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INFORMATION INDEPENDENCE AND COMMON KNOWLEDGE

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In Bayesian environments with private information, as described by the types of Harsanyi, how can types of agents be (statistically) disassociated from each other and how are such disassociations reflected in the agents' knowledge structure?

Conditions studied are (i) subjective independence (the opponents' types are independent conditional on one's own) and (ii) type disassociation under common knowledge (the agents' types are independent, conditional on some common-knowledge variable). Subjective independence is motivated by its implications in Bayesian games and in studies of equilibrium concepts.

We find that a variable that disassociates types is more informative than any common-knowledge variable. With three or more agents, conditions (i) and (ii) are equivalent. They also imply that any variable which is common knowledge to two agents is common knowledge to all, and imply the existence of a unique common-knowledge variable that disassociates types, which is the one defined by Aumann.

KEYWORDS: Bayesian games, independent types, common knowledge.

1. INTRODUCTION

THIS NOTE DEALS WITH the private information available to different agents in Bayesian environments, as described by the types of Harsanyi (1967/68), and with the notion of common knowledge, as described in Aumann (1976). In particular, we are interested in conditions of partial independence of agents' types and their relationship to common-knowledge variables. Such conditions help explain the environment's information structure: how types of agents are (statistically) associated with each other and what is common knowledge about such associations.

A central condition of interest is *subjective independence*: the opponents' types of every agent are (statistically) independent when the agent conditions on his own type. This condition is interesting for a variety of reasons.

First, from an empirical point of view, when agents test whether their opponents' types are independent, it is often done after they already know their own type. So in effect, they test conditions of *subjective*, and not full, independence.

From an analytical point of view, subjective independence plays an important role when individual agents make use of laws of large numbers that require independence. In a strategic game, for example, a player who knows his own type can use laws of large numbers to make accurate predictions about the aggregate behavior of large groups of opponents. Kalai (2004) showed that in one-simultaneous-move Bayesian games with many semianonymous play-

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ers, under subjective independence,[2] the equilibria are robust to structural changes.[3]

Yet even if the number of players is small, a player who knows his own type and repeatedly observes his opponents' choices can use laws of large numbers to predict the opponents' future actions. Kalai and Lehrer (1993) studied the equilibria of Bayesian repeated games and showed that subjective independence implies that every Bayesian equilibrium converges to a Nash equilibrium of the repeated game, as if the privately known types were common knowledge.

Gossner and Hörner (2006) studied repeated games with imperfect monitoring under subjective independence, where, conditional on each player's signal (or on a garbled version of it), the opponents' signals are independent. They showed that each player's punishment level, defined as his min–max payoff in the repeated game (where the min is taken over the independent strategy profiles of the opponents), equals the player's usual min–max payoff in mixed strategies of the stage game. Thus, in this context subjective independence allows for simple characterizations of individually rational payoffs, an important step in the analysis of folk theorems in repeated games with imperfect monitoring.

In different contexts, which do not involve laws of large numbers, subjective independence is important in the epistemological analysis of game-theoretic solutions. For example, Bernheim (1986) used subjective independence to justify the use of Nash equilibrium (for the case of more than two agents), and, more recently, Brandenburger and Friedenberg (2008) used it to differentiate rationalizability from iterated dominance.

With implications like those above, it is desirable to have a better understanding of the condition of subjective independence. How restrictive is it and in what types of situations does it arise? This note answers such questions.

A second important condition in this paper is *independence under common knowledge*: conditional on some common-knowledge variable, the types of all the agents are independent. This condition is important for two reasons: (i) it enables an analyst to break down a complex strategic environment into individual components, each defined by the realized value of the common-knowledge variable and (ii) conditional on each such realized value, the analyst may assume type independence. For example, a Bayesian game can be broken down to smaller strategically separable Bayesian subgames, each with independent types.

With three or more agents, this paper shows that subjective independence is equivalent to independence under common knowledge. Moreover, when there

[2]While the formal statement there is stated with full independence, the proof only uses subjective independence.

[3]The equilibria survive even if the simultaneous move assumption is relaxed to allow for sequential play, revisions of earlier choices, delegation, information transmission, communication, and more.

is independence under a common-knowledge variable, there is a unique variable with this property, namely, the variable identified by Aumann (1976). This suggests useful algorithms for disassociating types in Bayesian environments.

A direct proof of the equivalence of the two independence conditions above is long and tedious. The main body of the paper identifies general relationships between type independence and common knowledge. In addition to leading to a simpler proof of the equivalence above, these relationships offer better understanding of the structures of Bayesian environments.

In particular, we show that, in general, any variable that disassociates the types must be more informative than any common-knowledge variable. Moreover, subjective independence implies (i) that there can be no coalitional secrets (any variable that is common knowledge to two agents is common knowledge to all) and (ii) that there is essentially a unique way to disassociate types.

The next section offers examples to help with the underlying intuition. It is followed by a formal presentation that establishes the above results in a standard model where the agents hold a common-prior distribution over the possible environment's states, as is customary in much of the epistemologic literature and applications. The concluding section offers two additional items: (i) an elaboration on how to disassociate types and (ii) a discussion on the role of the common-prior assumption, showing that without it the main results may fail.

1.1. *Examples*

The following examples help illustrate the concepts.

EXAMPLE 1—Sun and Moods: In an n-person Bayesian environment, the sun may either *shine* or *not* and, conditional on the state of the sun, σ, agent i's mood, μ_i, is either *happy* or *depressed*. Each agent privately learns both whether it is sunny or not and his own mood (for example, "It is sunny and I am depressed"), and hence he may be one of four possible types.

Let $s(\cdot)$ be the probability distribution over the states of the sun (with positive probability on each) and let $m_i(\cdot|\sigma)$ be agent i's probability distribution over moods, given the state of the sun. Assume that the likelihood of any vector of types (σ, μ) is the product $s(\sigma)\prod_i m_i(\mu_i|\sigma)$ and that all these distributions are commonly known to the agents.

In the example above, the agents' types are not independent. For example, knowledge of one's type precludes possible types of the opponents. But the state of the sun *disassociates* the types in the sense that conditional on any state of the sun, the types are independent. Moreover, the state of the sun is *common knowledge*. Since in this situation a common-knowledge variable disassociates the types, the types are *independent under common knowledge*. One can also verify that in this example the agents' types are subjectively independent.

EXAMPLE 2—Sun, Ozone, and Moods: As in the previous example, the state of the sun, σ, can be shine or not shine, and each agent's mood, μ_i, can be happy or depressed. In addition, the ozone level, ζ, may be *good* or *bad*. So each state of the environment is described by a triple: $(\sigma, \zeta, \langle\mu_i\rangle_{i=1,\dots,n})$. However, each agent knows only the weather and his own mood; he does not know the ozone level. So he can be one of the same four types as in the previous example.

Pairs of sun–ozone combinations, (σ, ζ), are determined by a full-support joint distribution $k(\sigma, \zeta)$ and, conditional on any such pair, the n moods *are independently and identically distributed* with $0 < p(\mu_i|\sigma, \zeta) < 1$ denoting the probability that agent i's mood is μ_i. Assume that for every fixed sun state σ, the probability of a happy mood is strictly greater when the ozone level is *good* than when it is *bad*.

In the last example, the state of the sun is a common-knowledge variable. But it does not disassociate the moods (the moods are not independent conditional on it) and no common-knowledge variable disassociates the moods. To disassociate the moods, one needs to know both the state of the sun and the ozone level. Subjective independence fails too, since knowledge of the sun and one's own mood is not sufficient to disassociate the opponents' moods.

2. FORMAL PRESENTATION

A set $I = \{1, 2, \dots, n\}$ $(n \geq 2)$ describes the *agents* in a *Bayesian environment* $\mathcal{E} = (\Omega, P, T)$, where $T = T_I = (T_i)_{i\in I}$ is the profile of agents' *type functions*, each being a random variable defined over the probability space (Ω, P). For a group of players $J \subseteq I$, $T_J = (T_i)_{i\in J}$ denotes the profile of types of players in J.

Implicitly, it is assumed that there is common knowledge of the three components of the environment $\mathcal{E}$ and that when a state $\omega \in \Omega$ is drawn, each agent i is informed only of his own type $t_i = T_i(\omega)$.[4] However, through his knowledge of $\mathcal{E}$, each agent makes inferences about the possible types of his opponents, the inferences that the opponents may make, and so forth.

The space Ω may be finite or countable, and we assume without loss of generality that $P(\omega) > 0$ for every $\omega \in \Omega$ and that $P(t_i) > 0$ for every t_i in the range of each T_i, that is, T_i is onto its range. We do *not* assume that $P(t) > 0$ for all profiles of types t, which would be a severe restriction.

DEFINITION 1: A random variable Y is *finer than* (or refines) a random variable Z (alternatively Z is coarser than Y) if for every pair of states, ω and ω', $Z(\omega) \neq Z(\omega')$ implies that $Y(\omega) \neq Y(\omega')$.

[4]Readers who are used to epistemology models may prefer to think of the underlying partitions rather than the variables T_i. This would avoid the need for equivalence classes discussed below, but would be more cumbersome in notations and applications.

Two variables, Y and Z, are (informationally) *equivalent*, $Y \approx Z$, if they refine each other ($Z(\omega) = Z(\omega')$ if and only if $Y(\omega) = Y(\omega')$). Y *strictly refines* Z if Y refines Z and they are not equivalent.

Clearly, "Y refines Z" is transitive and $Y \approx Z$ is an equivalence relation on the set of random variables. In the discussion that follows, we are mostly interested in the equivalence classes of variables $[Z]$ instead of the variables Z themselves.

It is easy to see that under the equivalence above and the "Y refines Z" relation, the variables form a (complete) lattice. The finest variable is represented by the function $F(\omega) = \{\omega\}$ and the coarsest variable is represented by the function $K(\omega) = \Omega$.

2.1. *Conditional Independence: Disassociating Types*

DEFINITION 2: Consider a subgroup of agents $J \subseteq I$. The variables T_J are *independent* if for every vector of values t_J, $P(T_J = t_J) = \prod_{j\in J} P(T_j = t_j)$. The variables T_J are *independent conditional on* a variable D if for all possible values t_J and d, $P(T_J = t_J | D = d) = \prod_{j\in J} P(T_j = t_j | D = d)$. When this is the case, we say that D *disassociates* T_J.

When J is the group of all agents I, we may omit the subscript J, and simply say that the types are independent conditional on D and that D disassociates the types. In this case we may write

$$P(T = t) = \sum_d \prod_i P(T_i = t_i | D = d) P(D = d).$$

Clearly, if D disassociates the types of T, it disassociates the types of any subgroup of agents, but the converse is not true. Moreover, disassociating types is always possible (for example, by conditioning on the finest variable F) and often beneficial, but coarser disassociations are more desirable, since they help explain the relationships between the types with a smaller number of parameters.

PROPOSITION 1: *There exists a maximally coarse disassociation, that is, a disassociation D, such that any coarser disassociation is equivalent to it.*

PROOF: We apply Zorn's lemma to the set of disassociations, partially ordered by the "is coarser than" relation. We only need to prove that for any totally ordered family $(D_\alpha)_{\alpha\in\mathcal{D}}$ of disassociations, there exists a disassociation D that is coarser than any D_α. Define the equivalence relation $\omega \sim \omega'$ to mean that there exists an α such that $D_\alpha(\omega) = D_\alpha(\omega')$ and define the variable $D(\omega)$

to be the equivalence class of ω, $D(\omega) = \{\omega'; \omega \sim \omega'\}$. D is clearly coarser than any D_α. For every vector of values t_J and every ω,

$$P(T_J = t_J | D_\alpha(\omega)) = \frac{P(T_J = t_j, D_\alpha(\omega))}{P(D_\alpha(\omega))}.$$

Both $P(T_J = t_j, D_\alpha(\omega))$ and $P(D_\alpha(\omega))$ are increasing and converge, respectively, to $P(T_J = t_j, D(\omega))$ and to $P(D(\omega))$ on the chain defined by $\mathcal{D}$. Thus, all the equalities defining each D_α as a disassociation can be taken to the limit, and D is also a disassociation. *Q.E.D.*

DEFINITION 3: A type disassociation is *coarsest* if it is coarser than any other disassociation. Clearly, a coarsest disassociation must be unique (up to equivalence).

Although there always exists a maximally coarse disassociation, it may not be unique and the coarsest disassociation may not exist. We defer further discussion about the negative implications of this phenomenon to the concluding section.

2.2. *Common Knowledge*

DEFINITION 4: For a group of agents $J \subseteq I$, *a* variable C is J *common knowledge* if C is coarser than every T_j with $j \in J$. When C is I common knowledge, we simply say that it is common knowledge.[5]

Clearly, common-knowledge variables exist, since any constant variable is common knowledge for any group of agents. Also, common knowledge is coalitionally monotonic: if C is J common knowledge, it must also be L common knowledge for any $L \subseteq J$.

Unlike disassociations, in the case of common knowledge it is desirable to identify the maximally refined (rather than the maximally coarse) variables, since they describe the most detailed information that is commonly known to the agents.

DEFINITION 5: A common-knowledge variable C is *finest* if it refines every other common-knowledge variable. Clearly, a finest common-knowledge variable is unique (up to equivalence).

Unlike disassociations of types, for which the coarsest one may not exist, the finest common-knowledge variable always exists. An explicit description of it is offered by a construction due to Aumann:

[5]As we clarify below, this is equivalent to the standard definition of common knowledge described in Aumann (1976).

For a fixed group of agents $J \subseteq I$, let $\sim$ be the equivalence relation defined by $\omega \sim \omega'$ if there exists a chain $\omega_0 = \omega, \omega_1, \ldots, \omega_n = \omega'$ such that for every $k = 1, \ldots, n$, there exists an agent $j \in J$ who is the same type in states ω_{k-1} and ω_{k-1}: $T_j(\omega_{k-1}) = T_j(\omega_k)$. We define Aum_J by the equivalence classes for this relation: $\mathrm{Aum}_J(\omega) = \{\omega' : \omega' \sim \omega\}$. For notational simplicity we let $\mathrm{Aum} = \mathrm{Aum}_I$.

Formally, as defined above, Aum is a random variable with a range in the set of all subsets of Ω, but it also describes the common-knowledge partition defined by Aumann (1976).

The following lemma shows that Aum is common knowledge and it is the finest common-knowledge variable.

LEMMA 1: Aum *is the finest common-knowledge variable*.

PROOF: To see that Aum is common knowledge, note that under the construction above, $T_i(\omega) = T_i(\omega')$ implies $\mathrm{Aum}(\omega) = \mathrm{Aum}(\omega')$. Thus, Aum is refined by every T_i.

To see that Aum is the finest common knowledge, consider any common-knowledge variable C. For any ω, ω' such that $\mathrm{Aum}(\omega) = \mathrm{Aum}(\omega')$, consider a chain $\omega_0 = \omega, \omega_1, \ldots, \omega_n = \omega'$ such that for every $k = 1, \ldots, n$, there exists an i with $T_i(\omega_{k-1}) = T_i(\omega_k)$. For every k, $T_i(\omega_{k-1}) = T_i(\omega_k)$ implies $C(\omega_{k-1}) = C(\omega_k)$, so that $C(\omega) = C(\omega')$. Hence, Aum refines C.

Finally, if both Aum and Aum$'$ are finest, Aum refines Aum$'$ and Aum$'$ refines Aum. *Q.E.D.*

2.3. *Main Results*

THEOREM 1—Type Disassociations Refine Common Knowledge: *Any variable D that disassociates the types is finer than any common-knowledge variable*.

PROOF: We prove a stronger conclusion, namely, that D refines any C which is common knowledge to any group of two agents. This is stronger because if C is common knowledge, it must be common knowledge to any group of two or more agents, yet the converse is not true.

Notice, however, that if D disassociates the types of T, then it disassociates the types of any group of two agents. So it suffices to prove the theorem for the case of $n = 2$ agents.

Since Aum refines all common-knowledge variables, it suffices to show that if $D(\omega^1) = D(\omega^2)$, then $\mathrm{Aum}(\omega^1) = \mathrm{Aum}(\omega^2)$. Let $D(\omega^1) = D(\omega^2) = d$, and let $t_1 = T_1(\omega^1)$ and $t_2 = T_2(\omega^2)$. Then $P(T_1 = t_1 \text{ and } T_2 = t_2 | D = d) = P(T_1 = t_1 | D = d) P(T_2 = t_2 | D = d) > 0$. So there is a $\overline{\omega}$ with $T_1(\overline{\omega}) = t_1$ $(= T_1(\omega^1))$ and $T_2(\overline{\omega}) = t_2$ $(= T_2(\omega^2))$. In other words, ω^1 and ω^2 are equivalent in the sense of Aumann so that $\mathrm{Aum}(\omega^1) = \mathrm{Aum}(\omega^2)$. *Q.E.D.*

COROLLARY 1: *The only common-knowledge variable that may disassociate the types is* Aum.

DEFINITION 6: The types of T are subjectively independent if $P(T_{-i} = t_{-i}|T_i = t_i) = \prod_{j\neq i} P(T_j = t_j|T_i = t_i)$ for every agent i and every vector of types t (alternatively, T_i disassociates T_{-i} for every i).

Notice that the definition of subjective independence is global, rather than local, where one would require only subjective independence at individual states ω. Moreover, given the implicit common knowledge of P and T, if subjective indepence holds, then it is common knowledge: every agent can verify it for himself, for each one of his opponents, for their ability to verify it for their opponents, and so on.

DEFINITION 7: The types of T are independent under common knowledge if there is a common-knowledge variable that disassociates them.

THEOREM 2: *For $n \geq 3$ agents and any vector of types T, the following three conditions are equivalent*:

C1. *The types are subjectively independent*.

C2. *The types are independent under common knowledge*.

C3. Aum *disassociates the types*.

Moreover, *any of the conditions above implies the following statements*:

C4. *The coarsest disassociating variable exists*.

C5. *No Coalitional Secrets*: *Any variable Z is common knowledge to a group of agents J if and only if it is common knowledge to a group of agents K*, *where J and K are any two groups* (*overlapping or disjoint*) *with two or more agents each*.

PROOF: C2 $\Rightarrow$ C1. The following lemma is useful.

LEMMA 2: *If X_1 and (X_2, X_3) are independent conditional on X_4*, *then X_1 and X_2 are independent conditional on (X_3, X_4)*.

PROOF: $P(X_1|(X_2, X_3), X_4) = P(X_1|X_4)$ implies $P(X_1|X_3, X_4) = P(X_1|X_4)$, which implies the desired conclusion: $P(X_1|X_2, (X_3, X_4)) = P(X_1|X_3, X_4)$. *Q.E.D.*

Assume C2, that is, that T_i and $(T_k)_{k\neq i}$ are independent conditional on a common-knowledge variable Z. For any $j \neq i$, Lemma 2, with $X_1 = T_i$, $X_2 = (T_k)_{k\neq i,j}$, $X_3 = T_j$, and $X_4 = Z$, shows that T_i and $(T_k)_{k\neq i,j}$ are independent conditional on (T_j, Z). This proves that T_i and $(T_k)_{k\neq i,j}$ are independent conditional on T_j, since (T_j, Z) and T_j generate the same partition. This is true for every i, j, establishing subjective independence.

C1 $\Rightarrow$ C5. Let J be any group of at least two agents and take $i \notin J$. Since the family $(T_j)_{j\in J}$ is independent conditional on T_i, Theorem 1 implies that T_i refines Aum_J. Thus, $\mathrm{Aum}_{J\cup\{i\}} = \mathrm{Aum}_J$. Since this is true for any J, an induction argument shows that for any group J of at least two agents, $\mathrm{Aum}_J = \mathrm{Aum}$.

C1 $\Rightarrow$ C3. Let i, j, and k be three different agents. Since T_k and $(T_l)_{l\neq k,i}$ are independent conditional on T_i, $P(t_k|(t_l)_{l\neq k,i}, t_i) = P(t_k|(t'_l)_{l\neq k,i}, t_i)$ if $P((t_l)_{l\neq k,i}, t_i)P((t'_l)_{l\neq k,i}, t_i) > 0$. Similarly, $P((t_l)_{l\neq k,j}, t_j)P((t'_l)_{l\neq k,j}, t_j) > 0$ implies $P(t_k|(t_l)_{l\neq k,j}, t_j) = P(t_k|(t'_l)_{l\neq k,j}, t_j)$. Thus for any a in the range of $\mathrm{Aum}_{\{i,j\}}$, $P(t_k|(t_l)_{l\neq k}, a) = P(t_I|(t'_l)_{l\neq k}, a)$ holds whenever $P((t_l)_{l\neq k}, a)P((t'_l)_{l\neq k}, a) > 0$. Hence, T_k and $(T_l)_{l\neq k}$ are independent conditional on $\mathrm{Aum}_{\{i,j\}}$. Now using C1 and its implied C5, T_k and $(T_l)_{l\neq k}$ are independent conditional on Aum. Since this is true for any k, the family T is independent conditional on Aum.

C3 $\Rightarrow$ C2. This is immediate, since Aum is common knowledge.

C3 $\Rightarrow$ C4. This is immediate, since Aum disassociates the types and refines any variable that disassociates the types by Theorem 1. *Q.E.D.*

3. FURTHER ELABORATION

The first example—sun and moods (no ozone)—illustrates a situation where subjective independence and all the other C1–C5 conditions hold. The state of the sun disassociates the moods of the agents and as the Aum variable, any other disassociation must simply be its refinement. In other words, other than the sun, there is no explanation of how the moods of the agents are related to each other (including among subsets of agents), and this sun disassociation is also the finest common knowledge.

The second example—sun, ozone, and moods—illustrates a situation in which subjective independence and the equivalent conditions C1–C3 fail. The Aum variable is still the state of the sun, but it does not disassociate the moods. What can we say about disassociating the types in such situations? As it turns out, despite the failing of subjective independence, the main findings of this paper may still be helpful.

3.1. *Disassociating Types*

Continuing with the sun, ozone, and moods example, we observe first that the two-dimensional variable that describes the values of the sun and the ozone, (S, Z), does disassociate the types and that it is a maximally coarse variable with this property. To see that it is maximally coarse among all disassociations, assume to the contrary that there is a disassociation D that is strictly coarser than (S, Z). By Theorem 1, D must be strictly finer than Aum. This leaves only two such possible disassociations: D_1, which discloses the ozone levels when the sun is shining, but does not do so when the sun is not shining, or, similarly, D_2, which discloses the ozone levels when the sun is not shining, but does not do so when the sun is shining.

To show, for example, that D_1 is not a disassociation, restrict yourself to the event "the sun is not shining," which is common knowledge, and verify that conditional on this event the moods are not independent.

Do the state of the sun and the ozone level, (S, Z), offer the only way to disassociate the moods? No. For example, for every agent i, let $N^i = (S, M_{-i})$ describe the state of the sun and the vector of moods of all agents other than i. The reader may verify, using some of the main results described earlier, that every such N^i is a maximally coarse variable that disassociates the types. Moreover, N^i is not equivalent to (S, Z).

The nonexistence of the coarsest disassociation, or the existence of multiple maximally coarse disassociations, presents a conceptual difficulty. This is because unlike the example with the sun only, where there is a unique and full explanation of the mood relationships among the agents, now there are several competing such explanations. This means that an analyst, who wants to have one "natural" explanation of the mood relationships, has to introduce additional considerations. For example, Gossner, Laraki, and Tomala (2009) chose the decomposition that minimizes the entropy of the information not observed by the analyst.

As the discussion above suggests, type disassociation may be attempted by the following series of steps: First, identify the variable Aum, a task that involves only unions and intersections, and disregards probabilities. Then check to see if the types are independent conditional on Aum. If they are, then Aum is the *unique* explanation of type relationships (and it is also common knowledge). If the types are not independent conditional on Aum, then there are no common-knowledge disassociations and all disassociations must be strict refinements of Aum.

3.2. *Relaxing the Common-Prior Assumption*

In the model above, we (implicitly) assumed that the environment $\mathcal{E} = (\Omega, P, T)$ is common knowledge. Differential information was expressed only by the fact that when a state ω is realized, each agent i is informed only of the value of his own type $T_i(\omega)$. This formulation follows the original model of Aumann (1976) and most of the followup literature.

What happens to subjective independence if $\mathcal{E} = (\Omega, P, T)$ is *not* common knowledge to all the agents? There are different ways to model such relaxations, while still retaining coherent concepts of common knowledge and disassociations.

One simple relaxation is the following. Replace the triple $\mathcal{E} = (\Omega, P, T)$ by n triples $\mathcal{E}^k = (\Omega^k, P^k, T^k)$, $k = 1, 2, \dots, n$, where every such triple represents the beliefs of agent k about the complete setting. For every such k, impose on $\mathcal{E}^k$ the assumptions made in the paper on $\mathcal{E}$. In other words, each agent has a different model of the world, but he believes that all the agents have the same model as his. It follows immediately that all the results hold in every agent's mind.

However, the above relaxation is extreme in opposite directions. On the one hand, the triples $\mathcal{E}^k$, modeling the different agents' representations of the world, are completely unrelated to each other. One the other hand, within his own $\mathcal{E}^k$, every agent assumes full coincidence and common knowledge of the same environment.

A model that relaxes the common-prior assumption in a minimal sense and is less extreme than the one above is a generalized Harsanyi model (GHM; see Harsanyi (1967/68)). While, in general, a minimal relaxation may not go far enough, it is sufficient for our purpose here. As we show below, such a minimal relaxation already overturns the main result of the paper.

In this GHM there is common knowledge of the set of states Ω and of the vector of types T, but each agent k has a different prior probability P^k over the states of the environment. Moreover, it is assumed that the different P^k's are common knowledge. This means that each agent k is convinced that his prior P^k is correct and that the priors of his opponents, $P^{k'}$'s, are wrong. So in contrast to the model of Aumann (1976), they may agree to disagree, with disagreements that are common knowledge. The following is an example of such a GHM in which the main results of the paper fail.[6]

EXAMPLE 3—Three Mood Setters: There are nine states described by triples $\mu = (\mu_1, \mu_2, \mu_3)$, where each μ_i, taking the values happy (H) or depressed (D), describes the mood of agent i. Each agent is informed only of his own mood. Moreover, based on his individual prior distribution over moods P^i, he believes that he is the "mood setter." For example, agent 1's prior is (i) $P^1(\mu_1 = H) = P^1(\mu_1 = D) = 0.50$, and with each of X, Y, Z taking the values of H and D, he has (ii) $P^1(\mu_2 = X|\mu_1 = X) = P^1(\mu_3 = X|\mu_1 = X) = 0.90$ and (iii) $P^1(\mu_2 = Y \text{ and } \mu_3 = Z|\mu_1 = X) = P^1(\mu_2 = Y|\mu_1 = X)P^1(\mu_3 = Z|\mu_1 = X)$.

Condition (ii) represents the fact that agent 1 believes that he is a mood setter. Moreover, condition (iii) states his belief that his mood disassociates the moods of the others, which is subjective independence.

In addition to the fact that each agent believes that he is the mood setter, as described by symmetric distributions P^2 and P^3, there is implicit common knowledge that all three agents believe themselves to be the sole mood setter.

In the example above the only common-knowledge variable (or the Aum variable) is the coarsest variable, represented by any constant function. Thus, the condition of independence under common knowledge translates to unconditional type independence.

But under any P^i, $i = 1, 2, 3$, no agent believes that the types are independent. (For example, $P^1(\mu_j = H) = 0.50$ for $j = 1, 2, 3$, yet $P^1(H, H, H) =$

[6]We thank an associate editor for suggesting a different (more complex) example in which the same phenomenon is observed.

$0.5 \times (0.9)^2 \neq (0.5)^3$.) Given the common knowledge of the different priors, there is common knowledge that nobody believes the type independence, that is, independence under common knowledge fails.

Yet condition (iii) above means that there is subjective independence, where each agent believes that for any type of himself, his opponents are independent of each other. Again, given the common knowledge of the individual priors, there is also common knowledge of subjective independence, that is, every agent knows that conditional on his types, the opponents' types are independent, they all know that they all know that, and so forth.

Clearly, one special case of the GHM is the case of a common prior studied in this paper, in which subjective independence and independence under common knowledge are equivalent. It is not clear whether, within the GHM, the equivalence can be obtained under weaker conditions than a common prior.

REFERENCES

AUMANN, R. J. (1976): "Agreeing to Disagree," *The Annals of Statistics*, 4, 1236–1239. [1317,1319, 1322,1323,1326,1327]

BERNHEIM, B. D. (1986): "Axiomatic Characterizations of Rational Choice in Strategic Environments," *The Scandinavian Journal of Economics*, 88, 473–488. [1318]

BRANDENBURGER, A., AND A. FRIEDENBERG (2008): "Intrinsic Correlation in Games," *Journal of Economic Theory*, 141, 28–67. [1318]

GOSSNER, O., AND J. HÖRNER (2006): "When Is the Individually Rational Payoff in a Repeated Game Equal to the Minmax Payoff?" Discussion Paper 1440, CMS-EMS. [1318]

GOSSNER, O., T. LARAKI, AND R. TOMALA (2009): "Informationally Optimal Correlation," *Mathematical Programming B*, 116, 147–172. [1326]

HARSANYI, J. C. (1967/68): "Games With Incomplete Information Played by "Bayesian" Players, Parts I, II, and III," *Management Science*, 14, 159–182, 320–334, 486–502. [1317,1327]

KALAI, E. (2004): "Large Robust Games," *Econometrica*, 72, 1631–1666. [1317]

KALAI, E., AND E. LEHRER (1993): "Rational Learning Leads to Nash Equilibrium," *Econometrica*, 61, 1019–1945. [1318]

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