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Discussion Paper #1489
August 1, 2010

Preference for Randomization

Ambiguity Aversion and Inequality Aversion

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JEL classification: D81, D03

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Preference for Randomization*

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Abstract

In Anscombe and Aumann's (1963) domain, there are two types of mixtures. One is an *ex-ante mixture*, or a lottery on acts. The other is an *ex-post mixture*, or a state-wise mixture of acts. These two mixtures have been assumed to be indifferent under the *Reversal of Order axiom*. However, we argue that the difference between these two mixtures is crucial in some important contexts. Under *ambiguity aversion*, an ex-ante mixture could provide only *ex-ante hedging* but not *ex-post hedging*. Under

*The part of this paper on inequality aversion was originally circulated as Saito, K. (January 25, 2008) "Social Preference under Uncertainty," the University of Tokyo, COE Discussion Papers, No. F-217. The original paper was presented firstly on January 16, 2007 at Micro-workshop seminar in the University of Tokyo.

†I am indebted to my adviser Eddie Dekel for continuous guidance, support, and encouragement. I am grateful to Marciano Siniscalchi, Kyuongwon Seo, Peter Klibanoff, Jeff Ely, and Faruk Gul for many discussions which have led to the improvement of the paper. I would like to thank to my former adviser Michihiro Kandori for his guidance when I was at the University of Tokyo. I would also like to thank Massimo Marinacci, Fabio Maccheroni, Larry Epstein, Itzhak Gilboa, Elchanan Ben-Porath, Thibault Gajdos, Soo Hong Chew, Jacob S. Sagi, Uzi Segal, Edi Karni, Jürgen Eichberger, David Kelsey, Tomasz Strzalecki, Philippe Mongin, Pierpaolo Battigalli, John E. Roemer, Costis Skiadas, Ernst Fehr, Colin Camerer, Kenneth Binmore, James Andreoni, Alvaro Sandroni, Adam Dominiak, Nabil Al-Najjar, Christoph Kuzumics, Daisuke Nakajima, Kazuya Kamiya, Akihiko Matsui, Hitoshi Matsushima, and seminar participants at the University of Tokyo, Northwestern University, and RUD 2010 in Paris. I gratefully acknowledge financial support from the Center for Economic Theory of the Economics Department of Northwestern University.

inequality aversion, an ex-ante mixture could provide only *ex-ante equality* but not *ex-post equality*. We provide a unified framework that treats a preference for ex-ante mixtures separately from a preference for ex-post mixtures. In particular, two representations are characterized for each context. One representation for ambiguity aversion is an extension of Gilboa and Schmeidler's (1989) *Maxmin preferences*. The other representation for inequality aversion is an extension of Fehr and Schmidt's (1999) *Piecewise preferences*. In both representations, a single parameter characterizes a preference for ex-ante mixtures. For both representations, instead of the Reversal of Order axiom, we propose a weaker axiom, the *Indifference axiom*, which is a criterion, suggested in Raiffa's (1961) critique, for evaluating lotteries on acts. These models are consistent with much recent experimental evidence in each context.

KEYWORDS: Ambiguity; randomization; Ellsberg paradox; other-regarding preferences; inequality; maxmin utility.

JEL Classification Numbers: D81, D03.

1 Introduction

This paper investigates a *preference for randomization*. People exhibit such a preference as a form of *hedging* because of *ambiguity aversion*, as Raiffa (1961) suggests in his famous critique. Indeed, Dwenger, Kübler, and Weizsäcker (2010) have found such a preference in a field experiment. In addition, in a social context, people exhibit such a preference because of *inequality aversion*, as in the case of "Machina's (1989) mom" who prefers flipping a coin to decide how to allocate an indivisible good among her children.[1] Indeed, in some jurisdictions, a coin is flipped to decide between two candidates who obtain equal number of voters in an election and between two companies tendering equal prices for a project.[2]

Despite its importance, little work has been done on this preference for randomization. Recently, however, experimental researchers have begun to study such a preference in the

[1]Diamond (1967) proposes a similar argument for this preference for randomization.
[2]See Samaha (2010) for details.

contexts of both ambiguity and inequality aversion. [3] One important observation drawn from such experimental studies is that *timing* of randomization matters. The purpose of the present paper is to provide an axiomatic model that characterizes such a preference in both contexts in a unified way.

In one sense, the seminal paper by Anscombe and Aumann (1963) addresses the issue of timing of randomization. They consider two types of randomization depending on timing. One is an *ex-ante mixture*, or a lottery on payoff profiles, which is a randomization *before* a state realizes. For example, P in Figure 1 is the fifty-fifty ex-ante mixture of $(\$100, \$0)$ and $(\$0, \$100)$. This type of mixture is henceforth indicated by $\oplus$. The other is an *ex-*

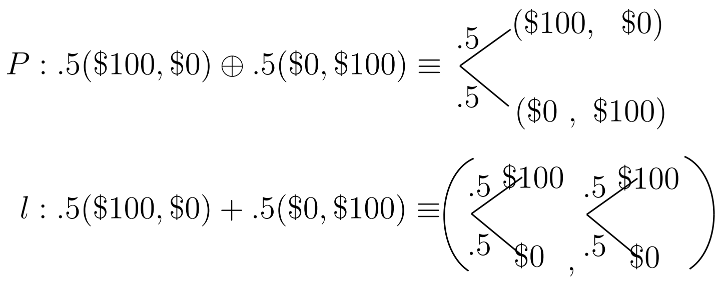

Figure 1: Ex-ante Mixture P and Ex-post Mixture l

post mixture, or a state-wise mixture of payoff profiles, which is a randomization *after* a state realizes. For example, l in Figure 1 is the fifty-fifty ex-post mixture of $(\$100, \$0)$ and $(\$0, \$100)$. This type of mixture is henceforth indicated by $+$, as is conventional literature.

Under each context, the difference between the mixtures is crucial, as will be explained in detail later: under *ambiguity aversion*, an ex-ante mixture could provide only *ex-ante hedging* but not *ex-post hedging.* This could be the reason why, in some experiments, some people, who prefer ex-post mixtures, are averse to ex-ante mixtures. Also, under *inequality aversion*, an ex-ante mixture could provide only *ex-ante equality* but not *ex-post equality.*

However, in Anscombe and Aumann (1963), the *Reversal of Order axiom* implies that an ex-ante mixture is indifferent with its ex-post mixture, i.e., $\alpha f \oplus (1-\alpha) g \sim \alpha f + (1-\alpha) g$

[3]For instance, in the context of ambiguity aversion, see Dominiak and Schnedler (2009), Spears (2009), and Dwenger et al. (2010). For examples, in the context of inequality aversion associated with other-regarding preferences, see Bolton and Ockenfels (forthcoming), Krawczyk and Le Lec (2008), and Kircher, Luding, and Sandroni (2009). These experiments are discussed in detail in Section 2

for any payoff profiles f and g, and, $\alpha \in [0,1]$. Hence, this axiom precludes the study of a preference for ex-ante mixtures separately from a preference for ex-post mixtures.

For the above reason, we do not assume the Reversal of Order axiom. Instead, we propose a new weaker axiom, the *Indifference axiom*. To see the difference between these axioms, notice that one way to justify the Reversal of Order axiom is by a *state-wise* evaluation: if you look at P state-wise, it offers the same lottery as l. As will be explained in Section 1.1, the criterion has been implicitly used by Raiffa (1961) in his famous critique of ambiguity aversion.

There is, however, another natural comparison between P and l: if you look at each payoff profile in the support of P, it offers nonconstant payoff profiles, namely $(\$100, \$0)$ and $(\$0, \$100)$, which would be less attractive than the constant payoff profile l under ambiguity aversion as well as under inequality aversion;[4] this way of evaluation is called *support-wise* evaluation. The Indifference axiom states that if two lotteries on payoff profiles are indifferent according to *both the state-wise and the support-wise* criteria, then the lotteries should be indifferent.

Using the Indifference axiom together with standard axioms, two preferences are characterized for each context respectively: for ambiguity aversion, we axiomatize *EAP Maxmin preference* shown as (2) in Section 1.1, which is an extension of Gilboa and Schmeidler's (1989) *Maxmin preferences*. For inequality aversion, on the other hand, we axiomatize *EAP Piecewise preferences* shown as (3) in Section 1.2, which is an extension of Fehr and Schmidt's (1999) *Piecewise preferences*.

Both representations have a similar structure. To see this, let P be any ex-ante mixture with finite support. Then, $P = P(f^1)f^1 \oplus \cdots \oplus P(f^n)f^n$, where $P(f^i)$ is the probability assigned to payoff profile f^i by P. Then, for each state s, the marginal distribution of P on s is $P_s = P(f^1)f^1_s + \cdots + P(f^n)f^n_s$, where f^i_s is the payoff at s in f^i. Hence, the payoff profile $(P_s)_s$, which offers P_s at each state s, summarizes an *ex-ante evaluation* of P before

[4] In a social context, under which inequality aversion matters, states are reinterpreted as individuals. So, the nonconstant payoff profiles entail ex-post inequality.

a state realizes. Given the notations, the general representation can be presented as follows:

$$V(P) = \delta\, U\big((P_s)_s\big) + (1-\delta)\int_{\mathscr{F}} U(f)dP(f), \tag{1}$$

where δ is a real number and U is a real-valued function on the set $\mathscr{F}$ of payoff profiles.

In the representation (1), the function U captures a preference for ex-post mixtures as in Maxmin preferences or in Piecewise preferences. Given U, on the other hand, the real number δ captures a preference for ex-ante mixtures in each context, as will be explained in Section 1.1 and 1.2, respectively.[5] To see this, note that δ is the relative weight between the first and the second terms, where the first term is a utility associated with the ex-ante evaluation, while the second term is a utility associated with the *ex-post evaluation* because, in the second term, each ex-post payoff profile in the support of P is evaluated by U separately.

Indeed, the representation (1) satisfies the Reversal of Order axiom if and only if $\delta = 1$. Given that our purpose is to develop a model which does not satisfy the Reversal of Order axiom, one might wonder why it does not suffice to consider the simpler special case in which $\delta = 0$. However, this special case trivially implies the *Independence axiom* on ex-ante mixtures so that there is *no* strict preference for ex-ante mixtures.

The remainder of Section 1 is organized as follows: Section 1.1 provides an overview of EAP Maxmin preferences; while, Section 1.2 provides an overview of EAP Piecewise preferences; finally, in Section 1.3, the related literature is discussed. Next, Section 2 reviews recent experimental evidence on a preference for ex-ante mixtures under the two types of aversion. After that, Section 3 introduces the setup. Then, EAP Maxmin preferences are characterized in Section 4, while in Section 5, EAP Piecewise preferences are characterized. In Section 6, EAP Maxmin and EAP Piecewise preferences are applied to games. Finally in Section 7, further relationships among the axioms of Anscombe and Aumann (1963), Seo (2009), and our model are investigated. All proofs are in the appendix.

[5] If U is linear, then both terms coincide with each other. So δ becomes trivial.

1.1 EAP Maxmin preferences

Ellsberg (1961) proposed the following thought experiment: consider two urns, one of which we call *objective* and the other of which we call *ambiguous*. Each urn contains 100 balls, each of which is either red or black. The objective urn contains 50 black and 50 red balls. There is no further information about the contents of the ambiguous urn. You first decide which urn you will draw from; then you bet on the color of the ball that you will draw, and you then draw a ball. If your bet turns out to be correct, you will get \$100. Typically, subjects strictly prefer the objective urn than the ambiguous urn. This behavior is called *ambiguity aversion*.

Raiffa (1961) criticizes ambiguity-averse preferences as follows: by flipping a coin to choose on which color in the ambiguous urn to bet, you can obtain an ex-ante mixture P that is shown in Figure 2. If you look at P *state-wise*, it offers the same lottery that the

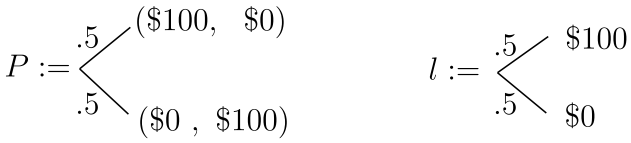

Figure 2: Flipping a Coin to Make a Decision

objective urn offers, namely, which shown as l in Figure 2. So, P and l should be indifferent. Hence, there is no reason why you strictly prefer the objective urn.

As Raiffa's (1961) argument suggests, some people might prefer flipping a coin and then deciding. One conceivable justification for such a preference is that ex-ante mixtures provide *hedging* in ex-ante expected payoffs. When a coin is flipped, the ex-ante expected payoff for each color becomes a constant \$50, although the decision maker finally ends up with the ambiguous bets ex post. We call this preference for ex-ante mixtures *ex-ante ambiguity aversion*. In contrast, conventional ambiguity aversion constitutes a preference for ex-post mixtures.[6] Henceforth, we call this conventional ambiguity aversion *ex-post ambiguity aversion*. Indeed, recent experiments, reported in Dominiak and Schnedler (2009) and Spears (2009), have found that subjects often have different attitudes toward ex-ante

[6]See Uncertainty Aversion axiom in Gilboa and Schmeidler (1989, p. 144) for an example.

and ex-post ambiguity.

Using the Indifference axiom together with standard axioms used in Gilboa and Schmeidler (1989), we characterize *Ex-ante/Ex-post (EAP) Maxmin preferences* that capture ex-ante ambiguity aversion and also, but separately, ex-post ambiguity aversion as follows:

$$V(P) = \delta \min_{\mu \in C} \int_S \Big(\int_{\mathscr{F}} u(f_s) dP(f) \Big) d\mu(s) + (1-\delta) \int_{\mathscr{F}} \Big(\min_{\mu \in C} \int_S u(f_s) d\mu(s) \Big) dP(f), \quad (2)$$

where S is the set of states, C is a subset of the set of all finitely additive probabilities on S, and u is a von Neumann-Morgenstern utility function.

In the representation (2), the set C of priors captures ex-post ambiguity aversion as in Gilboa and Schmeidler (1989). On the other hand, the relative weight δ between the first and the second termrs captures ex-ante ambiguity aversion, as will be formally shown in Section 4.4. To see this, observe that in the first term, minimum is taken outside of the integral not only with respect to ex-post mixtures but also with respect to ex-ante mixtures, in contrast to in the second term. So, in the first term, ex-ante mixtures provide hedging as much as ex-post mixtures, as opposed to in the second term.

1.2 EAP Piecewise Preferences

Another situation in which people typically prefer ex-ante mixtures is a social environment in which inequality matters. A classical example of "Machina's (1989) mom" captures the essence of such a preference as follows: a mother has one indivisible good, which worths, say, \$100. She has to give it either to her daughter or to her son. She is indifferent between the choices. In such a situation, she would prefer flipping a coin in order to make a decision; then she obtains an ex-ante mixture P on allocations over the children that is shown in Figure 2 again.

The rationale of such preferences would be that ex-ante mixtures provide *ex-ante equality*, or equality in ex-ante expected payoffs. In the example of "Machina's (1989) mom", when a coin is flipped, the ex-ante expected payoff for each child becomes the same \$50, although the mother finally ends up with the unequal allocations ex post. We call this preference for ex-ante mixtures *ex-ante inequality aversion.* In contrast, ex-post mixtures even provide *ex-post equality*, or equality in ex-post payoffs; so, we call a preference for ex-post mixtures *ex-post inequality aversion.*

Indeed, as will be discussed in Section 2.2, such preferences for ex-ante mixtures as well as for ex-post mixtures have been observed in many recent experiments on other-regarding preferences. One robust finding in the experimental studies is that such preferences are nonmonotonic with respect not only to ex–ante expected payoffs but also to ex–post payoffs. For example, in ultimatum games, almost half of recipients, on average, reject positive but unfair offers by dictators, which is less than 20 percent of a total prize; and thereby both of the recipients and dictators obtain nothing. (See Fehr and Schmidt (2005) for a survey.) Such behavior is inconsistent with Gilboa and Schmeidler's (1989) Maxmin preferences, which are monotonic with respect to ex-post payoffs.[7]

To describe such experimental evidence parsimoniously, using the Indifference axiom again, together with standard axioms, we characterize *Ex-ante/Ex-post (EAP) Piecewise preferences* that capture ex-ante inequality aversion and also, but separately, ex-post inequality aversion as follows:

$$\begin{aligned} V(P) = \delta\Bigg(E_P u(f_1) - \sum_{s\neq 1}\Big(\alpha_s \max\{E_P u(f_s) - E_P u(f_1), 0\} + \beta_s \max\{E_P u(f_1) - E_P u(f_s), 0\}\Big)\Bigg) \\ + (1-\delta)\int_{\mathscr{F}}\Bigg(u(f_1) - \sum_{s\neq 1}\Big(\alpha_s \max\{u(f_s) - u(f_1), 0\} + \beta_s \max\{u(f_1) - u(f_s), 0\}\Big)\Bigg)dP(f), \end{aligned} \tag{3}$$

where $1 \in S$ denotes the decision maker, α_s, β_s are nonnegative real numbers, u is a von Neumann-Morgenstern utility function, and $E_P u(f_s) = \int_{\mathscr{F}} u(f_s)dP(f)$. Under the assumption of the risk neutrality, EAP Piecewise preferences reduce to the utility function proposed by Fehr and Schmidt (1999), for degenerate lotteries on allocations.

In the representation (3), the nonnegative numbers α_s and β_s capture ex-post inequality aversion as in Fehr and Schmidt (1999). In particular, α_s and β_s are interpreted as indices

[7] As for a planner' social preferences, however, it would be reasonable to assume monotonicity with respect to ex-post payoffs. So, EAP Maxmin preferences would be consistent with such preferences. Indeed, it is easy to see that if $\delta \in (0,1)$, EAP Maxmin preferences can describe the choice of "Machina's (1989) mom".

of disutility from *envy* and *guilt* toward the individual s when the decision maker gets less and more, respectively than the individual s. Given this, the relative weight δ between the first and the second terms captures ex-ante inequality aversion, as will be formally shown in Section 5.4. To see this, note that the first term represents a concern about ex-ante equality, because the term depends on the differences in the expected utilities, while the second term captures a concern about ex-post equality, because the term depends on the differences in the ex-post utilities.

1.3 Related Literature

To our knowledge, no other axiomatic papers have studied ex-ante mixtures and ex-post mixtures in the context of a preference for randomization.

However, there are a few axiomatic papers which relax the Reversal of Order axiom. Among them, to the best of our knowledge, the first is Drèze (1987), which studies the issue in a context of games with moral hazard. A recent paper that is most closely related to the present paper is Seo (2009).[8] He does not assume the Reversal of Order axiom either. However, under Seo's key axiom, *Dominance*, the *Reduction of Compound Lotteries axiom* implies the Reversal of Order axiom. This means that, in Seo's model, the distinction between ex-ante mixtures and ex-post mixtures is impossible as long as we assume the standard assumption on the reduction.

In contrast, under the Indifference axiom, this incompatibility does not arise, because, as will be shown in Section 7, the Indifference axiom is weaker than the Reversal of Order axiom, thereby enabling the distinction between the two types mixtures, but it is still stronger than the Reduction of Compound Lotteries axiom. Indeed, there is a direct connection between the Indifference axiom and the Dominance axiom: under the Reduction of Compound Lotteries axiom, the Dominance axiom implies the Indifference axiom but not vice versa, as will be shown in Section 7.

In terms of applications, the present paper is related to a literature on game theory

[8]He assumes the Independence axiom on ex-ante mixtures, so that no preference for ex-ante mixture.

that studies ambiguity-averse players, in which mixed strategies correspond to lotteries on acts. The special cases of EAP Maxmin preferences and EAP Choquet preferences (Choquet counterpart of EAP Maxmin), where $\delta = 0$ or 1, have been used in the literature as follows:[9] Klibanoff (1996) and Lo (1996) have applied EAP Maxmin preferences with $\delta = 1$; Eichberger and Kelsey (2000) have applied EAP Choquet preferences with $\delta = 0$; Mukerji and Shin (2002) have applied EAP Choquet preferences with $\delta = 0$ as well as with $\delta = 1$. As these authors note, both assumptions $\delta = 1$ and $\delta = 0$ could provide unintuitively extreme predictions in some games, respectively. In such games, in Section 6, it will be shown that, $\delta \in (0, 1)$ could predict more reasonable behavior of ambiguity-averse players than $\delta = 0$ and 1.

Finally, the present paper is also related to the social choice literature regarding the trade-off between equality of opportunity and equality of outcome; these issues are addressed especially in Ben-Porath, Gilboa, and Schmeidler (1997) and Gajdos and Maurin (2004).[10] However, the models and motivations in both papers are different from ours. These papers have considered a social planner's preferences on matrices of real numbers that are utilities over a product space that consists of states and individuals. Hence, in their model, there is no conceptual counterpart of ex-ante mixtures.[11] In addition, our emphasis in this paper is other-regarding preferences, not a planner's social preferences, in response to recent rich experimental evidence on the former; the present is the first paper apart from Saito (2008) to provide an axiomatization of the utility function proposed by Fehr and Schmidt (1999).

2 Experiments

As noted, both EAP Maxmin and EAP Piecewise preferences are consistent with recent experimental evidence in each context.

[9] Since Choquet expected utilities with convex capacity have Maxmin representations, our axiomatization of EAP Maxmin preferences is also an axiomatize EAP Choquet preferences with convex capacities.

[10] Ben-Porath et al. (1997) do not provide an axiomatization. Gajdos and Maurin (2004) axiomatize a weaker representation than the one used in Ben-Porath et al. (1997). The representations and axioms proposed by Gajdos and Maurin (2004) are different from ours.

[11] Mixing two matrices in their model conceptually corresponds to an ex-post mixture in our model.

2.1 EAP Maxmin Preferences in Experiments

EAP Maxmin preferences can parsimoniously describe experimental evidence found by Dominiak and Schnedler (2009). They have studied relationship between attitudes toward ex-ante and ex-post ambiguity. The number in Table 1 shows the number of subjects who exhibited a corresponding attitude toward ex-ante and ex-post ambiguity.[12]

		Ex–post ambiguity		
		averse	neutral	
	averse	6	0	$\delta < 0$
Ex–ante ambiguity	neutral	17	12	$\delta = 0$
	loving	12	2	$\delta > 0$
		35	14	

Table 1: Attitudes toward Ex-ante and Ex-post Ambiguity

Dominiak and Schnedler's (2009) experimental result might be summarized by the following two points. First, subjects who are averse to ex-post ambiguity differ in their attitudes toward ex-ante ambiguity. This result is inconsistent not only with the Reversal of Order axiom but also with Raiffa's (1961) critique because his claim implies that all of the ex-post ambiguity-averse decision makers should be ex-ante ambiguity averse as well. Second, however, most of the ex-post ambiguity neutral subjects are ex-ante ambiguity neutral as well.

The first observation is explained by the heterogeneity of the parameter δ as follows: suppose EAP Maxmin preferences exhibit ex-post ambiguity aversion. Then, as will be shown in Section 4.5, the preferences exhibit ex-ante ambiguity aversion, neutrality, and loving, if and only if $\delta > 0$, $\delta = 0$, and $\delta < 0$, respectively, which is consistent with Table 1. The second observation is also consistent with EAP Maxmin preferences. As will be shown in Section 4.5, among EAP Maxmin preferences, ex-post ambiguity neutrality implies ex-ante ambiguity neutrality for any δ, which is also consistent with Table 1.

[12]The table excludes four subjects who exhibited ex-post ambiguity loving.

Spears (2009) independently conducted similar experiments to Dominiak and Schnedler (2009) and has obtained similar tendencies. On the other hand, in a field experiment, Dwenger et al. (2010) have found a significant evidence for ex-ante ambiguity aversion, which would suggest that $\delta > 0$.

2.2 EAP Piecewise Preferences in Experiments

EAP Piecewise preferences are also consistent with recent experimental evidence. Firstly, we discuss the experimental results in *probabilistic dictator games*, in which dictators allocate chances to win a prize, in contrast to standard dictator games, in which dictators allocate a prize itself. One of the robust findings in such experiments is that a substantial fraction of dictators shared chances to win, so that exhibited ex-ante inequality aversion. This finding is simply described by EAP Piecewise preferences with $\delta > 0$.[13] See, for the experiments, Karni, Salmon, and Sopher (2008), Bohnet, Greig, Herrmann, and Zeckhauser (2008), Bolton and Ockenfels (forthcoming), Krawczyk and Le Lec (2008), and Kircher et al. (2009).

Secondly, Kariv and Zame's (2009) experiment is also consistent with EAP Piecewise preferences. In their experiments, subjects are asked to divide a budget z into x and y such that $x+qy \leq z$, where q is a given price. After the decision, the payoff of the decision maker and a recipient are determined as x or y with the probability .5, so that what the decision maker obtains is an ex-ante mixture $.5(x, y) \oplus .5(y, x)$. Hence, the subjects are required to make decisions under a *veil of ignorance*.

One of their main findings is that most of the subjects did not allocate all funds to the cheaper element. This fact is also consistent with EAP Piecewise preferences. To see this, assume the risk neutrality, for simplicity. Then, the utility by the ex-ante mixture is as follows:

$$V\Big(.5(x, y) \oplus .5(y, x)\Big) \;=\; \frac{1}{2}\Big[(x + y) - (1 - \delta)(\alpha + \beta)|x - y|\Big]. \tag{4}$$

[13]With appropriate experiments, it would be easy to calibrate the value of δ.

So, when $(1-\delta)(\alpha+\beta)$ exceeds a certain level, in order to maximize the utility, the decision maker tries to equalize x and y even if the prices are not the same.[14]

Finally, EAP Piecewise preferences are also consistent with seemingly contradictory experimental results on *efficiency versus inequality*; recently, a number of papers have claimed that *efficiency*, or the sum of allocation across agents, has a stronger influence than inequality. In particular, Charness and Rabin (2002) report that in a dictator game, almost 50 percent of their subjects chose an efficient but unequal allocation (in which the dictator obtained 375 points and the receiver obtained 750 points) to an equal but inefficient allocation (in which each player obtained 400 points).[15] This behavior seems contradictory to any theory of ex-post inequality aversion including a theory provided by Fehr and Schmidt (1999).

The key fact that can explain this contradiction by using EAP Piecewise preferences is that in the experiments that are in favor of efficiency, each subject makes decisions as if he were a dictator, but the actual roles (i.e., dictator or receiver) are determined at random. Hence, the subjects are required to make decisions under *risk over roles.*

Under the risk over roles, each subject has to face a game with the other subjects because their decisions could determine the subject's payoff if a dictator is chosen among them. In a game that describes the aforementioned dictator game under the risk over roles, it will be shown in Section 6.2 that in an equilibrium, subjects with EAP Piecewise preferences choose the efficient allocation rather than the equal allocation because of the ex-ante equality.[16]

To understand this result intuitively, note that *risk over roles plays a role similar to that of veil of ignorance.* To see this, suppose that two subjects decide to allocate x to themselves and y to the other. Then, under the risk over roles, what they obtain is the ex-ante mixture

[14] Without loss of generality, consider the case where $q < 1$. If $1-\delta > \frac{q-1}{(\alpha+\beta)(q+1)}$, then EAP Piecewise preferences with such parameters are consistent with the experimental evidence that many subjects did not spend all the budget to the cheaper element, i.e., y.

[15] See, for other experiments, Engelmann and Strobel (2004, 2006)

[16] Indeed, this result is consistent with experimental evidence found by Bolton and Ockenfels (2006), which report that in three-person dictator games, under risk over roles, subjects tended to choose efficient but unequal allocations over equal but inefficient allocations.

$.5(x,y) \oplus .5(y,x)$. Thus, the utility of each subject is determined as in (4), in other words:

$$V\Big(.5(x,y) \oplus .5(y,x)\Big) \;=\; \frac{1}{2}\Big[(\text{“efficiency”}) - (1-\delta)(\alpha+\beta)(\text{“inequality”})\Big].$$

Hence, even if a subject cares about ex-post equality (i.e., α and β are positive), if he weighs ex-ante equality heavily enough (i.e., δ is larger than a certain level), then his utility from choosing the efficient allocation becomes larger than his utility from choosing the equal allocation, given that the other player chooses the same efficient allocation. Therefore, it looks *as if* the subjects with EAP Piecewise preferences care more about efficiency than about inequality.

3 Setup

For any topological space X, let $\Delta(X)$ be the set of distributions over X with finite supports. An element in $\Delta(X)$ is called a *lottery* on X. Let $\delta_x \in \Delta(X)$ denote a point mass on x.

Let S be a set of states and let Σ be an algebra of subsets of S. Let Z denote a set of outcomes. Both set S and set Z are assumed to be nonempty. A payoff profile f is called an *act* and defined to be a Σ-measurable function from S into $\Delta(Z)$ with finite range. For each act f, we write $f_s \in \Delta(Z)$, instead of $f(s)$. Let $\mathscr{F}$ be the set of all acts.

A preference relation $\succsim$ is defined on $\Delta(\mathscr{F})$. As usual, $\succ$ and $\sim$ denote, respectively, the asymmetric and symmetric parts of $\succsim$. A *constant act* is an act f such that $f_s = f'_s$ for all $s, s' \in S$. Elements in $\Delta(\mathscr{F})$ are denoted by P, Q, and R. For all $P \in \Delta(\mathscr{F})$, supp P is the support of P. Elements in $\mathscr{F}$ are denoted by f, g , and h. Elements in $\Delta(Z)$ are denoted by l, q, r and are identified as constant acts. For $f \in \mathscr{F}$, an element $l_f \in \Delta(Z)$ is a *certainty equivalent* for f if $f \sim l_f$.

Finally, ex-ante mixtures and ex-post mixtures are formally defined as follows:

Definition: For all $\alpha \in [0,1]$ and $P, Q \in \Delta(\mathscr{F})$, $\alpha P \oplus (1-\alpha)Q \in \Delta(\mathscr{F})$ is a lottery on acts such that $(\alpha P \oplus (1-\alpha)Q)(f) = \alpha P(f) + (1-\alpha)Q(f) \in [0,1]$ for each $f \in \mathscr{F}$. This operation is called an *ex-ante mixture*. For degenerate lotteries on acts, we write

$\alpha f \oplus (1-\alpha)g \in \Delta(\mathscr{F})$, instead of $\alpha\delta_f \oplus (1-\alpha)\delta_g$, for any $\alpha \in [0,1]$, and $f, g \in \mathscr{F}$.

DEFINITION: For all $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$, $\alpha f + (1-\alpha)g \in \mathscr{F}$ is an act such that $(\alpha f + (1-\alpha)g)(s)(z) = \alpha f(s)(z) + (1-\alpha)g(s)(z) \in [0,1]$ for each $s \in S$ and $z \in Z$. This operation is called an *ex-post mixture.*

4 EAP Maxmin Preferences

To characterize EAP Maxmin preferences, instead of Reversal of Order, we assume Indifference as well as the axioms used in Gilboa and Schmeidler (1989).

4.1 Axioms

The first six axioms are due to Gilboa and Schmeidler (1989). However, since Reversal of Order is not assumed, both Continuity and Certainty Independence are assumed for ex-ante mixtures and also, but separately, for ex-post mixtures.

AXIOM 1 (Weak Order): $\succsim$ is complete and transitive.

AXIOM 2 (Continuity):
(i) For all $P, Q, R \in \Delta(\mathscr{F})$, if $P \succ Q$ and $Q \succ R$, then there exist α and β in $(0,1)$ such that $\alpha P \oplus (1-\alpha)R \succ Q$ and $Q \succ \beta P \oplus (1-\beta)R$.
(ii) For all $f, g, h \in \mathscr{F}$, if $f \succ g$ and $g \succ h$, then there exist α and β in $(0,1)$ such that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$.

AXIOM 3 (Nondegeneracy): There exist $z_+, z_- \in Z$ such that $z_+ \succ z_-$.

AXIOM 4 (Monotonicity): For all $f, g \in \mathscr{F}$,

$$f_s \succsim g_s \text{ for all } s \in S \Rightarrow f \succsim g.$$

If a preference relation $\succsim$ satisfies the axioms above, then each act $f \in \mathscr{F}$ admits a certainty equivalent $l_f \in \Delta(Z)$.

Axiom 5 (Ex-post Ambiguity Aversion): For all $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$,

$$f \sim g \Rightarrow \alpha f + (1-\alpha) g \succsim f.$$

Mixing constant acts, ex-ante as well as ex-post, does not provide any hedging. Hence,

Axiom 6 (Ex-ante/Ex-post Certainty Independence):

(i) For all $\alpha \in (0,1]$, $P, Q \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$,

$$P \succsim Q \Leftrightarrow \alpha P \oplus (1-\alpha) l \succsim \alpha Q \oplus (1-\alpha) l.$$

(ii) For all $\alpha \in (0,1]$, $f, g \in \mathscr{F}$, and $l \in \Delta(Z)$,

$$f \succsim g \Leftrightarrow \alpha f + (1-\alpha) l \succsim \alpha g + (1-\alpha) l.$$

The final axiom is a weaker formalization of Raiffa's (1961) critique. As noted in Introduction, his argument corresponds to the state-wise criterion. First, to formalize the state-wise criterion, a preliminary concept is defined here:

Definition: For all $P \in \Delta(\mathscr{F})$ and $s \in S$,

$$P_s = P(f^1) f^1_s + \cdots + P(f^n) f^n_s,$$

where $P = P(f^1) f^1 \oplus \cdots \oplus P(f^n) f^n$.

In words, P_s is a reduced marginal distribution of P on s. Kreps (1988, p. 106) as well as Raiffa (1961) have proposed an act $(P_s)_{s \in S}$, which offers P_s at each state s, as a reasonable embedding of $P \in \Delta(\mathscr{F})$ to $\mathscr{F}$. Henceforth, we write $(P_s)_s$, instead of $(P_s)_{s \in S}$ for simplicity.

The next embedding corresponds to the support-wise criterion as follows: remember that $l_f \in \Delta(Z)$ is a certainty equivalent for an act f.

Definition: For all $P \in \Delta(\mathscr{F})$,

$$l_P = P(f^1) l_{f^1} + \cdots + P(f^n) l_{f^n},$$

where $P = P(f^1)f^1 \oplus \cdots \oplus P(f^n)f^n$.[17]

AXIOM 7 (Indifference): For all $P, Q \in \Delta(\mathscr{F})$,

$$\left\{ \begin{array}{ll} \text{(i)} \ (P_s)_s \sim (Q_s)_s; \text{ and} \\ \text{(ii)} \quad l_P \sim l_Q \end{array} \right) \Rightarrow P \sim Q.$$

Indifference states that if two lotteries on acts are indifferent according to the two criteria *jointly*, then the lotteries should be indifferent. As will be shown in Section 7, a stronger axiom without the condition (ii), which will be called *State-wise Indifference*, is equivalent to Reversal of Order. Since the stronger axiom, State-wise Indifference, corresponds to Raiffa's (1961) critique, Indifference would be interpreted as a weaker formalization of his critique.

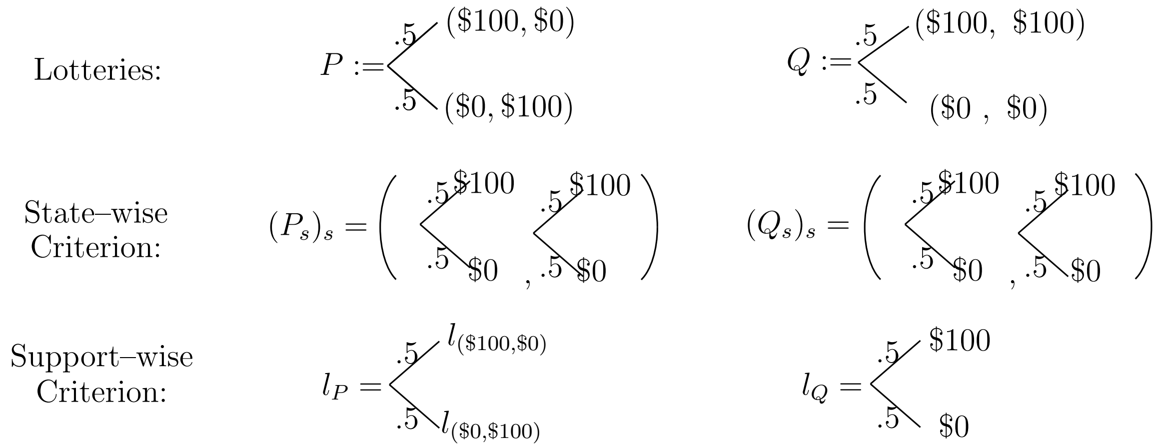

Figure 3: State-wise Criterion and Support-wise Criterion

To see formally the difference between Indifference and Raiffa's (1961) critique, consider two lotteries P and Q on acts in Figure 3; since $(P_s)_s = (Q_s)_s$, P and Q are indifferent according to State-wise Indifference. So, Raiffa would conclude that P and Q should be indifferent. According to the support-wise criterion, on the other hand, Q is better than P, because, under ambiguity aversion, $l_Q = (\$100, .5;\ \$0, .5) \succ (\$100, \$0) \sim (\$0, \$100) \sim l_P$. Hence, Indifference does *not* conclude that P and Q are indifferent.[18]

[17]Note that, in general, it is not true that $P \sim l_P$. However, for a degenerate lottery on acts, $f \sim l_{\delta_f} \equiv l_f$. So, there is no contradiction in the notations.

[18]If Indifference is strengthened to apply (i) and (ii) independently (that is, (i) *or* (ii) $\Rightarrow P \sim Q$), then

4.2 Representation

Before stating the result, we mention that the topology to be used on the space of finitely additive set functions on Σ is the weak* topology.

THEOREM 1: *For a preference relation $\succsim$ on $\Delta(\mathscr{F})$, the following statements are equivalent:*
(i) The preference relation satisfies Axioms 1–7.
(ii) There exist a real number δ, a nonempty convex closed set C of finitely additive probability measures on Σ, and a nonconstant mixture linear function $u : \Delta(Z) \to \mathbb{R}$, such that $\succsim$ is represented by the function $V : \Delta(\mathscr{F}) \to \mathbb{R}$ of the form

$$V(P) = \delta \min_{\mu \in C} \int_S \Big(\int_{\mathscr{F}} u(f_s) dP(f) \Big) d\mu(s) + (1-\delta) \int_{\mathscr{F}} \Big(\min_{\mu \in C} \int_S u(f_s) d\mu(s) \Big) dP(f).$$

DEFINITION: A preference relation $\succsim$ on $\Delta(\mathscr{F})$ is called an *Ex-ante/Ex-post (EAP) Maxmin* preference if it satisfies axioms in (i) of Theorem 1.

By Theorem 1, EAP Maxmin preferences can be represented by a triple (δ, C, u). Next, we give the uniqueness property of this representation.

COROLLARY 1: *The following two statements are equivalent:*
(i) Two triples (δ, C, u) and (δ', C', u') represent the same EAP Maxmin preference as in Theorem 1.
(ii) (a) $C = C'$, and there exist real numbers α and β such that $\alpha > 0$ and $u = \alpha u' + \beta$; and
(b) If C is nondegenerate, then $\delta = \delta'$.

4.3 Characterizations of δ

The parameter δ has a direct behavioral characterization in terms of *ex-ante ambiguity aversion* and *interim ambiguity aversion*:

AXIOM (Ex-ante Ambiguity Aversion): For all $\alpha \in (0, 1)$ and $f, g \in \mathscr{F}$,

$$f \sim g \Rightarrow \alpha f \oplus (1-\alpha) g \succsim f.$$

Anscombe and Aumann's (1963) subjective expected utility is obtained in Theorem 1.

Ex-ante ambiguity neutrality and *ex-ante ambiguity loving* are defined in the same way by changing the right-hand side of the definition to $\alpha f \oplus (1-\alpha)g \sim f$ and to $f \succsim \alpha f \oplus (1-\alpha)g$, respectively.

Axiom (Interim Ambiguity Aversion): For all $\alpha \in (0,1)$ and $f, g \in \mathscr{F}$,

$$\alpha f + (1-\alpha)g \succsim \alpha f \oplus (1-\alpha)g.$$

Interim ambiguity aversion means that an ex-post mixture is preferred over its ex-ante mixture. This is because an ex-post mixture provides hedging in ex-post utilities, whereas an ex-ante mixture provides hedging only in ex-ante expected utilities. In addition, *interim ambiguity neutrality* is defined in the same way by changing $\succsim$ to $\sim$, which is nothing but Reversal of Order among two acts.

Proposition 1: *Suppose $\succsim$ is an EAP Maxmin preference with nondegenerate C.*
(i) $\succsim$ exhibits ex-ante ambiguity aversion if and only if $\delta \geq 0$.
(ii) $\succsim$ exhibits interim ambiguity aversion if and only if $\delta \leq 1$.

Note that given the representation, it is easy to see that EAP Maxmin preferences with $\delta = 0$ and $\delta = 1$ satisfy ex-ante ambiguity neutrality and interim ambiguity neutrality, respectively.

4.4 Comparative Attitudes toward Ex-ante Ambiguity

We now study comparative attitudes toward ex-ante ambiguity.

Definition: Given two preference relations $\succsim_1$ and $\succsim_2$, $\succsim_1$ is said to be *more ex-ante ambiguity averse* than $\succsim_2$ if, for every $P \in \Delta(\mathscr{F})$ and every $f \in \mathscr{F}$,

$$P \succsim_2 f \Rightarrow P \succsim_1 f.$$

The next proposition shows that δ captures the attitude toward ex-ante ambiguity.

Proposition 2: *Suppose two EAP Maxmin preferences $\{\succsim_i\}_{i=1,2}$ are represented by $\{(\delta_i, C_i,$*

$u_i)\}_{i=1,2}$, where C_1 and C_2 are nondegenerate. Then, the following statements are equivalent:
(i) $\succsim_1$ is more ex-ante ambiguity averse than $\succsim_2$.
(ii) $\delta_1 \geq \delta_2$, $C_1 = C_2$, and there exist real numbers α and β such that $\alpha > 0$ and $u_1 = \alpha u_2 + \beta$.

Note that in (ii), both of the preferences coincide in C as well as in u under normalization. Therefore, Proposition 2 says that stronger ex-ante ambiguity aversion is characterized only by a *larger* value of δ. Therefore, δ can be interpreted as an *index of ex-ante ambiguity aversion.*

4.5 Relation between Attitudes toward Ex-ante and Ex-post Ambiguity

To conclude this section, implications of EAP Maxmin preferences on the relation between attitudes toward ex-ante and ex-post ambiguity are characterized. In particular, it will be shown that the implications of EAP Maxmin preferences are consistent with Dominiak and Schnedler's (2009) experimental evidence, which was summarized by two points in Table 1 as follows. Fistly, among ex-post ambiguity averse subjects, the attitude toward ex-ante ambiguity is quite heterogeneous; but secondly most ex-post ambiguity-neutral subjects are ex-ante ambiguity neutral as well. These results are formally described by EAP Maxmin preferences as follows:

PROPOSITION 3: *Suppose $\succsim$ is an EAP Maxmin preference.*
(i) (a) Suppose $\delta > 0$. Then, $\succsim$ exhibits ex-post ambiguity aversion if and only if $\succsim$ exhibits ex-ante ambiguity aversion.

(b) Suppose $\delta < 0$. Then, $\succsim$ exhibits ex-post ambiguity aversion if and only if $\succsim$ exhibits ex-ante ambiguity loving.

(c) Suppose $\delta = 0$. Then, $\succsim$ exhibits ex-ante ambiguity neutrality.
(ii) For any δ, if $\succsim$ exhibits ex-post ambiguity neutrality, then $\succsim$ exhibits ex-ante ambiguity neutrality.

Part (i) shows that the heterogeneity observed in the experiment can be described simply by whether or not δ is positive. Part (ii) shows that among EAP Maxmin preferences, ex-post ambiguity neutrality implies ex-ante ambiguity neutrality, as observed in the experiment.

5 EAP Piecewise Preferences

In this section, EAP Piecewise preferences are characterized. Accordingly, the set S of states is assumed to be finite and *reinterpreted as individuals* including a decision maker, who is denoted by $1 \in S$.

5.1 Axioms

The axioms for EAP Maxmin preferences are now modified to capture inequality aversion. No modification is necessary for Indifference, and the first two modifications required are minor.

As noted in Section 1.2, to capture inequality aversion, Monotonicity (Axiom 4) needs to be weakened as follows:

Axiom 4' (Substitution): For all $f, g \in \mathscr{F}$,

$$f_s \sim g_s \text{ for all } s \in S \Rightarrow f \sim g.$$

The second minor change is that the following axiom is assumed instead of Ex-post Ambiguity Aversion (Axiom 5).

Axiom 5' (Ex-post Inequality Aversion): Let $l_0 = \frac{1}{2}\delta_{z_+} + \frac{1}{2}\delta_{z_-}$. For all $s \neq 1$,
(i) $(l_0, (l_0)_{-s}) \succsim (z_+, (l_0)_{-s})$; and
(ii) $(l_0, (l_0)_{-s}) \succsim (z_-, (l_0)_{-s})$.[19]

Part (i) captures the disutility that results from *envy* toward the individual s when only the individual s is better off than the decision maker, while part (ii) captures the disutility

[19]For any lottery $l, r \in \Delta(Z)$ and $s \in S$, $(l, (r)_{-s})$ is an act which offers l for the individual s and offers r for the other individuals.

that results from *guilt* toward the individual s when only the individual s is worse off than the decision maker.

The final axiom that requires a modification is Ex-ante/Ex-post Certainty Independence (Axiom 6). Specifically, a new weaker version of comonotonicity needs to be defined. Remember that $1 \in S$ denotes the decision maker.

DEFINITION: Two acts $f, g \in \mathscr{F}$ are said to be *pointwise comonotonic* if for no $s \in S$, $f(s) \succ f(1)$ and $g(s) \prec g(1)$.

Suppose two acts f and g are pointwise comonotonic. Then, the rank of utilities of any individual with respect to the decision maker is not reversed between f and g.[20] Hence,

AXIOM 6' (Ex-ante/Ex-post Pointwise Comonotonic Independence):
(i) For all $\alpha \in (0, 1]$ and $P, Q, R \in \Delta(\mathscr{F})$ such that $(P_s)_s, (R_s)_s$, and $(Q_s)_s, (R_s)_s$ are each pointwise comonotonic,

$$P \succsim Q \Leftrightarrow \alpha P \oplus (1-\alpha)R \succsim \alpha Q \oplus (1-\alpha)R.$$

(ii) For all $\alpha \in (0, 1]$ and $f, g, h \in \mathscr{F}$ such that f, h, and g, h are each pointwise comonotonic,

$$f \succsim g \Leftrightarrow \alpha\ f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

As noted, no modification is necessary for Indifference. The interpretation of that axiom is straightforward here. The first criterion (i) corresponds to ex-ante equality, because each marginal distribution P_s yields ex-ante expected payoff of the individual s. The second criterion (ii) corresponds to ex-post equality, because each certainty equivalent l_f reflects the ex-post equality of f. Hence, Indifference means that if two lotteries P and Q on acts are indifferent in both ex-ante and ex-post equality, then P and Q should be indifferent.

[20]Schmeidler (1989, p. 586) has presented an interpretation of *comonotonicity* from the point of view of a *planner's social preferences* as follows: two income allocations f and g are comonotonic if the social rank of *any two agents* is not reversed between f and g. When we focus on an *agent's other-regarding preferences*, what is relevant to the agent is social rank *with respect to the agent himself*, not the social rank of *any two agents*.

5.2 Representation

THEOREM 2: *For a preference relation* $\succsim$ *on* $\Delta(\mathscr{F})$*, the following statements are equivalent:*
(i) The preference relation satisfies Axioms 1, 2, 3, 4', 5', 6', and 7.
(ii) There exist a real number δ*, nonnegative numbers* $\{\alpha_s, \beta_s\}_{s\neq 1}$*, and a nonconstant mixture linear function* $u : \Delta(Z) \to \mathbb{R}$ *such that* $\succsim$ *is represented by the function* $V : \Delta(\mathscr{F}) \to \mathbb{R}$ *of the form*

$$V(P) = \delta\Bigg(E_P u(f_1) - \sum_{s\neq 1}\Big(\alpha_s \max\{E_P u(f_s) - E_P u(f_1), 0\} + \beta_s \max\{E_P u(f_1) - E_P u(f_s), 0\}\Big)\Bigg)$$
$$+ (1-\delta)\int_{\mathscr{F}}\Bigg(u(f_1) - \sum_{s\neq 1}\Big(\alpha_s \max\{u(f_s) - u(f_1), 0\} + \beta_s \max\{u(f_1) - u(f_s), 0\}\Big)\Bigg)dP(f),$$

where $E_P u(f_s) = \int_{\mathscr{F}} u(f_s)dP(f)$*. Furthermore, the two quadruples* $(\delta, \alpha, \beta, u)$ *and* $(\delta', \alpha', \beta', u')$ *represent the same preference as in the above if and only if* $(\alpha, \beta) = (\alpha', \beta')$*,* $\delta = \delta'$ *if* $(\alpha, \beta) \neq \mathbf{0}$*, and there exist real numbers* a *and* b *such that* $a > 0$ *and* $u = au' + b$*.*

DEFINITION: A preference relation $\succsim$ on $\Delta(\mathscr{F})$ is called an *Ex-ante/Ex-post (EAP) Piecewise preference* if it satisfies axioms in (i) of Theorem 2.

5.3 Characterization of δ

The parameter δ has a direct behavioral characterization in terms of both *ex-ante inequality aversion* and *interim inequality aversion*, as follows:

AXIOM(Ex-ante Inequality Aversion):For all $s \neq 1$ and $l_+, l_- \in \Delta(Z)$ such that $l_+ \succ l_0 \succ l_-$,

$$(l_+, (l_0)_{-s}) \sim (l_-, (l_0)_{-s}) \Rightarrow \frac{1}{2}(l_+, (l_0)_{-s}) \oplus \frac{1}{2}(l_-, (l_0)_{-s}) \succsim (l_+, (l_0)_{-s}).$$

Ex-ante inequality aversion means that an ex-ante mixture of unequal allocations offsets the inequalities in the expected utilities. So, the ex-ante mixture would be more desirable.[21]

[21]Ex-ante inequality aversion is consistent with the experimental evidence, drawn from the probabilistic dictator games, that subjects who are indifferent between winning and losing tend to prefer flipping a coin to decide the winner.

In addition, *ex-ante inequality neutrality* is defined by changing $\succsim$ to $\sim$ in the right-hand side of the above definition.

Axiom (Interim Inequality Aversion): For all $s \neq 1$,

$$\frac{1}{2}(z_+, (l_0)_{-s}) + \frac{1}{2}(z_-, (l_0)_{-s}) \succsim \frac{1}{2}(z_+, (l_0)_{-s}) \oplus \frac{1}{2}(z_-, (l_0)_{-s}).$$

To interpret interim inequality aversion, recall that $l_0 = \frac{1}{2}\delta_{z_+} + \frac{1}{2}\delta_{z_-}$, so that the ex-post mixture in the left hand side is identical to constant act l_0 and provides ex-post equality. On the other hand, the ex-ante mixture in the right hand side could provide ex-ante equality but not ex-post equality. So, the ex-post mixture would be preferred over the ex-ante mixture. In addition, *interim inequality neutrality* is defined by changing $\succsim$ to $\sim$.

Corollary 2: *Suppose $\succsim$ is an EAP Piecewise preference with $(\alpha, \beta) \neq \boldsymbol{0}$.*
(i) $\succsim$ exhibits ex-ante inequality aversion if and only if $\delta \geq 0$.
(ii) $\succsim$ exhibits interim inequality aversion if and only if $\delta \leq 1$.

Note that given the representation, it is easy to see that EAP Piecewise preferences with $\delta = 0$ and $\delta = 1$ satisfy ex-ante inequality neutrality and interim inequality neutrality, respectively.

5.4 Comparative Attitudes toward Ex-ante Inequality

To conclude this section, comparative attitudes toward ex-ante inequality are characterized. As mentioned in Introduction, a preference for ex-ante mixtures is due to ex-ante inequality aversion in a social context, in contrast to ambiguous situations, in which such a preference is due to ex-ante ambiguity aversion. So, the same definition of being more ex-ante ambiguity averse in Section 4.4 is interpreted as the definition of being *more ex-ante inequality averse* in the context of inequality aversion.

Hence, results analogous to those derived from Proposition 2 in Section 4.4 also hold for inequality aversion.

Corollary 3: *Suppose two EAP Piecewise preferences $\{\succsim_i\}_{i=1,2}$ are represented by $\{(\delta_i, \alpha^i, \beta^i,$*

$u_i)\}_{i=1,2}$, where $(\alpha^1, \beta^1) \neq \mathbf{0} \neq (\alpha^2, \beta^2)$. Then the following statements are equivalent:
(i) $\succsim_1$ is more ex-ante inequality averse than $\succsim_2$.
(ii) $\delta_1 \geq \delta_2$, $(\alpha^1, \beta^1) = (\alpha^2, \beta^2)$, and there exist real numbers a and b such that $a > 0$ and $u_1 = au_2 + b$.

Note that in (ii), both of the preferences coincide in α and β as well as in u under a normalization. Therefore, Corollary 3 says that stronger ex-ante inequality aversion is characterized only by *larger* value of δ. Therefore, δ can be interpreted as an *index of ex-ante inequality aversion* in a social context.

6 Games

In preceding sections, we saw how EAP Maxmin and EAP Piecewise preferences are consistent with many experimental results, mainly on single-person decision making. In this section, EAP Maxmin and EAP Piecewise preferences are applied to games in order to see the implications of the models in strategic situations.

6.1 EAP Maxmin Preferences in Games

As noted in Introduction, the special cases of EAP Maxmin and EAP Choquet preferences, where $\delta = 0$ or 1 have been used in the game theory literature on ambiguity-averse players.[22] The following two symmetric games, Game I and Game II, suggest that $\delta \in (0, 1)$ would predict more realistic behavior of ambiguity-averse players than $\delta = 0$ and 1, respectively. The numbers in the games are von Neumann-Morgenstern utilities and x and ε are a positive numbers.

[22]To see the relationship between the literature and our model, fix a game and a player. Then, the player's pure strategy corresponds to an act; the set of strategies of the other players corresponds to the set of states; hence, the player's mixed strategies corresponds to ex-ante mixtures on acts.

$1\backslash 2$	d	e
a	$2x$	0
b	0	$2x$

Game I

$1\backslash 2$	d	e
a	$2x$	0
b	0	$2x$
c	$x-\varepsilon$	$x-\varepsilon$

Game II

For both of the games, when they are played for the first time, the symmetry makes it difficult for each player to have a unique prior probability over the opponent's strategies. So, in Game I, the ambiguity-averse players would prefer mixed strategies over pure strategies in order to hedge. In addition, in Game II, if a positive number ε is less than a certain threshold, player 1 would prefer strategy c, whose payoff is constant, over any mixed strategies between a and b.

EAP Maxmin preferences with $\delta \in (0,1)$ can describe these reasonable behaviors in a *strict* equilibrium in each game as opposed to with $\delta = 0$ and 1.[23] EAP Maxmin preferences with $\delta = 0$ show that in Game I, if a player is indifferent between the strategies, then there is no strict incentive to mix between them. In addition, EAP Maxmin preferences with $\delta = 1$ show that in Game II, for player 1, the fifty-fifty mix between a and b strictly dominates c for any small positive number ε.

6.2 EAP Piecewise Preferences in Games

In this section, we show that EAP Piecewise preferences can describe seemingly contradictory experimental results on efficiency versus inequality. As mentioned in Introduction, the key to resolving the putative contradiction is that it is only in the experiments that are strongly in favor of efficiency that subjects are under *risk over roles*. That is, in the experiments, each subject makes a decision as if he were a dictator, but actual roles (i.e., dictator or receiver) are determined at random.

[23]See Klibanoff (1996) for a definition of an equilibrium with ambiguity-averse players. He assumes $\delta = 1$ but the definition is easily applied to EAP Maxmin preferences with $\delta \neq 1$.

We study a Bayesian game that describes the dictator game from Charness and Rabin (2002), mentioned in Section 2.2. In the game, they report that about 50 percent of the subjects chose the efficient but unequal allocation rather than the equal but inefficient allocation. Assume, for simplicity, there exist two players $\{1, 2\}$ and two types of players as follows: *fair* (i.e., $\alpha^F, \beta^F > 0$ and $\delta^F > 0$) and *selfish* (i.e., $\alpha^S = 0 = \beta^S$). Player 1's set of actions is $\{(375, 750), (400, 400)\}$ and player 2's set of actions is $\{(750, 375), (400, 400)\}$, where the first and second coordinates show the material prizes for players 1 and 2, respectively. Denote the efficient but unequal allocation by Ef, and the equal but inefficient allocation by Eq. The game is described as Game I in Figure 4.

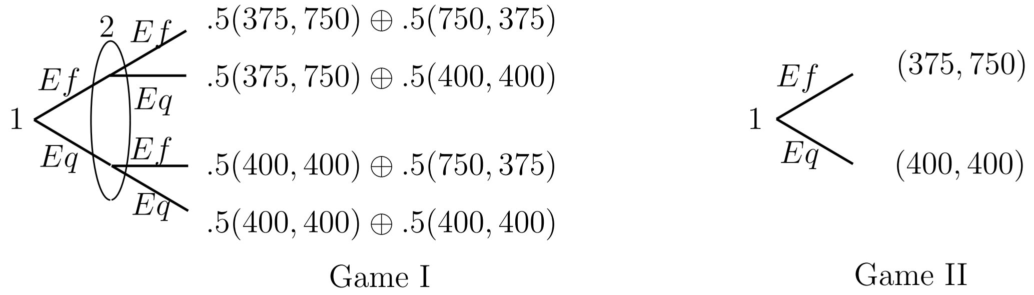

Figure 4: Dictator Games *with and without Risk over roles*

In Game I, the player's choice determines an allocation only if he turns out to be a dictator, in contrast to Game II, in which player 1's choice determines an allocation for sure. So, given that each role is determined with the probability .5, what a player obtains is a fifty-fifty ex-ante mixture on allocations. For example, if player 1 chooses action Ef and player 2 chooses Eq, players obtain an ex-ante mixture that gives $(375, 750)$ and $(400, 400)$ with the probability .5. Now the result can be stated as follows:

PROPOSITION 4: *Suppose*

(a) Players' preferences are EAP Piecewise with $u(z) = \log z$ *for all* $z \in \mathbb{R}_+$.

(b) There exist two types, fair (*i.e.*, $\alpha^F, \beta^F > 0$, *and* $\delta^F > 0$) *and selfish* (*i.e.*, $\alpha^S = 0 = \beta^S$).

Let $\alpha^F = .2$, $\beta^F = .9$, *and* $\delta^F = .85$. *Then the following results hold:*

(i) In Game I, there exists a Bayesian Nash equilibrium in which the fair type choose the efficient allocation (Ef), the selfish type choose the equal allocation (Eq), and the common prior probability on the fair type is .5.

(ii) In Game II, for both types, choosing the equal allocation (Eq) strictly dominates choosing the efficient allocation (Ef).

Note that in the result (i), the common prior probability on the fair type is consistent with the experimental evidence found by Charness and Rabin (2002), which report that about 50 percent of the subjects chose Ef.

Finally, to conclude this section, an implication of Proposition 4 on the use of risk over roles is discussed. Currently, in many experimental studies, subjects are required to make decisions under risk over roles. However, Proposition 4 shows that risk over roles induces subjects with EAP Piecewise preferences to choose the efficient allocations, even if they do not have a preference for efficiency itself. Indeed, this implication is consistent with the experimental evidence, found by Bolton and Ockenfels (2006), that under risk over roles, subjects tend to choose efficient but unequal allocations over equal but inefficient allocations.

7 Concluding Remarks on Axioms

To conclude the paper, the relationships among the key axioms used in Anscombe and Aumann (1963), Seo (2009), and our model are discussed. As noted, we will show that State-wise Indifference, which is a strengthening of Indifference by dropping the support-wise criterion, is equivalent with Reversal of Order. This result means that what makes the difference between Indifference and Reversal of Order is the support-wise criterion.

Based on the result above, we also show that Reversal of Order implies Indifference but not vice versa, and Indifference, in turn, implies Reduction of Compound Lotteries but not vice versa. This result clarifies the difference between Indifference and Seo's Dominance. This is because, as noted, under Dominance, Reversal of Order and Indifference become equivalent; so, it is impossible to distinguish between ex-ante and ex-post mixtures as long as we assume the standard axiom on the reduction.

First, Reversal of Order by Anscombe and Aumann (1963) is formally defined as follows:

Axiom (Reversal of Order): For all set $\{f^i\}_{i=1}^n$ of acts and set $\{\alpha_i\}_{i=1}^n$ of nonnegative numbers such that $\sum_{i=1}^{n} \alpha_i = 1$,

$$\alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n \sim \alpha_1 f^1 + \cdots + \alpha_n f^n.$$

As noted, Reversal of Order turns out to be equivalent to the following axiom:

Axiom (State-wise Indifference): For all $P, Q \in \Delta(\mathscr{F})$,

$$(P_s)_s \sim (Q_s)_s \Rightarrow P \sim Q.$$

Lemma 1: *Reversal of Order and State-wise Indifference are equivalent.*

Note that State-wise Indifference is a strengthening of Indifference by dropping the requirement of the support-wise criterion. Hence,

Corollary 4: *Reversal of Order implies Indifference.*

As the example illustrated by Figure 3 in Section 4 shows, the opposite of Corollary 4 is not true. However, Indifference implies Reversal of Order *among constant acts*. Formally,

Axiom (Reduction of Compound Lotteries): For all set $\{l^i\}_{i=1}^{n}$ of lotteries and set $\{\alpha_i\}_{i=1}^{n}$ of nonnegative numbers such that $\sum_{i=1}^{n} \alpha_i = 1$,

$$\alpha_1 l^1 \oplus \cdots \oplus \alpha_n l^n \sim \alpha_1 l^1 + \cdots + \alpha_n l^n.$$

Lemma 2: *Indifference implies Reduction of Compound Lotteries.*

As noted, Seo (2009) proposes an axiom of his own, *Dominance*, instead of Reversal of Order. To present the axiom, we must first introduce preliminary notations. For each $f \in \mathscr{F}$ and $\mu \in \Delta(S)$, $\Psi(f, \mu) = \mu(s_1) f_{s_1} + \cdots + \mu(s_{|S|}) f_{s_{|S|}} \in \Delta(Z)$.[24] In addition, for each $P \in \Delta(\mathscr{F})$ and $\mu \in \Delta(S)$, $\Psi(P, \mu) = P(f^1)\Psi(f^1, \mu) \oplus \cdots \oplus P(f^n)\Psi(f^n, \mu)$, where $P = P(f^1) f^1 \oplus \cdots \oplus P(f^n) f^n$. Now, his axiom can be stated as follows:

[24]Seo (2009) assumes that the set of states is finite.

Axiom (Dominance, Seo (2009)): For all $P, Q \in \Delta(\mathscr{F})$,

$$\Psi(P,\mu) \succsim \Psi(Q,\mu) \text{ for all } \mu \in \Delta(S) \Rightarrow P \succsim Q.$$

Under Dominance, Seo (2009, p. 1587, Lemma 5.1) shows the equivalence between Reduction of Compound Lotteries and Reversal of Order for $n = 2$. This equivalence can be extended immediately to any finite n. This observation together with Corollary 4 imply the following result:

Corollary 5: *Under Reduction of Compound Lotteries, Dominance implies Indifference.*

Therefore, under Reduction of Compound Lotteries, Dominance together with the axioms used in Theorems 1 and 2 (except Indifference) respectively imply EAP Maxmin and EAP Piecewise preferences with $\delta = 1$.

Appendix: Proofs

Section A provides a sketch of the proofs of the sufficiencies for Theorems 1 and 2. Section B provides the proofs for Lemmas. The proofs of Theorem 1 and related results are in Section C, while Section D presents the proofs of Theorem 2 and related results.

A Sketch of Proofs

First, a sketch of the proof for the sufficiency in Theorem 1 is provided. By the standard argument, there exists a function V representing $\succsim$ on $\Delta(\mathscr{F})$, which is unique up to positive affine transformation. Ex-post Ambiguity Aversion, Ex-ante/Ex-post Certainty Independence, and Indifference will show that V can be taken so that the restriction U of V on $\mathscr{F}$ has a Maxmin representation. That is, there exists a set C of priors and a mixture linear function u on $\Delta(Z)$ such that $U(f) = \min_{\mu \in C} \int_S u(f_s) d\mu(s)$.

Then, for all $P \in \Delta(\mathscr{F})$,

$$U((P_s)_s) = \min_{\mu \in C} \int_S \Big(\int_{\mathscr{F}} u(f_s) dP(f) \Big) d\mu(s); \quad U(l_P) = \int_{\mathscr{F}} \Big(\min_{\mu \in C} \int_S u(f_s) d\mu(s) \Big) dP(f). \quad (5)$$

Hence, it follows from Jensen's inequality that $U((P_s)_s) \geq U(l_P)$ for all $P \in \Delta(\mathscr{F})$. Define

$$\mathscr{C} = \Big\{ (u(l), u(l)) \in \mathbb{R}^2 \mid l \in \Delta(Z) \Big\}; \quad \mathscr{D} = \Big\{ \big(U((P_s)_s), U(l_P)\big) \in \mathbb{R}^2 \mid P \in \Delta(\mathscr{F}) \Big\}. \quad (6)$$

We now can show that $\mathscr{C}$ consists of the upper boundary of $\mathscr{D}$ as in Figure 5. In addition, if $(x, y) \in \mathscr{D}$, $(c, c) \in \mathscr{C}$, and $\alpha \in [0, 1]$, then $\alpha(x, y) + (1 - \alpha)(c, c) \in \mathscr{D}$.

Define a binary relation $\hat{\succsim}$ on $\mathscr{D}$: for all $(x, y), (x', y') \in \mathscr{D}$,

$$(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow V(P) \geq V(Q),$$

where $P, Q \in \Delta(\mathscr{F})$, $\big(U((P_s)_s), U(l_P)\big) = (x, y)$, and $\big(U((Q_s)_s), U(l_Q)\big) = (x', y')$. Indifference will show that $\hat{\succsim}$ is a well-defined binary relation. The purpose of the proof is to show that there exists a real number δ such that for any (x, y) and $(x', y') \in \mathscr{D}$, $(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow \delta x + (1 - \delta) y \geq \delta x' + (1 - \delta) y'$. Together with the definition of $\hat{\succsim}$, this implies that

$$V(P) \geq V(Q) \Leftrightarrow \delta U((P_s)_s) + (1 - \delta) U(l_P) \geq \delta U((Q_s)_s) + (1 - \delta) U(l_Q).$$

Since both V and U coincide with u on $\Delta(Z)$, it follows from the cardinal uniqueness of V that $V(P) = \delta U((P_s)_s) + (1 - \delta) U(l_P)$ for all P, as desired.

In the following, we sketch how to show the existence of the desired real number δ.[25] It will be shown that $\hat{\succsim}$ satisfies completeness, transitivity, monotonicity on $\mathscr{C}$, and *certainty*

[25]Note that, the continuity of $\hat{\succsim}$ does not follow directly from the continuity of $\succsim$. In addition, in $\mathbb{R}^2$, it is well-known that in general, additive linear representation requires more than Independence. (See Debrue (1960).) So, the standard argument might not show the existence of the desired δ directly.

independence:

$$(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow \alpha(x, y) + (1 - \alpha)(c, c) \mathrel{\hat{\succsim}} \alpha(x', y') + (1 - \alpha)(c, c). \tag{7}$$

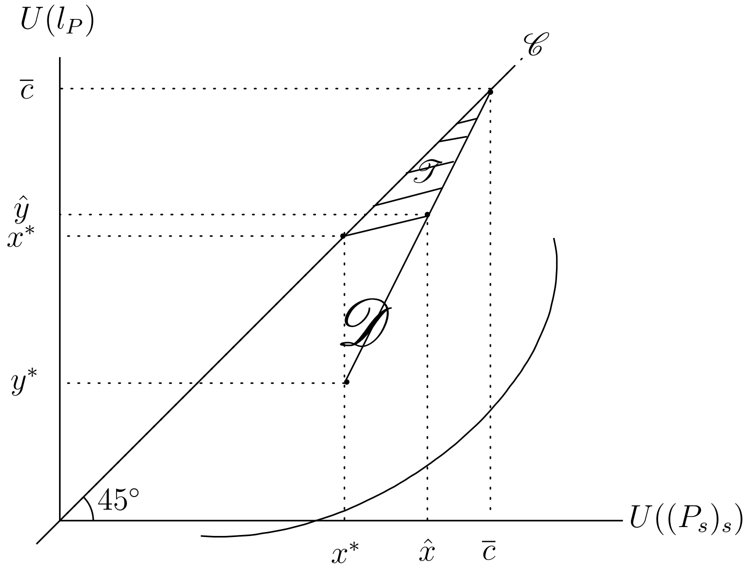

Figure 5: Indifference Curves of $\hat{\succsim}$.

If the set C of priors is degenerate, the existence of the δ is trivial.[26] So, suppose that C is nondegenerate. Then, there exist $f^*, g^* \in \mathscr{F}$ such that $\frac{1}{2}f^* + \frac{1}{2}g^* \succ f^* \sim g^*$. Let $P^* = \frac{1}{2}f^* \oplus \frac{1}{2}g^*$ and define $x^* = U((P^*_s)_s)$ and $y^* = U(l_{P^*})$. Then, $x^* > y^*$. Now, consider the case where $\frac{1}{2}f^* + (1-\alpha)g^* \succsim \frac{1}{2}f^* \oplus (1-\alpha)g^*$.[27] This implies that $(x^*, x^*) \mathrel{\hat{\succsim}} (x^*, y^*)$. Without loss of generality, assume there exists $\overline{c} > x^*$ such that $(\overline{c}, \overline{c}) \in \mathscr{D}$. Then, it follows from the monotonicity that $(\overline{c}, \overline{c}) \mathrel{\hat{\succ}} (x^*, x^*)$. Hence, Continuity of $\succsim$ will show the existence of $\overline{\alpha}$ such that $(x^*, x^*) \mathrel{\hat{\sim}} \overline{\alpha}(x^*, y^*) + (1-\overline{\alpha})(\overline{c}, \overline{c})$.

With this $\overline{\alpha}$, define $(\hat{x}, \hat{y}) = (\overline{\alpha}x^* + (1-\overline{\alpha})\overline{c}, \overline{\alpha}y^*(1-\overline{\alpha})\overline{c})$. Then, $(\hat{x}, \hat{y}) \sim (x^*, x^*)$. Now define $\mathscr{T}$ be a triangle, including the interior, which consists of the vertices $(\overline{c}, \overline{c})$, (x^*, x^*), and $(\hat{x}, \hat{y})$. It follows that $\mathscr{T} \subset \mathscr{D}$ and $\mathscr{T}$ is not degenerate. The certainty independence of $\hat{\succsim}$ and the Carathéodory's Theorem show that the indifference curves on $\mathscr{T}$ are parallel, as

[26]See Step 3 in the proof of Theorem 1 for details.

[27]In the other case where $\frac{1}{2}f^* \oplus (1-\alpha)g^* \succsim \frac{1}{2}f^* + (1-\alpha)g^*$, an analogous argument holds. See footnote 30 for details.

shown in Figure 5. Since $(x^*, x^*) \hat{\sim} (\hat{x}, \hat{y})$, the δ is determined to be $1 - (\hat{x} - x^*)/(\hat{x} - \hat{y})$. Finally, given that $\mathscr{C}$ consists of the upper boundary of both $\mathscr{D}$ and $\mathscr{T}$, the certainty independence of $\hat{\succsim}$ again will show that the indifference curves can be expanded over the whole domain $\mathscr{D}$. This completes the proof of Theorem 1.

In Theorem 2, the sufficiency of axioms is shown as follows. First, we show that $\succsim$ restricted on $\mathscr{F}$ has Fehr and Schmidt's (1999) utility representation. Given the representation on $\mathscr{F}$, the rest of the proof is the same as the proof of Theorem 1.

B Proof of Lemmas

B.1 Proof of Lemma 1

To see that Reversal of Order implies State-wise Indifference, fix $P, Q \in \Delta(\mathscr{F})$ such that $(P_s)_s \sim (Q_s)_s$ to show $P \sim Q$. Then, there exist sets $\{f^i\}_{i=1}^n$ and $\{g^j\}_{j=1}^m$ of acts and sets $\{\alpha_i\}_{i=1}^n$ and $\{\beta_j\}_{j=1}^m$ of nonnegative numbers such that $\sum_{i=1}^n \alpha_i = 1 = \sum_{j=1}^m \beta_j$, $P = \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n$, and $Q = \beta_1 g^1 \oplus \cdots \oplus \beta_m g^m$. Then, Reversal of Order shows $P \sim \alpha_1 f^1 + \cdots + \alpha_n f^n = (P_s)_s \sim (Q_s)_s = \beta_1 g^1 + \cdots + \beta_m g^m \sim Q$.

To see that State-wise Indifference implies Reversal of Order, fix any set $\{f^i\}_{i=1}^n$ of acts and set $\{\alpha_i\}_{i=1}^n$ of nonnegative numbers such that $\sum_{i=1}^n \alpha_i = 1$. Let $P = \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n$ and $Q = \alpha_1 f^1 + \cdots + \alpha_n f^n$ to show $P \sim Q$. Then for all $s \in S$, $P_s = \alpha_1 f_s^1 + \cdots + \alpha_n f_s^n = Q_s$, so that $(P_s)_s \sim (Q_s)_s$. Then, State-wise Indifference shows $P \sim Q$.

B.2 Proof of Lemma 2

To see that Indifference implies Reduction of Compound Lotteries, fix any set $\{l^i\}_{i=1}^n$ of lotteries and set $\{\alpha_i\}_{i=1}^n$ of nonnegative numbers such that $\sum_{i=1}^n \alpha_i = 1$. Let $P = \alpha_1 l^1 \oplus \cdots \oplus \alpha_n l^n$ and $Q = \alpha_1 l^1 + \cdots + \alpha_n l^n$ to show $P \sim Q$. Then, $P_s = \alpha_1 l^1 + \cdots + \alpha_n l^n = Q_s$ for all $s \in S$, so that condition (i) is satisfied. In addition, $l_P = \alpha_1 l^1 + \cdots + \alpha_n l^n = l_Q$, so that condition (ii) is also satisfied. Hence, Indifference implies $P \sim Q$.

C Proof of Theorem 1

The necessity of axioms is easy to check. Note that Monotonicity is imposed only on the set $\mathscr{F}$ of acts. So, EAP Maxmin preferences immediately satisfy the axiom since the preferences reduce into Gilboa and Schmeidler's (1989) Maxmin preferences on the restricted domain $\mathscr{F}$. To show Continuity, note that the set of finitely additive probabilities measures is compact under the product topology. So, the closed subset C is compact. Hence, the Berge's Maximum Theorem can be applied.

In the following, we will prove the sufficiency. Suppose that a preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem 1. Then, by Lemma 2, $\succsim$ satisfies Reduction of Compound Lotteries as well. The first step shows Reversal of Order between generic acts and constant acts, which will be used in the next step.

Step 1: For all $\alpha \in [0,1]$, $f \in \mathscr{F}$, and $l \in \Delta(Z)$, $\alpha f \oplus (1-\alpha)l \sim \alpha f + (1-\alpha)l$.

Proof of Step 1: Fix $\alpha \in [0,1]$, $f \in \mathscr{F}$, and $l \in \Delta(Z)$. Let $P = \alpha f \oplus (1-\alpha)l$ to show $P \sim \alpha f + (1-\alpha)l$. Then by Reduction of Compound Lotteries, for all $s \in S$, $P_s = \alpha f_s \oplus (1-\alpha)l \sim \alpha f_s + (1-\alpha)l$. So, by Monotonicity, $(P_s)_s \sim \alpha f + (1-\alpha)l$, so that the condition (i) in Indifference is satisfied. In addition, since $l_f \sim f$, Ex-post Certainty Independence shows $l_P = \alpha l_f + (1-\alpha)l \sim \alpha f + (1-\alpha)l$, so that the condition (ii) in Indifference is satisfied as well. Hence, Indifference shows $P \sim \alpha f + (1-\alpha)l$. ■

Step 2: There exists a function $V : \Delta(\mathscr{F}) \to \mathbb{R}$ such that

(i) V represents $\succsim$ on $\Delta(\mathscr{F})$,

(ii) for all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $V(\alpha P \oplus (1-\alpha)l) = \alpha V(P) + (1-\alpha)V(l)$,

(iii) V is unique up to positive affine transformation.

(iv) Let U be the restriction of V on $\mathscr{F}$. There exists a nonempty convex closed set C of finitely additive probability measures on Σ, and a mixture linear function $u : \Delta(Z) \to \mathbb{R}$ such that $U(f) = \min_{\mu \in C} \int_S u(f_s) d\mu(s)$.

Proof of Step 2: From the implication of the von Neumann-Morgenstern's Theorem, there exists a mixture linear function $u : \Delta(Z) \to \mathbb{R}$ representing $\succsim$ restricted to $\Delta(Z)$. In addition, u is unique up to positive affine transformation. So, choose u such that $u(z_+) = 1$

and $u(z_-) = -1$.

For an arbitrary $P \in \Delta(\mathscr{F})$, define

$$M_P = \{\alpha P \oplus (1-\alpha)l | l \in \Delta(Z) \text{ and } \alpha \in [0,1]\}.$$

Thus, M_P is the set of ex-ante mixtures of P and the constant acts. Using the von Neumann-Morgenstern's Theorem again, there is a function $V_P : M_P \to \mathbb{R}$ representing $\succsim$ restricted to M_P, which is linear with respect to the ex-ante mixtures. In addition, again, V_P is unique up to positive affine transformation. So, choose V_P such that $V_P(z_+) = 1$ and $V_P(z_-) = -1$.

For all $l, r \in \Delta(Z)$ $V_P(l) \geq V_P(r) \Leftrightarrow l \succsim r \Leftrightarrow u(l) \geq u(r)$. Hence, there exists an increasing function $v : u(\Delta(Z)) \to \mathbb{R}$ such that $V_P(l) = v(u(l))$ for all $l \in \Delta(Z)$. Moreover, since V_p and u are mixture linear, it follows from Reduction of Compound Lotteries that v is also mixture linear.[28] In addition, by the normalization, $v(1) = 1$ and $v(-1) = -1$. Hence, we can conclude that v is the identity function, so that $V_P(l) = u(l)$.

Now, we define a real valued function V on $\Delta(\mathscr{F})$ which represents $\succsim$ by $V(P) = V_P(P)$ for all $P \in \Delta(\mathscr{F})$. Note that V is well-defined, because if $R \in M_P \cap M_Q$, then $V_P(R) = V_Q(R)$. In addition, $V(\alpha P \oplus (1-\alpha)l) = \alpha V(P) + (1-\alpha)V(l)$ for all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$. Hence, parts (i), (ii), and (iii) hold.

Finally, to show (iv), let U be the restriction of V on $\mathscr{F}$. Fix $\alpha \in [0,1]$, $f \in \mathscr{F}$, and $l \in \Delta(Z)$. Then by Step 1, $\alpha f + (1-\alpha)l \sim \alpha f \oplus (1-\alpha)l$. Hence, $U(\alpha f + (1-\alpha)l) = V(\alpha f \oplus (1-\alpha)l) = \alpha V(f) + (1-\alpha)V(l) = \alpha U(f) + (1-\alpha)U(l)$, where the second equality is by Step 2 (ii). Hence, by Ex-post Ambiguity Aversion and Continuity (ii), part (iv) follows from Gilboa and Schmeidler (1989). ■

STEP 3: If C is degenerate then there exists a real number δ such that for all $P \in \Delta(\mathscr{F})$, $V(P) = \delta U((P_s)_s) + (1-\delta)\int_{\mathscr{F}} U(f)dP(f)$.

PROOF OF STEP 3: Suppose $C = \{\mu^*\}$ for some $\mu^* \in \Delta(S)$. Then for all $P \in \Delta(\mathscr{F})$, $U((P_s)_s) = \int_S \int_{\mathscr{F}} u(f_s)dP(f)d\mu^*(s) = \int_{\mathscr{F}} \int_S u(f_s)d\mu^*(s)dP(f) = U(l_P)$, where the second

[28] Choose $a, b \in u(\Delta(Z))$ and $\alpha \in [0,1]$ to show $v(\alpha a + (1-\alpha)b) = \alpha v(a) + (1-\alpha)v(b)$. There exist $l, r \in \Delta(Z)$ such that $u(l) = a$ and $u(r) = b$. Then by Reduction of Compound Lotteries, $v(\alpha a + (1-\alpha)b) = v(u(\alpha l + (1-\alpha)r)) = V_P(\alpha l + (1-\alpha)r) = V_P(\alpha l \oplus (1-\alpha)r) = \alpha V_P(l) + (1-\alpha)V_P(r) = \alpha v(a) + (1-\alpha)v(b)$.

equality is by the Fubini's Theorem. Therefore, $(P_s)_s \sim l_P$. Hence, Indifference shows that for all $P \in \Delta(\mathscr{F})$, $P \sim (P_s)_s \sim l_P$. Therefore, by Step 2, $V(P) = U((P_s)_s) = \int_{\mathscr{F}} U(f)dP(f)$. So, the result holds. ■

Henceforth, consider the case where C is nondegenerate. First, to make notations simple, for all $P \in \Delta(\mathscr{F})$, define

$$\zeta(P) = U\big((P_s)_s\big); \quad \eta(P) = U(l_P).$$

The next step shows the property of the functions ζ and η as follows:

STEP 4:
(i) For all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $\zeta(\alpha P \oplus (1-\alpha)l) = \alpha\zeta(P) + (1-\alpha)\zeta(l)$ and $\eta(\alpha P \oplus (1-\alpha)l) = \alpha\eta(P) + (1-\alpha)\eta(l)$.
(ii) For all $l \in \Delta(Z)$, $\zeta(l) = u(l) = \eta(l)$.
(iii) For all $P \in \Delta(\mathscr{F})$, $\zeta(P) \geq \eta(P)$.
PROOF OF STEP 4: Parts (i) and (ii) follow from Step 2 (iv). To show (iii), for all $x \in \mathbb{R}^{|S|}$ define $F : \mathbb{R}^{|S|} \to \mathbb{R}$ by $F(x) = \min_{\mu \in C} \int_S x_s \mu(s)$. Then F is concave. Therefore, for all $P \in \Delta(\mathscr{F})$, Jensen's Inequality (Hiriart-Urruty and Lemaréchal (1949, p. 76, Theorem 1.1.8)) shows $\zeta(P) = F\big(\big(\int_{\mathscr{F}} u(f_s)dP(f)\big)_{s\in S}\big) \geq \int_{\mathscr{F}} F\big(\big(u(f_s)\big)_{s\in S}\big)dP(f) = \eta(P)$. ■

Subsets $\mathscr{C}$ and $\mathscr{D}$ of $\mathbb{R}^2$ are defined by (6) in Section A. The next step shows that $\mathscr{C}$ and $\mathscr{D}$ are as in Figure 5 in Section A as follows:

STEP 5:
(i) $\mathscr{C} \subset \partial\mathscr{D}$, where $\partial\mathscr{D}$ is the boundary of $\mathscr{D}$.
(ii) For all $(x,y) \in \mathscr{D}$, $(c,c) \in \mathscr{C}$, and $\alpha \in [0,1]$, $\alpha(x,y) + (1-\alpha)(c,c) \in \mathscr{D}$.
PROOF OF STEP 5: By Step 4 (iii), for all $(x,y) \in \mathscr{D}$, $x \geq y$. Hence, $\mathscr{C} \subset \partial\mathscr{D}$. Now we will show (ii). Choose any $(x,y) \in \mathscr{D}$, $(c,c) \in \mathscr{C}$, and $\alpha \in [0,1]$. Then, there exist $P \in \Delta(\mathscr{F})$ and $l \in \Delta(Z)$ such that $(x,y) = (\zeta(P), \eta(P))$and $\zeta(l) = c = \eta(l)$. Hence, by Step 4 (i), $\zeta(\alpha P \oplus (1-\alpha)l) = \alpha\zeta(P) + (1-\alpha)\zeta(l) = \alpha x + (1-\alpha)c$ and $\eta(\alpha P \oplus (1-\alpha)l) = \alpha\eta(P) + (1-\alpha)\zeta(l) = \alpha y + (1-\alpha)c$. Therefore, $\alpha(x,y) + (1-\alpha)(c,c) \in \mathscr{D}$. ■

To define a binary relation $\hat{\succsim}$ on $\mathscr{D}$, first define $v : \mathscr{D} \to \mathbb{R}$ by for all $(x, y) \in \mathscr{D}$,

$$v(x, y) = V(P),$$

where $P \in \Delta(\mathscr{F})$ such that $\zeta(P) = x$ and $\eta(P) = y$.

STEP 6: v is well-defined, i.e., if $v(x, y) \neq v(x', y')$, then $(x, y) \neq (x', y')$.
PROOF OF STEP 6: Choose any $(x, y), (x', y') \in \mathscr{D}$ such that $v(x, y) \neq v(x', y')$. Assume to the contrary that $(x, y) = (x', y')$. Then, by definition, there exist $P, Q \in \Delta(\mathscr{F})$ such that $(\zeta(P), \eta(P)) = (x, y)$ and $(\zeta(Q), \eta(Q)) = (x', y')$. Hence, $(\zeta(P), \eta(P)) = (\zeta(Q), \eta(Q))$. Hence, $U((P_s)_s) = \zeta(P) = \zeta(Q) = U((Q_s)_s)$, so that the condition (i) in Indifference is satisfied. In addition, $U(l_P) = \int_{\mathscr{F}} U(f)dP(f) = \eta(P) = \eta(Q) = \int_{\mathscr{F}} U(f)dQ(f) = U(l_Q)$, so that the condition (ii) in Indifference is satisfied as well. Therefore, Indifference shows $v(x, y) = V(P) = V(Q) = v(x', y')$, which is a contradiction. Hence, $(x, y) \neq (x', y')$. ■

Now, define a binary relation $\hat{\succsim}$ on $\mathscr{D}$ by for all $(x, y), (x', y') \in \mathscr{D}$,

$$(x, y) \hat{\succsim} (x', y') \Leftrightarrow v(x, y) \geq v(x', y').$$

The next step shows the property of $\hat{\succsim}$ as follows:

STEP 7: $\hat{\succsim}$ satisfies completeness, transitivity, monotonicity on $\mathscr{C}$, and the certainty independence defined by (7) in Section A.
PROOF OF STEP 7: Since v is a well-defined real valued function, the completeness and transitivity are trivial. First, we will show the monotonicity on $\mathscr{C}$. Choose any $(c, c), (c', c') \in \mathscr{C}$. Then there exist $l, l' \in \Delta(Z)$ such that $u(l) = c$ and $u(l') = c'$. Hence, $(c, c) \hat{\succsim} (c', c') \Leftrightarrow v(u(l), u(l)) \geq v(u(l'), u(l')) \Leftrightarrow V(l) \geq V(l') \Leftrightarrow u(l) \geq u(l') \Leftrightarrow c \geq c'$.

Next, we will show the certainty independence. Choose any $(x, y), (x', y'), (c, c) \in \mathscr{D}$ and $\alpha \in [0, 1]$. By Step 5 (ii), $\alpha(x, y) + (1-\alpha)(c, c), \alpha(x', y') + (1-\alpha)(c, c) \in \mathscr{D}$. Then, there exist $P, Q \in \Delta(\mathscr{F})$ and $l \in \Delta(Z)$ such that $(x, y) = (\zeta(P), \eta(P))$, $(x', y') = (\zeta(Q), \eta(Q))$, and $(c, c) = (\zeta(l), \eta(l))$. By Step 4 (i), $\alpha(x, y) + (1-\alpha)(c, c) = (\zeta(\alpha P \oplus (1-\alpha)l), \eta(\alpha P \oplus (1-\alpha)l))$

and $\alpha(x',y')+(1-\alpha)(c,c)=(\zeta(\alpha Q\oplus(1-\alpha)l),\eta(\alpha Q\oplus(1-\alpha)l))$. Therefore,

$$\begin{aligned}
&(x,y)\,\hat{\succsim}\,(x',y')\\
&\Leftrightarrow v\big(\zeta(P),\eta(P)\big)\geq v\big(\zeta(Q),\eta(Q)\big)\\
&\Leftrightarrow V(P)\geq V(Q)\\
&\Leftrightarrow \alpha V(P)+(1-\alpha)V(l)\geq \alpha V(Q)+(1-\alpha)V(l)\\
&\Leftrightarrow V(\alpha P\oplus(1-\alpha)l)\geq V(\alpha Q\oplus(1-\alpha)l) && (\because \text{Step 2 (ii)})\\
&\Leftrightarrow v\big(\zeta(\alpha P\oplus(1-\alpha)l),\eta(\alpha P\oplus(1-\alpha)l)\big)\geq v\big(\zeta(\alpha Q\oplus(1-\alpha)l),\eta(\alpha Q\oplus(1-\alpha)l)\big)\\
&\Leftrightarrow \alpha(x,y)+(1-\alpha)(c,c)\,\hat{\succsim}\,\alpha(x',y')+(1-\alpha)(c,c).
\end{aligned}$$

■

Because of the nondegeneracy of C, there exist $f^*,g^*\in\mathscr{F}$ such that $f^*\sim g^*$ and $\frac{1}{2}f^*+\frac{1}{2}g^*\succ f^*$.[29] Define $(x^*,y^*)=\big(\zeta(\frac{1}{2}f^*\oplus\frac{1}{2}g^*),\eta(\frac{1}{2}f^*\oplus\frac{1}{2}g^*)\big)\in\mathscr{D}$. Hence, $x^*=\zeta(\frac{1}{2}f^*\oplus\frac{1}{2}g^*)=U(\frac{1}{2}f^*+\frac{1}{2}g^*)>U(f^*)=\eta(\frac{1}{2}f^*\oplus\frac{1}{2}g^*)=y^*$. By Nondegeneracy of $\succsim$, there exist $\overline{c}$ or $\underline{c}$ such that $\overline{c}>x^*$ or $x^*>\underline{c}$. By the mixture linearity of u, without loss of generality, assume $\overline{c}>x^*>\underline{c}$.

To define the set $\mathscr{T}$ as in Section A, the next step is proved.

Step 8:
(i) If $(x^*,x^*)\,\hat{\succsim}\,(x^*,y^*)$, then there exist $\overline{\alpha}>0$ such that $(x^*,x^*)\,\hat{\sim}\,\overline{\alpha}(x^*,y^*)+(1-\overline{\alpha})(\overline{c},\overline{c})$.
(ii) If $(x^*,y^*)\,\hat{\succsim}\,(x^*,x^*)$, then there exist $\underline{\alpha}>0$ such that $(x^*,x^*)\,\hat{\sim}\,\underline{\alpha}(x^*,y^*)+(1-\underline{\alpha})(\underline{c},\underline{c})$.

Proof of Step 8: We will show (i). By the monotonicity, $(\overline{c},\overline{c})\,\hat{\succ}\,(x^*,x^*)\,\hat{\succsim}\,(x^*,y^*)$. Then there exist $\bar{l}\in\Delta(Z)$ such that $u(\bar{l})=\overline{c}$ and $\bar{l}\succsim\frac{1}{2}f^*+\frac{1}{2}g^*\succsim\frac{1}{2}f^*\oplus\frac{1}{2}g^*$. Then by Continuity (i) of $\succsim$, there exists $\overline{\alpha}\in[0,1]$ such that $\frac{1}{2}f^*+\frac{1}{2}g^*\sim\overline{\alpha}(\frac{1}{2}f^*\oplus\frac{1}{2}g^*)\oplus(1-\overline{\alpha})\bar{l}$. Let $\hat{f}=\overline{\alpha}f^*+(1-\overline{\alpha})\bar{l}$ and $\hat{g}=\overline{\alpha}g^*+(1-\overline{\alpha})\bar{l}$. Then $\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g}\succ\hat{f}\sim\hat{g}$ and $\frac{1}{2}\hat{f}\oplus\frac{1}{2}\hat{g}\sim\frac{1}{2}f^*+\frac{1}{2}g^*$. Hence, $\overline{\alpha}(x^*,y^*)+(1-\overline{\alpha})(\overline{c},\overline{c})=(\zeta,\eta)(\frac{1}{2}\hat{f}\oplus\frac{1}{2}\hat{g})\hat{\sim}(\zeta,\eta)(\frac{1}{2}f^*+\frac{1}{2}g^*)=(x^*,x^*)$. Part (ii) is proved in the same way. ■

Henceforth, consider the case where $(x^*,x^*)\,\hat{\succsim}\,(x^*,y^*)$. Denote $\big(\overline{\alpha c}+(1-\overline{\alpha})x^*,\overline{\alpha c}+(1-\overline{\alpha})y^*\big)$ by $(\hat{x},\hat{y})$.[30] Then Step 8 shows $(\hat{x},\hat{y})\sim(x^*,x^*)$.

[29] Otherwise, $f\sim g\Rightarrow\frac{1}{2}f+\frac{1}{2}g\sim f$ for all $f,g\in\mathscr{F}$. This implies the subjective expected utility, so C becomes degenerate.

[30] In the other case where $(x^*,y^*)\succsim(x^*,x^*)$, denote $\big(\underline{\alpha c}+(1-\underline{\alpha})x^*,\underline{\alpha c}+(1-\underline{\alpha})y^*\big)$ by $(\tilde{x},\tilde{y})$. Then,

Define

$$\mathscr{T} = \Big\{(x,y) \in \mathbb{R}^2 \quad | \; x \geq y, \; \big\langle (x^* - \hat{x}, \hat{y} - x^*), (x,y) - (x^*, x^*) \big\rangle \geq 0, \\ \text{and } \big\langle (\overline{c} - \hat{x}, \hat{y} - \overline{c}), \quad (x,y) - (\overline{c}, \overline{c}) \big\rangle \geq 0 \Big\}, \tag{8}$$

where $\langle \cdot, \cdot \rangle$ is a inner product. The set $\mathscr{T}$ is a triangle including the interior which consists of the vertices $(\overline{c}, \overline{c})$, (x^*, x^*), and $(\hat{x}, \hat{y})$ as shown in Figure 5 in Section A.

Step 9: $\mathscr{T}$ is nondegenerate and $\mathscr{T} \subset \mathscr{D}$.

Proof of Step 9: Since $x^* > y^*$ and, in addition, $\overline{\alpha} > 0$, then $\hat{x} > \hat{y}$. Therefore, $(x^*, x^*) \neq (\hat{x}, \hat{y}) \neq (\overline{c}, \overline{c})$. Hence, $\mathscr{T}$ is not degenerate. Choose any $(x, y) \in \mathscr{T}$ to show $(x, y) \in \mathscr{D}$. Since $\mathscr{T}$ is the triangle, the Carathéodory's Theorem (Hiriart-Urruty and Lemaréchal (1949, p. 29, Theorem 1.3.6)) shows that there exist $\alpha, \beta \in [0, 1]$ such that $(x, y) = \alpha(\overline{c}, \overline{c}) + \beta(x^*, x^*) + (1 - \alpha - \beta)(\hat{x}, \hat{y})$. Now, let $c = \frac{\alpha}{\alpha+\beta}\overline{c} + \frac{\beta}{\alpha+\beta}x^*$. Then, $(x, y) = (\alpha + \beta)(c, c) + (1 - \alpha - \beta)(\hat{x}, \hat{y})$. Therefore, since $(\hat{x}, \hat{y}) \in \mathscr{D}$ and $(c, c) \in \mathscr{C}$, it follows from Step 5 (ii) that $(x, y) \in \mathscr{D}$. ■

The next step shows the existence of the desired real number δ on the restricted domain $\mathscr{T}$ as follows:

Step 10: There exists a real number δ such that for any $(x, y), (x', y') \in \mathscr{T}$, $(x, y) \mathrel{\hat{\succsim}} (x', y') \Leftrightarrow \delta x + (1 - \delta)y \geq \delta x' + (1 - \delta)y'$.

Proof of Step 10:

Substep 10.1: For all $(x, y) \in \mathscr{T}$, there exists a unique number $\alpha \in [0, 1]$ such that $(x, y) \sim \alpha(\overline{c}, \overline{c}) + (1 - \alpha)(x^*, x^*)$.

Proof of Substep 10.1: Choose any $(x, y) \in \mathscr{T}$. Since $\mathscr{T}$ is the triangle, the Carathéodory's Theorem, again, shows that there exist $\alpha, \beta \in [0, 1]$ such that $(x, y) = \alpha(\overline{c}, \overline{c}) + \beta(x^*, x^*) + (1 - \alpha - \beta)(\hat{x}, \hat{y})$. Since $(\hat{x}, \hat{y}) \hat{\sim} (x^*, x^*)$, the transitivity and the certainty independence shows $(x, y) \hat{\sim} \alpha(\overline{c}, \overline{c}) + (1 - \alpha)(x^*, x^*)$. Since $\overline{c} > x^*$, the monotonicity of $\hat{\succsim}$ on $\mathscr{C}$ shows that α is unique.

instead of the triangle $\mathscr{T}$ defined by (8), consider a triangle, including the interior, which consists of the vertices $(\tilde{c}, \tilde{c})$, $(\underline{c}, \underline{c})$, and (x^*, x^*). Then, the rest of the proof goes through exactly in the same way.

For all $(x,y) \in \mathscr{T}$, define $c(x,y) = \alpha\bar{c} + (1-\alpha)x^*$, where α is as in Substep 10.1.

SUBSTEP 10.2: For all $(x,y) \in \mathscr{T}$, $\dfrac{x - c(x,y)}{x-y} = \dfrac{\hat{x} - x^*}{\hat{x} - \hat{y}}$.

PROOF OF SUBSTEP 10.2: Choose any $(x,y) \in \mathscr{T}$. By the proof of Substep 10.1, there exist $\alpha, \beta \in [0,1]$ such that $(x,y) = \alpha(\bar{c},\bar{c}) + \beta(x^*,x^*) + (1-\alpha-\beta)(\hat{x},\hat{y})$. Then,

$$\begin{aligned} \frac{x - c(x,y)}{x-y} &= \frac{x - \alpha\bar{c} - (1-\alpha)x^*}{x-y} && (\because c(x,y) = \alpha\bar{c} + (1-\alpha)x^*) \\ &= \frac{\hat{x} - x^*}{\hat{x} - \hat{y}}. && (\because (x,y) = \alpha(\bar{c},\bar{c}) + \beta(x^*,x^*) + (1-\alpha-\beta)(\hat{x},\hat{y})) \end{aligned}$$

Define $\delta = 1 - \dfrac{\hat{x} - x^*}{\hat{x} - \hat{y}}$. Then Substep 10.2 and the definition of δ shows $c(x,y) = \delta x + (1-\delta)y$ for all $(x,y) \in \mathscr{T}$.

SUBSTEP 10.3: For any $(x,y), (x',y') \in \mathscr{T}$, $(x,y) \hat{\succsim} (x',y') \Leftrightarrow \delta x + (1-\delta)y \geq \delta x' + (1-\delta)y'$.

PROOF OF SUBSTEP 10.3: Choose any $(x,y), (x',y') \in \mathscr{T}$. Then

$$\begin{aligned} (x,y) \hat{\succsim} (x',y') &\Leftrightarrow (c(x,y), c(x,y)) \hat{\succsim} (c(x',y'), c(x',y')) && (\because \text{Substep 10.1}) \\ &\Leftrightarrow c(x,y) \geq c(x',y') && (\because \text{Step 7}) \\ &\Leftrightarrow \delta x + (1-\delta)y \geq \delta x' + (1-\delta)y'. && (\because \text{Substep 10.2}) \end{aligned}$$ ■

The next step shows the existence of the desired δ on the whole domain $\mathscr{D}$ as follows:

STEP 11: For all $(x,y), (x',y') \in \mathscr{D}$, $(x,y) \hat{\succsim} (x',y') \Leftrightarrow \delta x + (1-\delta)y \geq \delta x' + (1-\delta)y'$.

PROOF OF STEP 11: Choose any $(x,y), (x',y') \in \mathscr{D}$. Let $c^* = \frac{1}{2}\bar{c} + \frac{1}{2}x^*$. Since $\mathscr{T}$ is a nondegenerate triangle, there exists a positive number ε such that $\{(x,y) \in \mathscr{D} \mid \|(x,y) - (c^*,c^*)\| < \varepsilon\} \subset \mathscr{T}$. Hence, there exists $\alpha \in (0,1]$ such that $\alpha(x,y) + (1-\alpha)(c^*,c^*)$ and $\alpha(x',y') + (1-\alpha)(c^*,c^*)$ belong to $\{(x,y) \in \mathscr{D}'' \mid \|(x,y) - (c^*,c^*)\| < \varepsilon\} \subset \mathscr{T}$. Therefore,

$$\begin{aligned} (x,y) \hat{\succsim} (x',y') &\Leftrightarrow \alpha(x,y) + (1-\alpha)(c^*,c^*) \hat{\succsim} \alpha(x',y') + (1-\alpha)(c^*,c^*) && (\because \text{Step 7}) \\ &\Leftrightarrow \delta(\alpha x + (1-\alpha)c^*) + (1-\delta)(\alpha y + (1-\alpha)c^*) && (\because \text{Step 10}) \\ &\qquad \geq \delta(\alpha x' + (1-\alpha)c^*) + (1-\delta)(\alpha y' + (1-\alpha)c^*) \\ &\Leftrightarrow \delta x + (1-\delta)y \geq \delta x' + (1-\delta)y'. \end{aligned}$$ ■

STEP 12: For all $P, Q \in \Delta(\mathscr{F})$, $P \succsim Q \Leftrightarrow \delta\zeta(P) + (1-\delta)\eta(P) \geq \delta\zeta(Q) + (1-\delta)\eta(Q)$.

PROOF OF STEP 12: For all $P, Q \in \Delta(\mathscr{F})$, $P \succsim Q \Leftrightarrow V(P) \geq V(Q) \Leftrightarrow v(\zeta(P), \eta(P)) \geq v(\zeta(Q), \eta(Q)) \Leftrightarrow (\zeta(P), \eta(P)) \hat{\succsim} (\zeta(Q), \eta(Q)) \Leftrightarrow \delta\zeta(P) + (1-\delta)\eta(P) \geq \delta\zeta(Q) + (1-\delta)\eta(Q)$, where the last equivalence is by Step 11. ■

Step 12 shows that $\delta\zeta + (1-\delta)\eta$ represents $\succsim$ on $\Delta(\mathscr{F})$. Also by Step 4 (ii), $V = u = \delta\zeta + (1-\delta)\eta$ on $\Delta(Z)$. Since V is unique up to positive affine transformation, hence, $V = \delta\zeta + (1-\delta)\eta$. This completes the proof of Theorem 1.

C.1 Proof of Corollary 1

It is easy to see that (ii) implies (i). So, we will show that (i) implies (ii). Fix $\succsim$ on $\Delta(\mathscr{F})$. Let (δ, C, u) and (δ', C', u') represent $\succsim$ as in Theorem 1, then u and u' are affine representations of $\succsim$ restricted on $\Delta(Z)$. Hence, by the standard uniqueness results, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u = \alpha u' + \beta$. The uniqueness of C follows from Gilboa and Schmeidler (1989). So, $C = C'$.

To show $\delta = \delta'$, let V and V' be as in Theorem 1 defined by (δ, C, u) and (δ', C', u'), respectively. Let U and U' be the restrictions of V and V' on $\mathscr{F}$, respectively. Then, $U = \alpha U' + \beta$. Since C is nondegenerate, there exist $f^*, g^* \in \mathscr{F}$ such that $\frac{1}{2}f^* + \frac{1}{2}g^* \succ f^* \sim g^*$.

CASE 1: $\frac{1}{2}f^* \oplus \frac{1}{2}g^* \succsim \frac{1}{2}f^* + \frac{1}{2}g^*$. By Step 8 in the proof of Theorem 1, there exist $\hat{f}, \hat{g} \in \mathscr{F}$ such that $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ \hat{f} \sim \hat{g}$ and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim \frac{1}{2}f^* + \frac{1}{2}g^*$. Hence, $U(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) > U(\hat{f})$ and $U(\frac{1}{2}f^* + \frac{1}{2}g^*) = \delta U(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) + (1-\delta)U(\hat{f})$. So, $\delta = \frac{U(\frac{1}{2}f^* + \frac{1}{2}g^*) - U(\hat{f})}{U(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) - U(\hat{f})} = \frac{U'(\frac{1}{2}f^* + \frac{1}{2}g^*) - U'(\hat{f})}{U'(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) - U'(\hat{f})} = \delta'$, where the second equality holds because $U = \alpha U' + \beta$.

CASE 2: $\frac{1}{2}f^* + \frac{1}{2}g^* \succsim \frac{1}{2}f^* \oplus \frac{1}{2}g^*$. The proof is the same as Case 1.

C.2 Proof of Proposition 1

Suppose $\succsim$ is an EAP Maxmin preference represented by V as in Theorem 1 with nondegenerate C. Let U be the restriction of V on $\mathscr{F}$. Choose $\alpha \in [0, 1]$ and $f, g \in \mathscr{F}$. Then $V(\alpha f \oplus (1-\alpha)g) = \delta U(\alpha f + (1-\alpha)g) + (1-\delta)(\alpha U(f) + (1-\alpha)U(g))$. By the nondegeneracy of C, there exist $f^*, g^* \in \mathscr{F}$ such that $U(\frac{1}{2}f^* + \frac{1}{2}g^*) > U(f^*) = U(g^*)$.

To show (i), assume $f \sim g$. Then, $V(\alpha f \oplus (1-\alpha)g) \geq U(f) \Leftrightarrow \delta U(\alpha f + (1-\alpha)g) \geq \delta U(f) \Leftrightarrow \delta \geq 0$, where the last equivalence holds because $U(\alpha f + (1-\alpha)g) \geq U(f)$ and $U(\frac{1}{2}f^* + \frac{1}{2}g^*) > U(f^*)$. Hence, $\succsim$ exhibits ex-ante ambiguity aversion if and only if $\delta \geq 0$.

Part (ii) is proved as follows. $U(\alpha f + (1-\alpha)g) \geq V(\alpha f \oplus (1-\alpha)g) \Leftrightarrow (1-\delta)U(\alpha f + (1-\alpha)g) \geq (1-\delta)(\alpha U(f) + (1-\alpha)U(g)) \Leftrightarrow \delta \leq 1$, where the last equivalence holds in the same way as (i). Hence, $\succsim$ exhibits interim ambiguity aversion if and only if $\delta \leq 1$.

C.3 Proof of Proposition 2

Fix two EAP Maxmin preferences $\{\succsim_i\}_{i=1,2}$. Let (δ_i, C_i, u_i) represent $\succsim_i$ as in Theorem 1. Suppose C_i is nondegenerate. Let V_i be as in Theorem 1 defined by (δ_i, C_i, u_i). Let U_i be the restriction of V_i on $\mathscr{F}$.

First, we will prove that (i) implies (ii). Suppose $\succsim_1$ is more ex-ante ambiguity averse than $\succsim_2$. A straightforward argument shows $U_1 = U_2$. Hence, $C_1 = C_2$.[31] In the following, we will show $\delta_1 \geq \delta_2$.

Case 1: $\frac{1}{2}f^* \oplus \frac{1}{2}g^* \succsim_2 \frac{1}{2}f^* + \frac{1}{2}g^*$. By Step 8 in the proof of Theorem 1, there exist $\hat{f}, \hat{g} \in \mathscr{F}$ such that $\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g} \succ_i \hat{f} \sim_i \hat{g}$ and $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \sim_2 \frac{1}{2}f^* + \frac{1}{2}g^*$. Since $\succsim_1$ is more ex-ante ambiguity averse than $\succsim_2$, $\frac{1}{2}\hat{f} \oplus \frac{1}{2}\hat{g} \succsim_1 \frac{1}{2}f^* + \frac{1}{2}g^*$. Since $U_i(\frac{1}{2}\hat{f} + \frac{1}{2}\hat{g}) - U_i(\hat{f}) > 0$, therefore,

$$
\begin{aligned}
\delta_1 &\geq \frac{U_1(\frac{1}{2}f^*+\frac{1}{2}g^*)-U_1(\hat{f})}{U_1(\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g})-U_1(\hat{f})} && \big(\because \delta_1 U_1(\tfrac{1}{2}\hat{f} + \tfrac{1}{2}\hat{g}) + (1-\delta_1)U_1(\hat{f}) = V_1(\tfrac{1}{2}\hat{f} \oplus \tfrac{1}{2}\hat{g}) \geq U_1(\tfrac{1}{2}f^* + \tfrac{1}{2}g^*)\big) \\
&= \frac{U_2(\frac{1}{2}f^*+\frac{1}{2}g^*)-U_2(\hat{f})}{U_2(\frac{1}{2}\hat{f}+\frac{1}{2}\hat{g})-U_2(\hat{f})} && (\because U_1 = U_2) \\
&= \delta_2. && \big(\because \delta_2 U_2(\tfrac{1}{2}\hat{f} + \tfrac{1}{2}\hat{g}) + (1-\delta_2)U_2(\hat{f}) = V_2(\tfrac{1}{2}\hat{f} \oplus \tfrac{1}{2}\hat{g}) = U_2(\tfrac{1}{2}f^* + \tfrac{1}{2}g^*)\big)
\end{aligned}
$$

Case 2: $\frac{1}{2}f^* + (1-\frac{1}{2})g^* \succsim_2 \frac{1}{2}f^* \oplus (1-\frac{1}{2})g^*$. The proof is the same as Case 1.

Next, we will prove that (ii) implies (i). Suppose $\delta_1 \geq \delta_2$, $C_1 = C_2$, and there exist $\alpha > 0$, $\beta \in \mathbb{R}$ such that $u_1 = \alpha u_2 + \beta$. Then, $U_1 = \alpha U_2 + \beta$. Fix any $P \in \Delta(\mathscr{F})$ and $f \in \mathscr{F}$

[31] By normalization, $u_i(z_+) = 1$ and $u_i(z_-) = -1$ for all i. Let, $l_0 = \frac{1}{2}\delta_{z_+} + \frac{1}{2}\delta_{z_-}$. Then, $u_i(l_0) = 0$ for all i. Suppose to the contrary that $U_1 \neq U_2$. Then, without loss of generality, assume that there exists $f \in \mathscr{F}$ such that $U_1(f) > U_2(f)$. Moreover, by the constant linearity, without loss of generality, assume $1 > U_2(f) > 0$. Fix a positive number ε such that $\varepsilon < \min\{U_1(f) - U_2(f), 1 - U_2(f)\}$. Define $l = (U_2(f)+\varepsilon)\delta_{z_+} + (1-U_2(f)-\varepsilon)l_0$. Then $U_i(l) = U_2(f)+\varepsilon < U_1(f)$ for all i. Let $P = \delta_l$. Then, $P \succ_2 f$ but $f \succ_1 P$. This is a contradiction. Hence, $U_1 = U_2$, so that $C_1 = C_2$.

such that $P \succsim_2 f$ to show $P \succsim_1 f$.

Case 1: $U_2((P_s)_s) = \int_{\mathscr{F}} U_2(g)dP(g)$. Then $V_2(P) = U_2((P_s)_s) \geq U_2(f)$. Since $U_1 = \alpha U_2 + \beta$, $V_1(P) = U_1((P_s)_s) \geq U_1(f)$, as desired.

Case 2: $U_2((P_s)_s) \neq \int_{\mathscr{F}} U_2(g)dP(g)$. Since $U_1 = \alpha U_2 + \beta$, $U_1((P_s)_s) \neq \int_{\mathscr{F}} U_1(g)dP(g)$. Therefore,

$$\begin{aligned}
\delta_1 &\geq \delta_2 \\
&\geq \frac{U_2(f) - \int_{\mathscr{F}} U_2(g)dP(g)}{U_2((P_s)_s) - \int_{\mathscr{F}} U_2(g)dP(g)} \quad (\because \delta_2 U_2((P_s)_s) + (1-\delta_2)\int_{\mathscr{F}} U_2(g)dP(g) = V_2(P) \geq U_2(f)) \\
&= \frac{U_1(f) - \int_{\mathscr{F}} U_1(g)dP(g)}{U_1((P_s)_s) - \int_{\mathscr{F}} U_1(g)dP(g)} \quad (\because U_1 = \alpha U_2 + \beta)
\end{aligned}$$

Hence, $V_1(P) = \delta_1 U_1((P_s)_s) + (1-\delta_1)\int_{\mathscr{F}} U_1(g)dP(g) \geq U_1(f)$, as desired.

C.4 Proof of Proposition 3

Suppose $\succsim$ is an EAP Maxmin preference represented by V as in Theorem 1. Let U be the restriction of V on $\mathscr{F}$. It is easy to see that (ii) holds. So, we will show (i). To show (a), suppose $\delta > 0$. It is easy to see that ex-post ambiguity aversion implies ex-ante ambiguity aversion. To see the opposite direction, suppose that $\succsim$ exhibits ex-ante ambiguity aversion. Fix $\alpha \in [0,1]$ and $f, g \in \mathscr{F}$ such that $f \sim g$ to show $\alpha f + (1-\alpha)g \succsim f$. Since $\alpha f \oplus (1-\alpha)g \succsim f$, then $U(f) \leq V(\alpha f \oplus (1-\alpha)g) = \delta U(\alpha f + (1-\alpha)g) + (1-\delta)U(f)$, so that $\delta U(f) \leq \delta U(\alpha f + (1-\alpha)g)$. Since $\delta > 0$, then $U(\alpha f + (1-\alpha)g) \geq U(f)$. Part (b) is proved in the same way. It is easy to see that (c) holds.

D Proof of Theorem 2

The necessity of axioms is easy to check. Note that Substitution is imposed only on the set $\mathscr{F}$ of acts. So, EAP Piecewise preferences immediately satisfy the axiom. To show Continuity, note that EAP Piecewise preference is a weighted sum of max functions and a mixture linear function u.

In the following, we will prove the sufficiency. Suppose that a preference relation $\succsim$ on $\Delta(\mathscr{F})$ satisfies the axioms in Theorem 2. Then, by Lemma 2, $\succsim$ satisfies Reduction of Compound Lotteries as well. As noted in the sketch in Section A, after proving that $\succsim$ restricted on $\mathscr{F}$ has the piecewise utility representation proposed by Fehr and Schmidt (1999), the proof is the same as the proof of Theorem 1.

The first step shows Reversal of Order among pointwise comonotonic acts. First, for simplicity, a notation is introduced: for any set $\{f^i\}_{i=1}^n$ of acts and any set $\{\alpha_i\}_{i=1}^n$ of nonnegative numbers such that $\sum_{i=1}^n \alpha_i = 1$, define $\hat{\sum}\alpha_i f^i \equiv \alpha_1 f^1 \oplus \cdots \oplus \alpha_n f^n$ and $\sum \alpha_i f^i \equiv \alpha_1 f^1 + \cdots + \alpha_n f^n$.

Step 1: $\hat{\sum}_{i=1}^n \alpha_i f^i \sim \sum_{i=1}^n \alpha_i f^i$.

Proof of Step 1: By Induction on n. For $n = 1$, the statement is trivial.

First, we will prove the statement for $n = 2$. Fix $\alpha \in [0, 1]$ and $f^1, f^2 \in \mathscr{F}$. Let $P = \alpha f^1 \oplus (1-\alpha) f^2$ to show $P \sim \alpha f^1 + (1-\alpha) f^2$. Then by Reduction of Compound Lotteries, $P_s = \alpha f^1_s \oplus (1-\alpha) f^2_s \sim \alpha f^1_s + (1-\alpha) f^2_s$ for all $s \in S$. So, by Substitution, $(P_s)_s \sim \alpha f^1 + (1-\alpha) f^2$, so that the condition (i) in Indifference is satisfied. In addition, since f^1 and f^2 are pointwise comonotonic, Ex-post Pointwise Comonotonic Independence shows $l_P = \alpha l_{f^1} + (1-\alpha) l_{f^2} \sim \alpha f^1 + (1-\alpha) f^2$, so that the condition (ii) in Indifference is satisfied as well. Hence, Indifference shows $P \sim \alpha f^1 + (1-\alpha) f^2$.

Let $P \equiv \hat{\sum}_{i=1}^{n-1} \frac{\alpha_i}{1-\alpha_n} f^i$ and $g \equiv \sum_{i=1}^{n-1} \frac{\alpha_i}{1-\alpha_n} f^i$. Suppose the statement is true for $n-1$. Then $P \sim g$. Now, we will show the statement for n. Since any pair of acts among $\{f^i\}_{i=1}^n$ are pointwise comonotonic, any pair among $(P_s)_s, g$, and f^n are pointwise comonotonic. Therefore, $\hat{\sum}_{i=1}^n \alpha_i f^i \equiv (1-\alpha_n) P \oplus \alpha_n f^n \sim (1-\alpha_n) g \oplus \alpha_n f^n \sim (1-\alpha_n) g + \alpha_n f^n \equiv \sum_{i=1}^n \alpha_i f^i$, where the second equivalence is by Ex-ante Pointwise Comonotonic Independence and the third equivalence is by the statement for $n = 2$. ■

Step 2: There exists a function $V : \Delta(\mathscr{F}) \to \mathbb{R}$ such that

(i) V represents $\succsim$ on $\Delta(\mathscr{F})$,

(ii) for all $\alpha \in [0, 1]$ and $P, Q \in \Delta(\mathscr{F})$ such that $(P_s)_s$ and $(Q_s)_s$ are pointwise comonotonic, $V(\alpha P + (1-\alpha) Q) = \alpha V(P) + (1-\alpha) V(Q)$,

(iii) V is unique up to positive affine transformation.
(iv) Let u be the restriction of V on $\Delta(Z)$. Then $u : \Delta(Z) \to \mathbb{R}$ is a mixture linear function such that $u(z_+) = 1$, $u(z_-) = -1$, and $u(l_0) = 0$.

PROOF OF STEP 2: Since Ex-ante/Ex-post Pointwise Comonotonic Independence implies Ex-ante/Ex-post Certainty Independence, in the same way as Step 2 in the proof of Theorem 1, there exist a real valued function V on $\Delta(\mathscr{F})$ satisfying (i), (iii), (iv), and that for all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $V(\alpha P \oplus (1-\alpha)l) = \alpha V(P) + (1-\alpha)V(l)$.

Finally to show (ii), choose any $\alpha \in [0,1]$, $P, Q \in \Delta(\mathscr{F})$ such that $(P_s)_s$ and $(Q_s)_s$ are pointwise comonotonic. Let $n \in \mathbb{Z}_+$ such that $\frac{1}{n}V(P) \in [-1,1]$. Define $\hat{P} = \frac{1}{n}P \oplus (1-\frac{1}{n})l_0$ and $\hat{Q} = \frac{1}{n}Q \oplus (1-\frac{1}{n})l_0$. Then, $(\hat{P}_s)_s$ and $(\hat{Q}_s)_s$ are also pointwise comonotonic.

Since $[-1,1] \subset u(\Delta(Z))$, there exists $l \in \Delta(Z)$ such that $\hat{P} \sim l$. Then, Ex-ante Pointwise Comonotonic Independence shows $\alpha\hat{P} \oplus (1-\alpha)\hat{Q} \sim \alpha l \oplus (1-\alpha)\hat{Q}$. Therefore, by the constant linearity, $V(\alpha\hat{P} \oplus (1-\alpha)\hat{Q}) = V(\alpha l \oplus (1-\alpha)\hat{Q}) = \alpha V(l) + (1-\alpha)V(\hat{Q}) = \alpha V(\hat{P}) + (1-\alpha)V(\hat{Q}) = \frac{1}{n}\big(\alpha V(P) + (1-\alpha)V(Q)\big)$.

In addition, by the definitions of $\hat{P}$ and $\hat{Q}$, it follows from the constant linearity that $V(\alpha\hat{P} \oplus (1-\alpha)\hat{Q}) = V\big(\frac{1}{n}(\alpha P \oplus (1-\alpha)Q) \oplus \big(1-\frac{1}{n}\big)l_0\big) = \frac{1}{n}V(\alpha P \oplus (1-\alpha)Q)$. Therefore, $V(\alpha P \oplus (1-\alpha)Q) = \alpha V(P) + (1-\alpha)V(Q)$. ■

Let U be the restriction of V on $\mathscr{F}$. For all $s \neq 1$, define $\alpha_s = -U((z_+, (l_0)_{-s}))$ and $\beta_s = -U((z_-, (l_0)_{-s}))$.

STEP 3: $\{\alpha_s, \beta_s\}_{s \in S\setminus\{1\}}$ are nonnegative numbers such that for all $f \in \mathscr{F}$,

$$U(f) = u(f_1) - \sum_{s\neq 1}\Big(\alpha_s \max\{u(f_s) - u(f_1), 0\} + \beta_s \max\{u(f_1) - u(f_s), 0\}\Big).$$

PROOF OF STEP 3: By Ex-post Inequality Aversion, $\alpha_s, \beta_s \geq 0$ for all $s \neq 1$. Fix $f \in \mathscr{F}$. Define $K = \sup_{s\in S} |u(f_s) - u(f_1)|$. First, consider the case where $K = 0$, then $f_s \sim f_1$ for all $s \in S$. Then, by Substitution, $U(f) = u(f_1)$, as desired. Next, consider the case where $K \neq 0$. By definition, $K > 0$. Since S is finite, $K < \infty$. Define $\hat{f} = \frac{1}{K}f + \big(1 - \frac{1}{K}\big)l_0$. Then, by the definition of K, $u(\hat{f}_s) - u(\hat{f}_1) \in [-1,1]$ for all $s \in S$. Since $[-1,1] \subset u(\Delta(Z))$, for all

$s \in S$ there exists $l^s \in \Delta(Z)$ such that $u(l^s) = u(\hat{f}_s) - u(\hat{f}_1)$.

SUBSTEP 3.2: Any pair in $\{(l^s, (l_0)_{-s})\}_{s\neq 1}$ are pointwise comonotonic.

PROOF OF SUBSTEP 3.2: A straightforward argument will show the result.

SUBSTEP 3.3: $\frac{1}{|S|}\hat{f} \oplus \frac{|S|-1}{|S|} l_0 \sim \frac{1}{|S|}\hat{f}_1 \oplus \hat{\sum}_{s\neq 1}\frac{1}{|S|}(l^s, (l_0)_{-s})$.

PROOF OF SUBSTEP 3.3: Since $l_0 \in \Delta(Z)$ is pointwise comonotonic with any acts, Step 1 shows $\frac{1}{|S|}\hat{f} \oplus \frac{|S|-1}{|S|} l_0 \sim \frac{1}{|S|}\hat{f} + \frac{|S|-1}{|S|} l_0$. By Substep 3.2, any pair among $\{(l^s, (l_0)_{-s})\}_{s\neq 1}$ are pointwise comonotonic. Since $f_1 \in \Delta(Z)$ is also pointwise comonotonic with any acts, Step 1 again shows $\frac{1}{|S|}\hat{f}_1 \oplus \hat{\sum}_{s\neq 1}\frac{1}{|S|}(l^s, (l_0)_{-s}) \sim \frac{1}{|S|}\hat{f}_1 + \sum_{s\neq 1}\frac{1}{|S|}(l^s, (l_0)_{-s})$.

Now, let $g \equiv \frac{1}{|S|}\hat{f} + \frac{|S|-1}{|S|} l_0$ and $h \equiv \frac{1}{|S|}\hat{f}_1 + \sum_{s\neq 1}\frac{1}{|S|}(l^s, (l_0)_{-s})$ to show $g \sim h$. For all $s \in S$, $u(g_s) = \frac{1}{|S|}u(\hat{f}_s) = \frac{1}{|S|}\big(u(\hat{f}_1) + u(l^s)\big) = u(h_s)$, where the second equality holds because $u(l^s) = u(\hat{f}_s) - u(\hat{f}_1)$. Since $g_s \sim h_s$ for all $s \in S$, Substitution shows $g \sim h$. Since $g \sim \frac{1}{|S|}\hat{f} \oplus \frac{|S|-1}{|S|} l_0$ and $h \sim \frac{1}{|S|}\hat{f}_1 \oplus \hat{\sum}_{s\neq 1}\frac{1}{|S|}(l^s, (l_0)_{-s})$, then $\frac{1}{|S|}\hat{f} \oplus \frac{|S|-1}{|S|} l_0 \sim \frac{1}{|S|}\hat{f}_1 \oplus \hat{\sum}_{s\neq 1}\frac{1}{|S|}(l^s, (l_0)_{-s})$.

SUBSTEP 3.4: Let $\overline{S} = \{s \in S | \hat{f}_s \succ \hat{f}_1\}$ and $\underline{S} = \{s \in S | \hat{f}_1 \succ \hat{f}_s\}$.

(i) for all $s \in \overline{S}$, $V((l^s, (l_0)_{-s})) = -\alpha_s \max\{u(\hat{f}_s) - u(\hat{f}_1), 0\}$,

(ii) for all $s \in \underline{S}$, $V((l^s, (l_0)_{-s})) = -\beta_s \max\{u(\hat{f}_1) - u(\hat{f}_s), 0\}$,

(iii) for all $s \in S \setminus (\overline{S} \cup \underline{S})$, $V((l^s, (l_0)_{-s})) = 0$.

PROOF OF SUBSTEP 3.4: We will show (i). Fix $s \in \overline{S}$. Then $\hat{f}_s \succ \hat{f}_1$. So, $z_+ \succsim l^s \succ l_0$. Hence, $l^s \sim u(l^s)\delta_{z_+} + \big(1 - u(l^s)\big) l_0$. Hence, by Substitution, $(l^s, (l_0)_{-s}) \sim u(l^s)(z_+, (l_0)_{-s}) + \big(1 - u(l^s)\big) l_0$. Therefore, by Step 2 (ii), $V((l^s, (l_0)_{-s})) = u(l^s)V((z_+, (l_0)_{-s}))) = -\alpha_s u(l^s) = -\alpha_s \max\{u(\hat{f}_s) - u(\hat{f}_1), 0\}$. Part (ii) is proved in the same way. Finally, we will show (iii). For all $s \in S \setminus (\overline{S} \cup \underline{S})$, $l^s \sim l_0$, Hence, by Substitution, $(l^s, (l_0)_{-s}) \sim l_0$. So, $V((l^s, (l_0)_{-s})) = 0$.

Therefore,

$$
\begin{aligned}
U(f) &= KV(\hat{f}) && \big(\because \text{Step 2 (ii) \& } \hat{f} = \tfrac{1}{K}f + (1 - \tfrac{1}{K})l_0\big) \\
&= K|S|V\Big(\tfrac{1}{|S|}\hat{f} \oplus \tfrac{|S|-1}{|S|} l_0\Big) && (\because \text{Step 2 (ii)}) \\
&= K|S|V\Big(\tfrac{1}{|S|}\hat{f}_1 \oplus \hat{\textstyle\sum}_{s\neq 1}\tfrac{1}{|S|}(l^s, (l_0)_{-s})\Big) && (\because \text{Substep 3.3}) \\
&= K\Big(u(\hat{f}_1) + \textstyle\sum_{s\neq 1} V((l^s, (l_0)_{-s}))\Big). && (\because \text{Step 2 (ii) \& Substep 3.2})
\end{aligned}
$$

Since $u(\hat{f}_s) = \frac{1}{K}u(f_s)$ for all $s \in S$, substituting the results in Substep 3.4 yields the desired representation. ■

If $(\alpha, \beta) = \mathbf{0}$, then the theorem holds trivially. So, henceforth, we consider the case where $(\alpha, \beta) \neq \mathbf{0}$. To make notations simple, for all $P \in \Delta(\mathscr{F})$, define

$$\zeta(P) = U\big((P_s)_s\big); \quad \eta(P) = U(l_P).$$

The next step shows that the functions ζ and η have the same property as in Theorem 1.

Step 4:
(i) For all $\alpha \in [0,1]$, $P \in \Delta(\mathscr{F})$, and $l \in \Delta(Z)$, $\zeta(\alpha P \oplus (1-\alpha)l) = \alpha\zeta(P) + (1-\alpha)\zeta(l)$ and $\eta(\alpha P \oplus (1-\alpha)l) = \alpha\eta(P) + (1-\alpha)\eta(l)$.
(ii) For all $l \in \Delta(Z)$, $\zeta(l) = u(l) = \eta(l)$.
(iii) For all $P \in \Delta(\mathscr{F})$, $\zeta(P) \geq \eta(P)$.

Proof of Step 4: Parts (i) and (ii) hold in the same way as Step 4 in the proof of Theorem 1. To show (iii), fix $P \in \Delta(\mathscr{F})$. Fix $s \neq 1$. For all $x \in \mathbb{R}^{|S|}$, define $G_s : \mathbb{R}^{|S|} \to \mathbb{R}$ by $G_s(x) = -\alpha_s \max\{x_s - x_1, 0\} - \beta_s \max\{x_1 - x_s, 0\}$, where x_t is the t-th element of $x \in \mathbb{R}^{|S|}$. Then G_s is concave.[32] Define $F : \mathbb{R}^{|S|} \to \mathbb{R}$ by $F(x) = x_1 + \sum_{s\neq 1} G_s(x)$ for all $x \in \mathbb{R}^{|S|}$. Since F is a sum of concave functions, F is also concave. Therefore, for all $P \in \Delta(\mathscr{F})$, Jensen's Inequality shows $\zeta(P) = F\big(\big(\int_{\mathscr{F}} u(f_s)dP(f)\big)_{s\in S}\big) \geq \int_{\mathscr{F}} F\big(\big(u(f_s)\big)_{s\in S}\big)dP(f) = \eta(P)$. ■

Note that Ex-ante/Ex-post Pointwise Comonotonic Independence implies Ex-ante/Ex-post Certainty Independence. Given Step 4 above, the same argument as Step 5–12 in the proof of Theorem 1 shows the existence of the desired real number δ.

Finally, the next claim shows the uniqueness property of the representation.

Claim: The following two statements are equivalent:

[32] Remember (i) for any convex function ϕ, $\max\{\phi(\cdot), 0\}$ is convex; (ii) for any convex functions ϕ, ψ and nonnegative numbers a, b, $a\phi + b\psi$ is convex; (iii) for any convex function ϕ, $-\phi$ is concave. By (i), $\max\{x_s - x_1, 0\}$ and $\max\{x_1 - x_2, 0\}$ are convex. Then, by (ii), $\alpha_s \max\{x_s - x_1, 0\} + \beta_s \max\{x_1 - x_s, 0\}$ is convex. Finally, then by (iii), G_s is concave.

(i) Two triples $(\delta, \alpha, \beta, u)$ and $(\delta', \alpha', \beta', u')$ represent the same preference $\succsim$ as in Theorem 2.

(ii) (a) $(\alpha, \beta) = (\alpha', \beta')$ and there exist $a > 0, b \in \mathbb{R}$ such that $u = au' + b$; and

(b) If $(\alpha, \beta) \neq \mathbf{0}$ then $\delta = \delta'$.

Proof of Claim: It is easy to see that (ii) implies (i). So, we will show that (i) implies (ii). Choose any two triples $(\delta, \alpha, \beta, u)$ and $(\delta', \alpha', \beta', u')$ represent the same preference $\succsim$ as in Theorem 2. Given (a), part (b) is proved in the same way as Corollary 1. So, in the following, we will show (a). Since u and u' are affine representation of the restriction of $\succsim$ on $\Delta(Z)$, by the standard uniqueness results, there exist $a > 0, b \in \mathbb{R}$ such that $u = au' + b$. In addition, without loss of generality, assume $u(z_+) = 1$ and $u(z_-) = -1$.

Suppose to the contrary that $(\alpha, \beta) \neq (\alpha', \beta')$. Then, there exists at least one element $s \neq 1$ such that $\alpha_s \neq \alpha'_s$ or $\beta_s \neq \beta'_s$. Without loss of generality, assume $\alpha_s > \alpha'_s$. Let U and U' be the representations on $\mathscr{F}$ as in Step 3 defined by (α, β, u) and (α', β', u'), respectively. Take n large enough to hold $-\frac{\alpha_s}{n} \in [-1, 0]$. Since $[-1, 0] \subset u(\Delta(Z))$ and $U((z_+, (l_0)_{-s})) = -\alpha_s$, there exist $l \in \Delta(Z)$ such that $l \sim \frac{1}{n}(z_+, (l_0)_{-s}) + \frac{n-1}{n} l_0$. This shows $u(l) = U\big(\frac{1}{n}(z_+, (l_0)_{-s}) + \frac{n-1}{n} l_0\big) = -\frac{1}{n}\alpha_s$ and $au(l) + b = u'(l) = U'\big(\frac{1}{n}(z_+, (l_0)_{-s}) + \frac{n-1}{n} l_0\big) = b - \frac{1}{n}\alpha'_s a$. Since $\alpha > 0$, these equations show $\alpha_s = \alpha'_s$, which is a contradiction. Hence, $(\alpha, \beta) = (\alpha', \beta')$. ■

This completes the proof of Theorem 2.

D.1 Proof of Corollary 2

Suppose $\succsim$ is an EAP Piecewise preference represented by V as in Theorem 2. Since $(\alpha, \beta) \neq \mathbf{0}$. Without loss of generality, assume $\alpha_s > 0$ for some $s \neq 1$.

First, we will show (i). Choose any $l_+, l_- \in \Delta(Z)$ such that $l_+ \succ l_0 \succ l_-$ and $(l_+, (l_0)_{-s}) \sim (l_-, (l_0)_{-s})$. Then, $U(l_+, (l_0)_{-s}) = -\alpha_s u(l_+)$ and $U(l_-, (l_0)_{-s}) = \beta_s u(l_-)$. Since $(l_+, (l_0)_{-s}) \sim (l_-, (l_0)_{-s})$, then $-\alpha_s u(l_+) = \beta_s u(l_-)$. So, $\beta_s > 0$.

Case 1: $\frac{1}{2}u(l_+) + \frac{1}{2}u(l_-) \geq 0$.[33] Then $V(\frac{1}{2}(l_+, (l_0)_{-s}) \oplus \frac{1}{2}(l_-, (l_0)_{-s})) - U(l_+, (l_0)_{-s}) =$

[33] In this case, $V(\frac{1}{2}(l_+, (l_0)_{-s}) \oplus \frac{1}{2}(l_-, (l_0)_{-s})) = -\delta\alpha_s\big[\frac{1}{2}u(l_+) + \frac{1}{2}u(l_-)\big] - (1-\delta)\frac{1}{2}\big[\alpha_s u(l_+) - \beta_s u(l_-)\big]$.

$\delta\alpha_s\frac{1}{2}(u(l_+) - u(l_-))$. Hence, $\frac{1}{2}(l_+, (l_0)_{-s}) \oplus \frac{1}{2}(l_-, (l_0)_{-s}) \succsim (l_+, (l_0)_{-s})$ if and only if $\delta \geq 0$.

Case 2: $\frac{1}{2}u(l_+) + \frac{1}{2}u(l_-) \leq 0$. The proof is exactly the same as Case 1.

Next, we will show (ii). $V(\frac{1}{2}(z_+, (l_0)_{-s}) \oplus \frac{1}{2}(z_-, (l_0)_{-s})) = -\frac{1}{2}(1-\delta)(\alpha_s + \beta_s)$. Hence, $(l_0, (l_0)_{-s}) \succsim \frac{1}{2}(z_+, (l_0)_{-s}) \oplus \frac{1}{2}(z_-, (l_0)_{-s})$ if and only if $\delta \leq 1$.

D.2 Proof of Corollary 3

It is easy to see that (ii) implies (i) in the same way as in the proof of Proposition 2. So, we will show that (i) implies (ii). Fix two EAP Piecewise preferences $\{\succsim_i\}_{i=1,2}$. Let V_i be as in Theorem 2 defined by $(\delta_i, \alpha^i, \beta^i, u_i)$. Let U_i be the restriction of V_i on $\mathscr{F}$. The same straightforward argument as in the proof of Proposition 2 shows $U_1 = U_2$.[34] Hence, $(\alpha^1, \beta^1) = (\alpha^2, \beta^2)$.

Suppose $(\alpha^i, \beta^i) \neq \mathbf{0}$ to show $\delta_1 \geq \delta_2$. Without loss generality assume $\alpha^i_s \neq 0$ for some $s \neq 1$. Define $P = \frac{1}{2}(z_+, (l_0)_{-s}) \oplus \frac{1}{2}(z_-, (l_0)_{-s})$. Then, $V_i(P) = -\frac{1}{2}(1-\delta_i)(\alpha^i_s + \beta^i_s)$. There exists $n \in \mathbb{Z}_+$ such that $\frac{1}{n}V_2(P) \in [-1, 1]$. Define $\hat{P} = \frac{1}{n}P \oplus (1 - \frac{1}{n})l_0$. Since $[-1, 1] \subset u(\Delta(Z))$, there exists l such that $u_2(l) = V_2(\hat{P})$. Suppose $\succsim_1$ is more ex-ante inequality averse than $\succsim_2$. Then $u_1(l) \leq V_1(\hat{P})$. Therefore, in the same way as in the proof of Proposition 2, $\delta_1 \geq 1 + 2n\left(\frac{u_1(l)}{\alpha^1_s + \beta^1_s}\right) = 1 + 2n\left(\frac{u_2(l)}{\alpha^2_s + \beta^2_s}\right) = \delta_2$.

D.3 Proof of Proposition 4

Suppose (a) and (b) hold. Let $\alpha^F = .2$, $\beta^F = .9$, $\delta^F = .85$, and $u(z) = \log z$ for all $z \in \mathbb{R}_+$. Let V^t be as in Theorem 2 defined by $(\alpha^t, \beta^t, \delta^t, u)$ for all $t \in \{F, S\}$.

Part (ii) holds because $V^t(400, 400) = \log 400 > \log 375 \geq V^t(375, 750)$ for all t. In the following, we will show (i). The payoff function of player i with type t is denoted by Π^t_i. Let s^*_i be a strategy of player i such that the fair type (type F) play Ef and the selfish type (type S) play Eq. Given that the probability of fair type is .5, the payoffs of player i given

[34] See footnote 31 for details

the opponent strategy s_j^* is defined as follows:

$$\Pi_i^F(Ef|s_j^*) = V^F\big(.5(375,750) \oplus .25(750,375) \oplus .25(400,400)\big) \geq \delta^F\ 7.52 + (1-\delta^F)\ 5.89,$$
$$\Pi_i^F(Eq|s_j^*) = V^F\big(.5(400,400) \oplus .25(750,375) \oplus .25(400,400)\big) \leq \delta^F\ 7.5 + (1-\delta^F)\ 6.$$

Hence, if $\delta^F \geq .85$ then $\Pi_i^F(Ef|s_j^*) \geq \Pi_i^F(Eq|s_j^*)$. Also, since

$$\begin{aligned}\Pi_i^S(Ef|s_j^*) &= V^S\big(.5(375,750) \oplus .25(750,375) \oplus .25(400,400)\big) \leq 7.6,\\ \Pi_i^S(Eq|s_j^*) &= V^S\big(.5(400,400) \oplus .25(750,375) \oplus .25(400,400)\big) \geq 7.64,\end{aligned}$$

then, $\Pi_i^S(Eq|s_j^*) > \Pi_i^S(Ef|s_j^*)$. Therefore, (s_1^*, s_2^*) and the prior probability consist a Bayesian Nash equilibrium.

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