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SOME RESULTS ON GLOBAL ASYMPTOTIC STABILITY OF CONTROL SYSTEMS*

by

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Section 1

INTRODUCTION

In optimal control theory "Hamiltonian" systems of the form

$$(1.1)\qquad \dot{q} = -\frac{\partial H}{\partial k}$$

$$\dot{k} = \frac{\partial H}{\partial q}$$

arise in a natural way in the study of optimal control. Here $q \in R^n$, $k \in R^n$, "." denotes time derivative and $H: [0,\infty) \times R^n \times R^n \to R$ is the Hamiltonian. $H = H(t,q,k)$. See Lee and Markus [11] for a complete discussion of optimal control theory and the role played by (1.1).

It is of interest to study the asymptotic behavior of optimal control in infinite horizon control systems. Lee and Markus [11, p.396] obtain a global asymptotic stability result by placing conditions directly on the "reduced form" differential equation that emerges from the solution for optimal control. Unfortunately, one must "solve the problem" first before one can check whether this type of sufficient condition holds.

In this paper, we will obtain global asymptotic stability results for optimal control systems by placing restrictions on the matrix of partial derivatives of H, itself. Our methods of analysis are related to the basic work of Hartman and Olech [8], and Hartman [6], [7]; Markus and Yamabe [13]. Basically, what we will do in this paper is extend the methods of Hartman and Olech and others to systems of the form (1.1).

This is not a trivial exercise in generalizing the above mentioned works because substantially new techniques of analysis must be invented in order to deal with the "generic" saddle point character of system (1.1). And, furthermore, a building block of our technique will be an elegant method of obtaining part of Hartman and Olech's results due to Mas-Colell [12] a result not yet known in the mathematical literature.

Needless to say, some restrictions must be placed on (1.1) in order to obtain any results at all. We shall focus attention on the important subclass

$$\dot{q} = \rho q - \frac{\partial G}{\partial k} \equiv \rho q - G_k \tag{1.2}$$

$$\dot{k} = \frac{\partial G}{\partial q} \equiv G_q$$

where $G: R^n \times R^n \to R$ and $\rho \geqq 0$, $G = G(q,k)$. This subclass[1/] encompasses a large number of economic problems - Arrow and Kurz [1], Lee and Markus [11], Samuelson [16], Rockafellar [14], Cass [3], Koopmans [9], Kurz [10], Cass-Shell [4] are just a few of the distinguished economists and mathematicians that have studied (1.2).

An example of the type of theorem that we establish in this paper is

<u>Theorem</u>: Let $(\bar{q},\bar{k})$ be a rest point of (1.2). Put $z = (q,k) - (\bar{q},\bar{k})$ and

$$Q(q,k) \equiv \begin{bmatrix} G_{qq} & \rho/2\ I_n \\ \rho/2\ I_n & -G_{kk} \end{bmatrix}$$

If $[\rho q - G_k(q,k)]^T(k - \bar{k}) + [G_q(q,k)]^T(q - \bar{q}) = 0$ implies

$$z^T Q(q,k)\, z > 0 \quad \text{for all } (q,k) \neq (\bar{q},\bar{k}),$$

then all bounded solutions of (1.2) converge to $(\bar{q},\bar{k})$ as time tends to infinity.[2/] (Optimal paths will be bounded solutions of (1.2) under certain "regularity" conditions on the optimal control problem that generates (1.2).) Here "T" denotes transpose.

To our knowledge, no one has obtained the global asymptotic stability results in this paper for systems of the form (1.1) or (1.2). Cass-Shell [4] and Rockafellar [15] have stability results for optimal control, but their analyses use convexity methods. Our methods build on those of Hartman and Olech. Therefore, we obtain a set of results complementary to theirs. We will say more about comparison of our results with those of Hartman and Olech later.

In Section 2, we prove a general result on convergence of bounded trajectories. This theorem is then used to prove, in Section 3, a "Hartman-Olech" type of result for Hamiltonian systems. This theorem, however, is not similar to the most general results proved in [8].

For this reason, in Section 4, we prove a modified version of a result of [8]. The modified version of Hartman and Olech's methods lays out a new method of proving global asymptotic stability theorems for autonomous systems of differential equations. Our methods differ from Hartman and Olech's in two respects. First, Hartman and Olech assume that the rest point is unique and locally asymptotically stable. We assume instead that Hartman's $B(y)$ matrix (Hartman [7, p.542]) is negative definite at the rest point. This is a stronger assumption on the rest point than Hartman-Olech. Second, Hartman and Olech assume there is a <u>positive definite</u> matrix function $G(y)$ such that (in their notation)

$$(*) \qquad G(y)\,F(y)\,x = 0 \ \text{ implies } B(y)\,x \cdot x \leqq 0$$

for all x and all y, outside a neighborhood of the rest point $y = 0$. We relax the assumption that $G(y)$ is positive definite. We assume (*) only.

Since the original proof of Hartman and Olech relies strongly on the positive definiteness of G in order to use Riemannian geometry, we were, therefore, forced to modify the Hartman-Olech method of proof. In Section 5, we show how this result can be applied to study the convergence of solutions to certain optimal control problems. As a corollary, we obtain a convergence result for quasi-concave Hamiltonians.

The paper will close with suggestions on extending our methods to more general systems than (1.1).

Section 2

A GENERAL RESULT ON CONVERGENCE OF BOUNDED TRAJECTORIES

In this section, we shall present a general theorem that will generate Hartman and Olech's basic result [8 , p. 157, Theorem 2.3], our results [2] and many other results - all as simple corollaries. Furthermore, the general theorem will be stated and proved in such a way as to highlight a general Lyapunov method that is especially useful for the stability analysis of optimal paths generated by optimal control problems arising in capital theory. We need a lemma first. It is, basically, the result in [7 , p. 539].

Lemma 2.1 - Let $F(z)$ be continuous on an open set $E \subseteq R^m$, and such that solutions of

(*) $\dot{z} = F(z)$

are uniquely determined by initial conditions. Let $W(z)$ be a real valued function on E with the following properties:

(a) W is C^1 on E.

(b) $0 \leq \dot{W}(z)$ [where $\dot{W}(z)$ is the trajectory drivative of $W(z)$ for any $z \in E$.]

Let $z(t)$ be a solution of (*) for $t \geq 0$. Then the limit points of $z(t)$ for $t \geq 0$, in E, if any, are contained in the set $E_0 = \{z | \dot{W}(z) = 0\}$.

Proof: See Hartman [7, p. 539].

We can now prove

Theorem 2.1 - Let $f: R^m \to R^m$ be C^2. Consider the differential equation system

(2.3) $\dot{x} = f(x)$

Assume there is x such that $f(x) = 0$ (W.L.O.G. put $x = 0$) such that there is $V: R^m \to R$ satisfying:

(a) For all $x \neq 0$, $x \nabla^2 V(0) [J(0)x] < 0$.

(b) $\nabla V(0) = 0$,

(c) For all $x \neq 0$, $\nabla V(x) f(x) = 0$ implies $x \nabla^2 V(x) f(x) = 0$.

(d) For all $x \neq 0$, $x \nabla^2 V(x) f(x) = 0$ implies $\nabla V(x) J(x) x < 0$.

Then,

(α) $\nabla V(x) f(x) < 0$ for all $x \neq 0$,

(β) All trajectories that remain bounded for $t \geqq 0$ converge to 0.

Proof: Let $x \neq 0$, and put

$$g(\lambda) \equiv \nabla V(\lambda x) f(\lambda x). \tag{2.4}$$

We shall show that $g(1) < 0$ in order to obtain (α). We do this by showing that $g(0) = 0$, $g'(0) = 0$, $g''(0) < 0$, and $g(\bar{\lambda}) = 0$ implies $g'(\bar{\lambda}) < 0$ for $\bar{\lambda} > 0$. (At this point, the reader will do well to draw a graph of $g(\lambda)$ in order to convince himself that the above statements imply $g(1) < 0$.) Calculating we get

$$g'(\lambda) = x \nabla^2 V(\lambda x) f(\lambda x) + \nabla V(\lambda x) J(\lambda x)x \tag{2.5}$$

$$g''(\lambda) = x \left[\frac{d}{d\lambda} \nabla^2 V(\lambda x)\right] f(\lambda x) + x \nabla^2 V(\lambda x) [J(\lambda x)x]$$

$$+ x \nabla^2 V(\lambda x) [J(\lambda x)x] + \nabla V(\lambda x) \left[\frac{d}{d\lambda} J(\lambda x)\right]x. \tag{2.6}$$

Now $\lambda = 0$ implies $f(\lambda x) = 0$ so $g(0) = 0$. Also $g'(0) = 0$ from $f(0) = 0$, and (b). Furthermore, $f(0) = 0$, (b) imply

$$g''(0) = 2 x \nabla^2 V(0) [J(0)x] \tag{2.7}$$

But this is negative by (a). By continuity of g'' in λ, it must be true that there is $\varepsilon_0 > 0$ such that $g(\lambda) < 0$ for $\lambda \varepsilon [0, \varepsilon_0]$. Suppose now that there is $\lambda > 0$ such that $g(\lambda) = 0$. Then there must be a smallest $\bar{\lambda} > 0$ such that $g(\bar{\lambda}) = 0$. Also, $g'(\bar{\lambda}) \geqq 0$. Let us calculate $g'(\bar{\lambda})$, show that $g'(\bar{\lambda}) < 0$, and get an immediate contradiction. From (2.5)

$$(2.8) \qquad g'(\bar{\lambda}) = x \, \nabla^2 \, V(\bar{\lambda}x) \; f(\bar{\lambda}x) + \nabla \, V(\bar{\lambda}x) \; J(\bar{\lambda}x)x$$

Now $g(\bar{\lambda}) = 0$ implies $\nabla \, V(\bar{\lambda}x) \; f(\bar{\lambda}x) = 0$. But this, in turn, implies that $\bar{\lambda} \; x \; \nabla^2 \; V(\bar{\lambda}x) \; f(\bar{\lambda}x) = 0$ by (c). Finally, (d) implies that $\nabla \, V(\bar{\lambda}x) \; J(\bar{\lambda}x) \; (\bar{\lambda}x) < 0$. Thus, $g'(\bar{\lambda}) < 0$ - contradiction to $g'(\lambda) \geqq 0$. Therefore,

$$(2.9) \qquad \nabla \, V(x) \; f(x) < 0 \quad \text{for all } x \neq 0.$$

By Lemma 2.1, all the rest points of $\phi_t(x_0)$ satisfy $\nabla \, V(x) \; f(x) = 0$ and hence, $x = 0$ is the only candidate. But if $\phi_t(x_0)$ is bounded, $\phi_t(x_0)$ must have a limit point. Hence, $\lim_{t \to \infty} \phi_t(x_0) = 0$.

Q.E.D.

Note that to get global asymptotic stability results for bounded trajectories all one needs to do is find a V that is monotone on bounded trajectories and assume that $E_0 = \{x | \nabla \, V(x) \; f(x) = 0\} = \{0\}$. This result is important for global asymptotic stability analysis of optimal paths generated by control problems arising in capital theory. Also, Hartman-Olech [8] type results emerge as simple corollaries. Let us demonstrate the power of the theorem by extracting some corollaries.

<u>Corollary 2.1</u> - Let $f: R^m \to R^m$. Consider the ordinary differential equations $\dot{x} = f(x)$, $f(0) = 0$. If $J(x) + J^T(x)$ is negative definite for each x, then 0 is globally asymptotically stable.

Proof: Put $V = x^T x$. Then $\nabla \, V(x) = 2 \; x \; \nabla^2 \; V(x) = 2I$ where I is the n x n identity matrix. Assumption (2a) becomes

$$x \; J(0)x < 0 \quad \text{for all } x \neq 0.$$

But this follows because

$$2 \; x \; J(0)x = x \; (J(0) + J^T(0)) \; x < 0.$$

Assumption (2b) trivially holds since $\nabla \, V(x) = 2x$. Assumption (2c) amounts to $2x \; f(x) = 0$ implies $x(2I) \; f(x) = 0$ which obviously holds. (2d) obviously holds because $2x\, J(x)x < 0$ for all $x \neq 0$. It is obvious that rest points are L.A.S. since $J(0) + J^T(0)$ is a negative definite matrix. Thus, all bounded trajectories converge to 0, as $t \to \infty$. It is easy to use $V = x^T x$ decreasing in t in order to show that all trajectories are bounded. This ends the proof.

The following corollary is a stronger result than Hartman and Olech [8] in one way and weaker in another. We will explain the difference in more detail below.

Corollary 2.2 - (A. Mas-Colell [12]). Consider $\dot{x} = f(x)$, $f(0) = 0$. Assume that $x\, [J(0) + J^T(0)]\, x < 0$ for all $x \neq 0$, and

(2.10) $$x^T f(x) = 0 \text{ implies } x\, [J(x) + J^T(x)]\, x < 0 \text{ for all } x \neq 0.$$

Then 0 is globally asymptotically stable.

Proof: Let $V = x^T x$. We show that

(2.11) $$\frac{dV}{dt} = 2\, x^T f(x) < 0 \text{ for } x \neq 0.$$

Assumptions 2a,b,c,d of Theorem 2.1 are trivially verified. Therefore, $\frac{dV}{dt} < 0$, and the rest of the proof proceeds as in Corollary 2.1.

This type of result is reported in Hartman and Olech [8] and in Hartman's book [7]. In [7] and [8], 0 is assumed to be the only rest point and it is assumed to be locally asymptotically stable. On the one hand, Mas-Colell puts the stronger assumption: $x\, [J(0)]\, x < 0$ for $x \neq 0$ on the rest point. It is well known that negative real parts of the eigenvalues of $J(0)$ does not imply negative definiteness of $J(0) + J^T(0)$, but negative definiteness of $J(0) + J^T(0)$ does imply negative real parts for $J(0)$.

But on the other hand, Hartman and Olech [8] make the assumption: for all $x \neq 0$

$$w^T f(x) = 0 \text{ implies } w^T [J(x) + J^T(x)]\, w \leqq 0 \text{ for all vectors } w.$$

Note that Mas-Colell only assumes $x^T f(x) = 0$ implies $x^T [J(x) + J^T(x)]\, x < 0$. So he places the restriction on a much smaller set of w, but he requires the strong inequality. Furthermore, the proof of the Mas-Colell result is much simpler than that of Hartman and Olech.

It is possible to obtain general results of Hartman and Olech type from the theorem. For example.

Corollary 2.3 - Let G be a positive definite symmetric matrix, and let 0 be the unique rest point of $\dot{x} = f(x)$. Assume that for all $x \neq 0$,

$$x^T [G\, J(0)]\, x < 0,$$

and

$$x^T G\, f(x) = 0 \text{ implies } x^T [G\, J(x)]\, x < 0 \text{ for all } x \neq 0.$$

Then x = 0 is globally asymptotically stable for bounded trajectories.

Proof: Let $V(x) = x^T Gx$. Then,

$$\nabla V(x) = x^T [G + G^T] = 2\, x^T G$$

Also,

$$\nabla^2 V(x) = G + G^T = 2\, G.$$

The rest of the proof is now routine.

Corollary 2.3 is closely related to Hartman and Olech's [8 , Theorem 2.3, p.157] and Hartman's book [7 , Theorem 1.4, p.549]. Hartman and Olech also treat the case of G depending on x. We have not been able to obtain their result for non constant G as a special case of our theorem. Thus, their different methods of proof yield theorems that our methods presented in this section are unable to obtain. This leads us to believe that the original method of proof developed in [2] will be useful for developing Hartman and Olech type generalizations for non constant G for Modified Hamiltonian Dynamical Systems. We will do this below. We turn now to the study of Modified Hamiltonian Dynamical Systems.

Section 3

CONVERGENCE OF BOUNDED TRAJECTORIES OF M.H.D.S.

In this section we apply the results obtained in Section 2 to M.H.D.S. systems. We will assume that the M.H.D.S. has a singularity (q,k) and rewrite it as

(3.1) $$\dot{z}_1 = \rho(z_1 + \bar{q}) - H_2(z) \equiv F_1(z), \quad z \equiv (q,k) - (\bar{q},\bar{k})$$

$$\dot{z}_2 = H_1(z) \equiv F_2(z)$$

We may now state and prove

Theorem 3.1 - Let

(3.2) $$Q(z) = \begin{bmatrix} H_{11}(z) & \rho/2\ I \\ \rho/2\ I & -H_{22}(z) \end{bmatrix}$$

where I is the n x n identity matrix. Assume

(a) $0 = F(0)$ is the unique rest point of $\dot{z} = F(z)$

(b) For all $z \neq 0$,

(3.3) $$z_1^T F_2(z) + z_2^T F_1(z) = 0 \text{ implies } z\, Q(z)\, z > 0$$

(c) For all $w \neq 0$, $w\, Q(0)\, w > 0$.

Then all trajectories that are bounded for $t \geqq 0$ converge to 0 as $t \to \infty$

Proof: Let $V = z^T A z$ where

$$A = -\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

where I is the n x n identity matrix. Note that $z^T A z = -2 z_1^T z_2$. Since $\nabla^2 V(0) = A + A^T = 2A$ and $(wA)\ (J(0)w) = -w\, Q(0)w$, we have that (c) implies

(a) of Theorem 2.1. Also $\nabla V(z) = z^T (A + A^T) = 2 x^T A$, and hence, $\nabla V(0) = 0$. Hence (b) of Theorem 2.1 follows trivially. Now (c) of Theorem 2.1 amounts to $\nabla V(z) F(z) \equiv 2 z^T A F(z) = 0$ implies $z^T \nabla^2 V(z) F(z) \equiv 2 z^T A F(z) = 0$ which is trivially true. Furthermore, (d) amounts to $2 z^T A F(z) = 0$ implies

$$2(z^T A) J(z) z < 0. \tag{3.4}$$

But (3.4) is identical to (3.3) as an easy calculation will immediately show. Thus $\dot{V} < 0$ except at the rest point 0. The rest of the proof is routine by now. Q.E.D.

Section 4

A MODIFIED FORM OF THE HARTMAN AND OLECH RESULT

In Section 3, we obtained a quite general result on the global asymptotic stability of Modified Hamiltonian Systems as a corollary of Theorem 2.1. We also obtained a general Hartman-Olech type result in corollary 2.3. It was indicated, however, that not all of the Hartman and Olech's results could be obtained by the method used to prove Theorem 2.1. In particular, Hartman and Olech's Theorem 14.1 [8 , p.549] cannot be obtained from our Theorem 2.1 by the simple expedient of making the G in corollary 2.3 depend on x, and then generating the sufficient condition for Hartman-Olech Theorem 14.1 by cranking out the meaning of 2a-d of Theorem 2.1. The reader may convince himself by putting $V(x) \equiv x^T G(x)x$ and carrying out the calculations.

One might try to prove Hartman-Olech's Theorem 14.1 by trying different functions $g(\lambda)$ and carrying out the method of Theorem 2.1; e.g. put $V = xG(x)x$ and $g(\lambda) = \lambda \nabla V(\lambda) \cdot f(x)$, or $g(\lambda) \doteq \nabla V(\lambda) f(\lambda x)$, or $g(\lambda) = \nabla V(x) f(\lambda x)$, etc. None of these seems to give us Hartman and Olech's Theorem as the reader can see by reflection.

It is of interest to obtain an analogue of Hartman and Olech's Theorem for Modified Hamiltonian Systems. This will require an extension of Hartman and Olech's methods to Hamiltonian Systems because, as we agreed above, it will not just be a simple application of Theorem 2.1.

In order to do that we must first prove a modified form of the Hartman and Olech result. In the next section we show how this extension can be used to prove stability of bounded trajectories of certain M.H.D.S. The following result will be proved.

Theorem 4.1: Let $F: R^n \to R^n$ be a C^1 function. Consider the system

(4.1) $\dot{z} = F(z)$

and assume

(a) 0 is the unique rest point of (4.1)

(b) The linearization of $\dot{z} = F(z)$ at $z = 0$ has all eigenvalues with negative real parts.[3]

(c) There exists C^1 function $G: R^n \to M^S(R^n, R^n)$[4] such that for every $w \in R^n$, $w \neq 0$, for all $x \in R^n$, $w^T G(z) F(z) = 0$ implies $w^T B(z) w < 0$ where,
$B(z) = [G(z) J(z) + (G(z) J(z))^T + \dot{G}(z)]$ where
$\dot{G}(z)$ is the trajectory derivative of G; i.e.,
$\dot{G}(z) = \sum_{r=1}^{n} \frac{\partial G}{\partial z_r} F_r$.

(d) If K is a compact subset of R^n, then $\bigcup_{z \in K} \bigcup_{t \geq 0} \phi_t(z)$ is bounded.

A few words must be said about the relationship of Theorem 4.1 with Hartman and Olech's Theorem 14.1. Hartman and Olech assume (a) and the local asymptotic stability of the rest point (which is essentially (b)). They assume (c) outside a neighborhood of $z = 0$. We assume it everywhere including $z = 0$. They do not assume the boundedness of solutions as we do in (d). However, they assume that for each z, G(z) is positive definite. This is the assumption we drop.

The fact that G(z) is not assumed positive definite requires some crucial changes in the proof. The strategy used, however, is similar to Hartman and Olech's. The outline of the proof proceeds as follows. We construct an orthogonal field of trajectories as do Hartman and Olech. Then, we show that for $p > 0$ that the distance $r(x(o,q), x(p,q))$ between the points $x(o,q)$, $x(p,q)$

goes to 0 as $q \to \infty$, where the notation used here is exactly the same as in Hartman-Olech. As in Hartman-Olech, $p > 0$ corresponds to trajectories starting in the interior of the attractor set of $z = 0$, and $p = 0$ corresponds to a trajectory starting at a boundary point of the attractor. As in Hartman-Olech, $r(x(o,q), x(p,q)) \to 0$, $q \to \infty$ contradicts the assumption that the attractor of 0 has a boundary point. Thus, global asymptotic stability must obtain.

Our proof consists of estimating the quantity $r(x(o,q), x(p,q))$ from above and showing that it converges to 0 as $q \to \infty$. Now, Hartman and Olech do this by showing that the Riemannian arc length

$$\int_o^p [x_p \; G \; x_p]^{1/2} \, ds$$

decreases as q increases. For us, however, G is not necessarily positive definite so, therefore, the Hartman-Olech argument breaks down.

Our argument amounts to showing:

(1) $x_p \; G \; x_p > 0$ for $x_p^T \; G \; F = 0$, and, furthermore, $x_p(p,q) \; G(x(p,q)) \; x_p(p,q)$ decreases in q.

(2) There is $\varepsilon > 0$ such that there is $\beta > 0$ such that for $0 \leqq p \leqq \beta$, $q \geqq 0$

$$\int_o^p [x_p(p,o) \; G(x(p,o)) \; x_p(p,o)]^{1/2} \, dp$$

$$\geqq \int_o^p [x_p(p,q) \; G(x(p,q)) \; x_p(p,q)]^{1/2} \, dp$$

$$\geqq \varepsilon \int_o^p |x_p(p,q)| dp$$

(3) Step 2 allows us to bound the "Riemann arc length like" quantity

$$\int_o^p [x_p(p,q) \; G(x(p,q)) \; x_p(p,q)]^{1/2} \, dp$$

from below by the usual arc length

$$\varepsilon \int_o^p |x_p(p,q)| dp.$$

Therefore, from this point on, we may copy the rest of the Hartman and Olech proof - making only minor modifications.

The proof is built up by a sequence of lemmas. Before we get on with that, let us point out that $A \equiv A(0) \equiv \{z_0 | \phi_t(z_0) \to 0, t \to \infty\}$ is open. Following Hartman-Olech, we then assume that there exists a point $\bar{z} \in R^n - A$, $\bar{z} \in \bar{A}$ (the closure of the attractor), and show that the "distance" between the solution starting at $\bar{z}$ and certain solutions starting in the interior of the attractor goes to zero as $t \to \infty$.

Lemma 4.1 - If $z \in \bar{A}$ (the closure of the attractor), then $F^T(z)\ G(z)\ F(z) > 0$

Proof: By assumption (c), $w^T\ B(0)\ w < 0$ for all $w \neq 0$ because all $w \in R^n$ satisfy $w\ G(0)\ F(0) = 0$, since $F(0) = 0$. Since $B(z)$ is a continuous function of z, we have $B(z)$ quasi-negative definite for $|z| < \delta$ for some $\delta > 0$. By assumption (b) every neighborhood of zero contains a positively invariant neighborhood of zero; i.e., a neighborhood N s.t. $\bigcup_{t \geq 0} \phi_t(N) \subset N$. Let $\bar{N}$ be one such compact neighborhood such that $\bar{N} \subset B(0,\delta)$.[5/] Hence, for $z \in \bar{N}$ and $v \in R^n$, $v \neq 0$, $v\ B(z)\ v < -\varepsilon\ |v|^2$ for some $\varepsilon > 0$ since $\bar{N}$ is compact. Consider now $z \in A$. Then, there exists $T(z)$ such that $\phi_t(z) \in \bar{N}$ for $t \geq T(z)$. Suppose now $F^T(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z)) = 0$, for some t. Then,

$$\frac{d}{dt}\ [F^T(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z))] = [J(\phi_t(z))\ F(\phi_t(z))]^T\ G(\phi_t(z))\ F(\phi_t(z))$$

$$+ F^T(\phi_t(z))\ G(\phi_t(z))\ J(\phi_t(z))\ F(\phi_t(z))$$

$$+ F^T(\phi_t(z))\ [\ \frac{\partial G}{\partial \phi_t(z)}\ ]\ (\phi_t(z))\ F(\phi_t(z))$$

$$= F^T(\phi_t(z))\ B(\phi_t(z))\ F(\phi_t(z)) < 0 \text{ by assumptions (a) and (c).}$$

Hence if for some $t_0 \geq 0$, $F^T(\phi_{t_0}(z))\ G(\phi_{t_0}(z))\ F(\phi_{t_0}(z)) \leq 0$, then for all $t \geq t_0$, $F^T(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z)) \leq 0$. But for $t > T(z)$, $\phi_t(z) \in \bar{N}$, and hence, $F^T(\phi_t(z))\ B(\phi_t(z))\ F(\phi_t(z)) < -\varepsilon\ |F(\phi_t(z)|^2$.

Thus,

$$\frac{d}{dt} [F^T(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z))] = F^T(\phi_t(z))\ B(\phi_t(z))\ F(\phi_t(z))$$

$$< - \varepsilon\ |F(\phi_t(z))|^2 < - \eta\ |F^T(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z))| \text{ for some } \eta > 0.$$

Hence,

$$F^T(\phi_{t_0}(z))\ G(\phi_{t_0}(z))\ F(\phi_{t_0}(z)) < 0 \quad \text{for some } t_0 \text{ implies}$$

$$\lim_{t\to\infty} F^T(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z)) = - \infty$$

which is a contradiction to the boundedness of $\phi_t(z)$ and continuity of F and G. Suppose now $z \in \bar{A}$. Since $z = \lim_{k\to\infty} z_k$ for some $z_k \in A$, we must have $F^T(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z)) \geq 0$ for all $t \geq 0$. Also if $F^T(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z)) = 0$, again applying (*) we get $\frac{d}{dt} F(\phi_t(z))\ G(\phi_t(z))\ F(\phi_t(z)) < 0$, and hence, $F(\phi_{t+h}(z))\ G(\phi_{t+h}(z))\ F(\phi_{t+h}(z)) < 0$ for h small which is a contradiction.

<u>Lemma 4.2</u> - There exists $\bar{z} \in \partial A$ (∂A denotes the boundary of A), and $r > 0$ and C^1 function $\sigma: (-r,r) \to R^n$ such that $\sigma(p) \in A$ for all $p > 0$, $\sigma(0) = \bar{z}$, and $\sigma'(0)\ G(\bar{z})\ F(\bar{z}) = 0$.

Proof: If there exists $z \in \bar{A} - A$, then there exists $\bar{z} \in \partial A$ such that $d(\bar{z},0) = d_H(R^n - A, \{0\})$ where d_H denotes the Hausdorff distance. Hence for any $0 < p < 1$, $(1 - p)\ \bar{z} \in A$. Since, by Lemma 4.1, we have $G(\bar{z})\ F(\bar{z}) \neq 0$, therefore, the set $M = \{\bar{z} + x | x \perp G(\bar{z})\ F(\bar{z})\}$ is an n-1 dimensional affine subspace of R^n. Furthermore, $F(\bar{z}) \notin M$. Let t(p) be the "first time" at which the solution $\psi_t((1 - p)\bar{z})$ "hits" M; i.e., t(p) solves $\{\phi_{t(p)}[(1 - p)\bar{z}] - \bar{z}\}^T$ $G(\bar{z})\ F(\bar{z}) = 0$.

Let f: R x R → R be defined by

$$(t,p) \to \{\phi_t[(1-p)\bar{z}] - \bar{z}\}^T G(\bar{z}) F(\bar{z})$$

then,

$$f(0,0) = \{\phi_0[\bar{z}] - \bar{z}\}^T G(\bar{z}) F(\bar{z}) = 0$$

Furthermore,

$$\frac{df}{dt}(0,0) = F(\bar{z})^T G(\bar{z}) F(\bar{z}) \neq 0 \text{ by Lemma 4.1.}$$

Hence by the implicit function theorem, there exists $r > 0$ and a C^1 function $t(p)$ such that for $p \in (-r,r)$, $f(t,p) = 0$ iff $t = t(p)$. Let $\sigma(p) = \phi_{t(p)}[(1-p)\bar{z}]$. Then, $\sigma(p)$ is C^1, $\sigma(p) \notin A$ for $p > 0$, $\sigma(0) = \bar{z}$. Also $\sigma'(0)\, G(\bar{z})\, F(\bar{z}) = 0$ since $\sigma(p) \subset M$, M is an affine subspace, and by the definition of M.

Q.E.D.

From now on, we fix $u = \sigma'(0)$. Note that $F^T G(\bar{z})\, u = 0$. Associated with the curve $\sigma(p)$, there exists a two dimensional surface S that can be parametrized by $y(t,p) = \phi_t(\sigma(p))$, $|p| < r$, $0 \leq t < \infty$. Let us consider the differential equation transverse to the parameter arcs p = constant (i.e., the solutions of the original differential equation on S) determined by the relation $F^T G \frac{dy}{dp} = 0$ where $y = y(T,p)$ and $T = T(p)$. Let $y_p = \frac{\partial y}{\partial p}$ and $T = T(p,q)$ be the solution to

$$\text{(4.2)} \qquad \frac{dT}{dp} = -\frac{F^T G y_p}{F^T G F}, \quad T(0,q) = q$$

where all functions are calculated at $y(T,p)$. Note that by Lemma 4.1, we must have $F^T G F > 0$ for all $z \in \bar{A}$.

If a solution to (4.2) exists, we write $x(p,q) = y(T(p,q), p)$. Note that

$$\text{(4.3)} \qquad x_p = F\, T_p + y_p \text{ and } x_p^T G F = 0.$$

We may now state and prove certain preliminary results.

<u>Lemma 4.3</u> - There exists compact neighborhood $\bar{N}$ of 0 such that for $z \in \bar{N}$, and $z \neq 0$, $G(z)$ is positive definite.

Proof: As in the proof of Lemma 4.1, let $\bar{N}$ be a compact neighborhood of 0 such that if $z \in \bar{N}$, $v \neq 0$, $v\, B(z)\, v < -\varepsilon\, |v|^2$ and $\bigcup_{t=0}^{\infty} \phi_t(\bar{N}) \subseteq \bar{N}$. Clearly, we may choose $\bar{N} \subset A$, and hence, for $z \in \bar{N}$, $\lim_{t\to\infty} \phi_t(z) = 0$. For any $\mu \in R^n$, consider the variational system $\dot{y} = J(t)y$, $y(0) = \mu$ where $J(t)$ is the Jacobian of F evaluated at $\phi_t(z)$. Hence, $\lim_{t\to\infty} J(t) = J(0)$, and since by hypothesis $J(0)$ has all eigenvalues with negative real parts, $y(t) \to 0$ as $t \to \infty$. However,

$$\frac{d}{dt}\, y(t)\, G(\phi_t(z))\, y(t) = y(t)\, B(\phi_t(z))\, y(t) < -\varepsilon\, |y(t)|^2 < 0.$$

Hence if for some t, $y(t)\, G(\phi_t(z))\, y(t) \leq 0$, therefore, $\lim_{t\to\infty} y(t)\, G(\phi_t(z))\, y(t) < 0$ because $y(t)\, G(\phi_t(z))\, y(t)$ is decreasing in time. This contradicts $\lim_{t\to\infty} |y(t)| = 0$. Hence, $y(t)\, G(\phi_t(z))\, y(t) > 0$, for all $t \geqq 0$ and, in particular, $\mu\, G(z)\, \mu > 0$.

<u>Lemma 4.4</u> - Suppose $\tilde{z} \in A$, $\tilde{z} \neq 0$. Then for all $\mu \neq 0$ such that $\mu^T\, G(\tilde{z})\, F(\tilde{z}) = 0$ we must have $\mu^T\, G(\tilde{z})\, \mu > 0$.

Proof: Since $\tilde{z} \in A$, there exists $\tilde{t}$ such that $\phi_{\tilde{t}}(\tilde{z}) \in \bar{N}$. Consider the line segment $\Pi = \tilde{z} + \tau\, \mu$, $-\underline{\tau} < \tau < \bar{\tau}$. Since A is open, for $\underline{\tau}$ and $\bar{\tau}$ sufficiently close to zero, $\Pi \subset A$.

Let $S(\tilde{z}) = \{w \in R^n | w = y(t,z),\ z \in \Pi,\ t \geq 0\}$. Let $T(\tau,q)$ solve:

$$T_\tau = \frac{-F^T\, G\, y_\tau}{F^T\, G\, F} \tag{4.4}$$

$$T(0,q) = q$$

This differential equation is well defined since $F^T\, G\, F > 0$ by Lemma 4.1. For $0 \leq q < 2\,\tilde{t}$, (4.4) has a solution $T(\tau,q)$ for $0 \leq \tau < \tau'$ for some $\tau' > 0$. (cf. Hale [5] Theorem 3.1, page 18).

For $0 \leq \tau < \tau'$ and $0 \leq q < 2\,\tilde{t}$, write $v(\tau,q) = y(T(\tau,q),\tau)$. Note that $v_\tau = F\,T_\tau + y_\tau$, $v_\tau\,G\,F = 0$, and $v_\tau(0,0) = y_\tau(0,0) = \mu$, since

$$T_\tau(0,0) = \frac{-F^T\,G\mu}{F^T\,G\,F} = 0,$$

hence,

$$\frac{d}{dq}\; v_\tau^T\,(\tau,q)\;G(v(\tau,q))\;v_\tau(\tau,q) = T_q\;v_\tau^T(\tau,q)\;B(v(\tau,q))\;v_\tau(\tau,q) < 0 \tag{4.5}$$

By assumption (c) of Theorem 4.1 and because $T_q(\tau,q)$ solves a linear differential equation with $T_q(0,q) = 1$. If $\mu\,G(\tilde{z})\,\mu \equiv v_\tau^T(0,0)\;G(v(0,0))\;v_\tau(0,0) \leq 0$, we must have by (4.5) that

$$v_\tau^T(0,q)\;G(v(0,q))\;v_\tau(0,q) < 0 \quad \text{for all } q \geq 0$$

In particular, $v_\tau^T(0,\tilde{t})\;G(v(0,\tilde{t}))\;v_\tau(0,\tilde{t}) < 0$, but this contradicts the fact that $v(0,\tilde{t}) \equiv y(\tilde{t},\tilde{z}) \in \bar{N}$, and G is positive definite in $\bar{N}$.

<u>Corollary 4.1</u> - Suppose $\tilde{z} \in \bar{A}$, $\tilde{z} \neq 0$ and $\mu^T\,G(\tilde{z})\,F(\tilde{z}) = 0$. Then, $\mu^T\,G(\tilde{z})\,\mu > 0$.

Proof: By making the same construction as before, we define $\tilde{v}(\tau,q)$. Suppose $\mu^T\,G(\tilde{z})\,\mu \leq 0$, then for q small, $\tilde{v}_\tau^T(0,q)\;G(\tilde{v}(0,q))\;\tilde{v}_\tau(0,q) < 0$ and

$\tilde{v}_\tau^T(0,q)\;G(\tilde{v}(0,q))\;F(\tilde{v}(0,q)) = 0$. Since $\bar{A}$ is invariant, $\tilde{v}(0,q) \in \bar{A}$. Also since G and F are continuous functions, and by Lemma 4.1, $G(\tilde{v}(0,q))\;F(\tilde{v}(0,q)) \neq 0$, we can construct sequence $z_n \to \tilde{v}(0,q)$, $z_n \in A$, and sequence $\lambda_n \to v_\tau(0,q)$ such that $\lambda_n^T\,G(z_n)\;F(z_n) = 0$. Hence by Lemma 4.4, $(\lambda_n)^T\,G(z_n)\;F(z_n) < 0$, and hence,

$\tilde{v}_\tau(0,q)\;G(\tilde{v}(0,q))\;\tilde{v}_\tau(0,q) \leq 0$ which is a contradiction.

From now on, let $\bar{\bar{N}} \subset \bar{N}$ be a fixed compact invariant neighborhood of zero. We can now show that, in fact, G acts like a Riemannian metric on the directions orthogonal to GF.

<u>Lemma 4.5</u> - Let $q_0 = \sup\,\{q/T(p,q)$ is defined for all $0 \leq p \leq \beta$ and $x(p,q) \in \bar{\bar{N}}\}$. Then, there exists $\varepsilon > 0$ such that $|x_p(p,q)|^2 \leq \varepsilon\; x_p^T(p,q)\;G(x(p,q))\;x_p(p,q)$ for all $0 \leq p \leq \beta$, $0 \leq q < q_0$.

Proof: Suppose not. Then, there are sequences $\{p_n\}_{n=0}^{\infty}$ and $\{q_n\}_{n=0}^{\infty}$ such that

$$\lim_{n\to\infty} \frac{x_p^T(p_n,q_n)\ G(x(p_n,q_n))\ x_p(p_n,q_n)}{|x_p(p_n,q_n)|^2} = 0$$

Since by assumption (d) of Theorem 4.1, $x(p_n,q_n)$ is in a bounded set contained in $\bar{A}$, there is a subsequence which we still denote by (p_n,q_n) such that

$$\lim_{n\to\infty} x(p_n,q_n) = \tilde{z} \in \bar{A}.$$

Also, since

$$\left| \frac{x_p(p_n,q_n)}{|x_p(q_n,q_n)|} \right| = 1,$$

there exists yet another subsequence, that we still denote by (p_n,q_n), such that

$$\lim_{n\to\infty} \frac{x_p(p_n,q_n)}{|x_p(p_n,q_n)|} = v$$

Hence, $v^T\ G(\tilde{z})\ v = 0$. Also, $x_p^T(p_n,q_n)\ G(x(p_n,q_n))\ F(x(p_n,q_n)) = 0$ implies $v^T\ G(\tilde{z})\ F(\tilde{z}) = 0$. But this contradicts Corollary 4.1.

Q.E.D.

The fact that $x_p^T\ G\ x_p > \varepsilon\ |x_p|^2$ allows us to use

$$\left(\frac{1}{\varepsilon}\right)^{1/2} \int_o^{\beta} [x_p^T(p,q)\ G(x(p,q))\ x_p(p,q)]^{1/2}\ dp$$

as an upper bound to the arc length distance $d(x(\beta,q),\ x(o,q)) \equiv \int_o^{\beta} |x_p(p,q)|\ dp$ between $x(\beta,q)$ and $x(0,q)$. The proof of Theorem 4.1 from now on is exactly Hartman and Olech's. Thus, we simply give a sketch in the next few lines. If $\ell(p) = \sup\ \{t/y(t,p) \notin \bar{\bar{N}}\}$, then one can show that $\lim_{q\to q_0} T(p,q) = \ell(p)$, where the q_0 is the same as in the statement of the preceding lemma. One then uses the fact that

$$\frac{d}{dq} \; x_p^T (p,q) \; G(x(p,q)) \; x_p(p,q) < 0$$

to show that

$$d(x(p,q), \; x(0,q)) \leq \left(\frac{1}{\varepsilon}\right)^{1/2} \int_o^{\bar{p}} [x_p^T(p,0) \; G(x(p,0)) \; x_p(p,0)]^{1/2} dp$$

for $0 \leq \bar{p} \leq \beta$.

Hence,

$$\lim_{q \to q_0} d(x(\bar{p},q), \; x(0,q)) \leq \left(\frac{1}{\varepsilon}\right)^{1/2} \int_o^{\bar{p}} [x_p^T(p,0) \; G(x(p,0)) \; x_p(p,0)]^{1/2} dp$$

Since

$$|x_p^T(p,0) \; G(x(p,0)) \; x_p(p,0)| < K$$

for some $K > 0$ for $0 \leq p \leq \beta$, by choosing $\bar{p}$ small enough one can show that

$$\lim_{q \to q_0} d(x(p,q), \; x(0,q)) < d_H(\bar{\bar{N}}, \; \partial\bar{N})$$

where $d_H(\bar{\bar{N}}, \partial\bar{N})$ is the Hausdorff distance between $\bar{\bar{N}}$ and the boundary of $\bar{N}$. Hence,

$$\lim_{q \to q_0} x(0,q) \in \bar{N} \subset A, \text{ and hence, } x(0,q) \in A \text{ for q near } q_0.$$

This contradicts the assumption that $x(0,0) \notin A$. Theorem 4.1 is established.

Section 5

The Hartman and Olech Method for M.H.D.S.

In this section we show how the modification of Hartman and Olech's result proved in section 4 can be used to provide sufficiency conditions for global asymptotic stability of optimal solutions to control problems.

Consider the M.H.D.S.

$$\dot{q} = \rho q - H_2(q,k) \tag{5.1}$$

$$\dot{k} = H_1(q,k)$$

We will use the following assumption in what follows.

Assumption 5.1 - There exists just one rest point $(\bar{q},\bar{k})$ for (5.1). Put $z = (q,k) - (\bar{q},\bar{k})$ and rewrite (5.1) in the form

$$\dot{z}_1 = \rho(\bar{q} + z_1) - H_2(z) \equiv F_1(z) \tag{5.2}$$

$$\dot{z}_2 = H_1(z) \equiv F_2(z)$$

By abuse of notation, write $H(z) = H((\bar{q}, \bar{k}) + z)$.

In many optimal control problems, the variable q - called the co-state variable - can be written as a function $q = g(k)$. The function g is related to what is sometimes called a synthesizing function. See Lee and Markus [11] for an excellent discussion of optimal control problems.

Usually the analyst knows enough about his problem to establish things like (a) g(k) exists and is continuously differentiable; a.e., in k, (b) for any $k_0 \in R^n$ if $k(t|k_0)$ is an optimal path of the state variable with initial condition k_0, then $k(t|k_0)$ is bounded, (c) g(k) is the derivative of the value function W(k).

(Here W(k) equals the maximum value of the problem starting at time 0, initial condition k and having an infinite horizon), and (d) the value W solves a partial differential equation of Hamilton-Jacobi type.

Furtheremore, it is usually straightforward to do a local analysis of g(k) at rest points. But for arbitrary k it almost requires "solving the problem first" to verify things like detailed sign restrictions on objects like the Jacobian matrix of g, for example, other than general properties such as negative semi-definiteness, for example.

For many applications it turns out that $k(t|k_0)$ is bounded for all $t \geqq 0$ for all k_0. Also, except for "borderline cases" existence theory in optimal control leads to the following assumption of local asymptotic stability of a rest point in a natural way.

Assumption 5.2 - The rest point $0 \in R^{2n}$ of (5.2) is locally stable in the following sense: the linearization of

$$\dot{z}_2 = F_2(g(z_2), z_2) \tag{5.3}$$

at $z_2 = 0$ has all eigenvalues with negative real parts.

Note the abuse of notation in using g to denote the synthesis function in z coordinates when it is also used to denote the synthesis function in (q,k) coordinates. We will do this when the context is clear. The boundedness property that is naturally reflected in many applied problems and is essential for our method is

Assumption 5.3 - For all z_{20} the solution $\phi_t^2(z_{20})$ of (5.3) is bounded independent of $t \geqq 0$.

Let $\phi_t(z_0)$ denote the solution of (5.2) with initial condition z_0. We will always be working with solutions that are bounded independently of $t \geqq 0$. The synthesis function g is assumed to characterize bounded trajectories in the following sense

Assumption 5.4[6/] - There is continuously differentiable g with $g(0) = 0$ such that for all z_0, $\phi_t(z_0)$ is bounded independently of $t \geqq 0$ iff $\phi_t^1(z_0) = g[\phi_t^2(z_0)]$ for all $t \geqq 0$.

Here ϕ_t^1 denotes the first n coordinates of ϕ_t and ϕ_t^2 denotes the last n coordinates of ϕ_t. Assumption 5.4 allows us to reduce the study of optimal paths to the study of (5.3).

The set of z such that $z = (g(z_2), z_2)$ is a differentiable n dimensional manifold embedded in R^{2n} - call it λ. λ is positively invariant in the sense that $\phi_t(z_0) \in \lambda$ for all $t \geqq 0$, for all $z_0 \in \lambda$. We will also need

Assumption 5.5[7/] - Let K be a compact subset of λ. Then, $\bigcup_{z \in K} \bigcup_{t \geq 0} \phi_t(z)$ is bounded.

Let α: $R^{2n} \to M^S(R^{2n}, R^{2n})$ be a C^1 function. Let

$$B(z) = [(\alpha(z)\ J(z))^T + \alpha(z)\ J(z) + \dot{\alpha}(z)]$$

where J is the Jacobian of F and

$$\dot{\alpha} = \sum_{r=1}^{2n} \frac{\partial \alpha}{\partial z_r} F_r$$

which is the trajectory derivative of the matrix $\alpha(z)$.

We may now state the following theorem:

Theorem 5.1 - Suppose (5.2) obeys assumptions 5.1 to 5.5, and furthermore, the following property holds:

(*) For all $w_2 \in R^n$, $w_2 \neq 0$ for all $z \in R^{2n}$, we have $(J_g\ w_2, w_2)^T \alpha(z)\ F(z) = 0$

implies $(J_g\ w_2, w_2)^T\ B(z)\ (J_g\ w_2, w_2) > 0$.

Then all trajectories in the bounded manifold λ converge to 0 as $t \to \infty$.

We will prove this theorem by showing that the differential equation (5.3) satisfies the hypotheses of Theorem 4.1. Before we do that, however, we will make some comments on assumption (*).

Although assumption (*) uses information on the Jacobian matrix J_g of g, it is obviously implied by:

(*') For all z, $w^T a(z)\ F(z) = 0$ implies $w^T B(z)\ w > 0$ for all $w \neq 0$ which, of course, uses no information on g.

Also, the case

$$a(z) \equiv \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

is the one studied in Section 3. For then, $w^T\ B(z)w = w^T(2Q(z))w$.

In order to prove Theorem 5.1, it is clearly sufficient to prove the following lemma, whose proof is exceedingly elementary and tedious.

<u>Lemma</u>:[8/] Consider the differential equation (5.2) in R^{2n}. Let $g: R^n \to R^n$ be a "synthesis" function; i.e., if $q_o = g(k_o)$, then $q(t|k_o) = g(k(t|k_o))$ for all t where $(q(t|q_o), k(t|k_o)$ solves (5.2). Then if there exists symmetric matrix function $a(z)$ such that $w^T\ a(z)\ F(z) = 0 \Rightarrow w^T(a\ J + (aJ)^T + \dot{a})\ w > 0$ for all w of the form $w = (w_1, w_2)$, $w_1 = J_g\ w_2$, for all $z = (z_1, z_2)$ where $z_1 = g(z_2)$, where J is the Jacobian of F, and $\dot{a}$ is the trajectory derivative of a, and if

$$G = -(J_g^T\ a_{11}\ J_g + J_g^T\ a_{12} + a_{12}^T\ J_g + a_{22}),$$

then $w_2^T\ G\ F_2 = 0$ implies,

$$w_2^T\ (G\ \frac{dF_2}{dk} + (G\ \frac{dF_2}{dk})^T + \dot{G})\ w_2 < 0$$

where

$$\frac{dF_2}{dk} = \frac{d}{dk}\ F_2(g(k), k).$$

Proof: First note that

$$(J_g\ w_2,\ w_2)^T \begin{bmatrix} a_{11} & a_{12} \\ a_{12}^T & a_{22} \end{bmatrix} (J_g\ F_2,\ F_2) = 0$$

iff,

$$w_2(J_g^T\ a_{11}\ J_g + J_g^T\ a_{12} + a_{12}^T\ J_g + a_{22})\ F_2 = 0,$$

and hence,

by definition of G, iff $w_2\ G\ F_2 = 0$. We must show that

$$(J_g\ w_2,\ w_2)^T(\alpha\ J + (\alpha J)^T + \dot{\alpha})\ (J_g\ w_2,\ w_2) > 0$$

implies

$$w_2^T\ (G\ J_n + (G\ J_n)^T + \dot{G})\ w_2 < 0 \text{ where } J_n = \frac{dF_2}{dk} \equiv F_{21}\ J_g + F_{22}.$$

But, $(J_g\ w_2,\ w_2)^T\ (\alpha J + (\alpha J)^T + \dot{\alpha})(J_g\ w_2, w_2)$

$$= \begin{pmatrix} J_g\ w_2 \\ w_2 \end{pmatrix}^T \left[\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12}^T & \alpha_{22} \end{pmatrix}\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} + \begin{pmatrix} F_{11}^T & F_{21}^T \\ F_{12}^T & F_{22}^T \end{pmatrix}\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12}^T & \alpha_{22} \end{pmatrix} + \begin{pmatrix} \dot{\alpha}_{11} & \dot{\alpha}_{12} \\ \dot{\alpha}_{12}^T & \dot{\alpha}_{22} \end{pmatrix}\right]\begin{pmatrix} J_g\ w_2 \\ w_2 \end{pmatrix}$$

$$= \begin{pmatrix} J_g\ w_2 \\ w_2 \end{pmatrix}^T \left[\begin{pmatrix} \alpha_{11}\ F_{11} + \alpha_{12}\ F_{21} & \alpha_{11}\ F_{12} + \alpha_{12}\ F_{22} \\ \alpha_{12}^T\ F_{11} \quad \alpha_{22}\ F_{21} & \alpha_{12}^T\ F_{12} \quad \alpha_{22}\ F_{22} \end{pmatrix} + \begin{pmatrix} F_{11}^T\ \alpha_{11} + F_{21}^T\ \alpha_{12}^T & F_{11}^T\ \alpha_{12} + F_{21}^T\ \alpha_{22} \\ F_{12}^T\ \alpha_{11} \quad F_{22}^T\ \alpha_{12}^T & F_{12}^T\ \alpha_{12} \quad F_{22}^T\ \alpha_{22} \end{pmatrix} + \begin{pmatrix} \dot{\alpha}_{11} & \dot{\alpha}_{12} \\ \dot{\alpha}_{12}^T & \dot{\alpha}_{22} \end{pmatrix}\right]\begin{pmatrix} J_g\ w_2 \\ w_2 \end{pmatrix}$$

(5.4)
$$= w_2^T [J_g^T(a_{11} F_{11} + a_{12} F_{21} + F_{11}^T a_{11} + F_{21}^T a_{12}^T + \dot{a}_{11}) J_g$$

$$+ J_g^T(a_{11} F_{12} + a_{12} F_{22} + F_{11}^T a_{12} + F_{21}^T a_{22} + \dot{a}_{12})$$

$$+ (a_{12}^T F_{11} + a_{22} F_{21} + F_{12}^T a_{11} + F_{22}^T a_{12}^T + \dot{a}_{22}^T) J_g$$

$$+ (a_{12}^T F_{12} + a_{22} F_{22} + F_{12}^T a_{12} + F_{22}^T a_{22} + \dot{a}_{22})] w_2.$$

Also, using $J_n = F_{21} J_g + F_{22}$ we have

(5.5)
$$w_2^T (G J_n + (G J_n)^T + \dot{G}) w_2$$

$$\equiv w_2^T (J_g^T a_{11} J_g F_{21} J_g + J_g^T a_{11} J_g F_{22} + J_g^T a_{12} F_{21} J_g$$

$$+ J_g^T a_{12} F_{22} + a_{12}^T J_g F_{21} J_g + a_{12}^T J_g F_{22} + a_{22} F_{21} J_g$$

$$+ a_{22} F_{22} + J_g^T F_{21}^T J_g^T a_{11} J_g + F_{22}^T J_g^T a_{11} J_g + J_g^T F_{21}^T a_{12}^T J_g$$

$$+ F_{22}^T a_{12}^T J_g + J_g^T F_{21}^T J_g^T a_{12} + F_{21}^T J_g^T a_{12} + J_g^T F_{21}^T a_{22} + F_{22}^T a_{22}$$

$$+ J_g^T a_{11} J_g + J_g^T \dot{a}_{11} + J_g + J_g^T a_{11} \dot{J}_g + \dot{J}_g^T a_{12}$$

$$+ J_g^T \dot{a}_{12} + \dot{a}_{12}^T J_g + a_{12}^T \dot{J}_g + \dot{a}_{22})w_2$$

and rearranging we get

(5.6)
$$w_2^T [J_g^T a_{11}(J_g F_{21} J_g + J_g F_{22} + \dot{J}_g)(J_g^T F_{21}^T J_g^T + F_{22}^T J_g^T + \dot{J}_g^T)a_{11} J_g$$

$$+ J_g^T a_{12}(F_{21} J_g + F_{22}) + (J_g^T F_{21}^T + F_{22}^T) a_{12}^T J_g$$

$$+ \alpha_{12}^T(J_g\ F_{21}\ J_g + J_g\ F_{22} + \dot{J}_g) + (J_g{}^T\ F_{21}^T\ J_g{}^T + F_{22}^T\ J_g{}^T + \dot{J}_g{}^T)\ \alpha_{12}$$

$$+ \alpha_{22}(F_{21}\ J_g + F_{22}) + (J_g{}^T\ F_{21}^T + F_{22}^T)\ \alpha_{22} + J_g{}^T\ \dot{\alpha}_{11}\ J_g$$

$$+ J_g{}^T\ \dot{\alpha}_{12} + \dot{\alpha}_{12}^T\ J_g + \dot{\alpha}_{22})w_2$$

Note that

$$\text{(5.7)} \qquad \dot{q} = F_1(q(k),k) \equiv J_g(k)\ F_2(q(k),k) \text{ and hence,}$$

$$F_{11}\ J_g + F_{12} \equiv (D_k\ J_g)\ F_2 + J_g\ F_{21}\ J_g + J_g\ F_{22}$$

$$\equiv \dot{J}_g + J_g\ F_{21}\ J_g + J_g\ F_{22}$$

and hence,

$$\text{(5.8)} \qquad J_g{}^T\ F_{11}^T + F_{12}^T = \dot{J}_g{}^T + J_g{}^T\ F_{21}^T\ J_g{}^T + F_{22}^T\ J_g{}^T$$

substituting (5.7) and (5.8) into (5.4), we obtain

$$(J_g\ w_2,\ w_2)^T\ (\alpha\ J + (\alpha J)^T + \dot{\alpha})\ (J_g\ w_2,\ w_2)$$

$$\equiv w_2\ [J_g{}^T\ \alpha_{11}(\dot{J}_g + J_g\ F_{21}\ J_g + J_g\ F_{22})$$

$$+ (\dot{J}_g{}^T + J_g{}^T\ F_{21}^T\ J_g{}^T + F_{21}^T\ J_g{}^T)\ \alpha_{11}\ J_g$$

$$+ J_g{}^T\ \alpha_{12}(F_{21}\ J_g + F_{22}) + (J_g{}^T\ F_{21}^T + F_{22}^T)\ \alpha_{12}^T\ J_g$$

$$+ \alpha_{12}^T(\dot{J}_g + J_g\ F_{21}\ J_g + J_g\ F_{22}) + (\dot{J}_g{}^T\ J_g{}^T\ F_{21}^T\ J_g{}^T + F_{22}^T\ J_g{}^T)\ \alpha_{12}$$

$$+ \alpha_{22}(F_{21}\ J_g + F_{22}) + (J_g{}^T\ F_{21}^T + F_{22}^T)\ \alpha_{22}$$

$$+ J_g^T \; \dot{\alpha}_{11} J_g + J_g^T \; \dot{\alpha}_{12} + \dot{\alpha}_{12}^T J_g + \dot{\alpha}_{22}] \, w_2$$

i.e., (5.6). Q.E.D.

This identity shows us that assumption (c) of Theorem 4.1 is the same as assumption (*) here. Theorem 5.1 now follows. This is so because from Theorem 4.1 it follows that given any z_2, the solution $\phi_t^2(z_2)$ to $\dot{z}_2 = F_2(g(z_2), z_2)$ is such that $\lim_{t\to\infty} \phi_t^2(z_2) = 0$. Hence, $\lim_{t\to\infty} g(\phi_t^2(z_2)) = g(0) = 0$.

Theorem 5.1 allows us to prove stability theorems for quasi concave Hamiltonians.

<u>Corollary 5.1</u> - Assume the hypotheses 5.1 to 5.5 of Theorem 5.1, and that J_g is symmetric. Suppose that the Hamiltonian function satisfies: for all z of the form $z_1 = g(k) - g(\bar{k})$, $z_2 = k - \bar{k}$, $z = (z_1, z_2)$

(i) $c_1 H_{11}(z) c_1 \geqq \alpha |c_1|^2$

for all $c_1 \neq 0$ such that $c_1 H_1(z) = 0$

(ii) $c_2 [-H_{22}(z)] c_2 \geqq \beta |c_2|$

for all $c_2 \neq 0$ such that $c_2 [H_2(z) - \rho(z_1 + \bar{q})] = 0$

for some (α,β) in R^2 with $\alpha\beta > \rho^2/4$. Then global asymptotic stability of bounded trajectories holds.

Proof: Since in Theorem 5.1 z(t) is a bounded trajectory and, thus, $\dot{z}_1 = J_g \dot{z}_2$ we have

$$\dot{z}_1 = F_1(z) = \rho(z_1 + \bar{q}) - H_2(z) = J_g F_2(z) = J_g H_1(z).$$

Since

$c_1 = J_g c_2$, therefore,

$c_1^T F_2[z(t)] + c_2^T F_1[z(t)] = 0$

iff

$$(J_g c_2)^T F_2 + c_2^T J_g F_2 = 0.$$

Hence, assumption (*) of Theorem 5.1 holds for

$$\alpha(z) \equiv \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

where I is the n x n identity matrix iff

$$c_2^T F_1 = c_1^T F_2 = 0 \Rightarrow c\, Q\, c > 0.$$

By (i) and (ii) it follows by direct computation that

$$c\, Q[z(t)]\, c > 0 \quad \text{for all c satisfing}$$

$$c_2^T F_1 = c_1^T F_2 = 0.$$

Hence, GAS must hold.

FOOTNOTES

1/ Consider, for example, the problem

$$\text{maximize} \int_0^\infty e^{-\rho t} U(k,x)dt$$

$$\text{s.t. } \dot{k} = x,\ k(0) = k_0$$

This problem generates a system of the form (1.2). The Lee and Markus infinite horizon convex cost criteria problems [11, chapter 3] are of this form.

2/ Our hypothesis is, basically, some kind of curvature hypothesis on the "present value Hamiltonian" $G(q,k)$. A sufficient condition for it to hold is that the product $\lambda_1 \lambda_2$ of the smallest eigenvalues, λ_1, λ_2, of the symmetric matrices G_{qq}, $-G_{kk}$, respectively, along the optimum path be greater than $\rho^2/4$. This condition has considerable intuitive appeal. It is the basic stability hypothesis of Rockafellar [15].

3/ Actually, assumption (c) below applied to the rest point 0 can be used to show that if $\dot{z} = F(z)$ is locally asymptotically stable then all roots must have negative real parts. Assumption (b) is written here in this way to better separate the ideas used in the proof.

4/ Here $M^S(R^n,R^n)$ denotes the class of n x n symmetric matrices.

5/ $B(0,\delta) = \{z \in R^n \mid |z| \leq \delta\}$.

6/ Intuitively speaking, Assumption 5.4 amounts to existence of a C^2 value function W. Note that g is just the gradient of W when the gradient exists. Assumption 5.3 asks that the value W possess a bounded gradient.

7/ This assumption is relatively benign for economic problems. All that it means is that K remains bounded (infinite capital is not accumulated) and the value possess bounded derivative on compact sets.

8/ This Lemma does a simple but tedious job. Think of g as the derivative of the value function of an optimal control problem. Then,

$$(i) \qquad F_1(z) = \dot{z}_1 = \frac{d}{dt} g(z_2) = J_g(z_2)\, \dot{z}_2 = J_g(z_2)\, F_2$$

Using (i), the Lemma shows us (by a long computation) that the assumption:

$$w^T \alpha(z) F(z) = 0 \Rightarrow w^T (\alpha J + (\alpha J)^T + \dot{\alpha}) w > 0$$

for all $w = (J_g w_2, w_2) \neq 0$, for all $z = (g(z_2), z_2)$ translates into the assumption: there is a symmetric matrix $G(z)$ such that for all z_2, for all $w_2 \neq 0$

$$w_2^T G F_2 = 0 \Rightarrow w_2^T \left(G \frac{dF_2}{dk} + \left(G \frac{dF_2}{dk}\right)^T + \dot{G}\right) w_2 < 0.$$

This last allows Theorem 4.1 to be applied directly.

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