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MEASURE STRUCTURES FOR FUNCTION SPACES

by

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ABSTRACT

Let F be a set of Borel function from X to Y, where X, Y are copies of the unit interval. A probability distribution (mixed strategy) λ over F is said to be admissible if it has the following attribute: for any Borel probability measure μ over X and any Borel subset $E \subset Y$, the probability of $f(x)$ appearing in E when $f \in F$, $x \in X$ are chosen at random is uniquely determined by λ, μ. We consider the following question: Let $\mathcal{A}$ be a countable Boolean algebra of subsets of F, where each element in $\mathcal{A}$ is definable by a formula in the language of set theory. (We stipulate that F itself is also definable in set theoretic language.) Let $\underline{F}_{\mathcal{A}}$ be the smallest σ - ring over F containing $\mathcal{A}$. Given an arbitrary probability measure $\underline{\lambda}$ over $\underline{F}_{\mathcal{A}}$, can $\underline{\lambda}$ be extended to an admissible distribution? An affirmative answer is shown to be consistent with the Zermelo - Fraenkel axioms of set theory.

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1. We consider the case of a two-person game in which the players, 1 and 2, have an infinite number of pure strategies to choose from. Each of player 1's strategies consists of a function $f \in F$ such that $f\colon X \to Y$. Player 2 is free to select any point $x \in X$. The outcome of any such pair of strategies is then $f(x) \in Y$.

In games with a finite number of pure strategies, the mixed strategies are the various probability configuration over the set of pure strategies. The question then arises as to how one should define mixed strategies over F when F is infinite. This type of problem can be encountered in control theory and in some situations in the theory of stochastic processes. The typical optimization problem is often reducible to a "game against nature" in which the player seeks to maximize his minimum return, given a prescribed set of admissible control functions.

The problem of selecting a function $f\colon X \to Y$ at random for the case where X, Y are copies of the unit interval and F is a set of Borel functions from X to Y was investigated by R. J. Aumann in [A-2]. In Aumann's approach, F is randomized in the following manner: Let Ω, X, Y be copies of the unit interval with the usual Borel structure. Designate the Borel structures over Ω, X, Y by $\underline{\Omega}, \underline{X}, \underline{Y}$. Each $\omega \in \Omega \equiv [0,1]$ is associated with some $f_\omega \in F$. G is a measurable subset of F provided that G is of the form

$G \equiv \{f_\omega | \omega \in B\}$, where $B \subset \Omega$ is Borel. In this manner, a measure $\underline{F}$ over F is obtained. Let λ and μ be Borel probability distributions[1/] over Ω and X and let λ^* be the induced probability distribution defined over $\underline{F}$. The mixed strategies over F are taken to be the various induced probability distributions.

If λ^* and μ are to be regarded as mixed strategies, a natural requirement that they should fulfill is that the set $Q_o \overset{\text{def}}{\equiv} \{(f,x) \mid f \in F, x \in X, f(x) \in E\}$ be measurable with respect to $\lambda^* \times \mu$ for any Borel subset $E \subset Y$. That is, λ^* and μ should determine a probability value for the occurance of $f(x)$ in E when $f \in F$, $x \in X$ are chosen at random. However, if Q_o is measurable with respect to $\lambda^* \times \mu$, then $\{(\omega,x) | f_\omega(x) \in E\} \subset \Omega \times X$ ought to be measurable with respect to $\lambda \times \mu$. Thus, if $\Theta: \Omega \times X \to Y$ is defined by $\Theta(\omega,x) = f_\omega(x)$, then $\Theta^{-1}(E)$ must be measurable with respect to $\lambda \times \mu$ given any Borel $E \subset Y$. If for each $f \in F$ there exists an $\omega \in \Omega$ such that $f \equiv \Theta(\omega,\cdot)$, F is then called the <u>range</u> of Θ. The range coincides with the set of functions which can actually occur. Since players 1 and 2 are free to choose their mixed strategies λ^* and μ at will, we would like $\Theta^{-1}(E)$ to be measurable with respect to every permissible λ and μ. One way to insure this is to stipulate (as in [A-2]) that Θ be Borel. The class of F's that can be obtained when this requirement is imposed is then limited even further.

Let us at first assume, as in [A-2], that F is the range of a Borel measurable Θ. Given any Borel subset E of Y, we would then like to evaluate the probability of $f(x)$ occuring in E when λ^* and

[1/] Distributions which associate a probability measure for every Borel subset of Ω, X, respectively.

μ, the mixed strategies over F and X, are prescribed. This probability should depend only on λ^* and μ, and not on the λ which induces λ^*. If λ_1 and λ_2 are two different Borel probability distributions defined over Ω and both induce the same distribution λ^*, then the resulting probability measure for the set $\{(f,x) \mid f \in F,\ x \in X,\ f(x) \in E\}$ should in each case be the same. One possibility is that $\{(f,x) \mid f \in F,\ x \in X,\ f(x) \in E\}$ might belong to the σ-ring generated by sets of the form $C \times D$, where $C \in \underline{F}$ and $D \in \underline{X}$. If this is the case, then $\{(f,x) \mid f \in F,\ x \in X,\ f(x) \in E\}$ is, loosely speaking, measurable in the Borel sense; it then has a uniquely determined measure under $\lambda^* \times \mu$. This measure would coincide with the probability of $f(x)$ occurring in E. We refer to the conjecture that $\{(f,x) \mid f \in F,\ x \in X,\ f(x) \in E\}$ is "Borel measurable" in $\underline{F} \times \underline{X}$ for every Borel subset $E \subset Y$ and every Borel Θ as the identification space hypothesis. The truth or untruth of this hypothesis was left as an open problem in [A-2].

As we will see, counter-examples exist in which $\{(f,x) \mid f \in F,\ x \in X,\ f(x) \in E\}$ fails to satisfy the identification space hypothesis. An alternate method for assigning a probability value to $\{(f,x) \mid f \in F,\ x \in X,\ f(x) \in E\}$ will therefore be discussed. A second difficulty is connected with the limitation imposed by restricting ourselves to f's which are ranges of Borel Θ's.

In defining a probability distribution (mixed strategy) μ over $X \equiv \{0,1\}$ the conventional approach would be to assign values to μ initially over the rational closed sub-intervals. Under the assumption that μ is σ-finite, μ is then extendable in a unique manner to all

Borel subsets. [1/] We wish to obtain mixed strategies over function spaces in analogous fashion. Suppose then that F_o is some definable [2/] set of Borel functions from X to Y and that $\mathcal{O}_o = \{\mathcal{O}_o^{(1)}, \mathcal{O}_o^{(2)}, \ldots\}$ is a countable set of definable subsets of F. Let $\underline{F}_{\mathcal{O}_o}$ be the σ-field on F_o generated by $\mathcal{O}_o$, and let $\underline{\lambda}$ be a σ-finite probability measure over $\underline{F}_{\mathcal{O}_o}$. Given a Borel measure μ over X and a Borel subset E of Y, we are interested in determining a probability value under $\underline{\lambda}$, μ that f(x) occur in E when $f \in F$, $x \in X$ are chosen at random.

In order to adapt the methods of [A-2] to function spaces of this generality, a more extensive family of Θ's, $\Theta : \Omega \times X \to Y$, will have to be introduced. We do this in ℙ 4. A function space which is the range of a Θ of the type described there will be referred to as a measurable range.

If $(F_o, \underline{F}_{\mathcal{O}_o})$ is a measurable range then in accordance with the definition which will appear, there exists a $\Theta : \Omega \times X \to Y$ as described in ℙ4 of which F_o is the range, i.e. for each $f \in F_o$, there is an $\omega \in \Omega$ such that $F \equiv \Theta(\omega, \cdot)$. Moreover, a Borel probability measure, λ, over Ω exists which induces $\underline{\lambda}$. That is, if we let $\theta : \Omega \to F_o$ be the function defined by $\theta(\omega) = \Theta(\omega, \cdot)$, then for each $C \in \underline{F}_{\mathcal{O}_o}$, $\theta^{-1}(C)$ is measurable with respect to λ, and $\underline{\lambda}(C) = \lambda(\theta^{-1}(C))$. The measure $\underline{\lambda}$ may also be extended to subsets not necessarily included in $\underline{F}_o$: given any $D \subset F_o$ such that $\theta^{-1}(D)$ is measurable with respect to λ, let $\underline{\lambda}(D) = \lambda(\theta^{-1}(D))$. It will be shown that $\underline{\lambda} \times \mu$ then determines a unique probability value for $\{(f,x) \mid F \in F_o, x \in X, f(x) \in E\}$.

1/ [Hlm-2], p. 54.

2/ definable from a formula in the language of set theory. A precise formulation of the notion of definability will appear in ℙ 2.3.

We are therefore interested in conditions under which $(F_o, F_{\mathcal{T}_o})$ is a measurable range.

The question of whether every definable space of functions $(F_o, F_{\mathcal{T}_o})$ as described above is a measurable range cannot be resolved within Zermelo-Fraenkel set theory. We show, however, that an affirmative answer is consistent with the Zermelo-Fraenkel axioms.

In deriving the various results, we make considerable use of concepts stemming from mathematical logic. A survey of the necessary prerequisites is presented in the next section.

We express deep appreciation to Jonathan Stavi for many illuminating discussions on generic models. The proof of Lemma 4.7 is based upon an approach indicated by Professor Benjamin Weiss. We are grateful for his help.

2. We begin by introducing concepts, theorems, and notation which will be called upon in subsequent sections.

2.1. The analytical hierarchy. Let $\mathcal{R}$ be the set of reals. An analytic subset of $\mathcal{R}$ is the projection of a Borel subset of $\mathcal{R} \times \mathcal{R}$ onto $\mathcal{R}$. A co-analytic subset of $\mathcal{R}$ is the complement of an analytic subset. Analytic and co-analytic subsets of $\mathcal{R} \times \mathcal{R}$, $\mathcal{R} \times \mathcal{R} \times \mathcal{R}$, etc. are defined analogously. It is known ([Hf], p. 210) that a set $B \subset \mathcal{R}$ ($B \subset \mathcal{R} \times \mathcal{R}$, etc.) is Borel if it is simultaneously analytic and co-analytic.

Precise definitions of the notion of recursive relations may be found in [Sh]. Heuristically, a relation $R(x_1, \ldots x_m, i_1, \ldots, i_n)$

is recursive if a Turing machine can be programmed to decide within a finite number of steps whether $(x_1, \ldots, x_m, i_1, \ldots i_n)$ is in R for any $(x_1, \ldots, x_m, i_1, \ldots, i_n)$ belonging to R. Here $x_1, \ldots x_m$ are either reals or sequences of natural numbers; $i_1, \ldots, i_n$ are natural numbers. A relation $Q(x_1, \ldots, x_m, i_1, \ldots, i_n)$ is recursive in a parameter ξ_o, where ξ_o is a fixed infinite sequence of natural numbers, if $Q(x_1, \ldots, x_m, i_1, \ldots, i_n) \equiv Q_1(x_1, \ldots, x_m, i_1, \ldots, i_n, \xi_o)$ and Q_1 is recursive. A relation $R_o(x_1, \ldots, x_m, i_1, \ldots, i_n)$ is Σ_1^1 if $R_o(x_1, \ldots, x_m, i_1, \ldots, i_n) \equiv \exists\, x_{m+1} \forall i_{n+1}\, R_1(x_1, \ldots, x_m, x_{m+1}, i_1, \ldots, i_n, i_{n+1})$, where R_1 is recursive. $R_o(x_1, \ldots, x_m, i_1, \ldots, i_n)$ is Σ_1^1 in a parameter ξ_o if $R_o(x_1, \ldots, x_m, i_1, \ldots, i_n) \equiv \exists x_{m+1} \forall i_{n+1}\, R_1(x_1, \ldots, x_m, x_{m+1}, \xi_o, i_1, \ldots, i_n, i_{n+1})$ where R_1 is recursive. $R_o(x_1, \ldots, x_m, i_1, \ldots, i_n)$ is Π_1^1 (Π_1^1 in a parameter) if it is the negation or complement of a Σ_1^1-relation (Σ_1^1 relation in a parameter).

The following facts are known:

(1) A relation $R(x_1, \ldots, x_n)$ over n-ads of reals is recursive in a parameter if the set $\{(y_1, \ldots, y_n) \mid R(y_1, \ldots, y_n)\}$ is open.

(2) A relation $R(x_1, \ldots, x_n)$ is Σ_1^1 in a parameter if the set $\{(y_1, \ldots, y_n) \mid R(y_1, \ldots, y_n)\}$ is analytic.

(3)[3/] (a) There exist Π_1^1 relations $\mathcal{A}_2(x_1, x_2), \mathcal{A}_3(x_1, x_2)$ (where x_1 is a real number, x_2 is an infinite sequence of natural numbers) such that for every Borel set $B \subset \mathcal{R}$ there exists an infinite

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3/ See [So], p. 26 for proof.

sequence ξ of natural numbers such that for all real x,

$$x \in B \Leftrightarrow \mathcal{A}_2(x, \xi);$$

$$x \notin B \Leftrightarrow \mathcal{A}_3(x, \xi).$$

For any such ξ, B, we say that ξ codes B. We denote the set coded by ξ as B_ξ.

(b) There exists a Π^1_1 relation $\mathcal{A}_1(x)$, where x is an infinite sequence of natural numbers, such that $\mathcal{A}_1(x) \Leftrightarrow x$ codes a Borel set B.

(4) An analogy of (3) for Borel subsets of E^n. Every Borel set $B \subset E^n$ is coded by some sequence ξ of natural numbers. The coding may be carried out in a manner which would assure the existence for each Borel set $B \subset E^n$ of a code ξ, where ξ is an infinite sequence of 0's and 1's. That is, there exist Π^1_1 relations $\mathcal{A}_{(n,2)}(x_1, \ldots, x_n\, x_{n+1})$, $\mathcal{A}_{(n,3)}(x_1, \ldots, x_n, x_{n+1})$ such that for each Borel set $B \subset E^n$ there is a real number (a sequence of 0's and 1's) ξ such that

$$(x_1, \ldots, x_n) \in B \Leftrightarrow \mathcal{A}_{(n,2)}(x_1, \ldots, x_n, \xi);$$

$$(x_1, \ldots, x_n) \notin B \Leftrightarrow \mathcal{A}_{(n,3)}(x_1, \ldots, x_n, \xi).$$

There are, in fact, an uncountable number of sequences of 0's and 1's that code any Borel $B \subset E^n$.

A Σ^1_1 relation (Π^1_1, recursive relation) is a special case of a relation that is Σ^1_1 (Π^1_1, recursive) in a parameter. In all that follows, we shall make no distinction between relations that require a parameter and those that do not. Relations that are Σ^1_1 (Π^1_1, recursive) in a parameter will simply be referred to as Σ^1_1 (Π^1_1, recursive).

2.2. Models of Zermelo - Fraenkel Set Theory. Sets and classes are generally regarded as the most fundamental objects in mathematics. Real numbers, functions, limits, and other mathematical entities can all be defined in terms of sets. When this redefinition is carried out, theorems in the various branches of mathematics are transformed into statements about sets.

The original 19th century approach to set theory was intuitive, or "naive." A set was considered to be any collection of objects. This unrestricted approach soon led to contradictions or antinomies, such as the Russel paradox: if $\mathcal{U}$, the class of all sets, were itself a set, then $\mathcal{U}'$, the class of all sets which do not belong to themselves would also be a set; it is then easily seen that $\mathcal{U}' \in \mathcal{U}'$ implies $\mathcal{U}' \notin \mathcal{U}'$ and $\mathcal{U}' \notin \mathcal{U}'$ implies $\mathcal{U}' \in \mathcal{U}'$. To avoid paradoxes of this type rules had to be established for imposing greater limitation on the class of objects that could be regarded as sets. Various systems of axioms were proposed. Of these, the Zermelo-Fraenkel axioms received the widest acceptance. A list of the Zermelo-Fraenkel axioms may be found in [C]. We refer to these axioms (including the axiom of choice) as ZFC. Most, if not all theorems in classical analysis, when translated into statements about sets, can be proven entirely within ZFC. There are, however, many true statements abouts sets which cannot be proven within ZFC alone. This follows from Gödel's incompleteness theorem.

In dealing with a particular system of axioms, one of the problems of interest to logicians is the question of their independence. The oldest example of this sort of problem occurs in classical plane geometry. For many hundreds of years mathematicians were puzzled as to whether Euclid's parallel line postulate could or could not be proven from the remaining postulates. Put another way, could there exist a geometrical or mathematical universe in which Euclid's parallel line postulate would be untrue? This question was finally settled in the 19th century by Lobachewsky and independently by Bolyai.

A mathematical model of geometry was shown to exist in which all Euclidean axioms except for the parallel line postulate would be fulfilled. In recent years, mathematical models were constructed which provide important consistency and independence results in set theory. One of the most prominent of these was the proof by Paul Cohen that the continuum hypothesis is independent of ZFC. Precise definitions of mathematical models and of their relation to formalized languages and systems of axioms may be found in [C] or [R].

Let $\mathcal{T}$ be a non-empty subset of the axioms of ZFC and suppose that $\mathfrak{M}$ is a model of $\mathcal{T}$, i.e. that $\mathfrak{M}$ satisfies every axiom in $\mathcal{T}$ when interpreted in $\mathfrak{M}$. $\mathfrak{M}$ is an $\in$-model of $\mathcal{T}$ if the individual elements in $\mathfrak{M}$ are all sets and if the binary relation $\in$ denoted by the predicate $\bar{\in}$ in the language of $\mathcal{T}$ conicides with the relation "belongs to." $\mathfrak{M}$ is a transitive $\in$-model if $y \in \mathfrak{M}$ and $x \in y$ implies $x \in \mathfrak{M}$.

Although there are valid reasons for supposing that transitive $\in$-models of all of ZFC exist ([C], p. 78), their existence cannot be proven by means of ZFC alone. On the other hand, the existence of transitive $\in$-models of any finite subset of ZFC may be proven entirely within the framework of ZFC ([C], p. 82).

The existence of a transitive $\in$-model of $\mathcal{T}$ implies the existence of a minimal model of $\mathcal{T}$, i.e. a transitive $\in$-model consisting of those sets belonging to all transitive $\in$-models of $\mathcal{T}$ (the intersection of all transitive $\in$-models of $\mathcal{T}$). In view of the countability of $\mathcal{T}$, the minimal model would necessarily be countable.

If Φ is a sentence in the language of $\mathfrak{M}$, where $\mathfrak{M}$ is a transitive $\in$-model, we write $\mathfrak{M} \models \Phi$ if Φ is satisfied in $\mathfrak{M}$.

2.3. Definability. A set $A \subset \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R}$ is said to be definable from a set b if there exists a formula in the language of set theory, $\psi_0(y_1, y_2)$, such that $A \equiv \{z \mid z \in \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R}$ and $\psi_0(b,z)$ is true$\}$. In particular, if A is definable from c for some $c \in \mathcal{R}$, then A is said to be $\mathcal{R}$ - definable. If there exists a formula $\Phi(x)$ in the language of set theory such that $A \equiv \{z \mid z \in \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R}$ and $\Phi(z)$ is true$\}$ then A is said to be definable. Definability is clearly a special case of $\mathcal{R}$ - definability.

2.4. Inaccessible cardinals. We shall say that a set a is of accessible cardinality if any one of the following three conditions is observed:

(1) The cardinality of a (card a) is equal or less than $\aleph_0$

(2) There exists a set b such that card b < card a and card a $\leq$ card 2^b (Example: if a is of cardinality $2^{\aleph_0}$);

(3) There exists a set I such that card I < card a and for each set $c \in I$, card c < card a and a is equal to the union of the sets belonging to I. (Example: if a is of cardinality $\aleph_{\aleph_0}$).

If a does not fulfill any of the conditions, then a is said to be of inaccessible cardinality. If a is of accessible cardinality and every subset $b \subset a$ is also of accessible cardinality then a will be referred to as "small." Otherwise, (i.e. if there exists a subset $b \subseteq a$ such that b is of inaccessible cardinality) then a is said to be "large."

The proposition, "No inaccessible cardinals exist" is known to be consistent with ZFC ([C], p. 80). Nevertheless, there are convincing grounds for believing that inaccessible cardinals do exists. The reasoning is as follows: Let $\mathcal{B}$ be the class of all "small" ordinals. We inquire as to whether $\mathcal{B}$ is a set. There are, of course, many instances of classes which are not sets. If, for example, the class of all ordinals were a set, then it, too, would be an ordinal and would be included in itself. We would then be led to the well known Russel paradox. On the other hand, the hypothesis that $\mathcal{B}$ is a set would apparently not lead to this sort of contradiction - $\mathcal{B}$ would be much larger than any of its members; it would therefore not be a small cardinal and would consequently not be included in itself. Since the hypothesis that $\mathcal{B}$ is a set cannot be proven within ZFC, it is tempting to accept this as an additional postulate. Presumably this would be consistent with the axioms of ZFC. If $\mathcal{B}$ is a set, then $\mathcal{B}$ is an ordinal. Its cardinality would be inaccessible. By similar arguments one may adduce the existence of 2, 3, or many distinct inaccessible cardinals.

There are numerous consistency theorems which cannot be obtained by means of ZFC alone. The inaccessible cardinality axioms can sometimes be used to produce consistency results of this type.

If one postulates the existence of two distinct inaccessible cardinals, one may then prove the following:

$$\left.\begin{array}{l}\text{A countable transitive } \in\text{-model } \mathfrak{M} \text{ of ZFC exists} \\ \text{such that } \mathfrak{M} \models \text{ there exists an inaccessible cardinal.}\end{array}\right\} \quad (2.4.1)$$

Based upon (2.4.1), Solovay proved ([So]):

$$\left.\begin{array}{l}\text{There exists a countable transitive } \in\text{-model } \mathfrak{N} \text{ of ZFC} \\ \text{such that } \mathfrak{N} \models \text{" Every set of reals } A \subset \mathcal{R} \text{ which is definable} \\ \text{by means of a countable sequence of ordinals is Lebesgue is measurable} \\ \text{if, in addition } A \subset \mathcal{R} \text{ is uncountable, it includes a subset which is} \\ \text{a perfect set.}\end{array}\right\} \quad (2.4.2$$

The assertion in (2.4.2) will be used further ahead in proving the consistency result mentioned in $\mathbb{P}$ 1.

3. The present section deals with mixed strategies over function spaces which are ranges of a Borel $\Theta: \Omega \times X \to Y$. More general spaces of functions will be treated later on. We begin by reviewing in detail some of the definitions given in the introductory section, $\mathbb{P}$ 1.

3.1. Let $\Omega \equiv X \equiv Y$ be copies of the unit interval and let $\underline{\Omega} \equiv \underline{X} \equiv \underline{Y}$ be copies of the unit interval's Borel structure. We stipulate that F be a subset of Y^X, i.e. that every $f \in F$ be defined on X and have values in Y. Suppose Θ is a function whose domain of definition is $\Omega \times X$ and whose values are all in Y. F is said to be the range of Θ if for each $f_o \in F$ there exists an $\omega_o \in \Omega$ such that $f_o \equiv \Theta(\omega_o, \cdot)$ i.e. for some $\omega_o \in \Omega$, f_o concides with the function defined by Θ on X, when ω_o is fixed.

Given a function $\Theta: \Omega \times X \to Y$ of which F is the range, let $\theta: \Omega \to F$ be defined as follows: $\theta(\omega) \stackrel{def}{=} \Theta(\omega, \cdot)$. Denote by $\underline{F}$ the σ-field over F induced by θ: $C \in \underline{F}$ iff $\theta^{-1}(C) \in \underline{\Omega}$. Every Borel probability measure λ over Ω then induces a probability measure $\overset{*}{\lambda}$ over F. The mixed strategies over F considered in [A-2] are the various induced probability measures. The mixed strategies over X are the Borel distribution over that set.

Let $\underline{F} \otimes \underline{X}$ denote the σ-field generated by sets of the form $C \times D$, with $C \in \underline{F}$, $D \in \underline{X}$. In accordance with Aumann's conjecture (the identification space hypothesis) the set $G_E \overset{\text{def}}{\equiv} \{(f,x) \mid f \in F,$ $x \in X,\ f(x) \in E\}$ is included in the σ-field $\underline{F} \otimes \underline{X}$ for every Borel subset E of Y. If the hypothesis were correct, G_E would have a uniquely determined measure with respect to $\overset{*}{\lambda} \times \mu$. In what follows we give a counter-example which refutes this conjecture.

Lemma 3.1. Let $(C, \underline{C}, m_1)$, $(D, \underline{D}, m_2)$ be two σ-finite measure spaces and let $\underline{C} \otimes \underline{D}$ be the smallest σ-ring over $C \times D$ containing all sets of the form $C_1 \times D_1$, where $C_1 \in \underline{C}$, $D_1 \in \underline{D}$. Let $A_1 \subset C \times D$ and let $m_o(A_1)$ be the outer measure of A_1 in the space $(C \times D, \underline{C} \otimes \underline{D}, m_1 \times m_2)$. Then for every $\epsilon > 0$ there exists a sequence of rectangles $\{C_i \times D_i\}$ such that $C_i \in \underline{C}$, $D_i \in \underline{D}$ and $\cup C_i \times D_i \supset A_1$ and $m_1 \times m_2 (\cup C_i \times D_i) < m_o(A_1) + \epsilon$.

Proof: $m_o(A_1) = \inf\limits_{\substack{A \in \underline{C} \otimes \underline{D} \\ A \supset A_1}} m_1 \times m_2(A)$. Consequently there exists a set $A_2 \in \underline{C} \otimes \underline{D}$ such that $A_2 \supset A_1$ and $m_1 \times m_2(A_2) < m_o(A_1) + \frac{\epsilon}{2}$. The existence of a sequence of rectangles $\{C_i \times D_i\}$ such that $\cup C_i \times D_i \supset A_2$ and $m_1 \times m_2(\cup C_i \times D_i) < m_1 \times m_2(A_2) + \frac{\epsilon}{3} < m_o(A_1) + \epsilon$ may be proven by induction on the Borel rank of A_2.

Counter-Example (A): Let $\Omega \equiv X \equiv Y$ be copies of the unit interval and let $\underline{\Omega} \equiv \underline{X} \equiv \underline{Y}$ be copies of the usual Borel structure defined over Ω, X, and Y. We choose a 1 - 1 Borel function h which takes the unit interval onto itself such that h is measure preserving and ergodic, i.e., $h\colon [0,1] \to [0,1]$ and for every Lebesgue measurable $A \subset [0,1]$, $m(A) = m(h(A)) = m(h^{-1}(A))$ and $m(h(A) \cap A) < m(A)$ whenever $0 < m(A) < 1$. (Examples of measure preserving ergodic transformations may be found in [Hlm - 1].) For each fixed $t_1 \in [0,1]$, let $h_{t_1} \equiv \{(x, t_1) | h^n(t_1) = x$ for some integer $-\infty < n < \infty\}$ (for positive n, $h^n(t) = \underbrace{h\,(h(\cdots h(t))\cdots)}_{n \text{ iterations}}$, $h^{-n}(t) = \underbrace{h^{-1}(h^{-1}(\cdots h^{-1}(t))\cdots)}_{n \text{ iterations}}$; $h^0(t) = t$). h_t is referred to as the orbit of t. We define $\Theta : \Omega \times X \to Y$ as follows:

$$\Theta(\omega, x) = \begin{cases} 1 & \text{if } h^n(\omega) = x \text{ for } -\infty < n < \infty; \\ 0 & \text{otherwise.} \end{cases}$$

Θ is easily seen to be a Borel function (it is expressible in terms of h by means of both Π_1^1 and Σ_1^1 statements.) For each fixed $\omega_1 \in \Omega$, let $f_{\omega_1} \equiv \Theta(\omega_1, \cdot)$. For every $\omega \in \Omega$, let $\theta(\omega) = f_\omega$. θ associates each $\omega \in \Omega$ with a function $f_\omega : X \to Y$. Let $F = \theta(\Omega)$ and let $\underline{F}$ be the family of subsets $C \subset F$ such $\theta^{-1}(C) \in \underline{\Omega}$. Denote by $\underline{F} \times \underline{X}$ the smallest σ-ring generated by sets of the form $C \in \underline{F}$, $D \in \underline{X}$. Let λ and μ be copies of the usual Lebesgue measure over Ω and X respectively and let λ^* be the measure induced by λ over F. If the identification space hypothesis were true in the strict sense, then $\{(f,x) | f \in F, x \in X, f(x) = 1\} \in \underline{F} \otimes X$. If it were true in the weaker, Lebesgue sense, then there would have to exist $A_1, A_2 \in \underline{F} \otimes \underline{X}$ such that $A_1 \subset \{(f, x) \mid f \in F, x \in X, f(x) = 1\} \subset A_2$ and $\lambda^* \times \mu(A_1) = \lambda^* \times \mu(A_2)$. We shall show that the hypothesis is violated even in the Lebesgue sense.

We first observe that for $C \in \underline{F}$, either $\lambda^*(C) = 0$ or $\lambda^*(C) = 1$. If this were not the case then for some $D \in \underline{F}$, $0 < \lambda^*(D) = \lambda(\theta^{-1}(D)) < 1$.

Denote $\theta^{-1}(D)$ by $\hat{D}$. Since $0 < \lambda(\hat{D}) < 1$, $\lambda\,(h(\hat{D}) - \hat{D}) > 0$ because of ergodicity. Thus $h(\hat{D}) - \hat{D}$ is non-empty. Let $\omega \in h(\hat{D}) - \hat{D}$ and let $\omega_1 = h^{-1}(\omega)$. Then $\omega_1 \in \hat{D}$ and since $\omega = h(\omega_1)$, both ω and ω_1 have the same orbit, i.e. $h_\omega = h_{\omega_1}$. Consequently, $\theta(\omega) = \theta(\omega_1)$ and thus ω must be in $\theta^{-1}(D) = \hat{D}$, a contradiction.

For any $Z_1 \subset F \times X$, let $m_o(Z_1)$ denote the outer measure of Z_1 under $\lambda^* \times \mu$, i.e. $m_o(Z_1) = \inf\limits_{\substack{Z \in \underline{F} \otimes \underline{X} \\ Z \supset Z_1}} \lambda^* \times \mu(Z)$. We shall see that

$m_o(\{(f,x) \mid f \in F,\ x \in X,\ f(x) = 1\})$ and $m_o(\{(f,x) \mid f \in F,\ x \in X,\ f(x) = 0\})$ are both 1.

Suppose that $m_o(\{(f,x) \mid f \in F,\ x \in X,\ f(x) = 1\}) = c < 1$. Then by Lemma 3.1 there exists a sequence of rectangles $\{C_i \times D_i\}$ where $C_i \in F$, $D_i \in \underline{X}$, $\{(f,x) \mid f \in F,\ x \in X,\ f(x) = 1\} \subset \overset{\infty}{\cup} \{C_i \times D_i\}$ and $\lambda^* \times \mu(\bigcup_{i=1}^{\infty} \{C_i \times D_i\}) = c_1 < 1$. Let $Q \overset{def}{=} \{i \mid \lambda^*(C_i) = 1\}$, $T \overset{def}{=} \bigcup_{i \in Q} D_i$.

If $\mu(T) = 1$, then $\lambda^* \times \mu\,(\bigcup_{i \in R} \{C_i \times D_i\}) = 1$, in contradiction to the assumption that $\lambda^* \times \mu(\bigcup_{i=1}^{\infty} \{C_i \times D_i\}) = c_1 < 1$. Thus $\mu(T^{comp}) = c_2 > 0$. Let $U \overset{def}{=} \bigcup_{i \in Q^{comp}} C_i$, $V \overset{def}{=} U^{comp}$. Then $\lambda^*(V) = 1$, and for every (f_1, x_1), if $f_1 \in V$ and $x_1 \in T^{comp}$ then $(f_1, x_1) \notin \overset{\infty}{\cup} \{C_i \times D_i\}$. Since

$\{(f,x) \mid f \in F,\ x \in X,\ f(x) = 1\} \subset \bigcup_{i=1}^{\infty} \{C_i \times D_i\}$, it follows that $f_1(x_1) = 0$ for every $f_1 \in V$, $x_1 \in T^{comp}$. Let $T' = \bigcup_{-\infty<n<\infty} h^n\,(T^{comp})$. Since h is ergodic, $\mu\,(T') = 1$. Let $\hat{V} = \theta^{-1}(V)$. Then $\lambda(\hat{V}) = 1$. Since λ and μ are both copies of the Lebesgue measure over $[0,1]$ and $\lambda(\hat{V}) = \mu(T') = 1$, it follows that there exists a real number t such that for some $\omega_o \in \hat{V}$, $x_o \in T'$, ω_o and x_o are both identical to t. For any such ω_o, x_o, $\Theta(\omega_o, x_o) = 1$. $x_o = h^n(x_1)$ for some $x_1 \in T^{comp}$ and some integer n. $-\infty < n < \infty$. For any such x_1, $\Theta(\omega_o, x_1)$ is also 1, because like x_o, x_1 is also on the orbit of ω_o. Thus $f\omega_o(x_1) = \Theta(\omega_o, x_1) = 1$. However, $f\omega_o \in V$, $x_1 \in T^{comp}$ so by what was shown previously $f\omega_o(x_1)$ must be 0, a contradiction. We thus see that $m_o(\{(f,x)|f \in F, x \in X, f(x) = 1\}) = 1$. In the same fashion we can prove that $m_o(\{(f,x)|f \in F, x \in X, f(x) = 0\}) = 1$. We conclude that in the present case the hypothesis fails even in the Lebesgue sense.

3.2. In view of the preceding counter-example, we seek alternate means for determining a probability value under λ^*, μ for $\{(f,x)|f \in F, x \in X, f(x) \in E\}$ where $E \subset Y$ is Borel, λ^* is an induced distribution over F and μ is a Borel probability measure over X.

<u>Lemma 3.2.1.</u> Let C be a Borel subset of $\Omega \times X$ and let μ be a Borel probability measure over X. Then for any rational r_1, r_2, $\Theta \le r_1 \le r_2 \le 1$, the set $\{\omega | r_1 \le \mu(\{x|(\omega,x) \in C\}) \le r_2\}$ is Borel.

<u>Proof</u>: By induction on the Borel rank of C. In the case where C is of the form $A_1 \times A_2$ where $A_1 \subset \Omega, A_2 \subset X$, and A_1, A_2 are both rational intervals, the truth of the lemma is immediate. If $C = C_1 \cup \ldots \cup C_n$ where each C_i is of the form $A_1^{(i)} \times A_2^{(i)}$, with $A_1^{(i)} \subset \Omega, A_2^{(i)} \subset X$, and $A_1^{(i)}, A_2^{(i)}$ are rational intervals, then the lemma follows by induction on n. Given any particular Borel set $D \subset \Omega \times X$, if the lemma is true for D then

it is true for D^{comp}. Finally, suppose $D_1, D_2, \ldots$ is any countable sequence of subsets of $\Omega \times X$ for which the lemma holds and assume that $D_1 \subset D_2 \subset \ldots$. Let $D = \bigcup_{i=1}^{\infty} D_i$. If $\omega_1 \in \Omega$ is such that $r_1 \leq \mu(\{x | (\omega_1, x) \in D\}) \leq r_2$ then for every ε there is an n such that for all $i > n$, $r_1 - \varepsilon \leq \mu(\{x | (\omega_1, x) \in D_i\}) \leq r_2$. In other words, $\{\omega | r_1 \leq \mu(\{x | (\omega, x) \in D\}) \leq r_2 \equiv \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i>n} \{\omega | r_1 - \frac{1}{m} \leq \mu(\{x | (\omega, x) \in D_i\}) \leq r_2\}$. For every natural m, $\{\omega | r_1 - \frac{1}{m} \leq \mu(\{x | (\omega, x) \in D_i\} \leq r_2\}$ is Borel by hypothesis on D_i; consequently the set $\bigcap_{i>n} \{\omega | r_1 - \frac{1}{m} \leq \mu(\{x | (\omega, x) \in D_i\}) \leq r_2\}$ is Borel for every natural m, n, since it is the intersection of a countable sequence of Borel sets. Similarly the intersection $\bigcap_{m=1}^{\infty} (\bigcup_{n=1} \bigcap_{i>n} \{\omega | r_1 - \frac{1}{m} \leq \mu(\{x | (\omega, x) \in D_i\}) \leq r_2\}$ involves a countable sequence of Borel sets and is therefore Borel. Thus $\{\omega | r_1 \leq \mu(\{x | (\omega, x) \in D\}) \leq r_2\}$ is Borel. The family of sets for which the lemma holds is therefore closed under complements and countable unions and consequently contains all Borel subsets of $\Omega \times X$.

Lemma 3.2.2. Let $\Theta : \Omega \times X \to Y$ be a Borel function and let B be a Borel subset of Y. Let r_1, r_2 be rational numbers, $0 \leq r_1 \leq r_2 \leq 1$, and let μ be a probability measure on X. Then the set $\{\omega | r_1 \leq \mu(\{x | (\omega, x) \in B\}) \leq r_2\}$ is Borel measurable.

Proof: Since Θ is a Borel function, $\Theta^{-1}(B) \subset \Omega$ is Borel. By the preceeding lemma $\{\omega | r_1 \leq \mu(\{x | (\omega, x) \in \Theta^{-1}(B)\}) \leq r_2\}$ is Borel. However, $\{\omega | r_1 \leq \mu(\{x | (\omega, x) \in \Theta^{-1}(B)\}) \leq r_2\} \equiv \{\omega | r_1 \leq \mu(\{x | \Theta(\omega, x) \in B\}) \leq r_2\}$, which completes the proof.

Lemma 3.2.3. Using the foregoing lemmas we consider an alternate method for evaluating the probability measure of $\{(f, x) | f(x) \in E \subset Y, f \in F, x \in X\}$

under λ^*, μ, where F is the range of a Borel $\Theta: \Omega \times X \to Y$ and where λ^*, the induced distribution over F, and μ, the Borel probability measure over X, are prescribed.

Let r, s be arbitrary rational numbers, $0 \leq r \leq s \leq 1$. In accordance with Lemma 3.2.2, the set $O_{(r,s)} \overset{\text{def}}{\equiv} \{\omega | r \leq \mu(\{x | \Theta(\omega, x) \in E\}) \leq s\} \subset \Omega$ is Borel. Let $\tilde{O}_{(r,s)} = \theta(O_{(r,s)})$. It is easily seen that $\theta^{-1}(\tilde{O}_{(r,s)}) \equiv O_{(r,s)}$, i.e. for any $\omega \in \Omega$, if $\theta(\omega) \in \tilde{O}_{(r,s)}$ then $\omega \in O_{(r,s)}$. Thus, since $\theta^{-1}(\tilde{O}_{(r,s)}) \equiv O_{(r,s)}$ and $O_{(r,s)}$ is Borel measurable, $\tilde{O}_{(r,s)} \in \underline{F}$ and is therefore measurable with respect to λ^*. Denote the measure of $\tilde{O}_{(r,s)}$ with respect to λ^* by $\lambda^*_{(r,s)}$.

Suppose, now, that $f \in F$ and $x \in X$ are chosen at random. The probability of f belonging to $\tilde{O}_{(r,s)}$ is then $\lambda^*_{(r,s)}$. For any f selected from $\tilde{O}_{(r,s)}$, the probability of $f(x)$ belonging to E lies between r and s. Thus the probability of choosing f and x so that $f \in \tilde{O}_{(r,s)}$ and $f(x) \in E$ would evidently lie between $\lambda^*_{(r,s)} \cdot r$ and $\lambda^*_{(r,s)} \cdot s$. Let $r_0, r_1, \ldots, r_n$ be a sequence of rational numbers in $[0,1]$ such that $r_0 \leq r_1 \leq \ldots \leq r_n$, $r_0 = 0, r_n = 1$, $r_{i+1} - r_i = \frac{1}{n}$ for all $0 \leq i \leq n-1$. By the same mode of reasoning, the probability of choosing f, x so that $f(x) \in E$ would evidently be between $\lambda^*_{(r_0,r_1)} \cdot r_0 + \lambda^*_{(r_1,r_2)} \cdot r_1 + \ldots \lambda^*_{(r_{n-1},r_n)} \cdot r_{n-1}$ and $\lambda^*_{(r_0,r_1)} \cdot r_1 + \lambda^*_{(r_1,r_2)} \cdot r_2 + \ldots + \lambda^*_{(r_{n-1},r_n)} \cdot r_n$. Using the Cauchy-Schwartz inequality we evaluate the difference between these two bounds:

$$(\lambda^*_{(r_0,r_1)} \cdot r_1 + \lambda^*_{(r_1,r_2)} \cdot r_2 + \ldots + \lambda^*_{(r_{n-1},r_n)} \cdot r_n) - (\lambda^*_{(r_0,r_1)} \cdot r_0 + \lambda^*_{(r_1,r_2)} \cdot r_1 + \ldots + \lambda^*_{(r_{n-1},r_n)} \cdot r_{n-1}) = \lambda^*_{(r_0,r_1)} \cdot (r_1 - r_0) + \lambda^*_{(r_1,r_2)} \cdot (r_2 - r_1) + \ldots + \lambda^*_{(r_{n-1},r_n)} \cdot (r_n - r_{n-1}) \leq$$

$$\sqrt{(\lambda^*_{(r_0,r_1)})^2 + (\lambda^*_{(r_1,\ \ r_2)})^2 + \ldots + (\lambda^*_{(r_{n-1},r_n)})^2} \cdot$$

$$\sqrt{(r_1 - r_0)^2 + (r_2 - r_1)^2 + \ldots + (r_n - r_{n-1})^2} \leq$$

$$1 \cdot \sqrt{(r_1 - r_0)^2 + (r_2 - r_1)^2 + \ldots + (r_n - r_{n-1})^2} = \sqrt{\frac{n}{n^2}} =$$

$$\sqrt{\frac{1}{n}}$$

Thus as $n \to \infty$, the difference between the two quantities tends toward zero. Hence the probability p of choosing f, x such that $f(x) \in E$ would evidently be $p = \lim_{n\to\infty} \lambda^*_{(r_0,r_1)} \cdot r_0 + \ldots + \lambda^*_{(r_{n-1},r_n)} \cdot r_n$.

This value clearly depends on λ^* alone and does not depend on the λ through which λ^* is induced.

3.3. Despite its apparent plausibility, the method for evaluating $\lambda^* \times \mu(\{(f,x) \mid f \in F, x \in X, f(x) \in E\})$ depicted above cannot be applied to every possible $F \subset Y^X$. Situations exist in which no reasonable probability evaluation can be ascribed to $\{(f,x) \mid f \in F, x \in X, f(x) \in E\}$. We consider the following case.

<u>Counter-Example (B)[4/]</u>: Let α_1 be the set of all countable ordinals and assume the truth of the continuum hypothesis. Then there is a one to one function h_o from $X = [0,1]$ onto α_1. For each $x_1 \in X$, let $A_{x_1} \equiv \{y \mid y \in [0,1]$ such that the ordinal $h_o(y)$ is less than $h_o(x_1)\}$. Let $f_{[x_1]} : X \to Y$ be the characteristic function of A_{x_1}.($f_{[x_1]}(x) = 1$ if $x \in A_{x_1}$ and 0 otherwise.) Note that if $x_1 \neq x_2$ then $f_{[x_1]} \not\equiv f_{[x_2]}$. Let F be the set of functions of the form $f_{[x_1]}$, i.e. $F \equiv \{f \mid f \equiv f_{[x_1]}$ for some $x_1 \in [0,1]\}$. We now define a measure structure, $\underline{F}$, over F. For each $f \in F$, let

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4/ This example grew out of a private discussion with Shmuel Berger.

$W(f)$ be the unique x in X such that $f \equiv f_{[x]}$. Given any rational interval (a,b) where $0 \leq a < b \leq 1$, let $G_{(a,b)} \equiv \{f | W(f) \in (a,b)\}$. We let $\underline{F}$ be the smallest σ-finite measure space generated by sets of the form $G_{(a,b)}$. For each set $G_{(a,b)} \subset F$, let $\overline{\lambda}(G_{(a,b)}) = b - a$, and for every rational $x \in [0,1]$, let $\overline{\lambda}(\{f_{[x]}\}) = 0$. $\overline{\lambda}$ is then extendable to a uniquely definable probability measure, $\hat{\lambda}$, over all of $\underline{F}$ ([Hlm - 2], p.54). Clearly $\hat{\lambda}$ is non-atomic, i.e., $\hat{\lambda}(\{f\}) = 0$ for all $f \in F$. Let μ be the usual Lebesgue measure over $X \equiv [0,1]$. We seek to evaluate the probability that $f(x) = 1$, when $f \in F$, $x \in X$ are chosen at random under these conditions. For any fixed $x_1 \in X$, the set $\{f | f \in F, f(x_1) = 0\}$ is countable. Thus the probability that $f(x_1) = 1$ is 1. This is true for any $x_1 \in X$. It should then follow that the probability measure for $\{(f,x) | f \in F, x \in X, f(x) = 1\}$ is 1. On the other hand for every fixed $f_o \in F$, $f_o(x) = 1$ for only a countable number of x's. Thus the probability that $f_o(x) = 1$ is 0 for any particular $f_o \in F$. From this point of view, the probability that $f(x) = 1$ must be zero.

4. The foregoing section, ℙ 3, dealt mainly with function spaces which were ranges of Borel Θ's. Given a Borel function $\Theta: \Omega \times X \to Y$, we showed that if $F \equiv \{f | f: X \to Y$ and $f \equiv \Theta(\omega, \cdot)$ for some $\omega \in \Omega\}$ and if $\underline{F}$ is the family of subsets C of F such that $\theta^{-1}(C)$ is a Borel subset of Ω (where $\theta: \Omega \to F$ is defined by $\theta(\omega) \equiv \Theta(\omega, \cdot)$), then for any induced probability distribution λ^* over F and any Borel probability measure μ over X, λ^* and μ determine a unique probability value for $\{(f,x) | f \in F, x \in X, f(x) \in E\}$ whenever $E \subset Y$ is Borel. The methods for evaluating this measure were described

in $\mathbb{P}$ 3.2 . We seek to extend these methods to more general types of function spaces.

Let F be a set of Borel functions from X to Y and let $\underline{F}$ be a σ-field of subsets of F. The space $(F,\underline{F})$ will be referred to as a <u>measurable range</u> if there exists a (not necessarily Borel) function $\Theta: \Omega \times X \to Y$ with the following properties:

(1) If $\theta: \Omega \to F$ is defined by $\theta(\omega) \equiv \Theta(\omega,\cdot)$, then $\theta(\Omega) = F$;

(2) For any probability measure $\underline{\lambda}$ defined over $\underline{F}$, there exists a measure λ over $\underline{\Omega}$ (the Borel subset of Ω) such that

(a) given any $C \in \underline{F}$, $\theta^{-1}(C)$ is measurable with respect to λ;

(b) $\lambda(\theta^{-1}(C)) \equiv \underline{\lambda}(C)$;

(c) for any Borel probability measure μ over X, $\Theta : \Omega \times X \to Y$ is measurable with respect to $\lambda \times \mu$, i.e. for any Borel subset $B \subset Y$, $\Theta^{-1}(B)$ is measurable with respect to $\lambda \times \mu$.

Function spaces which are measurable ranges will be treated in much the same way as in the case where they are ranges of Borel Θ's. If $(F,\underline{F})$ is a measurable range, the mixed strategies over F are those distributions which are induced by Borel λ's defined over Ω. Greater generality is thereby achieved both in the class of function spaces we can randomize as well as in the set of mixed strategies that may be obtained.

In determining a probability measure (or mixed strategy) μ over $X \equiv [0,1]$, one usually starts by defining μ over the rational sub-intervals of X. μ is assumed to be σ-finite over the Boolean algebra $\mathcal{A}'$ consisting of finite unions and intersections of rational closed intervals. The Caratheodory extension theorem (see [Hlm-2], p. 54) then asserts that μ can be uniquely extended over the entire Borel structure generated by $\mathcal{A}'$. Although there are many other Boolean algebras of subsets over which μ might initially be defined, the algebra $\mathcal{A}'$ is the most easily expressible in set theoretical language and therefore a natural one with which to begin.

If we pursue the same approach in obtaining mixed strategies, $\underline{\lambda}$, over a set of functions, F_1, we would start by defining $\underline{\lambda}$ over a countable family of subsets of F_1, where each such subset is definable from F_1 by means of a formula in the language of set theory. Suppose, therefore, that $\mathcal{A}_1$ is a Boolean algebra of subsets of F_1 and that each element in $\mathcal{A}_1$ is definable from F_1 by a set theoretic formula. Assume in addition that F_1 itself is definable from a real number and that each $f \in F_1$ is a Borel function from X into Y. Let $\underline{\lambda}$ be a σ-finite measure defined over $\mathcal{A}_1$. $\underline{\lambda}$ can then be extended to a unique σ-finite measure defined over $\underline{F}_{\mathcal{A}_1}$, where $\underline{F}_{\mathcal{A}_1}$ is the σ-field generated by $\mathcal{A}_1$. We are then interested in the possibility of determining a probability measure under $\underline{\lambda}$ and μ for the set $\{(f,x) \mid f \in F_1, x \in X, f(x) \in E\}$, where E is any Borel subset of Y.

If $(F_1, \underline{F}_{\mathcal{A}_1})$ is a measurable range then there exists a $\Theta_1\colon \Omega \times X \to Y$ of which F_1 is the range; also a Borel λ over Ω exists such that for each $C \in \underline{F}_{\mathcal{A}_1}$, $\lambda(\theta_1^{-1}(C)) = \underline{\lambda}(C)$, where $\theta_1\colon \Omega \to F_1$ is given by $\theta_1(\omega) = \Theta_1(\omega, \cdot)$. For any subset $D \subset F_1$ (not necessarily in $\underline{F}_{\mathcal{A}_1}$) such that $\theta^{-1}(D)$ is measurable with respect to λ, let $\lambda^*(D) = \lambda(\theta^{-1}(D))$. The induced measure λ^* clearly coincides with $\underline{\lambda}$ over the measure structure $\underline{F}_{\mathcal{A}_1}$. It will be shown in $\mathbb{P}$ 4.9 that λ^* and μ determine a unique probability measure for $\{(f,x) \mid f \in F, x \in X, f(x) \in E\}$ given any Borel subset E of Y. Thus, if $(F_1, \underline{F}_{\mathcal{A}_1})$ is a measurable range, $\underline{\lambda}$ may be extended to a (not necessarily unique) probability measure λ^* for which $\lambda^* \times \mu$ $(\{(f,x) | f \in F, x \in X, f(x) \in E\})$ is uniquely defined.

The question of whether a function space $(F_1, \underline{F}_{\mathcal{A}_1})$ as described above is in general a measurable range cannot be resolved within ZFC. In fact, within the model L of ZFC constructed by Gödel ([C], pp. 87-88), the proposition[5/] "For any definable set, F, of Borel functions from X to Y and any countable algebra $\mathcal{A}$ consisting of definable subsets of F, the space $(F, \underline{F}_{\mathcal{A}})$ is a measurable range" is false.[6/] We shall show, however, that in the countable model $\mathcal{N}$ of ZFC described in [So][7/] the proposition is true. We remark that in order to construct the model $\mathcal{N}$, one assumes the existence of inaccessible cardinals.

In all that follows, the symbol $\mathcal{N}$ will denote the model $\mathcal{N}$ of [So]. The reader is referred to [So] for a full description of this model.

5/ This proposition should be viewed as a meta-statement, not necessarily expressible within the language of ZFC.

6/ The reason is a follows: In L, there exists a well ordering over the reals which is definable by a set theoretical formula. The continuum hypothesis is also true in this model. Consequently there exists within L a definable set of Borel functions F_2 from X to Y and a countable Boolean algebra $\mathcal{A}_2$ consisting of definable subsets of F_2 such that $(F_2, \underline{F}_{\mathcal{A}_2})$ has all of the properties depicted in Counter-example (B). It is easily seen that no function space bearing these properties can be a measurable range.

7/ This model was originally introduced by Azriel Levy [Le].

4.1. Lemma ([So], Theorem 2 (2), p. 3). In $\mathcal{N}$, the following statement is true:

Let A be any subset of $\Omega \equiv [0,1]$ definable from a countable sequence of ordinals; then A is measurable with respect to the usual Lebesgue measure m, i.e. there exists Borel sets $A_1, A_2 \subset \Omega$ such that $A_1 \subset A \subset A_2$ and $m(A_1) = m(A_2)$.

(In particular, since every real number c is expressible as a countable sequence of 0's and 1's, it follows that if A is any $\mathcal{R}$-definable set of reals belonging to $\mathcal{N}$ (i.e. a set of reals which is definable from some real c in $\mathcal{N}$) then A is definable from a countable sequence of ordinals and thus $\mathcal{N} \models$ "A is Lebesgue measurable.")

4.2 Lemma. In $\mathcal{N}$, the following statement is true:

Let λ be a Borel probability measure defined over $\Omega \equiv [0,1]$ and let A be any subset of $\Omega \equiv [0,1]$ which is definable from a countable sequence of ordinals. Then A is measurable with respect to λ, i.e. there exists Borel subsets A_1, A_2 of Ω such that $A_1 \subset A \subset A_2$ and $\lambda(A_1) = \lambda(A_2)$.

Proof: With no loss in generality assume that $\lambda(\{\omega\}) = 0$ for every $\omega \in \Omega$. Let $\Omega' \equiv [0,1]$ be another copy of the unit interval and let m be the usual Lebesgue measure defined over Ω'. We define a mapping $g_o\colon \Omega \to \Omega'$: $g_o(\omega) \overset{\text{def}}{=}$ the unique $t \in \Omega'$ such that $t = \lambda([0,\omega))$, where $\lambda([0,\omega))$ is the measure under λ of the half open interval $[0,\omega) \subset \Omega$. g_o is easily seen to be onto. The image under g_o of every open interval is a closed or open interval. By induction on Borel ranks one proves that for any $C \subset \Omega$, C is Borel iff $g_o(C)$ is a Borel subset of Ω'. For any open interval $(\omega,\bar{\omega})$ in Ω, $0 \leq \omega < \bar{\omega} \leq 1$,

$$m(g_o((\omega\ ,\bar{\omega}))) = m((g_o(\omega),\ g_o(\bar{\omega})) = g_o(\bar{\omega}) - g_o(\omega) = \lambda(\omega,\bar{\omega})).$$

Similarly, by induction on Borel ranks,

$$\lambda(B) = m(g_o(B))$$

for any Borel subset $B \subset \Omega$. Suppose that A is any subset of Ω definable from a countable sequence of ordinals. Then $g_o(A)$ is likewise definable from a countable sequence of ordinals; it is therefore Lebesgue measurable with respect to m, in accordance with Lemma 4.1. Hence there exist Borel subsets A_1, A_2, of Ω' such that $A_1 \subset g_o(A) \subset A_2$ and $m(A_1) = m(A_2)$. Thus $\lambda(g_o^{-1}(A_1)) = \lambda(g_o^{-1}(A_2))$ and $g_o^{-1}(A_1) \subset A \subset g_o^{-1}(A_2)$. Therefore A is measurable with respect to λ.

Corollary: In $\mathcal{N}$, the following statement in true:

Let $\bar{m}$ be a Borel probability measure defined over $\Omega \times X \equiv [0,1] \times [0,1]$ and let A be any subset of $\Omega \times X$ definable from a countable sequence of ordinals. Then A is measurable with respect to $\bar{m}$.

4.3. Lemma ([So] Theorem 2 (4), p. 3). In $\mathcal{N}$, the following statement is true:

For every uncountable set of reals A definable from a countable sequence of ordinals, there is a perfect set A_1 such that $A_1 \subset A$.

4.4. Lemma. In $\mathcal{N}$, the following statement is true:

Let F be a non-empty set of Borel functions from X to Y such that F is definable from a countable sequence of ordinals, and let $\tilde{F}$ be the set of reals which code[8/] functions in F. Then there exists a function,

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8/ We recall the definition of codes for Borel subsets of E^n appearing in IP 2.1, (4). A Borel function from X to Y in a special instance of a Borel subset of X $\times$ Y; consequently for any $f \in F$, there is a real parameter $\xi \in \mathcal{R}$ which codes f. There are, in fact, an uncountable number of reals in $\mathcal{R}$ which code any given $f \in F$.

g, $g: \Omega \to \tilde{F}$ which takes $\Omega \equiv [0,1]$ onto $\tilde{F}$, and where g is definable from a countable sequence of ordinals.

Proof: Since every $f \in F$ is coded by an uncountable number of reals, $\tilde{F}$ is thus uncountable. Moreover, $\tilde{F}$ is definable from a countable sequence of reals, since F has this property. Hence by Lemma 4.3, there is a perfect set of reals C such that $C \subset \tilde{F}$. Let $C_1 \subset C$ be a countable set of reals which is dense in C. C_1 is clearly $\mathcal{R}$-definable. Let g_2 be a 1 - 1 order preserving mapping from the **rationals** in Ω into C_1. We define a 1 - 1 function g_1 from Ω into C: For rational x, let $g_1(x) = g_2(x)$, for irrational x, let $g_1(x) = \sup\limits_{\substack{z \text{ rational} \\ z < x}} g_2(x)$.

We then define $g: \Omega \to \tilde{F}$ as follows:

$$g(x) = \begin{cases} g_1(x) & \text{if for some } y \notin \tilde{F} \text{ and for some non-negative integer } n, \\ & x = g_1^n(y) = \underbrace{g_1 \cdot g_1 \cdot \ldots \cdot g_1}_{n \text{ iterations}}(y); \\ x & \text{otherwise} \end{cases}$$

Using considerations of the Cantor-Bernstein theorem, we find that y takes $\Omega \equiv [0,1]$ one-to-one onto $\tilde{F}$. Clearly g is definable from a countable sequence of ordinals.

4.5. Lemma ([So], Theorem 2 (5), p. 3). In $\mathcal{N}$, the following statement is true:

Let $A \subseteq \mathcal{R}^2$ be a subset of $\mathcal{R} \times \mathcal{R}$ definable from a countable sequence of ordinals and such that for all $x \in \mathcal{R}$ there exists a $y \in \mathcal{R}$ such that $\langle x,y \rangle \in A$. Then there exists a Borel function $h: \mathcal{R} \to \mathcal{R}$ such that $\langle x,h(x) \rangle \in A$ for almost all x in $\mathcal{R}$ (almost all with respect to the usual Lebesgue measure.)

4.6 Lemma. In $\mathcal{N}$, the following statement is true:

Let m_o be a Borel probability measure over $[0,1]$ and let $A \subset [0,1] \times [0,1]$ be definable from a countable sequence of ordinals; suppose that the projection of A onto the unit interval is of measure 1 with respect to m_o. Then there exists a Borel function $\hat{h}: [0,1] \to [0,1]$ such that $\langle x,\hat{h}(x)\rangle \in A$ for almost all $x \in [0,1]$ (almost all, under the measure m_o).

Proof: We assume, with no loss in generality, that $m_o(\{x\}) = 0$ for every $x \in [0,1]$. Let m be the usual Lebesgue measure. We define $h_o: [0,1] \to [0,1]$ in the same way that g_o was defined in the proof of Lemma 4.2: for any $t \in [0,1]$, $h_o(t) =$ the unique t' in $[0,1]$ such that $t' = m_o([0,t))$. h_o then takes $[0,1]$ onto $[0,1]$. Moreover, $C \subset [0,1]$ is Borel iff $h_o(C)$ is Borel; if $C \subset [0,1]$ is Borel then $m_o(C) = m(h_o(C))$. This is proven by transfinite induction on the Borel rank of C.
Let $\hat{A} = \{(x,y) | x \in [0,1], y \in [0,1]$, and for some $x_1 \in [0,1]$, $h(x_1) = x$ and $(h_o^{-1}(x), y) \in A\}$. $\hat{A}$ is clearly definable from a countable sequence of ordinals. Denote by A_p and $\hat{A}_p$ the projection of A and $\hat{A}$, respectively, onto $[0,1]$. $m_o(A_p) = 1$, by hypothesis. Since $\hat{A}_p = h_o(A_p)$ and since h_o carries Borel sets onto Borel sets of the same measure, it follows that $\hat{A}_p$ is measurable with respect to m. In addition $m(\hat{A}_p) = m_o(h_o^{-1}(\hat{A}_p)) = m_o(A_p) = 1$. It then easily follows from Lemma 4.5 that there is a Borel function $h: [0,1] \to [0,1]$ such that for almost every $x \in [0,1]$ (under m), $\langle x,h(x)\rangle \in \hat{A}$. Define $\hat{h}: [0,1] \to [0,1]$ as follows: $\hat{h}(x) \stackrel{\text{def}}{=} h(h_o(x))$. $\hat{h}$ then possess all of the asserted properties.

4.7. Lemma. In $\mathcal{N}$, the following statement is true:

Let $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$ be a countable set of subsets of Ω, where each set $\mathcal{A}_i$ is definable from a countable sequence of ordinals and where $\mathcal{A}$ is closed under complements, finite unions, and finite intersections. Let $\Sigma_{\mathcal{A}}$ be the σ-ring generated by $\mathcal{A}$ and let λ be any σ-finite probability measure over $\Sigma_{\mathcal{A}}$. Then λ may be extended to a σ-finite probability measure λ_1 over all subsets of Ω definable from a countable sequence of ordinals.

<u>Proof</u>: The proof is based upon a method used in [Lb], where an analogous result is obtained.[9/]

For $\mathcal{A}_i \in \mathcal{A}$, let $\mathcal{A}_i^{(1)} = \mathcal{A}_i$, $\mathcal{A}_i^{(0)} = \Omega - \mathcal{A}_i$. Note that for any infinite sequence of 0's and 1's, $s = s_1, s_2, \ldots$, the intersection $\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots$ is in $\Sigma_{\mathcal{A}}$. Let S be the set of all infinite sequences of 0's and 1's. We refer to a subset $U \subset S$ of the form $U \equiv \{s | s = (s_1, s_2, \ldots) \in S,\ s_1 = j_1,\ s_2 = j_2,\ \ldots\ s_n = j_n\}$, where each $j_i = 0$ or 1, as an open interval in S. Let $\underline{S}$ be the σ-field over S generated by the open intervals. We consider the following probability measure m_1, over $\underline{S}$. Given any open interval $U \subset S$ of the form $U = \{s | s = (s_1, s_2, \ldots) \in S,\ s_1 = j_1,\ s_2 = j_2, \ldots, s_n = j_n\}$, let $m_1(U) = \lambda(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots \cap \mathcal{A}_n^{(s_n)})$. m_1 is clearly σ-finite over the algebra of finite unions and intersections of intervals, since λ is σ-finite for the corresponding sets in $\Sigma_{\mathcal{A}}$. Hence, by the Caratheodory extension theorem, m_1 is uniquely extendable to all sets in $\underline{S}$. Applying Lemma 4.2 to $(S, \underline{S}, m_1)$ we receive that every subset of S definable from a countable sequence of ordinals is measurable with respect to m_1.

9/ We are grateful to Professor Benjamin Weiss for calling our attention to this approach.

We define a correspondence Ψ from the subsets of S to the subsets of Ω: for each $C \subset S$, let $\Psi(C) \overset{\text{def}}{=} \{\omega \mid$ There exists a sequence $s = (s_1, s_2, \ldots) \in C$ such that $\omega \in \mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots\}$. It follows by transfinite induction on Borel ranks that for any $C \in \underline{S}$, $\Psi(C) \in \Sigma_{\mathcal{A}}$ and $m_1(C) = \lambda(\Psi(C))$. Also, for every $W \in \Sigma_{\mathcal{A}}$ there is a $\overline{W}$ in $\underline{S}$ such that $\Psi(\overline{W}) = W$.

Let $D \overset{\text{def}}{=} \{s \mid s = (s_1, s_2, \ldots) \in S$ and $\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots$ is non-empty$\}$, $\overline{E} \overset{\text{def}}{=} \{(s,\omega) \mid s = (s_1, s_2, \ldots) \in S,\ \omega \in \Omega,\ \omega \in \mathcal{A}_1^{(s_1)} \subset \mathcal{A}_2^{(s_2)} \cap \ldots\}$. Since D is definable from a countable sequence of ordinals, it is measurable with respect to m_1. We assert that $m_1(D) = 1$. If not, then there exist Borel sets $V_1, V_2 \in \underline{S}$ such that $m_1(V_1 \cap V_2) = 0$, $V_1 \supset D$, $V_2 \supset S - D$, and $m_1(V_2) > 0$. Then $m_1(V_2 - V_1) > 0$. However, $m_1(V_2 - V_1) = \lambda(\Psi(V_2 - V_1))$, while $\lambda(\Psi(V_2 - V_1)) = 0$ since $\Psi(V_2 - V_1) \subset \Psi(\Omega - D)$, which is empty, a contradiction.

Since $m_1(D) = 1$, we may apply Lemma 4.6 and thereby derive the existence of a Borel function $h: S \to \Omega$ such that for almost all $s \in S$ (almost all with respect to the measure m_1), $(s, h(s)) \in \overline{E}$. Thus there exists a Borel set $G \subset S$ such that $m_1(G) = 1$ and for all $s \in G$, $(s, h(s)) \in \overline{E}$. Let $H = \{\omega \mid \omega \in \Omega$ and $\omega = h(s)$ for some $s \in G\}$. Since every Borel set is coded by means of a Π_1^1 formula and some real parameter, G is $\mathcal{R}$-definable. Consequently H, too, is $\mathcal{R}$-definable. Note that for almost all $s = (s_1, s_2, \ldots) \in S$ (under the measure m_1) H contains a single representative from $\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots$; that is, m_1 $(\{s \mid s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H$ consists of a single point$\}) = 1$.

We now define λ_1. For every rational closed sub-interval $[a,b] \subset \Omega$, let $\lambda_1([a,b]) \overset{\text{def}}{=} m_1(\{s \mid s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H \cap [a,b] \neq \emptyset\})$. The set $\{s \mid s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H \cap [a,b] \neq \emptyset\}$ is definable from a countable sequence of ordinals and is therefore measurable with respect to m_1. Consequently λ_1 is well defined. Also λ_1 is a σ-finite probability measure over the Boolean algebra consisting of complements, finite intersections and finite unions of rational closed intervals. This follows from the fact that m_1 has this property for the corresponding Boolean algebra of subsets of S. Thus λ_1 is extendable to a uniquely defined probability measure over all Borel subsets of Ω. Using the fact that $\lambda_1([a,b]) = m_1(\{s \mid s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H \cap [a,b] \neq \emptyset\}$ we obtain by transfinite induction that $\lambda_1(C) = m_1(\{s \mid s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H \cap C \neq \emptyset\}$ for every Borel subset $C \subset \Omega$. In accordance with Lemma 4.2, every subset of Ω definable from a countable sequence of ordinals is measurable with respect to λ_1.

To prove that λ_1 is an extension of λ, it is sufficient to prove that $\lambda_1(\mathcal{A}_i) = \lambda(\mathcal{A}_i)$ for every $\mathcal{A}_i$ in $\mathcal{A}$. It would then follow by transfinite induction that $\lambda_1(B) = \lambda(B)$ for every $B \in \Sigma_{\mathcal{A}}$.

By definition of m_1, $\lambda(\mathcal{A}_i) = m_1(\{s \mid s = (s_1, s_2, \ldots) \in S$ and $s_i = 1\})$. Clearly $\{s \mid s = (s_1, s_2, \ldots) \in S$ and $s_i = 1\} = \{s \mid s = (s_1, s_2, \ldots) \in S$ and $s_i = 1$ and $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H \neq \emptyset\} \cup \{s \mid s = (s_1, s_2, \ldots) \in S$ and $s_i = 1$ and $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H = \emptyset\}$. However for almost all $s = (s_1, s_2, \ldots) \in S$ (under the measure m_1) $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H \neq \emptyset$. Therefore, $m_1(\{s \mid s = (s_1, s_2, \ldots) \in S$ and $s_i = 1$ and $(\mathcal{A}_1^{(s_1)} \cap \mathcal{A}_2^{(s_2)} \cap \ldots) \cap H = \emptyset\} = 0$ and thus $\lambda(\mathcal{A}_i) = m_1(\{s \mid s = (s_1, s_2, \ldots) \in S$ and $s_i = 1$

and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap \ldots) \cap H \neq \emptyset\}$. Since $\mathcal{O}_i$ is by hypothesis definable from a countable sequence of ordinals, it is measurable with respect to λ_1. Hence there exist Borel subsets A_1, A_2 such that $A_1 \subset \mathcal{O}_i \subset A_2$ and $\lambda_1(A_1) = \lambda_1(A_2)$. Thus $\lambda_1(A_1) = m_1(\{s | s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap \ldots) \cap H \cap A_1) \neq \emptyset\}) = \lambda_1(A_2) = m_1(\{s | s = (s_1, s_2, \ldots)$ $\in S$ and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap \ldots) \cap H \cap A_2) \neq \emptyset\})$. Since $\{s | s = (s_1, s_2, \ldots)$ $\in S$ and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap \ldots) \cap H \cap A_1 \neq \emptyset\} \subset \{s | s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap \ldots) \cap H \cap \mathcal{O}_i \neq \emptyset\} \subset \{s | s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap \ldots) \cap H \cap A_2 \neq \emptyset\}$ we receive $m_1(\{s | s = (s_1, s_2, \ldots)$ $\in S$ and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap \ldots) \cap H \cap \mathcal{O}_i \neq \emptyset\} = \lambda_1(A_1) = \lambda_1(A_2) = \lambda_1(\mathcal{O}_i)$. However $m_1(\{s | s = (s_1, s_2, \ldots) \in S$ and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap \ldots) \cap H \cap \mathcal{O}_i \neq \emptyset\}) =$

$m_1(\{s | s = (s_1, s_2, \ldots) \in S$ and $s_i = 1$ and $(\mathcal{O}_1^{(s_1)} \cap \mathcal{O}_2^{(s_2)} \cap H \neq \emptyset\}) =$ $\lambda(\mathcal{O}_i)$. Thus $\lambda(\mathcal{O}_i) = \lambda_1(\mathcal{O}_i)$, and the proof is complete.

4.8. Theorem. In $\mathcal{N}$, the following statement is true:

Let F be a set of Borel functions from X to Y, where F is definable from a countable sequence of ordinals, and let $\mathcal{O} = \{\mathcal{O}_1, \mathcal{O}_2, \ldots\}$ be a countable set of subsets of F, where each $\mathcal{O}_i$ is definable from a countable sequence of ordinals and where $\mathcal{O}$ is closed under complements, finite unions, and finite intersections. Let $\underset{\sim}{F}_{\mathcal{O}}$ be the σ-field generated by $\mathcal{O}$. Then $(F, \underset{\sim}{F}_{\mathcal{O}})$ is a measurable range.

Proof: Each $f \in F$ comprises a Borel subset of $\mathcal{R}^2$. Hence by IP 2.1 (4), the functions in F are each coded by parameters in $\mathcal{R}$. Let $\tilde{F}$ be the set of parameters in $\mathcal{R}$ which code functions in F. $\tilde{F}$ is then definable from a countable sequence of ordinals. Hence, by Lemma 4.4 there exists a function $g \colon \Omega \to \tilde{F}$ definable from a countable sequence of ordinals which takes Ω onto $\tilde{F}$. For each ω, let $f_{g(\omega)}$ be the function in F coded by $g(\omega)$. Let $\Theta: \Omega \times X \to Y$ be defined as follows. $\Theta(\omega,x) = f_{g(\omega)}(x)$. Then Θ is definable from a countable sequence of ordinals and F constitutes the range of Θ.

Let $\theta: \Omega \to F$ be defined by $\theta(\omega) = \Theta(\omega,\cdot)$. For each $\mathcal{A}_i \in \mathcal{A}$, let $\hat{\mathcal{A}}_i = \theta^{-1}(\mathcal{A}_i)$. Similarly, let $\hat{\mathcal{A}}$ be the set of subsets of Ω, $\{\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \ldots\}$, and let $\underline{F}_{\hat{\mathcal{A}}}$ be the σ-field generated by $\hat{\mathcal{A}}$.

Given any probability measure $\underline{\lambda}$ over $\underline{F}_{\mathcal{A}}$, let λ be the corresponding probability measure over $\underline{F}_{\hat{\mathcal{A}}}$. In accordance with Lemma 4.7, λ may be extended to a probability measure, λ_1, defined over all subsets of Ω, that are definable from a countable sequence of ordinals. Let μ be any Borel probability measure over X, and let E be an arbitrary Borel subset of Y. Since E is coded by a real number and since Θ is definable by a countable sequence of ordinals, $\Theta^{-1}(E) \subset \Omega \times X$ also has this property. Thus, by Corollary 4.2 $\Theta^{-1}(E)$ is measurable with respect to $\lambda_1 \times \mu$. $(F, \underline{F}_{\mathcal{A}})$, thus possesses all attributes of measurable ranges, which completes the proof.

4.9. Given a function $\Theta: \Omega \times X \to Y$, where Θ is definable from a countable sequence of ordinals and a Borel probability measure λ over Ω (where Θ, λ are assumed to be in $\mathcal{N}$), let F be the range of Θ and let $\theta: \Omega \to F$ be defined by $\theta(\omega) = \Theta(\omega,\cdot)$. Let $\underline{F}$ be the set of subsets of F that are definable from a countable sequence of ordinals.[10/] Since in $\mathcal{N}$, every subset of Ω that is definable from a countable sequence of ordinals is measurable with respect to λ, $\theta^{-1}(C)$ is measurable under λ for every $C \in \underline{F}$. Thus λ induces a uniquely

[10/] The notion of definability from a countable sequence of ordinals is expressible by a set-theoretical formula (see [M-S]).

defined measure λ^* over $\underline{F}$: $\lambda^*(C) = \lambda(\theta^{-1}(C))$. For any Borel measure μ over X and any rational r_1, r_2, if $E \subset Y$ is Borel then the set $\{f | r_1 \leq \mu(\{x | f(x) \in E\}) \leq r_2\}$ is in $\underline{F}$. Hence $\{f | r_1 \leq \mu(\{x | f(x) \in E\}) \leq r_2\}$ is measurable with respect to λ^*. We can therefore use the methods of ℙ 3.2.3 to determine the probability under $\lambda^* \times \mu$ that $f(x) \in E$ when $f \in F$, $x \in X$ are chosen at random: $\lambda^* \times \mu(\{(f,x) | f \in F, x \in X, f(x) \in E\}) = \lim_{n\to\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \lambda^*(\{f | \frac{m-1}{n} \leq \mu(\{x | f(x) \in E\}) \leq \frac{m}{n}\}) \cdot \frac{m-1}{n}$. Note that this expression depends only on λ^*, and not on the Borel probability distribution, λ, through which λ^* is induced. That is, if two different Borel probability measures λ_1, λ_2 over Ω induce the same probability distribution λ^* over $\underline{F}$, the probability under λ^*, μ of $f(x)$ occurring in E does not depend upon which of these distributions is used. Since $\Theta^{-1}(E)$ is measurable with respect to $\lambda \times \mu$, it easily follows from Fubini's theorem ([Hlm-2], pp. 147-148) that

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \lambda^*(\{f | \frac{m-1}{n} \leq \mu(\{x | f(x) \in E\}) \leq \frac{m}{n}\}) \cdot (\frac{m-1}{n}) =$$

$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \lambda(\{\omega | \omega \in \Omega \text{ and } \frac{m-1}{n} \leq \mu\{x | \Theta(\omega,x) \in E\} \leq \frac{m}{n}\}) \cdot (\frac{m-1}{n}) =$$

$$\lambda \times \mu(\Theta^{-1}(E)).$$

The measurability of $\Theta^{-1}(E)$ with respect to $\lambda \times \mu$ implies that the **anomaly** of Counter-example (B)(in which integration along the Ω-axis produces a different result from that achieved when integrating over the X-axis) cannot occur in the present case.

We conclude that in η, if $(F, \underline{F}_\sigma)$ satisfies the hypothesis of Theorem 4.8 then for any probability measure $\underline{\lambda}$ over $\underline{F}_\sigma$, $\underline{\lambda}$ can be extended to a probability measure λ^* defined over all subsets of F which are definable from a countable sequence of ordinals. Given a Borel probability measure μ over X and a Borel subset E of Y, $\lambda^* \times \mu$ determines a unique probability value for the occurance of $f(x)$ in E.

5. The validity in η of Theorem 4.8 and of the statements in $\mathbb{P}$ 4.9 does not of course imply that these assertions are true in the real world. Nevertheless the countable models in which these statements are fulfilled are likely to provide accurate replicas of many competitive situations that occur. In decision problems that take place in everyday life the set of choices available to an individual is at most countably infinite. The possible coalitions which an individual may join are usually definable by some simple phrase in ordinary language,eg. oil producing nations, Southerners, etc. The number of such phrases, being countable, the set of coalitions open to an individual must therefore be countable. The same holds true in most actual game situations, including differentiable games. The strategies available to a player are usually definable by a formal expression in mathematical language; hence the number of available strategies would be countable. Even if the player were to employ a random device to help him decide on forthcoming moves, his set of options is countable; no more than a countable number of these devices lie at his disposal. Although the number of infinite sequences that could conceivably result through the use of a random device is uncountable, the sequence of results that actually occurs can be viewed as predetermined. Thus the countability of a model such as η does not seriously restrict its applicability to usual competitive situations.

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