $m = 1$ and so the error is $o(\Delta)$.

Lemma 1 considers a situation where player 1's outside option is incompatible with commitment type 2 ($v_1 > 1 - \theta_2$), while player 2's outside option is compatible with commitment type 1 ($v_2 < 1 - \theta_1$). In this situation, player 1 would rather take his outside option v_1 than trade with commitment type 2 who demands θ_2. So, mimicking the commitment type does not improve player 2's bargaining share and can delay a possible agreement. This implies that player 2 will reveal rationality immediately. If player 1 prefers his Rubinstein share $u_1^*(\Delta)$ to θ_1, then he does not have an incentive to build a reputation either and the players will agree on their Rubinstein shares in the first period. If on the other hand $u_1^*(\Delta) < \theta_1$, that is, if player 1 has an incentive to build a reputation, then player 1 will propose θ_1 in each period. Once player 2 reveals her rationality, the continuation bargaining game is a game with one-sided incomplete information and in this continuation player 1 can secure a payoff close to θ_1, if Δ is sufficiently small (Myerson (1991)). The configuration of outside options considered in Lemma 1 has not been analyzed in previous literature. We provide the complete proof in Appendix C. Also, we provide an analysis of the bargaining game with one-sided incomplete information in Appendix B.

Lemma 1 (One-sided reputation)**.** *Suppose that $c_1 > 0$, $c_2 > 0$, $\theta_1 > v_1$, player 1's outside option is **incompatible** with $1-\theta_2$ and player 2's outside option is **compatible** with $1-\theta_1$, i.e., $v_1 > 1-\theta_2$ and $v_2 < 1-\theta_1$. If $\theta_1 > u_1^*(\Delta)$, then*

(*i*) *Player 1 always proposes θ_1,*

(*ii*) *Player 2 reveals rationality in period 1 or period 2,*

(*iii*) *$U_1(\sigma|n) \approx \theta_1$ and $U_2(\sigma|n) \approx 1-\theta_1$;*

if $\theta_1 < u_1^*(\Delta)$, then $U_1(\sigma|n) = u_1^*(\Delta)$ and $U_2(\sigma|n) = u_2^*(\Delta)$; in any PBE σ of $\Gamma(\Delta, c, v)$.

Lemma 2 considers a situation where both outside options are dominated by the commitment type demands. In this case both players would rather trade with a commitment type than take the outside option. This game is identical to the bargaining game analyzed by Abreu and Gul (2000). Let,

$$\lambda_i \equiv \lim_{\Delta\to 0}\left(\lambda_i(\Delta) = \frac{(1-e^{-r_j\Delta})(1-\theta_i)}{\Delta(\theta_i+\theta_j-1)}\right) = \frac{r_j(1-\theta_i)}{(\theta_i+\theta_j-1)}, \tag{5}$$

$$T_i \equiv -\ln c_i/\lambda_i \tag{6}$$

$$T = T_i \min\{T_j/T_i, 1\} \text{ and} \tag{7}$$

$$b_i \equiv c_i c_j^{-\lambda_i/\lambda_j} \text{ for } T_i > T_j. \tag{8}$$

Abreu and Gul (2000) showed that all PBE of the bargaining game converge to a war of attrition where each agent reveals their rationality with constant hazard rate λ_i, both agents complete their revealing at time T, and the "weaker" agent, i.e, the agent i with the larger T_i, will concede with positive probability equal to $1-b_i$ at time zero. In the war of attrition, both players are indifferent between revealing rationality immediately to their opponent or continuing to resist, after time zero. This implies that the payoff to the normal type is equal to $1-\theta_j$ which is the payoff obtained by yielding immediately to the commitment type after time zero. Consequently, the bargaining game payoff of the strong player j is approximately $(1-b_i)\theta_j + b_i(1-\theta_i)$. Also, the probability that the strong player j concedes at time zero is equal to zero, i.e., $1-b_j = 0$. Consequently, the bargaining game payoff of the weak player i is approximately equal to $(1-b_j)\theta_i + b_j(1-\theta_j) = 1-\theta_j$. These findings are summarized in the following lemma.

Lemma 2 (Two-sided reputation)**.** *Suppose that $c_1 > 0$, $c_2 > 0$. If both player 1 and player 2's outside options are **compatible** with $1-\theta_2$ and $1-\theta_1$ respectively, i.e., $v_1 < 1-\theta_2$ and $v_2 < 1-\theta_1$, then $U_i(\sigma) \approx (1-b_j)\theta_i + b_j(1-\theta_j)$ with error $o(\sqrt{\Delta})$ in any PBE σ of $\Gamma(\Delta, c, v)$.*

Proof. See Compte and Jehiel (2002) Proposition 3 or Abreu and Gul (2000) Proposition 4. □

Lemma 3 considers a situation where each player's outside option exceeds the payoff from trading with a commitment type. Under this scenario, the incentive to mimic the commitment type is eliminated for both players since their opponent never accepts the demand of a commitment type. However, once both players reveal rationality, the unique PBE of the bargaining game results in the Rubinstein outcome. This result, established in Compte and Jehiel (2002), is summarized in the following lemma.

Lemma 3. *Suppose that $c_1 > 0$, $c_2 > 0$, $v_1 < u_1^*(\Delta)$ and $v_2 < u_2^*(\Delta)$. If both player 1 and player 2's outside options are **incompatible** with $1-\theta_2$ and $1-\theta_1$ respectively, i.e., $v_1 > 1-\theta_2$ and $v_2 > 1-\theta_1$, then the players receive their Rubinstein payoffs, i.e., $U_1(\sigma|n) = u_1^*(\Delta)$ and $U_2(\sigma|n) = u_2^*(\Delta)$, in any PBE σ of $\Gamma(\Delta, c, v)$*

Proof. See Compte and Jehiel (2002) Proposition 5. □

5. Unbalanced markets and one-sided reputation

In this section we focus on unbalanced markets ($L_1 < (1-z_2)L_2$), i.e., markets where there are more player 2s than agent 1s entering the market looking for a trade. We present Theorem 1 and Corollary 1. Theorem 1 shows that equilibrium play is characterized by one-sided reputation building in an unbalanced market. Corollary 1 generalizes this result to an unbalanced market with multiple commitment types. For all our results we assume that the time between offers, Δ, is small.

First we prove an intermediate result, Lemma 4, which shows that player 1 receives at least θ_1 in the bargaining game, player 1's equilibrium value, v_1, is at least $e^{-r_1 t(\Delta)}\theta_1$ and player 2's equilibrium value v_2 is equal to $x(\Delta)$ in any search equilibrium. The argument for $v_2 = x(\Delta)$ is straightforward: a fraction of the player 2s must leave the market without trading by taking their exogenous outside option in order for the market to remain in steady state. Consequently, player 2 must be indifferent between leaving the market, which delivers a payoff equal to $x(\Delta)$, and remaining active in the market. In equilibrium, the market tightness m_2 is sufficiently smaller than 1 and ensures that player 2's value is equal to $x(\Delta)$. In particular, as $x(\Delta)$ approaches zero, m_2 also approaches zero and the measure of class 2 agents in the market grows arbitrarily large.

The argument to show that player 1 receives at least θ_1 in the bargaining game is more subtle: if $x(\Delta)$ is sufficiently small, then $v_2 = x(\Delta) < 1-\theta_1$, i.e., player 2's outside option is compatible with $1-\theta_1$. Also, we show that c_2/c_1 is small, if the exogenous outside option x is sufficiently small. However, if $v_2 < 1-\theta_1$ and if c_2/c_1 is sufficiently small, then player 1's bargaining game payoff is at least θ_1 (Lemma D.1 in the appendix). The key step in this argument is to show that c_2/c_1 is small, that is, commitment type 2s are under-represented in the market compared to commitment type 1s. Roughly the intuition is as follows: commitment type 1 does well in equilibrium since player 2 is willing to trade with him. Consequently, commitment type 1 will not leave the market without trading, if $x(\Delta)$ is sufficiently small. Commitment type 2, in contrast, always voluntarily leaves after one period in the market. This is because commitment type 2 is strictly worse off than player 2 and player 2's payoff is equal to $x(\Delta)$. Hence the measure of commitment type 2s in the market is equal to z_2L_2. However, there must be an arbitrarily large measure of player 2s in the market in order for the market tightness parameter m_2 to be sufficiently small. Consequently, c_2 and also c_2/c_1 are arbitrarily small, if $x(\Delta)$ is sufficiently small.

Lemma 4. *Suppose that the market is unbalanced. In any search equilibrium $v_2(\sigma) = x(\Delta)$. Moreover, for any $\epsilon > 0$, there is a positive cutoff $\bar{\Delta}$ such that if $\Delta < \bar{\Delta}$, then*

(*i*) *Payoffs in the bargaining game are $U_1(\sigma) \geq \theta_1 - \epsilon$ and $U_2(\sigma) \leq 1 - \theta_1 + \epsilon$,*
(*ii*) *Consequently, player 1's value is $v_1(\sigma) \geq e^{-r_1 t(\Delta)}(\theta_1 - \epsilon)$,*

in any search equilibrium σ of $E(\Delta, L, z)$.

The conclusions of Lemma 4 require that both Δ and $x(\Delta)$ are small. If $t(\Delta)$ is also sufficiently small, then we can further characterize equilibrium behavior. Lemma 4 demonstrated that $v_1(\sigma) \geq e^{-r_1 t(\Delta)}(\theta_1 - \epsilon)$. If $t(\Delta)$ and ϵ are sufficiently small, then agent 1's equilibrium value dominates conceding to his opponent's commitment type, $v_1(\sigma) \geq e^{-r_1 t(\Delta)}(\theta_1 - \epsilon) > 1 - \theta_2$, in any equilibrium (recall that $\theta_1 + \theta_2 > 1$). Consequently, player 2's outside option, $v_2(\sigma) = x$, is compatible with $1-\theta_1$, but player 1's outside option is incompatible with $1-\theta_2$. Hence, the bargaining stage-game involves one-sided reputation building as in Lemma 1, if $\theta_1 > u_1^*(\Delta)$. In particular, player 2 reveals rationality immediately whereas player 1 always proposes θ_1 and builds a reputation. Following this line of reasoning, the following theorem characterizes equilibrium behavior in an unbalanced market.

Theorem 1. *Suppose that the market is unbalanced. There is a cutoff $\bar{\Delta}$ such that if $\Delta < \bar{\Delta}$ and $\theta_1 > u_1^*(\Delta)$, then*

(i) *Player 1 always proposes θ_1,*

(ii) *Player 2 reveals rationality in period 1 or period 2,*

(iii) $U_1(\sigma|n) \approx \theta_1$ *and* $U_2(\sigma|n) \approx 1-\theta_1$;

and if $\theta_1 < u_1^*(\Delta)$, then $U_1(\sigma|n) = u_1^*(\Delta)$ and $U_2(\sigma|n) = u_2^*(\Delta)$; in the bargaining stage of any search equilibrium σ of $E(\Delta, L, z)$.

The equilibrium behavior with commitment types (as characterized in Theorem 1) contrasts with equilibrium behavior without any commitment types (as characterized in section 3). In every equilibrium player 1 fully mimics the commitment type and insists on θ_1. This is because player 2 is always willing to trade with commitment type 1. More subtly, the equilibrium bargaining behavior does not involve two-sided reputation building (as in Lemma 2), even though there are commitment types on both sides of the market. Player 1's equilibrium payoff, and thus his endogenous outside option, strictly exceeds accepting the inflexible demand of commitment type 2. As in Lemma 1 this precludes player 2 from building a reputation. In particular the theorem shows,

(i) Player 1's equilibrium value, which is approximately $e^{-r_1 t(\Delta)}\theta_1$, strictly exceeds his equilibrium payoff without commitment types $e^{-r_1 t(\Delta)}u_1^*(\Delta)$, and also the commitment type demand $1-\theta_2$. This implies that only player 1 builds a reputation.

(ii) Since the market tilts the bargaining power in the bargaining stage game towards player 1, the queue length required to make player 2 willing to take the exogenous outside option is reduced. Consequently, the overall efficiency in the market is improved compared to the complete information benchmark. Viewed from an alternative angle, player 1's equilibrium value increases to θ_1 while player 2's equilibrium value, which equals $x(\Delta)$, remains unchanged. This implies that overall efficiency is improved. Notice that a greedier commitment type, i.e., a higher θ_1, results in a greater efficiency gain.

(iii) The inefficiency in the bargaining stage is minimal. On the equilibrium path player 2 immediately reveals rationality and the number of periods of delay, in a game with one-sided incomplete information, is at most κ.

Theorem 1 considers a market with only one commitment type on each side. Suppose instead an agent i is one of finitely many commitment types in a set T_i. Let θ_i^n denote the inflexible demand of type $n \in \{1, ..., |T_i|\}$ of class i; let z_i^n denote the fraction of class i agents entering the market in each period who are of type n; and redefine $z_i = \sum_n z_i^n$. So, as before, $L_i(1-z_i)$ is the measure of rational agents of class i entering the market in each period. Let $\bar{\theta}_i = \max_{\{\theta_i^n \in T_i\}}\{\theta_i^n\}$ and $\underline{\theta}_i = \min_{\{\theta_i^n \in T_i\}}\{\theta_i^n\}$. We assume that $\underline{\theta}_1 + \underline{\theta}_2 > 1$, that is, the demands of no two commitment types are compatible. The following corollary shows that player 1 will mimic his most greedy commitment type and will receive a payoff arbitrarily close to the inflexible demand of his most greedy commitment type in any equilibrium.

Corollary 1. *Suppose that the market is unbalanced. There is a positive cutoff $\bar{\Delta}$ such that if $\Delta < \bar{\Delta}$ and $\bar{\theta}_1 > u_1^*(\Delta)$, then $v_2(\sigma) = x(\Delta)$, $v_1(\sigma) > 1 - \theta_2$ and*

(i) In the bargaining stage-game player 1 always proposes $\bar{\theta}_1$,

(ii) Player 2 reveals rationality in period 1 or period 2,

(iii) Payoffs in the bargaining game are $U_1(\sigma|n) \approx \bar{\theta}_1$ and $U_2(\sigma|n) \approx 1 - \bar{\theta}_1$,

and if $\bar{\theta}_1 < u_1^*(\Delta)$, then $U_1(\sigma|n) = u_1^*(\Delta)$ and $U_2(\sigma|n) = u_2^*(\Delta)$; in any search equilibrium σ of $E(\Delta, L, z)$.

6. Balanced markets and two-sided reputation

In this section we switch our focus to balanced markets ($L_1 > L_2(1-z_2)$) and we present Theorem 2, Corollary 2, Theorem 3 and Corollary 3. Theorem 2 shows that the outside options of all normal types are compatible with the demands of the commitment types in any search equilibrium, Corollary 2 generalizes this result to markets with multiple commitment types, and Theorem 3 describes a particular search equilibrium. In the equilibrium that we present in Theorem 3, player 1 and player 2 play a war of attrition, there are substantial delays in reaching an agreement, and agents opt-out from bargaining relationships on the equilibrium path. In this equilibrium, inefficiencies and delays remain substantial, even at the limit of complete rationality (Corollary 3).

Recall that in an unbalanced market the equilibrium values for the long-side of the market are determined by market forces. More precisely, for a steady state to exist a portion of the long-side must voluntarily leave the market and so must receive value no more than $x(\Delta)$. In a balanced market on the other hand, flow demand and supply are possibly equal and may place no restrictions on the equilibrium values of agents. Consequently, a balanced market leaves room for a richer set of outcomes in the bargaining stage.

We say that a market is *generic*, if the entering measure of normal types on the two sides are unequal, i.e., if $(1-z_1)L_1 \neq (1-z_2)L_2$. In a generic and balanced market, the endogenous outside option of both normal types must be compatible with the demands of the commitment types (Theorem 2). The intuition for the result is as follows: Suppose that neither normal type trades with a commitment type. This assumption implies that $v_i \geq 1 - \theta_j > x(\Delta)$ for $i \in \{1, 2\}$. Because $v_i > x(\Delta)$ for $i \in \{1, 2\}$, neither normal type leaves the market without trading. However, the assumption that normal types only trade with each other and the genericity assumption ($L_1(1-z_1) \neq L_2(1-z_2)$) makes a steady state impossible. Consequently, $v_i \leq 1-\theta_j$ for some $i \in \{1,2\}$. Suppose that player 1's value is strictly less than $1-\theta_2$ and player 2's value is strictly greater than $1-\theta_1$. In this case, player 1 trades with both player 2 and commitment type 2 with certainty (see Lemma 1). However, player 2 only trades with player 1 because $v_2 > 1-\theta_1$. This implies that for $x(\Delta)$ sufficiently small, the values of both player 2 and commitment type 2 are strictly greater than $x(\Delta)$. Consequently, neither player 2 nor commitment type 2 will leave the market without trading. Also, all commitment type 1s receive value equal to zero in equilibrium and so leave the market voluntarily without trading. This implies that a flow of $(1-z_1)L_1$ must accommodate the trades of a flow of L_2. However, this is not possible since $L_2 \geq L_1$ by assumption. A more subtle argument is needed to show that the values cannot satisfy $v_i > 1-\theta_j$ and $v_j = 1-\theta_i$. We defer this argument to the appendix.

Theorem 2. *Suppose that the market is generic and balanced. There is a positive cutoff $\bar{\Delta}$ such that if $\Delta < \bar{\Delta}$, then $v_1(\sigma) \leq 1-\theta_2$ and $v_2(\sigma) \leq 1-\theta_1$ in any search equilibrium σ of $E(\Delta, L, z)$.*

In Theorem 2 we restricted attention to the generic case where $L_1(1-z_1) \neq L_2(1-z_2)$. If $L_1(1-z_1) = L_2(1-z_2)$, then an efficient equilibrium exists. In this efficient equilibrium normal agents receive their Rubinstein payoffs and the commitment types are never traded. However, the inefficient equilibrium we present below in Theorem 3 remains an equilibrium even when $L_1(1-z_1) = L_2(1-z_2)$.[10]

Equilibria in a generic and balanced markets contrast both with equilibria in markets with complete information and equilibria in unbalanced markets (Theorem 1). In particular,

(i) The inflexible commitment type demands (θ_1 and θ_2) determine upper bounds on equilibrium values, i.e., $v_1 \leq 1-\theta_2$ and $v_2 \leq 1-\theta_1$. This implies that $v_1/e^{-r_1 t(\Delta)} + v_2/e^{-r_2 t(\Delta)} < 1$ for $t(\Delta)$ close to zero. Consequently, in a balanced market all equilibria entail significant inefficiency. In contrast, in a market with complete information there is an efficient equilibrium where the players receive the Rubinstein payoffs.

(ii) The inflexible commitment type demands determine a lower bound on the magnitude of inefficiency in the market. This lower bound is independent of the entering proportion of commitment types. Hence inefficiency remains substantial

[10] There are also other inefficient equilibria when $L_1(1-z_1) = L_2(1-z_2)$.

even in the limiting case of complete rationality (i.e., for any small z_1 and z_2). This contrasts with models of two-sided incomplete information, such as Abreu and Gul (2000) or Compte and Jehiel (2002), where efficiency is restored in the limiting case of complete rationality.

In Corollary 2 we provide conditions under which the findings of Theorem 2 extend to markets with multiple commitment types. Suppose that the commitment types are ordered according to increasing greediness, i.e, if $n < k$, then $\theta_i^n < \theta_i^k$. Suppose that $z_2 < z_1$ and let τ_1 denote the smallest index such that $1 - z_1 + \sum_{n=1}^{\tau_1} z_1^n = 1 - z_2$, if such a type exists. If such a type does not exist, then let $\tau_1 = 0$. Note that τ_1 is the least greedy commitment type that equates flow entry by each side of the market.

In the first part of Corollary 2, we assume that the type space is sufficiently rich and $\tau_1 > 0$. Consequently, the total flow by commitment type 1s who are less greedy than τ_1 plus player 1s $(1 - z_1 + \sum_{n=1}^{\tau_1} z_1^n)$ equals the entry flow of player 2s $(1 - z_2)$. Under this assumption we show that an equilibrium exists where behavior is governed by one-sided reputation building in the bargaining stage and hence efficiency is restored. This market exhibits dynamics similar to the unbalanced market as characterized by Theorem 1. We refer to this as the case of a "fine type space" since, had the type distributions over commitment types been atomless with support $[\underline{\theta}_1, \bar{\theta}_1]$, then this condition would be automatically satisfied.

The second part of the corollary deals with the case of a coarse type space, that is, at least one commitment type is required on each side to equate the flow entry of the two sides ($\tau_1 = 0$). In this case, the corollary shows that the findings of Theorem 2 remains valid, and the two normal types are compatible with the demands of the least greedy commitment types.

Corollary 2. *Suppose that the market is generic and balanced.*

(*i*) ***Fine type space and one-sided reputation.*** *Assume that $\tau_1 > 0$ and $\theta_1^{\tau_1} > u_1^*(\Delta)$. There is a cutoff $\bar{\Delta} > 0$ such that if $\Delta < \bar{\Delta}$, then there exists a search equilibrium σ of $E(\Delta, L, z)$ such that*

(a) $v_2(\sigma) \leq 1 - \theta_1^{\tau_1}$,

(b) *In the bargaining stage-game player 1 always proposes $\theta_1^{\tau_1}$,*

(c) *Player 2 reveals rationality in period 1 or period 2,*

(d) *Payoffs in the bargaining game are $U_1(\sigma) \approx \theta_1^{\tau_1}$ and $U_2(\sigma) \approx 1 - \theta_1^{\tau_1}$, and player 1's value is $v_1(\sigma) \approx e^{-r_1 t(\Delta)} \theta_1^{\tau_1}$.*

(*ii*) ***Coarse type space and two-sided reputation.*** *Assume that $\tau_1 = 0$. If $\Delta < \bar{\Delta}$, then $v_1(\sigma) \leq 1 - \underline{\theta}_2$ and $v_2(\sigma) \leq 1 - \underline{\theta}_1$ in any search equilibrium σ of $E(\Delta, L, z)$.*

6.1. **An inefficient equilibrium.** As demonstrated in Theorem 2, all search equilibria involve substantial inefficiency ($v_1 \leq 1 - \theta_2$ and $v_2 \leq 1 - \theta_1$). Inefficiency can stem from a small market tightness parameter (i.e., a large queue length) for one side of the market. Alternatively, the market may be inefficient because it takes substantial time for player 1 and 2 to reach an agreement when they bargain (i.e., because of delays in bargaining).

In what follows, we assume that $L_1 = L_2$ and we focus on the case where Δ and $t(\Delta)$ are arbitrarily small. In Theorem 3, we construct an equilibrium where the market tightness parameter is equal to one. Hence, inefficiency does not stem from long queue lengths in this equilibrium. If the market tightness parameter is equal to one, then as $t(\Delta)$ becomes small, the search market becomes frictionless for any agent: An agent can ensure finding a bargaining partner from the opposite side arbitrarily fast. Consequently, in this market all inefficiency is a result of informational asymmetries that lead to delays in bargaining.

In the equilibrium that we present in Theorem 3, both players build a reputation on the equilibrium path and there are delays resulting from a war of attrition a-la Abreu and Gul (2000). In this equilibrium, player 1 and player 2 always trade, player 2 always trades with commitment type 1, but player 1 opts out with positive probability against commitment type 2. In order for a simpler exposition, we describe the equilibrium at the limit where there are no frictions, i.e., $\Delta = t(\Delta) = x(\Delta) = 0$. In particular, we present the equilibrium using a continuous time war of attrition. The assumption $\Delta = 0$ is inessential and is used solely to simplify exposition. In Appendix F, we validate our use of continuous time by establishing that there exists a sequence of equilibria for games where $t(\Delta) \geq \Delta > 0$ (Theorem F.1) and also that these equilibria converge to the continuous time characterization we discuss here, as $\Delta \to 0$ and $t(\Delta) \to 0$ (Theorem F.2).

6.1.1. *The bargaining stage game as a continuous time war of attrition with opt-out.* In the continuous time war of attrition each player chooses to yield, insist or opt-out at each time t. So, a pure strategy for each player can be represented as a choice of a real time $t \in [0, \infty]$ and action $a \in \{Yield,\ Opt - out\}$. If the player chooses $t = \infty$ this represents that the player never yields or opts out. The commitment type's (inflexible) strategy is to never yield to an opponent.

If player i yields at time t before player j yields or opts out, then player i and player j receive $(1 - \theta_j)e^{-r_i t}$ and $\theta_j e^{-r_j t}$, respectively. If both players yield at the same time t, then player i and player j receive $e^{-r_i t}(\frac{\theta_i + 1 - \theta_j}{2})$ and $e^{-r_j t}(\frac{\theta_j + 1 - \theta_i}{2})$, respectively. If player i yields and player j opts out at time t, then we assume that the players trade and player

i and player j receive $(1-\theta_j)e^{-r_i t}$ and $\theta_j e^{-r_j t}$, respectively. If player i opts out at time t before player j yields, then the players receive their outside options v_1 and v_2.

A mixed strategy for player i in the bargaining game can be summarized by two cumulative distribution functions F_i and α_i such $F_i(\infty)+\alpha_i(\infty)\leq 1$ where $F_i(t)$ is the total probability with which player i and the commitment type i yield at or before time t, and $\alpha_i(t)$ is the probability with which player i and the commitment type i opt-out at or before time t.

6.1.2. *Equilibrium with selective break-ups.* We term a search equilibrium **"an equilibrium with selective break ups"** (SBU equilibrium) for the limit market $E(0,L,z)$ where $\Delta = t(\Delta) = x(\Delta) = 0$ if

- (i) Values $v_1 = 1-\theta_2$ and $v_2 = 1-\theta_1$,
- (ii) If $\alpha_i(t) > 0$ for some $t \geq 0$, then $\alpha_j(t) = 0$ for all t,
- (iii) If $\alpha_i(t) > 0$ for some $t \geq 0$, then $F_i(t) = F_i(t')$ for any $t' > t$.

Condition (i) and condition (ii) are satisfied by any search equilibrium asymptotically as $\Delta \to 0$ and $t(\Delta) \to 0$. Condition (iii) ensures tractable dynamics that are directly comparable to previous work (Abreu and Gul (2000) and Compte and Jehiel (2002)). We discuss the dynamics of the equilibrium in detail below.

Condition (i) states that the players are indifferent between yielding and opting out in the bargaining stage. Notice, Theorem 2 implies that $v_i \leq 1-\theta_j$ for both players in any search equilibrium. Also, a player can guarantee $1-\theta_j$ in the bargaining game by yielding immediately to his/her opponent. Thus $v_i \geq e^{-t(\Delta)r_i}(1-\theta_j)$ for both players and moreover if $t(\Delta) = 0$, then $v_i \geq 1-\theta_j$. So, condition (i) is satisfied asymptotically as $\Delta \to 0$ and $t(\Delta) \to 0$.

Condition (ii) states that at most one of the two players opts out in the bargaining stage. This condition is satisfied in any search equilibrium when Δ is positive but sufficiently small. This is because at least one of the two players is weaker in the bargaining stage and is willing to yield to the commitment type. For this player, yielding strictly dominates opting out and waiting for $t(\Delta)$ periods for another bargaining partner. For further detail see the development in Theorem F.1 and Theorem F.2.

Condition (iii) requires that if player i opts out with positive probability at some time t, then he does not yield at any time after t, including t. This condition ensures that both players yield according to an atomless density as in Abreu and Gul (2000).

As an initial step in describing a SBU equilibrium, we consider a bargaining game where opting out is not allowed. Let $\hat{c} = (\hat{c}_1, \hat{c}_2)$ denote the commitment type probabilities, if both player 1 and player 2 trade with commitment type 1 and 2 with probability 1 (i.e., if $p_{cn} = p_{nc} = 1$). More precisely, $\hat{c}_1 = z_1/(1-z_2)$ and $\hat{c}_2 = z_2/(1-z_1)$. Let $\hat{\Gamma}(0,c)$

denote the bargaining game where opting out is not allowed. In this game, equilibrium play follows Lemma 2: At time zero, the weaker player yields with an atom and after time zero the two players yield to each other continuously at constant hazard rates λ_1 and λ_2. Recall $\lambda_i = r_j(1-\theta_i)/(\theta_i+\theta_j-1)$ and if $\ln(\hat{c}_2/\lambda_2) > \ln(\hat{c}_1/\lambda_1)$, then player 1 is *strictly weaker* than player 2. Suppose that $\ln \hat{c}_2/\lambda_2 > \ln \hat{c}_1/\lambda_1$, that is, player 1 is the strictly weaker player and yields with an atom at time zero in $\hat{\Gamma}(0,\hat{c})$.

We now describe the dynamics of an SBU equilibrium σ. Behavior in the SBU equilibrium is identical to equilibrium play in $\hat{\Gamma}(0,\hat{c})$ except that neither player yields with positive probability at time zero. Instead player 1 opts out after his opponent completes her yielding. In particular, a SBU equilibrium satisfies the following:

(a) The players yield at constant rates λ_1 and λ_2, and neither player yields with positive probability at time zero. Consequently, $F_i(t) = 1 - e^{-\lambda_i t} \;\; \forall i \in \{1,2\}$.

(b) The players finish yielding concurrently at a common time T.

(c) Player 2 never opts out and trades with player 1 and commitment type 1 with probability one, i.e., $p_{nn} = p_{cn} = 1$. Consequently, the following equation holds:

$$1 - e^{-\lambda_2 T} = n_2 = 1 - c_2. \tag{9}$$

This equation ensures that the total probability that agent 2 yields by time T equals the total probability that agent 2 trades with a commitment type, i.e., the probability that agent 2 is player 2. Also, the common time $T = -\ln c_2/\lambda_2$ which we obtain by solving equation (9) for T.

(d) The weaker player in the game $\hat{\Gamma}(0,\hat{c})$, player 1 in this case, yields at rate λ_1 until time T and then opts out. Consequently, the following equation holds:

$$1 - e^{-\lambda_1 T} = p_{nc}(1 - c_1). \tag{10}$$

This equation ensures that the total probability that agent 1 yields by time T equals the total probability that agent 1 trades with commitment type 2, i.e., the probability that agent 1 is player 1 and trades with commitment type 2. Also, the probability that player 1 trades with the commitment type 2, p_{nc}, is equal to $(1 - e^{-\lambda_1 T})/(1 - c_1)$ which we obtain by solving equation (10) for p_{nc}.

(e) The steady state frequencies c_1 and c_2 are determined by the steady state equations and the trade probabilities $p_{nn} = p_{cn} = 1$ and $p_{nc} \leq 1$. Given these match probabilities, the steady state equations are as follows:

$$c_1 = \frac{z_1}{1 - z_2}, \tag{11}$$

$$c_2 = \frac{z_2}{z_2 + (1 - z_1 - z_2)\, p_{nc}}. \tag{12}$$

The following theorem shows that a SBU equilibrium exists and is unique, if the entering fraction of commitment types are sufficiently small. Let

$$z^* = \min\{\lambda_1, \lambda_2\}/(\lambda_1 + \lambda_2). \tag{13}$$

Theorem 3. *Suppose that $L_1 = L_2$. If $z_1 < z^*$ and $z_2 < z^*$, then there exists a unique SBU equilibrium for $E(0, L, z)$.*

6.2. **The limiting case of complete rationality.** We now turn to characterizing the limit outcome for the market as the entering measure of commitment types, $z_1L_1 + z_2L_2$, converges to zero. Theorem 2 and Theorem 3 only require that there is a strictly positive measure of commitment types entering the market. Hence the conclusion of Theorem 2 holds, i.e., $v_1 \leq 1 - \theta_2$ and $v_2 \leq 1 - \theta_1$, and the conclusion of Theorem 3 holds, i.e., a unique SBU equilibrium exists, even as the entering measure of commitment types converges to zero.

Corollary 3 below further shows that the probability that player 1 trades with commitment type 2 goes to one and the steady state frequency of commitment types converges to zero, in a SBU equilibrium as the entering measure of commitment types goes to zero. Recall that the market tightness parameter is equal to one in any SBU equilibrium and therefore finding a bargaining partner is asymptotically costless. Consequently, finding a normal type as a bargaining partner is asymptotically costless in a SBU equilibrium. However, even though the market is (asymptotically) frictionless and free of commitment types inefficiency in the bargaining game remains substantial ($v_1 \leq 1-\theta_2$ and $v_2 \leq 1-\theta_1$). Hence, Corollary 3 leads us to conclude that access to the market exacerbates bargaining inefficiencies caused by irrational types, instead of forcing outcomes closer to efficiency, even as we approach the frictionless limit of complete rationality.

Corollary 3. *Suppose that $L_1 = L_2$. Let σ^k denote a SBU equilibrium for $E(0, L, z^k)$ and let c^k denote the steady state frequency of commitment types in search equilibrium σ^k. If the entering measure of commitment types converges to zero, i.e., $\lim_k z_1^k L_1 + z_2^k L_2 = 0$, then $\lim_k p_{nc}^k = 1$ and the steady state frequency of commitment types converges to zero, i.e., $\lim_n c_1^k = \lim_n c_2^k = 0$.*

Corollary 3 stands in sharp contrast to previous literature. Abreu and Gul (2000) show that one of the two players is asymptotically strictly stronger and if player i is asymptotically strictly stronger, then the two players trade immediately without any delay, the equilibrium payoff of the stronger side is θ_i and the equilibrium payoff of the weaker player is $1-\theta_i$, at the limit.[11] Hence, inefficiency disappears but incomplete information still has an impact on the division of the surplus, at the limit. In an SBU equilibrium, in contrast, inefficiency remains substantial even at the limit.

In an SBU equilibrium, the probability that player 1 opts out against commitment type 2, i.e., $1-p_{nc}$, equates the strength of player 1 and player 2 in the bargaining stage and thereby ensures that neither player yields with an atom at the start of bargaining. Notice that p_{nc} cannot converge to zero, as z_1 and z_2 converge to zero. This is because if p_{nc} were to converge to zero, then player 1 and player 2 would not be equally strong in the bargaining stage. As z_1 and z_2 converge to zero, c_2 goes to zero and the common time that the two players complete yielding goes to infinity. This follows from equations (9) and (11) and the fact that p_{nc} does not go to zero. Consequently, as z_1 and z_2 converge to zero, the "amount" of opt-out required to equate the bargaining strength of the two player becomes arbitrarily small, i.e., p_{nc} converges to one.

[11]In particular, if $\lambda_1 < \lambda_2$ and if the commitment type probabilities approach to zero at the same rate, then player 2 is the stronger player, asymptotically. If the commitment type probabilities converge at different rates, then generically one of the two players is strictly stronger, asymptotically.

Appendix A. Proofs for Section 3

Proof of the Complete Information Benchmark. Let $\bar{u}_1$ denote the best payoff player 1 receives in the bargaining stage game in any equilibrium. Consequently, $v_1(\sigma) \leq \bar{u}_1$ in any equilibrium σ. So, $\underline{u}_2 \geq 1 - e^{-r_1\Delta}\bar{u}_1$. In any period where player 2 proposes she can guarantee $1 - e^{-r_1\Delta}\bar{u}_1$. In other words, player 2's worst payoff in a period she proposes $\underline{u}_2 \geq 1 - e^{-r_1\Delta}\bar{u}_1$. This is because in any period where player 1 responds he can expect at most $\max\{e^{-r_1\Delta}\bar{u}_1, e^{-r_1 t(\Delta)}v_1(\sigma)\}$ by rejecting the offer or opting-out. However, $\max\{e^{-r_1\Delta}\bar{u}_1, e^{-r_1 t(\Delta)}v_1(\sigma)\} = e^{-r_1\Delta}\bar{u}_1$, because, $v_1(\sigma) \leq \bar{u}_1$ and $e^{-r_1\Delta} \geq e^{-r_1 t(\Delta)}$.

Observe that $\bar{u}_1 \leq 1 - e^{-r_2\Delta}\underline{u}_2$. This is because player 2 can always reject player 1's offer and guarantee $\underline{u}_2$. Consequently, we have the following system of inequalities

$$\bar{u}_1 \leq 1 - e^{-r_2\Delta}\underline{u}_2 \tag{14}$$

$$\underline{u}_2 \geq 1 - e^{-r_1\Delta}\bar{u}_1 \tag{15}$$

$$\bar{u}_2 \leq 1 - e^{-r_1\Delta}\underline{u}_1 \tag{16}$$

$$\underline{u}_1 \geq 1 - e^{-r_2\Delta}\bar{u}_2 \tag{17}$$

Solving this system implies that $U_1(\sigma) = \underline{u}_1 = \bar{u}_1 = u_1^*(\Delta) > v_1(\sigma)$ and $U_2(\sigma) = u_2^*(\Delta)$, in any equilibrium σ.

For item (ii) observe that in order for a steady state, some of the class i agents must leave the market voluntarily and so $v_i(\sigma) = x(\Delta)$. Also,

$$v_i(\sigma)/e^{-r_i t(\Delta)} = m_i u_i^*(\Delta) + (1 - m_i)v_i(\sigma) \geq m_i u_i^*(\Delta).$$

Consequently, $u_i^*(\Delta) > x(\Delta)$ implies that $m_i < 1$ and $m_j = 1$. So, $v_j(\sigma)/e^{-r_j t(\Delta)} = u_j^*(\Delta)$.

Item (iii): $m_i = 1$ for at least one of the agents i. Consequently, $v_i(\sigma)/e^{-r_i t(\Delta)} = u_i^*(\Delta)$ for this i. If $m_i = 1$, then $m_j \leq 1$. Consequently, $v_j(\sigma)/e^{-r_j t(\Delta)} \leq u_j^*(\Delta)$. Also, it is easily verified that $m_1 = m_2 = 1$ and $v_i(\sigma)/e^{-r_i t(\Delta)} = u_i^*(\Delta)$ is an equilibrium. □

Appendix B. The bargaining game with one-sided incomplete information

In this section, we summarize various findings by Myerson (1991), Abreu and Gul (2000), and Compte and Jehiel (2002) for the bargaining stage-game $\Gamma(\Delta, c, v)$, where $c_1 > 0$, $c_2 = 0$ and $v_i < 1 - \theta_j$. We assume that player 1 is potentially a commitment type, player 2 is known to be rational with probability one, and both players' outside options are worse than yielding to the commitment types. Our development closely follows Compte and Jehiel (2002) Appendix A where all the stated results can be found.

Lemma B.1. *Suppose that $c_1 > 0$, $c_2 = 0$, $v_1 < 1 - \theta_2$ and $v_2 < 1 - \theta_1$. Then $U_1(\sigma) \approx \theta_1$ and $U_2(\sigma) \approx 1 - \theta_1$ in any PBE σ of $\Gamma(\Delta, c, v)$.*

Proof. See the development in Myerson (1991), Chapter 8, Theorem 8.4; or Abreu and Gul (2000) Lemma 1; or Compte and Jehiel (2002), Proposition 2. □

Let $\gamma_i = e^{-r_i \Delta}$. Compte and Jehiel (2002), Appendix A, define

$$v^n = [\gamma_1]^{2n+1}\theta_1 \tag{18}$$

$$\rho = \frac{1-[\gamma_1]^2}{1-[\gamma_2]^2}, \pi^0 = 1-\mu^0 \text{ and} \tag{19}$$

$$w^0(\mu) = \max\{(1-\mu)(1-v^0) + \mu\gamma_2(1-\theta_1), 1-\theta_1\} \tag{20}$$

$$\mu^0 : \gamma_2 w^0(\mu^0) = 1-\theta_1 \text{ if } \gamma_2 w^0(0) \geq 1-\theta_1, \text{ and } \mu^0 = 0 \text{ otherwise.} \tag{21}$$

Let N be the largest integer for which

$$[\gamma_1]^{2N}\theta_1 > u_2^*(\Delta)$$

Consider the sequence $\{\pi^n, \mu^n, w^n, w^n(.)\}_{0 \leq n \leq N}$ defined recursively by

$$\pi^{n+1} = \frac{w^n}{w^n + \rho v^n}, \tag{22}$$

$$\mu^{n+1} = \Pi_{k \leq n+1}(1-\pi^k), \tag{23}$$

$$w^{n+1} = \pi^{n+1}(1+(\rho-1)v^n), \tag{24}$$

$$w^{n+1}(\mu) = (1-\mu/\mu^n)(1-v^{n+1}) + (\mu/\mu^n)[\gamma_2]^2 w^n. \tag{25}$$

In this sequence, $w^n = w^n(\mu^n) = w^{n-1}(\mu^n)$. The following lemma shows that player 2's equilibrium payoff is a continuous and non-decreasing function of player 1's reputation level μ. The strength of this lemma is that it shows player 2's equilibrium payoff is independent of which equilibrium is played and which history has been reached in the game. It is completely determined by player 1's reputation level and by whether player 2 is a proposer or a responder.

Lemma B.2. *Suppose that $\mu > \mu^N$. Let $u_2(\mu)$ be the function that coincides with $w^n(\mu)$ on each interval $(\mu^{n+1}, \mu^n]$, $n \in \{0, .., N-1\}$.*

- *(i) Player 1 proposes θ_1 in all odd periods,*
- *(ii) In any even period, if player 1's reputation level $\mu \in (\mu_{n+1}, \mu_n)$, then player 2 proposes v_n,*
- *(iii) In any even period if player 1 is the commitment type with probability μ, then player 2's equilibrium payoff is equal to $u_2(\mu)$,*
- *(iv) In any odd period where player 1 has proposed θ_1, if player 1 is the commitment type with probability μ, then player 2's equilibrium payoff is equal to* $\max\{1-\theta_1, e^{-r_2\Delta}u_2(\mu)\}$

in any PBE σ of $\Gamma(\Delta, c, v)$.

Proof. See Compte and Jehiel (2002), Proposition 10. □

Note that $\mu^N \to 0$ as $\Delta \to 0$. The previous lemma also pins down player 1's payoff at all point in the game except the cut-off reputation level μ^n. This is because player 2 always offers v^n when $\mu \in (\mu^{n+1}, \mu^n)$ and player 1 always randomizes between accepting and rejecting. Consequently, in any even period where player 1's reputation is $\mu \in (\mu^{n+1}, \mu^n)$, player 1's equilibrium payoff is v^n. When $\mu = \mu^n$, player 2 may offer v^n or v^{n-1}. Compte and Jehiel (2002) construct PBE where player 2 randomizes between v^n and v^{n-1} if $\mu = \mu^n$. If $\mu = \mu^n$, then for any $q \in [0, 1]$ there is a PBE where player 2 offers v^n with probability q and offers v^{n-1} with probability $1 - q$. Therefore, if $\mu = \mu^n$, then player 1's equilibrium payoff set is the convex and closed interval $[v^n, v^{n-1}]$.

Lemma B.3. *There exists a PBE for the game $\Gamma(\Delta, c, v)$. The equilibrium payoff set for player 1, viewed as a (possibly multi-valued) function of μ is an upper-hemi-continuous compact and convex valued correspondence.*

Proof. See the above discussion and Compte and Jehiel (2002), Proposition 11. □

Appendix C. Proof of Lemma 1

Proof of Lemma 1. *Step 1. There exists a period T such that in period $T + 1$ agent 2 is known to be commitment type 2 with certainty and player 1 opts out with certainty by time $T + 1$.*

For player 1 to not opt-out or not reveal rationality for another K/Δ periods, the probability that player 2 concedes (i.e., reveals rationality by either offering something other than $1 - \theta_2$ to player 1, or by accepting $1 - \theta_1$) to player 1's demands, denoted p, must satisfy

$$1 - \theta_2 \leq p + (1 - p)e^{-r_1 K}.$$

This is because player 1 can guarantee at least $1 - \theta_2 < v_1(\sigma)$ today by opting out or by revealing rationality. Also, player 1 can get at most one if player 2 concedes, and player 1 can get at most one as a continuation payoff after the K/Δ periods. Choose K sufficiently large so that $1 - \theta_2 > e^{-r_1 K}$. So,

$$1 > p > \frac{1 - \theta_2 - e^{-r_1 K}}{1 - e^{-r_1 K}} > 0.$$

Choose, N such that

$$\frac{c_2}{(1 - \frac{1 - \theta_2 - e^{-r_1 K}}{1 - e^{-r_1 K}})^N} > 1.$$

Consequently, $\frac{c_2}{(1-p)^N} > 1$ and thus by period NK/Δ a rational player 1 is sure that player 2 is the commitment type in any equilibrium. Consequently, if $T + 1 > NK/\Delta$, then player 1 will opt-out with probability 1.

Step 2. Appendix B implies the following: If in history h^t player 2 has revealed rationality by proposing something different than θ_2 and player 1 is the commitment type with probability $\mu_2(h^t) > c_1$, then $U_1(\sigma) \approx \theta_1$ and $U_2(\sigma) \approx 1 - \theta_1$. Also, in any period where player 2 proposes, player 2's payoff is unique as a function of player 1's reputation level.

The steps that follow show that player 2 will either accept θ_1 in period one, or will reveal rationality by proposing something other than $1 - \theta_2$ in period two.

Step 3. If in an even period player 1 is known to be rational, then player 2 offers player 1 the Rubinstein payoff $e^{-r_1\Delta}u_1^*(\Delta)$*, and player 1 accepts.*

If player 2 asks for something other than θ_2 and reveals rationality, then player 1 can reject the offer and secure the Rubinstein payoff $u_1^*(\Delta)$ in the next period. Consequently, in the event that player 2 asks for something other than θ_2, then she will receive $1 - e^{-r_1\Delta}u_1^*(\Delta)$.

Let $\bar{u}$ denote the highest continuation payoff for player 2 at the start of any period where she proposes for any reputation level. If player 2 asks for θ_2, then player 1 will reject, because his outside option is larger than θ_2. Also, in the next period where player 1 proposes he will never offer anything above $e^{-r_2\Delta}\bar{u}$. This is because player 1 will always opt-out against the commitment type and player 2 will always accept $e^{-r_2\Delta}\bar{u}$. So, player 2's best payoff if she proposes θ_2 is $e^{-r_2 2\Delta}\bar{u}$. But this implies that $\bar{u} \leq \max\{e^{-r_2 2\Delta}\bar{u}, (1 - e^{-r_1\Delta})u_1^*(\Delta)\}$ and so $\bar{u} \leq (1 - e^{-r_1\Delta})u_1^*(\Delta)$. Thus player 2 should reveal rationality and offer player 1 the Rubinstein split.

Step 4. If player 1 offers something different than θ_1*, then player 1 offers* $u_2^*(\Delta)$*.*

If player 1 reveals rationality, then there will be an agreement in the next period and player 2 will receive $1 - e^{-r_1\Delta}u_1^*(\Delta)$. Consequently, player 2 will accept $e^{-r_2\Delta}(1 - e^{r_1\Delta}u_1^*) = u_2^*(\Delta)$ and no less than this. So player 1 will offer $u_2^*(\Delta)$ to player 2 and receive $u_1^*(\Delta) > e^{-r_1\Delta}u_1^*(\Delta)$ for himself, if he is to reveal rationality by deviating from θ_1.

Step 5. Let $S = \sup\{s : \exists h^s \text{ s.t. } z_i(h^s) \geq c_i \text{ and } \Pr\{\text{player 1 proposes } u_2^*(\Delta) \text{ in period } s | h^s\} > 0\}$*. By Step 1* $S \leq T$*. Also, in period* T*, player 1 will only offer* θ_1*. This is because the normal type of player 2 will always accept* θ_1 *in period* T *as player 1 will opt-out with certainty after period* T*.*

Step 6. In period S*, if player 1 instead offers* θ_1*, then player 2's payoff after this offer is uniquely determined and is at least* $1 - \theta_1$ *and is at most* $1 - \theta_1 + \kappa(1 - e^{-r\Delta})$*.*

If player 2 reveals rationality, then her payoff is uniquely determined as a function of player 1's reputation level, by Lemma B.2.

In the continuation game, player 2 either accepts θ_1 in period S or reveals rationality in period $S + 1$. This is because player 1 only offers θ_1 and never accepts θ_2. Consequently, any trade occurs through player 2 either accepting θ_1 or revealing rationality. However, player 1's reputation never decreases in the continuation. Also, player 2's continuation payoff, following player 2's revelation of rationality, is uniquely determined and decreasing in player 1's reputation level. So, player 2 does not gain from delaying revealing rationality after period $S + 1$. Once player 2 reveals rationality in period $S + 1$, and player 1 can guarantee a continuation payoff equal to $e^{-r_1\Delta}(\theta_1 - \kappa(1 - e^{-r\Delta}))$, by Lemma B.1. Hence, player 2's payoff is at most $1 - \theta_1 + (\kappa + 1)(1 - e^{-r\Delta})$. Also, player 2 can guarantee $1 - \theta_1$ by simply accepting the offer in S.

Step 7. Player 1 prefers to offer θ_1 *instead of* $u_2^*(\Delta)$ *in period* S*. Consequently, player 1 never offers anything but* θ_1*.*

If player 1 offers $u_2^*(\Delta)$, then the normal type player 2 will accept and if there is a rejection, then player 1 will opt-out in period $S + 1$. If player 1 offers θ_1 in period S, then normal player 2 will either accept or reveal rationality in period $S+1$. If player 2 does not reveal rationality, then player 1 will opt-out in period $S+1$. Consequently, player 1's payoff against the commitment type is identical regardless of whether he offers θ_1 or $u_2^*(\Delta)$. Player 1's payoff once player 2 has revealed rationality is $\theta_1 - \kappa(1-e^{-r\Delta})$ by Lemma 1. So, if player 1 offers θ_1, then her payoff against player 2 is at least $e^{-r_1\Delta}(\theta_1 - \kappa(1-e^{-r\Delta})$ which exceeds $u_1^*(\Delta)$.

Step 8. Player 2 reveals rationality in period 2 (i.e., the first period she proposes) or accepts θ_1 in period 1.

This is because player 1 always offers θ_1. Consequently, there is no incentive for player 2 to delay revealing rationality. Since player 2 reveals either in the first period or the second period, player 1's payoff is at least $e^{-r_1\Delta}(\theta_1 - \kappa(1-e^{-r\Delta}))$. □

Appendix D. Proofs of Lemma 4, Theorem 1 and Corollary 1

Lemma D.1. *Suppose that $v_2 < 1-\theta_1$. For every $\epsilon > 0$ and $\xi > 0$, there exists $\bar{\Delta} > 0$ and $\bar{c} > 0$ such that if $\Delta < \bar{\Delta}$, $c_1 > \xi$ and $c_2 < \bar{c}$, then $U_1(\sigma) \geq \theta_1 - \kappa(\xi)(1-e^{-r_2\Delta}) - \epsilon$ in any PBE σ of $\Gamma(\Delta, c, v)$.*

Proof. Step 1. There exists a period T, that is independent of c_2, such that if player 1 has not revealed rationality by period T, then he is known to be the commitment type with probability 1.

Follows immediately from Appendix C, Step 1.

Step 2. Player 2 must reveal rationality by period $T+1$.

If player 1 has not revealed rationality by time T, then player 2 knows that player 1 is the commitment type with probability 1. So, player 2 will reveal rationality by period $T+1$.

Step 3. Let p_t denote the total probability that player 2 reveals rationality in period t by either accepting or proposing something other than θ_2, after any history where $\mu_1(h^t) \geq c_1$. By the previous step and Bayes' rule

$$\frac{c_2}{\prod_{t=1}^{T+1}(1-p_t)} = 1.$$

Step 4. For every ϵ, there exists a $\bar{c}$ such that if $c_2 < \bar{c}$, then $p_1 + p_2 \geq 1-\epsilon$.

Suppose not. Then there exists a sequence $c_2^k \to 0$ and a $\epsilon > 0$ such that $p_1 + p_2 < 1-\epsilon$. This implies, by the previous step, that there exists $2 < m \leq T+1$ and a subsequence of equilibria σ^k such that $p_m^k \to 1$. If player 2 reveals rationality, then player 1's payoff is at least $\theta_1 - \kappa(1-e^{-r\Delta})$, by Lemma B.1. Consequently, there exists an index K such that for all equilibria σ^k with $k > K$ we have $1-\theta_2 < e^{-r_1\Delta}p_m^k(\theta_1 - \kappa(1-e^{-r\Delta}))$. This implies that player 1 will not reveal rationality in period $m-1$ in any equilibrium σ^k for all $k > K$. Player 2's payoff at any even period, where player 2 is known to be rational with certainty, is a non-increasing function of the reputation level μ_1 of player 1. This is also true for all odd periods. In equilibrium σ^k, $\mu_1(h^{m-2}) \leq \mu_1(h^m)$. Consequently, player 2's payoff from revealing rationality in period $m-2$ is strictly greater than player 2's payoff from not revealing rationality and then revealing rationality with probability p_m^k in period m. However, this is a contradiction.

The bound on player 1's equilibrium payoff $U_1(\sigma)$ follows from the fact that player 2 reveals rationality in period 1 or period 2 with probability of at least $1-\epsilon$. $\square$

Proof of Lemma 4. In the following development, we assume Δ is as small as we need and consequently that $x(\Delta)$ is sufficiently small.

Step 1. In any search equilibrium $v_2(\sigma) = x(\Delta)$ and consequently $m_2 < 1$ and $m_1 = 1$.

In order for the steady state equations to hold some of the class 2 agents must be leaving the market without trading. This implies that player 2's value $v_2(\sigma) = x(\Delta)$. In the bargaining stage game, player 2 can guarantee $1-\theta_1$. So in any equilibrium $v_2(\sigma)/e^{-r_2 t(\Delta)} \geq m_2(1-\theta_1) + (1-m_2)v_2(\sigma)$. Consequently, $x(\Delta) = v_2(\sigma) \geq \frac{e^{-r_2 t(\Delta)} m_2(1-\theta_1)}{1-e^{-r_2 t(\Delta)}(1-m_2)}$. This implies that $m_2 \leq \frac{x(1-e^{-r_2 t(\Delta)})}{(1-\theta_1)e^{-r_2 t(\Delta)} - e^{-r_2 t(\Delta)} x(\Delta)}$. Consequently, m_2 is arbitrarily close to zero for $x(\Delta)$ small. However, $m_2 < 1$ implies that $m_1 = 1$.

Step 2. In any search equilibrium $C_1 \geq L_1 z_1$ and $C_2 = z_2 L_2$.

$C_1 \geq L_1 z_1$ because $L_1 z_1$ is the number of commitment type 1s that enter the market in each period. Commitment type 2 does strictly worse than player 2. This is because player 2 can do at least as well as the commitment type 2 against player 1 by using an identical strategy. Also, player 2 can trade with commitment type 1 and obtain $1-\theta_1$ in these meetings. If the value of player 2 is less than or equal to $x(\Delta)$, then the payoff for commitment type 2 is strictly less than $x(\Delta)$. Consequently, all of these types, who are in the unmatched population at the end of a period, will choose to voluntarily exit instead of waiting $t(\Delta)/\Delta$ periods for a possible match. So $C_2 = z_2 L_2$.

Step 3. Take a sequence of $\Delta^k \to 0$ and let σ^k denote a search equilibrium for $E(\Delta^k, L, z)$. For any sequence of search equilibria σ^k, $N_2^k \to \infty$, $n_2^k \to 1$ and $c_2^k \to 0$. Also, there exists $\xi > 0$ such that, for all $\Delta^k < \hat{\Delta}$, $c_1^k \geq \xi$.

If $\Delta \to 0$, then $x(\Delta) \to 0$. If $x(\Delta) \to 0$, then $m_2 \leq \frac{x(\Delta)(1-e^{-r_2 t(\Delta)})}{e^{-r_2 t(\Delta)}(1-\theta_1) - e^{-r_2 t(\Delta)} x(\Delta)} \to 0$. Also $C_1 + N_1 \geq L_1$ and $C_2 = z_2 L_2$ for any $x(\Delta)$. Consequently, if $x(\Delta^k) \to 0$, then $m_2^k \to 0$ and so $N_2^k \to \infty$, $n_2^k \to 1$ and $c_2^k \to 0$.

We argue that $p_{nn} \geq 1 - \frac{\theta_1}{1-x(\Delta)}$. In the bargaining stage game player 1 does not opt-out in the first period. This is because if player 1 opts out in the first period, then the bargaining relationship is less valuable than being unmatched in the economy. This implies that $v_1 \leq e^{-r_1 t(\Delta)} v_1$, which is not possible. Player 2 can guarantee $1-\theta_1$ by immediately offering θ_1 to player 1. The best that player 2 can hope for is to receive 1 if there is no break-up and to receive $x(\Delta)$ if there is a break-up. Consequently,

$$1 - \theta_1 \leq \Pr\{op\} x(\Delta) + 1 - \Pr\{op\}$$

where $\Pr\{op\}$ is the total probability of an opt-out. Hence, the total probability of an opt-out is at most $\frac{\theta_1}{1-x(\Delta)}$. So, $p_{nn} > 1 - \frac{\theta_1}{1-x(\Delta)}$.

Notice that $N_1 \leq \frac{(1-z_1)L_1}{p_{nn}n_2}$ and $c_1 \geq \frac{z_1 L_1}{z_1 L_1 + N_1}$. Substituting and using $p_{nn}n_2 < 1$ gives

$$c_1 \geq \frac{z_1 L_1 p_{nn} n_2}{z_1 L_1 p_{nn} n_2 + (1-z_1)L_1} \geq z_1 p_{nn} n_2.$$

For $x(\Delta)$ sufficiently small, n_2 is close to 1 and $1 - \frac{\theta_1}{1-x(\Delta)}$ is close to $1-\theta_1$. Consequently, for any $\xi < z_1(1-\theta_1)$, we can choose $\hat{\Delta}$ such that for all $\Delta < \hat{\Delta}$, $z_1 n_2 p_{nn} > \xi > 0$ and $c_1 > \xi > 0$.

Consequently, by Step 3 and by Lemma D.1, there exists $\bar{\Delta} < \hat{\Delta}$ such that $U_1(\sigma) > \theta_1 - \epsilon$ for all $\Delta^k < \bar{\Delta}$ completing the argument. □

Proof of Theorem 1. Pick $\bar{\Delta}$ small such that $v_1(\sigma) \geq e^{-r_1 t(\bar{\Delta})}\theta_1 > 1-\theta_2$. So, if $\Delta < \bar{\Delta}$, then $v_1(\sigma) > 1-\theta_2$. However, if $v_1(\sigma) > 1-\theta_2$, $v_2(\sigma) = x(\Delta) < 1-\theta_1$, then the conditions of Lemma 1 are satisfied and the Lemma implies items (i) through (iii). □

Proof of Corollary 1. Let $C_i = \sum_{n \in T_i} C_i^n$ likewise $c_i = \sum_{n \in T_i} c_i^n$. The following are immediate consequences of Lemma 4: In any equilibrium $v_2(\sigma) = x(\Delta)$ and consequently $m_2 < 1$ and $m_1 = 1$. In any equilibrium $C_1^n \geq L_1 z_1$ for any $n \in T_1$ and $C_2 = z_2 L_2$. Take a sequence of $\Delta^k \to 0$ and let σ^k denote a search equilibrium. For any sequence of search equilibria σ^k, $N_2^k \to \infty$, $n_2^k \to 1$ and $c_2^k \to 0$. Also, there exists $\xi > 0$ such that, for all $\Delta^k < \bar{\Delta}$, $(c_1^n)^k \geq \xi$ for any $n \in T_1$.

We argue that if $\Delta^k < \bar{\Delta}$, then $v_1(\sigma^k) > 1 - \underline{\theta}_2$ for any equilibrium σ^k.

If $U_1(\sigma) > (1-\epsilon|T_2|)(\bar{\theta}_1 - \kappa(1-e^{-r\Delta}) - \epsilon)$, then for $x(\Delta)$ sufficiently small $v_1(\sigma) \geq n_2 U_1(\sigma) \geq (1-\xi|T_2|)(\bar{\theta}_1 - \kappa(1-e^{-r\Delta}) - \epsilon)$ since n_2 can be made arbitrarily close to 1. Pick ϵ such that $(1-\epsilon|T_2|)(\bar{\theta}_1 - \kappa(1-e^{-r\Delta}) - \epsilon) > 1 - \underline{\theta}_2$. Let $B \subset T_2$ denote the set of types for player 2 such that for any $n \in B$ the probability that player 2 mimics type n is larger than ϵ, conditional on player 1 mimicking $\bar{\theta}_1$ in period 1, in equilibrium σ. Suppose that the set B is non-empty. In any subgame where player 1 chooses to mimic $\bar{\theta}_1$ in period 1 and player 2 chooses to mimic $n \in B$, Lemma D.1 implies that $U_1(\sigma) > \theta_1 - \kappa(1-e^{-r\Delta}) - \epsilon$ for all $\Delta < \bar{\Delta}$.

Conditional on player 1 mimicking type $\bar{\theta}_1$ the probability that player 2 either mimics a type in B or reveals rationality in period 1 or period 2 is at least $(1-\epsilon|T_2|)$ by the definition of the set B. If player 1 chooses $\bar{\theta}_1$ and player 2 reveals rationality then player 1's payoff is at least $\bar{\theta}_1 - \kappa(1-e^{-r\Delta})$. Consequently, player 1 can secure payoff of at least $(1-\epsilon|T_2|)(\bar{\theta}_1 - \kappa(1 - e^{-r\Delta}) - \epsilon)$ by mimicking $\bar{\theta}_1$.

If $v_1(\sigma) > 1 - \underline{\theta}_2$, $v_2(\sigma) = x(\Delta) < 1 - \bar{\theta}_1$, then player 1 can always choose to mimic type $\bar{\theta}_1$ by proposing $\bar{\theta}_1$ in period 1. In the continuation game all the conditions of Lemma 1 are satisfied and the Lemma implies items (i) through (iii). □

Appendix E. Proofs of Theorem 2, Theorem 3, Corollary 2 and Corollary 3

Proof of Theorem 2. By the assumption of the theorem $(1-z_1)L_1 \neq (1-z_2)L_2$. We further assume in this proof $(1-z_1)L_1 < (1-z_2)L_2$. This further assumption is without loss of generality.

Step 1. Player i trades with commitment type j, for some i, and so $v_i \leq 1 - \theta_j$.

Suppose not, i.e., $p_{nc} = p_{cn} = 0$. If $p_{nc} = p_{cn} = 0$, then $v_i \geq 1 - \theta_j$ for all $i \in \{1, 2\}$. Because $v_i \geq 1 - \theta_j > x$ the normal types will not leave the market voluntarily and all exit must occur through trade. The steady state equations imply:

$$(1 - z_1)L_1 = N_1 m_1 n_2 p_{nn}$$
$$(1 - z_2)L_2 = N_2 m_2 n_1 p_{nn}$$

However, $N_1 m_1 n_2 p_{nn} = N_2 m_2 n_1 p_{nn}$ and $(1 - z_1)L_1 \neq (1 - z_2)L_2$ leads to a contradiction.

Step 2. Step 1 implies that $v_i \leq 1 - \theta_j$ *for some* i*. Suppose that* $v_i < 1 - \theta_j$ *and* $v_j > 1 - \theta_i$*.* This configuration of outside options is covered by Lemma 1 which implies that both player j and commitment type j trade with certainty with player i, i.e., $p_{cn} = p_{nn} = 1$, and receive a payoff close the θ_i against player i. However, this implies that the commitment type j will only leave the market through trade with player i for sufficiently small $x(\Delta)$. The steady state equations imply

$$(1 - z_j)L_j = m_j N_j p_{nn} n_i \tag{26}$$
$$z_j L_j = m_j C_j p_{cn} n_i \tag{27}$$
$$L_j = m_j n_i (N_j p_{nn} + C_j p_{cn}) \tag{26+27}$$

The steady state equation for the player i implies

$$L_i(1 - z_i) = m_i N_i (n_j p_{nn} + c_j p_{cn}) = m_j n_i (N_j p_{nn} + C_j p_{cn}) = L_j.$$

However, by assumption $L_1 > L_2(1 - z_2)$ and $L_2 > L_1(1 - z_1)$ contradicting the above equality.

Step 3. Suppose that $v_1 = 1 - \theta_2$ *and* $v_2 > 1 - \theta_1$*.* This implies that player 2 will never trade with commitment type 1. Also, player 2 will only leave the market through trade since $v_2 > 1 - \theta_1 > x(\Delta)$. So,

$$(1 - z_2)L_2 = m_2 N_2 p_{nn} n_1$$
$$(1 - z_1)L_1 \geq m_1 N_1 p_{nn} n_2 = m_2 N_2 p_{nn} n_1$$

However, this implies that $(1 - z_1)L_1 \geq (1 - z_2)L_2$ which contradicts that $(1 - z_1)L_1 < (1 - z_2)L_2$.

Step 4. Suppose that $v_1 > 1 - \theta_2$ *and* $v_2 = 1 - \theta_1$*.* The proof of this step is somewhat lengthy so first we sketch the argument. For a steady state to exist some commitment type 1s must leave the market without trading, i,e, their value from remaining in the market must be equal to x. To provide incentives for this p_{cn} needs to be sufficiently small. However, if p_{cn} is sufficiently small compared to p_{nn}, then the market is populated in large part by commitment types. This, however, would imply that player 2's payoff is also small and close to $x(\Delta)$, contradicting $v_2 = 1 - \theta_1$.

Observe that the equilibrium value for player 1

$$v_1/e^{-r_1 t(\Delta)} \leq m_1 p_{nn} + (1 - m_1 p_{nn}) v_1$$
$$v_1 \leq \frac{e^{-r_1 t(\Delta)} p_{nn} m_1}{1 - e^{-r_1 t(\Delta)}(1 - p_{nn} m_1)} \leq \frac{p_{nn} m_1}{1 - e^{-r_1 t(\Delta)}}$$

So, $v_1 > 1 - \theta_2$ implies that $m_1 \geq (1-\theta_2)(1 - e^{-r_1 t(\Delta)})$ and $p_{nn} \geq (1-\theta_2)(1 - e^{-r_1 t(\Delta)})$.

The following are the steady state equations for class 1 agents

$$N_1 m_1 n_2 p_{nn} = (1 - z_1) L_1$$
$$C_1 m_1 n_2 p_{cn} = L_2(1 - z_2) - L_1(1 - z_1)$$

Let $\alpha = \frac{L_2(1-z_2) - L_1(1-z_1)}{(1-z_1)L_1} > 0$. Dividing the first equation by the second equation and using $n_1 + c_1 = 1$ gives

$$n_1 = \frac{p_{cn}}{\alpha p_{nn} + p_{cn}} \tag{28}$$

$$c_1 = \frac{\alpha p_{nn}}{\alpha p_{nn} + p_{cn}} \tag{29}$$

Consequently,

$$c_1 \geq \frac{\alpha(1 - e^{-r_1 t(\Delta)})(1-\theta_2)}{\alpha(1 - e^{-r_1 t(\Delta)})(1-\theta_2) + 1} \equiv \underline{c}.$$

If $c_1 \geq \underline{c} > 0$, then there exists time T, which is independent of $x(\Delta)$, such that the normal types trade or opt-out with probability 1 by time T. This is proved in Appendix C as a part of the proof of Lemma 1.

If commitment type 1 trades with player 2, then the expected payoff to the commitment type 1 conditional on trading is at least $e^{-r_1 T}\theta_1$. This is because after time T player 2 knows with certainty that her opponent is the commitment type. At this point player 2 will either immediately opt-out or immediately trade with commitment type 1. Also, in any trade commitment type 1 receives θ_1.

In order for the steady state equations to hold commitment type 1 needs to be indifferent between voluntarily leaving and remaining in the market. Commitment type 1 receives at most θ_1 from player 2 so

$$x(\Delta)/e^{-r_1 t(\Delta)} \geq m_1 p_{cn} e^{-r_1 T}\theta_1 + (1 - m_1 p_{cn}) x(\Delta), \text{ which implies}$$
$$p_{cn} \leq \frac{(1 - e^{-r_1 t(\Delta)})x}{m_1 e^{-r_1 t(\Delta)}(e^{-r_1 T}\theta_1 - x)}.$$

Combining the upper bound for p_{cn} given in the above equation with steady state equation (28) for n_1 implies

$$n_1 \leq \frac{(1 - e^{-r_1 t(\Delta)})x(\Delta)}{\alpha p_{nn} m_1 e^{-r_1 t(\Delta)}\left(e^{-r_1 T}\theta_1 - x(\Delta)\right) + (1 - e^{-r_1 t(\Delta)})x(\Delta)} \leq \frac{(1 - e^{-r_1 t(\Delta)})x(\Delta)}{\alpha p_{nn} m_1 e^{-r_1 t(\Delta)}\left(e^{-r_1 T}\theta_1 - x(\Delta)\right)}.$$

The following gives a bound for player 2's equilibrium payoff:

$$v_2/e^{-r_2 t(\Delta)} \leq n_1 p_{nn} + c_1 p_{cn}(1-\theta_1) + (1-(n_1 p_{nn} + c_1 p_{cn}))v_2$$

Equations (28) and (29) implies that $c_1 p_{cn} = \alpha n_1 p_{nn}$ substituting gives

$$\begin{aligned} v_2/e^{-r_2 t(\Delta)} &\leq n_1 p_{nn}(1+\alpha(1-\theta_1)) + (1-n_1 p_{nn}(1+\alpha))v_2 \\ v_2 &\leq \frac{e^{-r_2 t(\Delta)} n_1 p_{nn}(1+\alpha(1-\theta_1))}{1-e^{-r_2 t(\Delta)} + e^{-r_2 t(\Delta)} n_1 p_{nn}(1+\alpha)} \leq \frac{1+\alpha}{1-e^{-r_2 t(\Delta)}} n_1 p_{nn}. \end{aligned}$$

Using the bounds on n_1 and m_1 implies

$$v_2 \leq \frac{1+\alpha}{1-e^{-r_2 t(\Delta)}} \frac{x(\Delta)}{\alpha(1-\theta_2)e^{-r_1 t(\Delta)}\left(e^{-r_1 T}\theta_1 - x(\Delta)\right)} \leq \frac{x(\Delta)}{1-e^{-r_2 t(\Delta)}} \frac{1+\alpha}{\alpha(1-\theta_2)e^{-r_1 t(\Delta)}\left(e^{-r_1 T}\theta_1 - x(\Delta)\right)}.$$

However, $\lim_{\Delta\to 0} x(\Delta)/(1-e^{-r_2 t(\Delta)}) = 0$, by assumption. Consequently, if Δ is sufficiently small, then the right hand is also small and $v_2 < 1-\theta_1$ leading to a contradiction. □

Proof of Corollary 2. Part (i): In this equilibrium, player 2 never opts out against commitment types that are less greedy than τ_1, including τ_1. Player 2 never trades with the commitment types greedier than τ_1. Player 1's outside option $e^{-r_1 t(\Delta)} v_1$ is strictly larger than yielding to any commitment type. All commitment type 2s and all commitment type 1s who are greedier than τ_1, voluntarily leave the market since they are never traded. In this case, the bargaining game is governed by Lemma 1. Consequently, player 1's payoff is at least $\underline{\theta}_1 - o(\Delta)$ and player 2's equilibrium value is at most $1-\underline{\theta}_1 + o(\Delta)$. We can ensure that $v_1 > 1-\underline{\theta}_2$, by choosing Δ sufficiently small. Also, pick the market tightness parameter for side 1 equal to one, i.e., $m_1 = 1$. Again, it is straightforward to choose $t(\Delta)$ sufficiently small so that v_2 is strictly greater than $\theta_1^{\tau_1+1}$. Also, we can pick $m_2 \leq 1$ to ensure that $v_2 \leq \underline{\theta}_1$.

Part (ii): Theorem 2 immediate implies that $v_i \leq 1-\underline{\theta}_j$ for all i and j. □

Proof of Theorem 3. Suppose that, without loss of generality, player 1 is the player who opts out according to condition (ii) in the definition of a SBU equilibrium. Let $T^* = \inf\{t : \alpha_1(t) > 0\}$ and let $T^* = \infty$ if $\{t : \alpha_1(t) > 0\} = \emptyset$. T^* denotes the first point in time where player 1 opts out. First, suppose that $T^* = \infty$, then Proposition 1 in Abreu and Gul (2000) implies that both agents concede at constant hazard rates λ_1 and λ_2 and complete yielding by a common time $\hat{T} < \infty$. Now suppose that $T^* < \infty$. Condition (iii) requires that player 1 does not yield after time T^*. Therefore, player 2's continuation payoff at any time $t > T^*$ is at most $1-\theta_1$, so she has no incentive to delay yielding beyond T^* and thus must complete yielding by time T^*. Player 1's continuation payoff at any time $t > T^*$ is at most $1-\theta_2$ because player 2 completes her yielding by time T^*. Consequently, player 1 has no incentive to delay opting out or yielding beyond time T^* and thus must complete yielding and opting out by time T^*. However, again Proposition 1 in Abreu and Gul (2000) implies that the two agents concede at constant hazard rates λ_1 and λ_2. Also, the previous discussion implies that both agents complete yielding by time T^*. Consequently, in any SBU equilibrium both players concede at constant hazard rates λ_1 and λ_2 and complete yielding by some common time $T < \infty$.

Condition (i) requires that neither player yields with an atom at time zero in a SBU equilibrium. Because if player j yields with positive probability at time zero, then $v_i(\sigma) = U_i(\sigma) > 1 - \theta_j$ which would contradict condition (i). Consequently, in a SBU equilibrium $F_1(t) = 1 - e^{-\lambda_1 t}$ and $F_2(t) = 1 - e^{-\lambda_2 t}$ for $t \leq T$. Since player 2 does not opt-out by assumption and yields at rate λ_2, she completes her yielding at time $T = -\ln c_2/\lambda_2$. Consequently, both players complete yielding by time $T = -\ln c_2/\lambda_2$ and equations (9) and (10) are satisfied in a SBU equilibrium. Also, a steady state implies that equations (11) and (12) are satisfied in a SBU equilibrium.

Recall z^* is such that $z^* \leq \frac{\lambda_1}{\lambda_2+\lambda_1} \leq 1 - z^*$. Below we show that there exists a unique $p_{nc} \in [0,1]$ such that yielding equations (9) and (10) and steady state equations (11) and (12) are satisfied. Also, we show that there exists a $p_{nc} \in [0,1]$ such that equations (9) - (12) are satisfied, only if player 1 is the weaker player in $\hat{\Gamma}(0,\hat{c})$. Consequently, this is the unique SBU equilibrium, if player 1 is assumed to be the weaker player in $\hat{\Gamma}(0,\hat{c})$.[12]

Suppose that player 1 is weaker in $\hat{\Gamma}(0,\hat{c})$ and define $b_2(p_{nc})$ using the following equation:

$$1 - b_2(p_{nc})e^{-\lambda_2 T(p_{nc})} = 1 - c_2(p_{nc}).$$

Substituting in for $T(p_{nc})$ and $c_2(p_{nc})$ gives the following expression for $b_2(p_{nc})$

$$b_2(p_{nc}) = \frac{z_2}{z_2 + p_{nc}(1 - z_1 - z_2)} \left(\frac{1 - z_2}{(1 - z_2) - p_{nc}(1 - z_1 - z_2)} \right)^{\frac{\lambda_2}{\lambda_1}}.$$

A necessary condition for equilibrium is $b_2(p_{nc}) = 1$ since player 2 does not yield at time zero. We show that there exists p_{nc} such that $b_2(p_{nc}) = 1$. Notice if $p_{nc} = 0$, then $b_2(p_{nc}) = 1$. However, $p_{nc} = 0$ is not an equilibrium. This is because $p_{nc} = 0$ implies that $c_2 = 1$ and thus $n_2 = 0$. This implies that player 1 neither trades with commitment type 2 ($p_{nc}c_2 = 0$) nor player 2 ($p_{nn}n_2 = 0$). Hence, $v_1 = 0$. But this violates the condition that $v_1 = 1 - \theta_2$.

Rewrite the expression for $b_2(p_{nc})$ as follows:

$$b_2(p_{nc}) = f(k(p_{nc}))^{1/\lambda_1}$$

where $k(p_{nc}) = z_2 + p_{nc}(1 - z_1 - z_2)$ and

$$f(k) = z_2^{\lambda_1}(1 - z_2)^{\lambda_2} k^{-\lambda_1}(1 - k)^{-\lambda_2}.$$

If $p_{nc} = 1$, then $b_2(1) \geq 1$ (since player 1 is weaker in $\hat{\Gamma}(0,\hat{c})$). The function $f(k)$ is strictly convex and minimized at $k = \frac{\lambda_1}{\lambda_2+\lambda_1} \in (0,1)$. Let

$$p^* = \frac{\frac{\lambda_1}{\lambda_1+\lambda_2} - z_2}{1 - z_1 - z_2} \tag{30}$$

[12]Since either player 1 or player 2 is the weaker player in $\hat{\Gamma}(0,\hat{c})$, our argument establishes that there is a unique SBU equilibrium.

so that $k(p^*) = \frac{\lambda_1}{\lambda_2+\lambda_1}$. Our assumption that $z_i < z^*$ implies that $p^* \in (0,1)$. Also, notice $b_2(p^*) < b_2(0) = 1 \leq b_2(1)$. Consequently, the convexity of f implies that f is decreasing for $p_{nc} < p^*$ and increasing for $p_{nc} > p^*$. This implies that $f(k(p_{nc})) < 1$ for all $p_{nc} < p^*$ and there exists a unique $p_{nc} \in (p^*, 1]$, such that $b_2(p_{nc}) = 1$. Note that if player 2 is strictly weaker in $\hat{\Gamma}(0, \hat{c})$, then $b_2(1) < 1$ and so there does not exist $p_{nc} \in (0,1]$ such that $b_2(p_{nc}) = 1$. Hence the equilibrium is unique. □

Proof of Corollary 3. First, without loss of generality, we pick a subsequence where player 1 is the weaker player in $\hat{\Gamma}(0, \hat{c}^k)$ for all k. The argument for Theorem 3 implies that $p^k_{nc} \geq p{*}^k$ for all k, where $p{*}^k$ is defined in equation (30) in the proof of Theorem 3. Notice $\lim_k p{*}^k = \lambda_1/(\lambda_1+\lambda_2)$. Consequently, $\lim_k p^k_{nc} \geq \lambda_1/(\lambda_1+\lambda_2) > 0$. Again the argument for Theorem 3 gives

$$b_2(p^k_{nc}) = \frac{z^k_2}{z^k_2 + p^k_{nc}(1 - z^k_1 - z^k_2)} \left(\frac{1 - z^k_2}{(1 - z^k_2) - p^k_{nc}(1 - z^k_1 - z^k_2)} \right)^{\frac{\lambda_2}{\lambda_1}} = 1.$$

for all k. Consequently, $\lim_k b_2(p^k_{nc}) = 1$. If $\lim_k b^k_2(p^k_{nc}) = 1$, then the previous equation implies that $\lim_k p^k_{nc} = 0$ or $\lim_k p^k_{nc} = 1$. However, $\lim_k p^k_{nc} \geq \lambda_1/(\lambda_1+\lambda_2) > 0$ implies that $\lim_k p^k_{nc} = 1$. Also, $\lim_k p^k_{nc} = 1$ and equations (11) and (12) imply that $\lim_k c^k_1 = \lim_k c^k_2 = 0$. □

Appendix F. SBU equilibria: existence and convergence

Let $f_1(t)$ denote the probability that player 1 reveals rationality in period t. A ς-SBU equilibrium is a search equilibrium σ with the following properties

- (i) Player 2 trades with commitment type 1 with probability 1,
- (ii) Player 1 opts out with positive probability in a period t only if $\sum_{s>t} f_1(s) < \varsigma$
- (iii) $v_1 = 1 - \theta_2$ and $v_2 < 1 - \theta_1$.

F.1. Existence.

Theorem F.1. *Assume that $t(\Delta) \geq K\sqrt{\Delta}$ for some $K > \kappa$ where the constant κ is defined as in Lemma 2, $\lim_{\Delta\to 0} t(\Delta) = 0$, and suppose that z^* is defined as in equation (13). There exists Δ^* such that if, $\Delta < \Delta^*$, $z_1 < z^*$ and $z_2 < z^*$, then there exists $\varsigma(\Delta)$ such for all $\varsigma < \varsigma(\Delta)$ a ς-SBU equilibrium exists.*

Proof. We define an "*alternative*" bargaining game, prove that a search equilibrium exists if the players play this *alternative* game in the bargaining stage, and show that this equilibrium is also an equilibrium for the original search economy.

Step 1. The alternative game.

Given exogenous payoff function $w : \mathbb{N} \to \mathbb{R}^2$, exogenous total break-up probability a, vector of outside options v, and vector of commitment type probabilities c we define the *alternative* game $\hat{\Gamma}(a, c, v, w)$. In the *alternative* game player 1 moves first in the odd periods and player 2 moves first in the even periods. The player that moves first has two actions available,

$\{R(eveal), I(nsist)\}$. If the player that moves first chooses R, then the game ends and payoffs are realized. If the player that moves first chooses I, then the follower picks action from $\{R(eveal), I(nsist)\}$. If she choses R, then the game ends and otherwise the game progresses to the next period. Also, at any node in period t where player 1 moves the game ends with probability $\alpha_1(t)$. The opt-out probability $\alpha_1(t)$ is a function of a and strategies. We define $\alpha_1(t)$ in Step 3. The function $w : \mathbb{N} \to \mathbb{R}^2$ determines payoffs (before discounting) to each player from revealing and being revealed to, at any period t, after a play of R by the player who speaks first. In a period where j moves first, a typical element of w, denoted $w(t) = (w_i^j, w_j^j)(t)$ where $w_i^j(t)$ is the payoff to i from j revealing in period t. If the player that speaks second, player j, reveals, then player i receives payoff θ_i and player j receives payoff $1-\theta_i$. If there is a break-up in a period, then the agents receive $e^{-r_i t(\Delta)} v_i$ as their payoff. In this game the commitment types never opt-out or take action R, and player i is the commitment type with probability c_i.

The alternative game is interpreted as follows: the strategy insist corresponds to player i asking for θ_i and rejecting an offer of θ_j by player j in the original game. Reveal corresponds to player i proposing something different than θ_1 but on an equilibrium path for the game with one-sided incomplete information in the original game. The exogenous continuation payoffs w are chosen from the set of equilibrium payoff vectors for the game with one-sided incomplete information. The exogenously given opt-out probability a is incorporated into the game so that player 1 only opts-out against the commitment type.

Step 2. Strategies in the alternative game. Let $\mathcal{F} = \{F : \mathbb{N} \to [0,1], F \text{ non-decreasing}\}$, that is $\mathcal{F}$ is the set of all sub-probability distribution functions over the set of natural numbers. Let $F(\infty) = \lim_{t\to\infty} F(t)$. Let f denote the density of F, i.e., $f(t) = F(t) - F(t-1)$. A strategy for player 1 is a function F_1 such that $F_1 \in \mathcal{F}$, and $\sum_t f_1(t) \leq (1-c_1)(1-a)$. A strategy for player 2 is a function $F_2 \in \mathcal{F}$ such that $\sum_t f_2(t) \leq (1-c_2)$.

Step 3. For any $F \in \mathcal{F}$ for player 1 let t_ς denote the first period such that $F(t) \geq F(\infty) - \varsigma$. For any exogenously given total opt-out probability $a \in [0,1]$ let

$$\alpha_1(t, F, a) = \begin{cases} a\frac{F(t_\varsigma)-(F(\infty)-\varsigma)}{\varsigma} & \textit{for } t = t_\varsigma, \\ a\frac{f(t)}{\varsigma} & \textit{for } t > t_\varsigma, \\ 0 & \textit{for } t < t_\varsigma. \end{cases}$$

Step 4. Utilities in the alternative game. Suppose player i uses strategy F_i. Define $\alpha_2(.) = 0$. In the following we drop the dependence of α_1 on F_1 and a when this does not cause any ambiguity. In this game the payoff to player i from revealing at time t where player i is the player to propose

$$U_i(F, a, c, v, w, t) = \sum_{s<t} \gamma_i^s (f_j(s) w_i^j(s) + \alpha_j(s) v_i) + (1 - F_j(t-1) - \sum_{s<t} \alpha_j(s)) \gamma_i^t w_i^i(t).$$

The payoff to player i from revealing at time t where player i is the player to respond

$$U_i(F,a,c,v,w,t) = \sum_{s\le t} \gamma_i^s (f_j(s) w_i^j(s) + \alpha_j(s) v_i) + (1 - F_j(t) + \sum_{s\le t} \alpha_j(s)) \gamma_i^t (1-\theta_j).$$

Step 5. The fixed point operator Φ*.* Define correspondence Φ such that $(F', a', c', v', w') \in \Phi(F,a,c,v,w)$ if and only if

$$a' = \begin{cases} 1-p^* & \text{if } v_1 < 1-\theta_2, \\ 0 & \text{if } v_1 > 1-\theta_2, \\ [0, 1-p^*] & \text{otherwise.} \end{cases}$$

$$F_1' \in \arg\max_{\{\hat{F}_1 \in F : \sum_t \hat{f}_1(t) \le (1-a)(1-c_1)\}} \sum_{t\ge 0} \gamma_1^t \hat{f}_1(t) U_1(F,a,c,v,w,t)$$
$$F_2' \in \arg\max_{\{\hat{F}_2 \in F : \sum_t \hat{f}_2(t) \le 1-c_2\}} \sum_{t\ge 0} \gamma_2^t \hat{f}_2(t) U_2(F,a,c,v,w,t)$$

$$v_1' = e^{-r_1 t(\Delta)} \sum_{t\ge 0} \gamma_1^t \left(\frac{f_1(t)}{1-c_1} U_1(F,a,c,v,w,t) + \frac{\alpha_1(t,F_1,a) v_1}{1-c_1} \right)$$
$$v_2' = e^{-r_2 t(\Delta)} \min\{1-\theta_1, \sum_{t\ge 0} \gamma_2^t \frac{f_2(t)}{1-c_2} U_2(F,a,c,v,w,t)\}$$

$$c_1' = \frac{p(F,a) z_1}{p(F,a) z_1 + 1 - (z_1 + z_2)}$$
$$c_2' = \frac{p(F,a) z_2}{p(F,a) z_2 + (1 - z_1 - z_2)(1-a)},$$

where p^* is defined in equation (30) in the proof of Theorem 3, $p(F,a) = \max\{p_{nn}(F,a), 1-\epsilon\}$ and $p_{nn}(F,a)$ denotes the probability that player 1 and player 2 trade, given revelation probabilities F_1 and F_2 and the opt-out probability $\alpha_1(F_1,a)$. The constant ϵ is chosen sufficiently small so that in the continuous time game considered in Theorem 3, if c_1 and c_2 are calculated using $p_{nn} = 1-\epsilon$ and $p_{nc} = p^*$, player 1 is the stronger player and $b_2(p^*, \epsilon) < 1$.

Also, let $\mu_i(F,a,t)$ denote the probability that player i is a commitment type given that player i has not revealed rationality in history h^t. The posterior probability μ_i is obtained using Bayes' rule conditioning on strategies $(F_1, F_2, \alpha_1(F_1,a))$. Notice $\mu_i(F,a,t)$ is a continuous function of (F,a). Let

$$(w_i^j, w_j^j)(t)' = \{U_i(\mu_i(F,a,t)), U_j(\mu_i(F,a,t))\},$$

where $\{U_i(\mu_i(F,a,t)), U_j(\mu_i(F,a,t))\}$ denotes the set of perfect equilibrium payoff vectors in the bargaining game with one-sided incomplete where player i reputation level is $\mu_i(F,a,t) > 0$. Recall that $(U_1(\mu_1), U_2(\mu_1))$ is an upper-hemi continuous, convex and compact valued correspondence (as a function of μ_1) by Lemma B.3.

The correspondence Φ, as defined above is clearly upper hemi-continuous, compact and convex-valued (in the product topology). Consequently, Glicksberg's fixed point theorem implies that a fixed point, (F, a, c, v, w) exists.

Step 6. The remaining steps show that if ς and Δ are sufficiently small, then $(F, \alpha_1(F_1, a), c)$ is an equilibrium, and v is the vector of values in this equilibrium, of the economy where the bargaining stage game is the original bargaining game and the continuation equilibrium once one player has revealed is chosen from the set of equilibria of the game with one-sided incomplete information such that payoffs are according to w.

Step 7. Player 1's value $v_1 \leq 1 - \theta_2$. Also, $a > 0$.

If $v_1 > 1 - \theta_2$, then $a = 0$. If $a = 0$, then this game is identical to the bargaining game without the options of opting-out. In this case, player 1 was chosen as the weaker player who concedes with probability $1 - b_1 \geq 0$. This implies that player 1's payoff in the bargaining stage is $1 - \theta_2$ with error $o(\sqrt{\Delta})$. Consequently, by Lemma 2 and because $t(\Delta) > K\sqrt{\Delta}$, for $\Delta > 0$ we have $v_1 < 1 - \theta_2$. However, this implies that $a = 1 - p^* > 0$, a contradiction.

Step 8. We will show $v_2 \geq (1-\theta_1) - C(1 - e^{-r_2 t(\Delta)})$ for some constant C which is independent of Δ. Player 2 can always reveal rationality immediately in period 1. This guarantees that player 2 will trade with the commitment type of class 1 in period 1. So, $v_2/e^{-r_2 t(\Delta)} \geq c_1(1 - \theta_1) + v_2(1 - c_1)$. Rearranging

$$v_2 \geq \frac{e^{-r_2 t(\Delta)}(1 - \theta_1)c_1}{1 - (1 - c_1)e^{-r_2 t(\Delta)}} = 1 - \theta_1 - \frac{(1 - \theta_1)(1 - e^{-r_2 t(\Delta)})}{1 - (1 - c_1)e^{-r_2 t(\Delta)}} \geq 1 - \theta_1 - \frac{1 - e^{-r_2 t(\Delta)}}{c_1}$$

However, the formulation of the fixed point operator Φ implies that $c_1 \geq \frac{z_1/(1-\epsilon)}{z_1/(1-\epsilon)+1-(z_1+z_2)}$. Consequently, $v_2 \geq (1 - \theta_1) - C(1 - e^{-r_2 t(\Delta)})$ for some constant C which is independent of Δ.

Step 9. The probability that player 1 and player 2 trade, $p_{nn}(F, a, c) \geq 1 - C\Delta$, where C is a constant independent of Δ. Consequently, commitment type probabilities $c_1 \geq \underline{c}_1 = \frac{z_1/(1-C\Delta)}{z_1/(1-C\Delta)+1-(z_1+z_2)}$ and $c_2 \geq \underline{c}_2 = \frac{z_2/(1-C\Delta)}{z_2/(1-C\Delta)+(1-z_1-z_2)}$.

Player 2 will complete her yielding by the latest in period t_ς for sufficiently small ς. This is because the probability that player 1 yields by at most ς in any of these periods. Consequently, player 2 will do strictly better by completing yielding in period $t_\varsigma + 1$ than in any period $t > t_\varsigma + 1$.

Suppose that player 2 reveals rationality with probability $p > C\Delta$ in period t_ς. Observe that player 1 reveals with positive probability in period t_ς by the definition of this period.

Suppose that period t_ς is a period where player 1 is proposing. Instead of revealing rationality in t_ς, player 1 can wait until $t_\varsigma + 1$, and reveal rationality with certainty then, if player 2 has not revealed yet. This strategy can not do any better than revealing rationality in t_ς. This implies that

$$1 - \theta_2 + \Delta\kappa \geq \gamma_1(1 - p)(1 - \theta_2) + \theta_1 p.$$

This inequality cannot hold for C sufficiently large.

Suppose that period t_ς is a period where player 1 is responding. However, since player 2 is revealing with probability p in t_ς player 1 will not reveal in periods $t_\varsigma - 1$ or $t_\varsigma - 2$, if C is sufficiently large. Since he does not reveal, his reputation level does not change in the two prior periods. This implies that player 2 is better of revealing in period $t_\varsigma - 2$, if such a period exists. If such a period does not exist, then player 1 must be revealing with probability of at least $F_1(\infty) - \varsigma$ in period t_ς. Notice for Δ small $e^{-r_2 t(\Delta)}$ is close to 1 and so v_2 is close $1 - \theta_2$. This implies, however, that if player 1 is revealing with probability $F_1(\infty) - \varsigma$ in period t_ς, then player 2 will do better by not revealing in period t_ς.

Player 2 can not reveal with probability more than $C\Delta$ in period $t_\varsigma + 1$ either. If player 2 was to reveal with probability greater than $C\Delta$, then player 1 will not reveal in period t_ς. However, this contradicts the definition of period t_ς which requires that player 1 reveal with positive probability in this period. Consequently, the probability that player 2 reveals in period $t \geq t_\varsigma$ is at most $2C\Delta$. Redefining C implies that $p_{nn} \geq 1 - C\Delta$.

Step 10. The bounds in the operator Φ are not binding, $a \in (0, 1 - p^*)$, $v_1 = 1 - \theta_2$, and $v_2 < 1 - \theta_1$. Consequently, $(F, \alpha_1(F_1, a), c)$ is an equilibrium, and v is the vector of values in this equilibrium, of the economy where the bargaining stage game is the original bargaining game and the continuation equilibrium once one player has revealed is chosen from the set of equilibria of the game with one-sided incomplete information such that payoffs are according to w.

Suppose that $v_1 < 1 - \theta_2$, then $a = 1 - p^*$. Revelations need to occur at rate λ_1 and λ_2 by Lemma 2. However, p^* is chosen such that if revelations occur at rate λ_1 and λ_2, then for Δ small player 2 is the player that reveals with a jump in the first two periods. Moreover, player 2's probability of revelation approaches $1 - b_2 > 0$. This would imply that $v_1 > 1 - \theta_2$ for sufficiently small Δ. Consequently, $v_1 = 1 - \theta_2$ and $a \in (0, 1 - p^*)$. Notice that since $t(\Delta)$ is larger than $\kappa\sqrt{\Delta}$, player 2 needs to be the player that reveals rationality with a jump in the first two periods, i.e., $b_2 < 1$ and $b_1 = 1$ as defined in Lemma 2. This however, implies that player 2's value $v_2 < 1 - \theta_2$. □

F.2. **Convergence of discrete time ς-SBU equilibria to a continuous time SBU equilibrium.** Let z^* denote the constants defined in equation (13) and let $\Delta^* > 0$ be the cutoff chosen in Theorem F.1. Suppose that $z_1 < z^*$ and $z_2 < z^*$. Let $(F_1^n, \alpha^n, F_2^n, v^n, c^n)$ denote a sequence of ς^n-SBU equilibria for the economy where the period length is $\Delta^n > 0$ where $\Delta^n < \Delta^*$. Suppose $\lim_{n\to\infty} \Delta^n = 0$ and consequently $t(\Delta)^n \to 0$. Also, suppose that $\lim_{n\to\infty} \varsigma^n = 0$. Such a sequence of equilibria exists by Theorem F.1.

Remark F.1. *By construction $c_i^n \geq \underline{c}_i$. By Step 1, of Appendix C there exists a time T such that $\frac{F(T)^n}{1-\underline{c}_1} + \sum_{t \leq T} \alpha(t)^n = 1$ and $\frac{F_2^n(T)}{1-\underline{c}_2} = 1$, for all n. Hence, the sub-probability distributions (F_1^n, α^n, F_2^n) have uniformly bounded support $[0, T]$. Consequently, Helly's theorem (Billingsley (1995), Theorem 25.9) implies that $(F_1^n, \alpha^n, F_2^n, v^n, c^n)$ has a convergent subsequence. Let (F_1, α, F_2, v, c) denote a sub-sequential limit.*

Theorem F.2. *The limit (F_1, α, F_2) is a SBU equilibrium for the continuous time bargaining stage-game where the vector of commitment type probabilities is c, v is the equilibrium values given that the SBU equilibrium (F_1, α, F_2) is played in the bargaining stage-game, and the vector c satisfies the steady state equations.*

Proof. Step 1. F_1 and F_2 do not have common discontinuity points. Also, $G = F_1 + \alpha$ and F_2 do not have common discontinuity points.

Step 2. Let $U_1^n = \int\int U_1(t,k)dG^n(t)dF_2^n(k)$ and $U_2^n = \int\int U_2(t,k)dF_1^n(t)dF_2^n(k)$ where

$$U_i(t,k) = \begin{cases} \theta_i & \text{if } t > k, \\ 1-\theta_j & \text{if } t < k, \\ 1/2 & \text{if } t = k. \end{cases}$$

F_1, F_2 and G do not have common discontinuity points consequently Billingsley (1995) Theorem 29.2 and Exercise 29.2 implies that $\lim U_1^n = \int\int U_1(t,k)dG(t)dF_2(k)$ and $\lim U_2^n = \int\int U_2(t,k)dF_1(t)dF_2(k)$. Also, $v_1 = \frac{\lim U_1^n}{1-c_1}$ and $v_2 = \frac{\lim U_2^n}{1-c_2}$.

Step 3. The functions (F_1, α, F_2) comprise a SBU equilibrium for the continuous time war of attrition with opt-outs.

The vector c^n and a^n satisfy the steady state equations for all n. Hence, c and a satisfy the steady state equations.The probability that player 1 and player 2 trades $p_{nn}^n \geq 1 - C\Delta^n$ where C is independent of Δ^n. Hence, $\lim_n p_{nn}^n = 1$. The values $v_1^n = 1-\theta_2$ and $v_2^n \leq 1-\theta_1$ for all n. Hence, $v_1 = 1-\theta_2$ and $v_2 \leq 1-\theta_1$. However, v_2 cannot be strictly less than $1-\theta_1$ since she can guarantee $1-\theta_1$ in the bargaining game and search is costless at the limit. So, $v_2 = 1-\theta_1$ and condition (i) is satisfied. Condition (ii) is satisfied by construction since player 2 never opts out. Condition (iii) is also satisfied since the probability that an opt-out and a revelation occur in the same period is at most ς^n in a ς^n-SBU equilibrium and $\lim_{n\to\infty} \varsigma^n = 0$.

We now show that (F_1, α) and F_2 are mutual best responses. F_1 does not jump at T and $p_{nc} > p^*$ by construction. In the continuous time war of attrition, if player 1 is behaving according to F_1, α, then for each ϵ, there is a N such that for all $n > N$, F_2^n is an ϵ best response to F_1, α and consequently, since ϵ is arbitrary F_2 is a best response to F_1, α. Also, the symmetric argument is true for player 2 showing that F_1, α is a best response to F_2. Proving that F_1, α and F_2 is an equilibrium. Since the war of attrition has a unique equilibrium with $p_{nc} > p^*$, F_1, α and F_2 coincide with this equilibrium. This argument is identical to Abreu and Gul (2000), proof of Proposition 4, on page 114 where a more detailed proof may be found. □

References

Abreu, D., and F. Gul (2000): "Bargaining and Reputation," *Econometrica*, 68(1), 85–117.

Abreu, D., and D. Pearce (2007): "Bargaining, reputation, and equilibrium selection in repeated games with contracts," *Econometrica*, 75(3), 653–710.

Billingsley, P. (1995): *Probability and Measure.* John Wiley and Sons, New York, NY.

Chatterjee, K., and L. Samuelson (1987): "Bargaining with two-sided incomplete information: An infinite horizon model with alternating offers," *Review of Economic Studies*, 54(2), 175–192.

Compte, O., and P. Jehiel (2002): "On the Role of outside Options in Bargaining with Obstinate Parties," *Econometrica*, 70(4), 1477–1517.

Fudenberg, D., and D. Levine (1989): "Reputation and equilibrium selection in games with a patient player," *Econometrica*, 57, 759–778.

——— (1992): "Maintaining a reputation when strategies are imperfectly observed," *Review of Economic Studies*, 59, 561–579.

Kambe, S. (1999): "Bargaining with Imperfect Commitment," *Games and Economic Behavior*, 28(2), 217–237.

Kreps, D., and R. Wilson (1982): "Reputation and Imperfect Information," *Journal of Economic Theory*, 27, 253–279.

Mailath, G., and L. Samuelson (2006): *Repeated Games and Reputations: Long-run Relationships.* Oxford University Press, Oxford, UK.

Milgrom, P., and J. Roberts (1982): "Predation, Reputation, and Entry Deterrence," *Journal of Economic Theory*, 27, 280–312.

Myerson, R. (1991): *Game Theory: Analysis of Conflict*. Harvard University Press.

Osborne, M., and A. Rubinstein (1990): *Bargaining and markets*. Academic Press San Diego.

Perry, M., and P. Reny (1993): "A Non-cooperative Bargaining Model with Strategically Timed Offers," *Journal of Economic Theory*, 59, 50–50.

Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica*, 50(1), 97–109.

Rubinstein, A., and A. Wolinsky (1985): "Equilibrium in a Market with Sequential Bargaining," *Econometrica*, 53(5), 1133–1150.

——— (1990): "Decentralized Trading, Strategic Behaviour and the Walrasian Outcome," *The Review of Economic Studies*, 57(1), 63–78.

Samuelson, L. (1992): "Disagreement in markets with matching and bargaining," *The Review of Economic Studies*, pp. 177–185.

Serrano, R., and O. Yosha (1993): "Information revelation in a market with pairwise meetings: the one sided information case," *Economic Theory*, 3(3), 481–499.

Shaked, A., and J. Sutton (1984): "Involuntary Unemployment as a Perfect Equilibrium in a Bargaining Model," *Econometrica*, 52(6), 1351–1364.

Sobel, J. (1991): "Durable goods monopoly with entry of new consumers," *Econometrica*, 59(5), 1455–1485.

Sutton, J. (1986): "Non-Cooperative Bargaining Theory: An Introduction," *Review of Economic Studies*, 53, 709–24.

Watson, J. (1998): "Alternating-Offer Bargaining with Two-Sided Incomplete Information," *The Review of Economic Studies*, 65(3), 573–594.

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