Discussion Paper No. 146

ENTRANCE-EXIT DISTRIBUTIONS FOR SEMIREGERATIVE PROCESSES

by

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May 1975

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Abstract

Let X be a standard Markov process, and let S be a perfectly additive increasing process with conditionally independent increments given the paths of X. Then, (X,S) is a Markov additive process. Let C be the random time change associated with S, and put $Z_t^- = X(C_{t-})$, $Z_t^+ = X(C_t)$, $R_t^- = t - S(C_{t-})$, $R_t^+ = S(C_t) - t$. When the state space of X is finite, GETOOR [*Adv. Appl. Prob.* (1974)] has recently obtained the joint distribution of these variables in terms of a triple Laplace transform. Here, the same is obtained explicitly by using renewal theoretic arguments along with the results on Lévy systems of (X,S) given in ÇINLAR [Z. Wahrscheinlichkeitstheorie (1975)]. These results are useful in reliability theory and in the boundary theory of Markov processes.

AMS Subject Classification Codes

Primary 60J50

Secondary 60J25

Keywords

Markov processes; boundary theory; entrance-exit results; regenerative systems; Markov additive processes.

ENTRANCE-EXIT DISTRIBUTIONS

FOR SEMIREGENERATIVE PROCESSES[1]

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1. INTRODUCTION

Let $(X,S) = (\Omega, \underline{\underline{M}}, \underline{\underline{M}}_t, X_t, S_t, \theta_t, P^x)$ be a Markov additive process. Here $X = (\Omega, \underline{\underline{M}}, \underline{\underline{M}}_t, X_t, \theta_t, P^x)$ is a standard Markov process with state space $(\mathbb{E}, \underline{\underline{E}})$ augmented by a point Δ (see BLUMENTHAL and GETOOR (1968) for the definition and other general terminology), and (S_t) is an increasing right continuous process with $S_0 = 0$ and $S_t = S_{\zeta-}$ for $t \geq \zeta = \inf\{u: X_u = \Delta\}$ which is adapted to $(\underline{\underline{M}}_t)$, is perfectly additive (that is, $S_{t+u}(\omega) = S_t(\omega) + S_u(\theta_t\omega)$ for all t, u, and ω) and is such that

$$(1.1)\qquad P^x\{X_u \circ \theta_t \in A,\ S_u \circ \theta_t \in B \mid \underline{\underline{M}}_t\} = P^{X(t)}\{X_u \in A,\ S_u \in B\}$$

for all $x \in \mathbb{E}_\Delta$, $t,u \in \mathbb{R}_+$, $A \in \underline{\underline{E}}_\Delta$, $B \in \underline{\underline{R}}_+$ (see ÇINLAR (1972) for the precise definition). In general $\underline{\underline{M}}_t$ is larger than $\underline{\underline{K}}_t = \sigma(X_s;\ s \leq t)^{\sim}$. If $\underline{\underline{M}}_t = \underline{\underline{K}}_t$ for all t, then (S_t) becomes an additive functional of X. Otherwise, (S_t) has conditionally independent increments given $\underline{\underline{K}}_\infty$ (this becomes an F-process according to NEVEU (1961a) when $\mathbb{E}$ is finite).

Define

$$(1.2)\qquad C_s = \inf\{t: S_t > s\},$$

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(1.3) $\quad Z_s^- = X_{C_s-}, \quad Z_s^+ = X_{C_s}, \quad R_s^- = s - S_{C_s-}, \quad R_s^+ = S_{C_s} - s$

for all $s \geq 0$. We are interested in the joint distribution of these variables. When $\mathbb{E}$ is a singleton, (S_t) becomes an increasing Lévy process and then our results reduce to those given by KINGMAN (1973) and KESTEN (1969). When $\mathbb{E}$ is finite, GETOOR (1974) computed the triple Laplace transform

$$\int dt\, e^{-\alpha t}\, P^i\{Z_t^- = j,\ Z_t^+ = k,\ R_t^- \in du,\ R_t^+ \in dv\} e^{-\beta u - \gamma v}.$$

Here we give an explicit derivation of the distribution of (1.3) in general by using renewal theoretic methods along with results on Lévy systems, and a change of variable formula out of GETOOR and SHARPE (1973b).

We will assume throughout that (S_t) is quasi-left-continuous. Then, it was shown in ÇINLAR (1975) that there is a Lévy system (H,L,a) which specifies the jump structure of (X,S) along with the conditional law of (S_t) given $\underline{\underline{K}}_\infty = \sigma(X_s,\ s \geq 0)^\sim$. Here, (H_t) is a continuous additive functional of X, L is a transition kernel from $(\mathbb{E},\underline{\underline{E}})$ into $(\mathbb{E},\underline{\underline{E}})\times(\mathbb{R}_+,\underline{\underline{R}}_+)$, and a is a positive Borel function on $\mathbb{E}$ (see the next section for the precise meaning). Define

(1.4) $$U(x,f) = E^x \int_0^\infty f(X_t,S_t)\,dH_t$$

for $x \in \mathbb{E}$ and f positive Borel on $\mathbb{E}\times\mathbb{R}_+$. The following proposition is preparatory to the theorem after it, which is our main result.

(1.5) PROPOSITION. Let $\mathbb{A} = \{a > 0\}$. There exists a transition kernel $(t,x,A) \to u_t(x,A)$ from $(\mathbb{R}_+,\underline{\underline{R}}_+)\times(\mathbb{E},\underline{\underline{E}})$ into $(\mathbb{E},\underline{\underline{E}})$ vanishing on $\mathbb{E}\setminus\mathbb{A}$ such that

(1.6) $$U(x,A\times B) = \int_B u_t(x,A)dt$$

for all $x \in \mathbb{E}$, $A \subset \mathbb{A}$ Borel, $B \in \underline{\underline{R}}_+$.

(1.7) MAIN THEOREM. For any positive Borel measurable function f on $\mathbb{E}\times\mathbb{E}\times\mathbb{R}_+\times\mathbb{R}_+$,

(1.8) $$E^x[f(Z_t^-,Z_t^+,R_t^-,R_t^+)] = \int_{\mathbb{E}} u_t(x,dy)a(y)f(y,y,0,0)$$

$$+ \int_{\mathbb{E}\times[0,t]} U(x,dy,ds) \int_{\mathbb{E}\times\mathbb{R}_+} L(y,dz,t-s+du)f(y,z,t-s,u)$$

for every $x \in \mathbb{E}$ and (Lebesgue) almost every $t \in \mathbb{R}_+$. If $f(\cdot,\cdot,\cdot,0) = 0$ identically, then the same is true for every t. □

A number of further results will be given in Section 3 along with the proofs. For the time being we note the interesting deficiency of the preceding theorem: the equality is shown to hold only for almost every t. The difficulty lies with computing the probabilities of events contained in $\{R_t^- = 0,\ R_t^+ = 0\}$. When $\mathbb{E}$ is a singleton, this becomes the infamous problem resolved by KESTEN (1969). Using his result, we are able to resolve the matter when X is a regular step process (and therefore, in particular, if $\mathbb{E}$ is countable). The matter is simpler for the event $\{R_t^- > 0,\ R_t^+ = 0\}$; in general, the qualifier "almost every t" cannot be removed. But again, we do not have a complete solution.

To see the significance of the seemingly small matter, and also to justify the title of this paper, consider a semiregenerative process $(Z_s;M)$ in the sense of MAISONNEUVE (1975b). Here (Z_s) is a process, M is a right closed random set, and (Z_s) enjoys the strong Markov property at all stopping times whose graphs are contained in M. This is a slightly different, but equivalent, formulation of the "regenerative systems" of MAISONNEUVE (1974). Then, M is called the <u>regeneration set</u>, and

(1.9) $L_s = \sup\{t \leq s: t \in M\}$, $N_s = \inf\{t > s: t \in M\}$

are the <u>last</u> time of regeneration before t and the <u>next</u> time of regeneration after t. In accordance with the terminology of the boundary theory of Markov processes, the processes

(1.10) $(Z_s^-) = (Z_{L_s^-})$, $(Z_s^+) = (Z_{N_s})$

are called, respectively, the exit and the entrance processes. Under reasonable conditions on M (see JACOD (1974) and also MAISONNEUVE (1975b) for the precise results) it can be shown that there exists a Markov additive process (X,S) such that the entities defined by (1.2),(1.3),(1.9), (1.10) are related to each other as follows:

(1.11) $M = \{s: S_t = s \text{ for some } t \in \mathbb{R}_+\}$,

(1.12) $R_s^- = s - L_s$; $R_s^+ = N_s - s$;

(1.13) $Z_s^- = Z_{L_s^-}$; $Z_s^+ = Z_{N_s}$.

In other words, R_s^- and R_s^+ are the "backward and forward recurrence times" and Z_s^- and Z_s^+ are the exit and entrance states. So, $R_s^+ = 0$ implies that $s \in M$, that is, that s is a time of regeneration.

Conversely, starting with a Markov additive process (X_t,S_t), if (Z_t^+) is defined by (1.3) and M by (1.11), the pair $(Z_t^+;M)$ is semiregenerative in the sense of MAISONNEUVE (1975b).

It was noted by MAISONNEUVE (1974) that semiregenerative processes may be studied by using the techniques of NEVEU (1961a),(1961b) and PYKE (1961a,b) via Markov additive processes. We are essentially doing just this by bringing in "renewal theoretic" thinking together with the results on Markov additive processes obtained in ÇINLAR (1972) and (1975). In fact, our techniques may be used to obtain most of the results given by MAISONNEUVE (1974),(1975a),(1975b) and by JACOD (1974), but we have limited ourselves to results which are extensions of their work. Moreover, these results are related to the last exit – first entrance decompositions for Markov processes of GETOOR and SHARPE (1973a,b). We are planning to show the precise connections later; roughly, they were working with the conditional expectations of our additive process (S_t).

2. PRELIMINARIES

Let $(X,S) = (\Omega,\underline{\underline{M}},\underline{\underline{M}}_t,X_t,S_t,\theta_t,P^x)$ be a Markov additive process as in the introduction. It was shown in ÇINLAR (1975) that there is a Lévy system (H',L') for (X,S), where H' is a continuous additive functional of X and L' is a transition kernel, such that

$$(2.1)\qquad E^x\Big[\sum_{u\le t} f(X_{u-},X_u,S_u - S_{u-})I_{\{X_{u-}\neq X_u\}\cup\{S_{u-}\neq S_u\}}\Big]$$

$$= E^x \int_0^t dH'_s \int_{\mathbb{E}\times\mathbb{R}_+} L'(X_s,dy,ds)f(X_s,y,s)$$

for all x, t and all positive $\underline{\underline{E}}\times\underline{\underline{E}}\times\underline{\underline{\mathbb{R}}}_+$ measurable functions f. This was shown for X Hunt with a reference measure; but the work of BENVENISTE and JACOD (1973) shows that the same is true for arbitrary standard Markov processes X.

The process S can be decomposed as (see ÇINLAR (1972) for this)

$$(2.2) \qquad S = A + S^d + S^f$$

where A is a continuous additive functional of X, S^d is a pure jump increasing additive process which is continuous in probability with respect to $P^x(\cdot|\underline{\underline{K}}_\infty)$, and S^f is a pure jump increasing additive process whose jumps coincide with those of X and therefore the jump times are fixed by X. We define

$$(2.3) \qquad H_t = H'_t + A_t + t,$$

and let the positive $\underline{\underline{E}}$-measurable functions a and h be such that

$$(2.4) \qquad A_t = \int_0^t a(X_s)dH_s; \quad H'_t = \int_0^t h(X_s)dH_s;$$

(that this is possible follows from BLUMENTHAL and GETOOR (1968, Chapter V)). Define

$$(2.5) \qquad L(x,\cdot) = h(x)L'(x,\cdot);$$

$$(2.6) \qquad L^d(x,B) = L(x,\{x\}\times B); \quad L^f(x,A\times B) = L(x,(A\setminus\{x\})\times B);$$

$$(2.7) \qquad K(x,A) = L^f(x,A\times \mathbb{R}_+); \quad F(x,y,B) = \frac{L^f(x,dy\times B)}{K(x,dy)};$$

(in fact one starts from F and K and defines L^f; see ÇINLAR (1975) for the derivations).

Then, (H,K) is a Lévy system for X alone; (H,L^f) is a Lévy system for the Markov additive process (X,S^f); (H,a) defines A by (2.4); and (H,L^d) defines the conditional law of S^d given $\underline{\underline{K}}_\infty = \sigma(X_s;\ s \geq 0)^{\sim}$ by

$$(2.8) \quad E^x[\exp(-\lambda S_t^d)|\underline{\underline{K}}_\infty] = \exp[-\int_0^t dH_s \int_{\mathbb{R}_+} L^d(X_s,du)(1 - e^{-\lambda u})].$$

Finally, if τ is a jump time of X, then $F(X_{\tau-},X_\tau,\cdot)$ is the conditional distribution of the magnitude of the jump of S^f at τ given $\underline{\underline{K}}_\infty$.

We call (H,L,a) <u>the</u> Lévy system of (X,S). The following random time change reduces the complexity of the future computations. Define

$$(2.9) \qquad G_t = \inf\{s: H_s > t\},$$

$$(2.10) \qquad \hat{X}_t = X(G_t), \quad \hat{S}_t = S(G_t), \ldots,$$

and define $\hat{C}_t$, $\hat{Z}_t^-$, $\hat{Z}_t^+$, $\hat{R}_t^-$, $\hat{R}_t^+$ by (1.2) and (1.3) but from $(\hat{X},\hat{S})$. Then we have the following

(2.11) PROPOSITION. $(\hat{X},\hat{S})$ is a Markov additive process with a Lévy system $(\hat{H},L,a)$ where $\hat{H}_t = t \wedge \hat{\zeta}$. Moreover,

$$(2.12) \qquad (\hat{Z}_t^-,\hat{Z}_t^+,\hat{R}_t^-,\hat{R}_t^+) = (Z_t^-,Z_t^+,R_t^-,R_t^+).$$

PROOF is immediate from the definitions involved since H is strictly increasing and continuous (which makes G continuous and strictly increasing); see also ÇINLAR (1975) Proposition (2.35).

Note that the potential U defined by (1.4) is related to $(\hat{X},\hat{S})$ by

$$(2.13) \qquad U(x,f) = E^x \int_0^\infty f(X_t,S_t)dH_t = E^x \int_0^\infty f(\hat{X}_t,\hat{S}_t)dt.$$

In view of (2.11) and (2.13), it is advantageous to work with $(\hat{X},\hat{S})$. We will do this throughout the remainder of this paper, but will also drop "^" from the notation. In other words, we may, without loss of any generality, assume that the Lévy system (H,L,a) is such that $H_t = t \wedge \zeta$.

Notations. In addition to the usual notations it will be convenient to introduce the following: For any $t \in \mathbb{R}_+ = [0,\infty)$ we write $\mathbb{R}_t = (t,\infty)$, the set of all real numbers to the right of t; and $\mathbb{B}_t = [0,t]$, the set of all numbers before t; for $B \subset \mathbb{R}_+$ we write $B - t = \{b - t \geq 0: b \in B\}$ and $B + t = \{b + t: b \in B\}$.

For any topological space G we write $\underline{\underline{G}}$ for the set of all its Borel subsets; we write $f \in p\underline{\underline{G}}$ to mean that f is a positive $\underline{\underline{G}}$-measurable function on G. If N is a transition kernel from $(E, \underline{\underline{E}})$ into $(F,\underline{\underline{F}})\times(G,\underline{\underline{G}})$, we write $N(x,dy,du)$ instead of $N(x,d(y,u))$, and write $N(x,f,g)$ instead of $N(x,h)$ whenever h has the form $h(y,u) = f(y)g(u)$, that is

$$(2.15)\qquad N(x,f,g) = \int_{F\times G} N(x,dy,du)f(y)g(u).$$

If N is a transition kernel from $(E, \underline{\underline{E}})$ into $(E, \underline{\underline{E}})\times(\mathbb{R}_+,\underline{\underline{R}}_+)$, and if $f \in p\underline{\underline{E}}\times\underline{\underline{R}}_+$, we define the "convolution" of N and f by

$$(2.16)\qquad N * f(x,t) = \int_{E\times\mathbb{B}_t} N(x,dy,du)f(y,t-u).$$

If M and N are two transition kernels from $(E,\underline{\underline{E}})$ into $(E,\underline{\underline{E}})\times(\mathbb{R}_+,\underline{\underline{R}}_+)$, their convolution is defined by

$$(2.17)\qquad M * N(x,h) = \int_{E\times\mathbb{R}_+} M(x,dy,du) \int_{E\times\mathbb{R}_+} N(y,dz,ds)h(z,u+s),\ h \in p\underline{\underline{E}}\times\underline{\underline{R}}_+.$$

The convolution operation is associative: $M * (N * f) = (M * N) * f$, but in general not commutative.

3. PROOF OF THE MAIN THEOREM

Let (X,S) be a Markov additive process as in the introduction, and let (H,L,a) be its Lévy system. We may and do assume $H_t = t \wedge \zeta$ without any loss of generality (see the preceding section).

Let $b > 0$ be fixed, and define

$$(3.1) \qquad T = \inf\{t\colon S_t - S_{t-} > b\}.$$

The following auxiliary result is of interest in itself. Recall that $\mathbb{R}_b = (b,\infty)$.

(3.2) PROPOSITION. For all $x \in \mathbb{E}$ and $f \in p\underline{\underline{E}}\times\underline{\underline{R}}_+$,

$$E^x[f(X_{T-},S_{T-})] = E^x \int_0^T f(X_t,S_t)L(X_t, \mathbb{E}, \mathbb{R}_b)dt.$$

PROOF. Define

$$(3.3) \qquad S_t^b = A_t + \sum_{u\leq t} (S_u - S_{u-})I_{\{S_u - S_{u-} \leq b\}}.$$

Then, $S_t = S_t^b$ on $\{T > t\}$ and $S_{T-} = S_{T-}^b = S_T^b$. Moreover, given $\underline{\underline{K}}_\infty = \sigma(X_s,\ s \geq 0)^\sim$, T is conditionally independent of X and S^b, and

$$M_t \equiv P^x\{T > t|\underline{\underline{K}}_\infty\} = \exp[-\int_0^t L^d(X_s, \mathbb{R}_b)ds]\prod_{u\leq t} F(X_{u-},X_u, \mathbb{R}_b).$$

(see ÇINLAR (1972)). Hence, with $\underline{\underline{K}}_\infty^b = \sigma(X_t,S_t^b;\ t \geq 0)^\sim$. we have

$$(3.4) \qquad E^x[f(X_{T-},S_{T-})|\underline{\underline{K}}_\infty^b] = \int_{\mathbb{R}_+} f(X_{t-},S_{t-}^b)(-dM_t)$$

$$= \int_0^\infty f(X_{t-},S_{t-}^b)M_tL^d(X_t, \mathbb{R}_b)dt + \sum_{t\in\mathbb{R}_+} f(X_{t-},S_{t-}^b)M_{t-}F(X_{t-},X_t, \mathbb{R}_b).$$

Now, the process $W_t = f(X_{t-}, S^b_{t-})M_{t-}$ is predictable; therefore, by theorems on Lévy systems,

$$\text{(3.5)}\quad E^x[\sum_t W_t F(X_{t-}, X_t, \mathbb{R}_b)] = E^x \int_0^\infty W_t K(X_{t-}, dy) F(X_{t-}, y, \mathbb{R}_b)$$

$$= E^x \int_0^\infty W_t \, L^f(X_{t-}, \mathbb{E}, \mathbb{R}_b)dt.$$

Putting this into (3.4) while taking expectations, and noting that X_{t-}, S^b_{t-}, M_{t-} can be replaced by X_t, S^b_t, M_t since their jump times are countable, we get

$$E^x[f(X_{T-}, S_{T-})] = E^x \int_0^\infty f(X_t, S^b_t) M_t (L^d(X_t, \mathbb{R}_b) + L^f(X_t, \mathbb{E}, \mathbb{R}_b))dt$$

$$= E^x \int_0^\infty f(X_t, S^b_t) L(X_t, \mathbb{E}, \mathbb{R}_b) I_{\{T > t\}} dt$$

$$= E^x \int_0^T f(X_t, S_t) L(X_t, \mathbb{E}, \mathbb{R}_b)dt$$

as desired. □

The following is immediate from the strong Markov property for (X,S); see ÇINLAR (1972), p. 103.

(3.6) PROPOSITION. Let τ be a stopping time of $(\underline{\underline{M}}_t)$ and define

$$Q(x,f) = E^x[f(X_\tau, S_\tau)].$$

Then, for any $f \in p\,\underline{\underline{\mathbb{E}}} \times \underline{\underline{R}}_+$ and $x \in \mathbb{E}$,

$$U(x,f) = E^x \int_0^\tau f(X_t, S_t)dt + Q * U(x,f).$$ □

The next result is essentially the second statement of the main theorem (1.7).

(3.7) PROPOSITION. Let $b > 0$, $A \times B \times C \times D \in \underline{\underline{E}} \times \underline{\underline{E}} \times \underline{\underline{R}}_+ \times \underline{\underline{R}}_+$, and

(3.8) $$\Gamma_t = \{Z_t^- \in A,\ Z_t^+ \in B,\ R_t^- \in C,\ R_t^+ \in b + D\}.$$

Then,

(3.9) $$P^x(\Gamma_t) = \int_{A \times \mathbb{B}_t} U(x,dy,ds) 1_C(t-s) L(y,B,t+b+D-s)$$

for every $x \in \mathbf{E}$ and every $t \in \mathbb{R}_+$ (recall that $\mathbb{B}_t = [0,t]$).

PROOF. Define T as in (3.1), and put

(3.10) $$f(x,t) = P^x(\Gamma_t); \quad g(x,t) = P^x(\Gamma_t; S_T > t).$$

Then, by the strong Markov property for (X,S) at T, and by the additivity of S which implies $S_{T+u} = S_T + S_u \circ \theta_T$,

(3.11) $$\begin{aligned} f(x,t) &= g(x,t) + P^x(\Gamma_t;\ S_T \leq t) \\ &= g(x,t) + E^x[f(X_T, t-S_T);\ S_T \leq t] \\ &= g(x,t) + \int_{\mathbb{E} \times \mathbb{B}_t} Q(x,dy,du) f(y,t-u) \end{aligned}$$

where

(3.12) $$Q(x,k) = E^x[k(X_T,S_T)], \quad k \in p\underline{\underline{E}} \times \underline{\underline{R}}_+.$$

Next consider $g(x,t)$. Since T is the time of first jump with magnitude greater than b, on $\{S_T > t\}$ we have $C_t = T$, and $Z_t^- = X_{T-}$, $Z_t^+ = X_T$, $R_t^- = t - S_{T-}$, and $R_t^+ = S_T - t$. So,

(3.13) $$g(x,t) = P^x\{X_{T-} \in A,\ X_T \in B,\ S_{T-} \in t - C,\ S_T \in t+b+D\}.$$

The stopping time T is totally inaccessible since S is quasi-left-continuous. Therefore, by the results of WEIL (1971) on conditioning on the strict past $\underline{\underline{M}}_{T-}$, we have

$$(3.14)\qquad P^x\{X_T \in B,\ S_T - S_{T-} \in b + D' | \underline{\underline{M}}_{T-}\} = \frac{L(X_{T-}, B, b + D')}{L(X_{T-}, \mathbb{E}, \mathbb{R}_b)}$$

for any $B \in \underline{\underline{E}}$ and $D' \in \underline{\underline{R}}_+$; here we used the additivity of S, so that $S_T - S_{T-}$ is conditionally independent of $\underline{\underline{M}}_{T-}$ given X_{T-}. Putting (3.14) into (3.13) we obtain

$$(3.15)\qquad g(x,t) = E^x[h(X_{T-}, t - S_{T-})/L(X_{T-}, \mathbb{E}, \mathbb{R}_b)]$$

where

$$(3.16)\qquad h(y,u) = 1_A(y)1_C(u)L(y,B,u + b + D).$$

In view of Proposition (3.2), (3.15) implies

$$(3.17)\qquad g = V * h$$

where h is as defined by (3.16) and

$$(3.18)\qquad V(x,k) = E^x \int_0^T k(X_t, S_t)dt, \qquad k \in p\,\underline{\underline{\mathbb{E}}} \times \underline{\underline{R}}_+.$$

Putting (3.17) into (3.11) we obtain

$$(3.19)\qquad f = V * h + Q * f,$$

and by Proposition (3.6), we have

$$(3.20)\qquad U * h = V * h + Q * U * h.$$

It is now clear from (3.19) and (3.20) that

$$(3.21)\qquad f = U * h$$

is a solution to (3.19). Note that $U * h(x,t)$ is exactly the right hand side of (3.9). Therefore, the proof will be complete once we show that $U * h$ is the only solution to (3.19). To show this, let f' and f'' be two bounded solutions to (3.19); then $k = f' - f''$ satisfies

$$(3.22) \qquad k = Q * k.$$

Let Q_n be defined recursively by $Q_1 = Q$, $Q_{n+1} = Q * Q_n$ through the formula (2.17). Then, (3.22) implies $k = Q_n * k$, and

$$(3.23) \qquad |k(x,t)| = |Q_n * k(x,t)| \leq c Q_n(x, \mathbf{E}, \mathbb{B}_t)$$

for every n, where c is the bound for k. On the other hand, T is totally inaccessible, and its iterates defined by

$$(3.24) \qquad T_0 = 0, \; T_{n+1} = T_n + T \circ \theta_{T_n}$$

are all stopping times, and $S_{T_{n+1}} \geq S_{T_n} + b$ by the definition of T. Hence,

$$(3.25) \qquad Q_n(x, \mathbf{E}, \mathbb{B}_t) = P^x\{S_{T_n} \leq t, X_{T_n} \in E\} \to 0$$

as $n \to \infty$ for any fixed $t \in \mathbb{R}_+$. This implies through (3.23) that $k(x,t) = 0$ for all x and t; that is, the only bounded solution of (3.22) is $k = 0$, and therefore (3.19) has exactly one solution. □

(3.26) COROLLARY. For any $f \in p\,\underline{\underline{E}} \times \underline{\underline{E}} \times \underline{\underline{\mathbb{R}}}_+ \times \underline{\underline{\mathbb{R}}}_+$ such that $f(\cdot,\cdot,\cdot,0) = 0$ the equation (1.8) holds for every $x \in \mathbf{E}$ and every $t \in \mathbb{R}_+$. Moreover, for all x and t again,

$$(3.27) \qquad E^x[f(Z_t^-, Z_t^+, R_t^-, R_t^+);\; Z_t^- = Z_t^+]$$
$$= \int_{\mathbf{E} \times \mathbb{B}_t} U(x,dy,ds) L^d(y, t-s+du) f(y,y,t-s,u);$$

$$(3.28)\qquad E^x[f(Z_t^-,Z_t^+,R_t^-,R_t^+);\ Z_t^-\neq Z_t^+]$$

$$= \int_{E\times\mathbb{B}_t} U(x,dy,ds)L^f(y,dz,t-s+du)f(y,z,t-s,u).$$

PROOF. In Proposition (3.7) let $D=\mathbb{R}_a$. Then, as $b\downarrow 0$, $L(y,B,t-s+b+D)$ increases to $L(y,B,t-s+D)$; and hence, by the monotone convergence theorem, (3.9) remains true with $b=0$ in (3.8) and (3.9) both. Now, by the usual monotone class arguments, (3.9) holds true for arbitrary Borel subsets D of $\mathbb{R}_0 = (0,\infty)$. This is equivalent to the first statement by the monotone class theorem again. The second statement, namely that concerning (3.27) and (3.28), is immediate from the first statement coupled with the definitions (2.6). □

The preceding corollary is the second statement of Theorem (1.7). There remains the problem of computing the probabilities of the subsets of $\{R_t^+ = 0\}$, namely the event that t is a time of regeneration. To that end, we start computing the distribution of (Z_t^-,Z_t^+).

(3.29) PROPOSITION. For any $f\in p\underline{\underline{E}}\times\underline{\underline{E}}$, $x\in\mathbb{E}$, and $\lambda\in\mathbb{R}_+$,

$$(3.30)\qquad E^x\int_0^\infty \lambda e^{-\lambda t}\, f(Z_t^-,Z_t^+)dt$$

$$= \int_{\mathbb{E}\times\mathbb{R}_+} U(x,dy,ds)\Big[a(y)f(y,y)\lambda e^{-\lambda s} + \int_{\mathbb{E}\times\mathbb{R}_+} L(y,dz,du)f(y,z)(1-e^{-\lambda u})\Big].$$

PROOF. Let e_λ be the mapping $x\to e^{-\lambda x}$, and recall the definition (1.2) of (C_t). By the general change of variable formula given by GETOOR and SHARPE (1973, p.551), for any $g\in p\underline{\underline{R}}_+$,

$$(3.31)\qquad \int_{(0,\infty)} g(t)d(1-e_\lambda(S_t)) = \int_{(0,S_\infty)} g(C_t)\lambda e^{-\lambda t}\,dt.$$

Taking $g(t) = f(X_{t-},X_t)$ above, recalling (1.3), and noting that for $t \geq S_\infty$ we have $C_t = \infty$ and $X_\infty = \Delta$ and $h(X_{t-},X_t) = 0$ by the usual conventions, we get

$$(3.32) \qquad \ell hs(3.30) = E^x \int_0^\infty f(X_{t-},X_t)d(1 - e_\lambda(S_t)),$$

where "ℓhs(•)" stands for the "left-hand side of (•)." By the generalized version of FUBINI's theorem,

$$(3.33) \qquad E^x[\int_0^\infty f(X_{t-},X_t)d(-e_\lambda(S_t))|\underline{\underline{K}}_\infty] = -\int_0^\infty f(X_{t-},X_t)dM_t,$$

where

$$(3.34) \qquad M_t = E^x[e_\lambda(S_t)|\underline{\underline{K}}_\infty]$$

$$= \exp[-\lambda \int_0^t a(X_s)ds - \int_0^t L^d(X_s,1-e_\lambda)ds] \prod_{s \leq t} F(X_{s-},X_s,e_\lambda)$$

(see ÇINLAR (1972) for this formula). Now (3.32),(3.33),(3.34) yield

$$(3.35) \qquad \ell hs(3.30) = E^x \int_0^\infty f(X_{t-},X_t)M_t(\lambda a(X_t) + L^d(X_t,1-e_\lambda))dt$$

$$+ E^x \sum_t f(X_{t-},X_t)M_{t-} F(X_{t-},X_t,1 - e_\lambda).$$

The process (M_{t-}) is predictable; therefore, by theorems on Lévy systems, the second term on the right-hand side is equal to

$$E^x \int_0^\infty dt\, M_{t-} \int_{\mathbb{E}} K(X_{t-},dy)f(X_{t-},y)F(X_{t-},y,1 - e_\lambda)$$

$$= E^x \int_0^\infty dt\, M_t \int_{\mathbb{E}} L^f(X_t,dz,1 - e_\lambda)f(X_t,z)$$

by the definitions (2.6) and (2.7). Putting this in (3.35) and noting (3.34), we see that

$$(3.36)\qquad \ell hs(3.30) = E^x \int_0^\infty g(X_t)M_t dt = E^x \int_0^\infty g(X_t)e_\lambda(S_t)dt = U(x,g,e_\lambda)$$

where

$$(3.37)\qquad g(y) = f(y,y)(\lambda a(y) + L^d(y,1-e_\lambda)) + \int L^f(y,dz,1-e_\lambda)f(y,z)$$

$$= \lambda a(y)f(y,y) + \int L(y,dz,1-e_\lambda)f(y,z).$$

With this g, $U(x,g,e_\lambda)$ is precisely the right-hand side of (3.30); and thus the proof is complete. □

Next we consider the problem of inverting the Laplace transform (3.30). Note that g defined by (3.37) can be written as

$$(3.38)\quad g(y) = \lambda\{a(y)f(y,y)\int_0^\infty \varepsilon_0(du)e^{-\lambda u} + \int_0^\infty [\int_{\mathbb{E}} L(y,dz,\mathbb{R}_u)f(y,z)]e^{-\lambda u}du\},$$

which has the form $\lambda \int n(y,du)e^{-\lambda u}$. Putting this in (3.36) we see that

$$U(x,g,e_\lambda) = \lambda \int U * n(x,du)e^{-\lambda u},$$

and this is equal to the right-hand side of (3.30). Inverting the Laplace transforms on both sides of (3.30), we obtain

$$(3.39)\qquad \int_B E^x[f(Z_t^-,Z_t^+)]dt = \int_{\mathbb{E}} U(x,dy,B)a(y)f(y,y)$$

$$+ \int_B dt \int_{\mathbb{E}\times\mathbb{B}_t} U(x,dy,ds) \int_{\mathbb{E}} L(y,dz,\mathbb{R}_{t-s})f(y,z)$$

for every $B \in \underline{\underline{R}}_+$.

We are now ready to give the

(3.40) PROOF of Proposition (1.5). Choose f such that $h(y) = f(y,y)$ is strictly positive, and let $\mathbb{A} = \{a > 0\}$. Now the first term on the right side of (3.39) is $U(x,a\cdot h,B)$, and clearly this is at most equal to the left-hand side. It follows that the measure $B \to U(x,a\cdot h,B)$ is absolutely continuous with respect to the Lebesgue measure. Let $u_t(x,ah)$ be its Radon-Nikodym derivative with respect to the Lebesgue measure. Since X is standard, its state space $(\mathbb{E}, \underline{\underline{E}})$ is locally compact with a countable base. Therefore, it is possible to choose this derivative such that $(t,x) \to u_t(x,ah)$ is $\underline{\underline{R}}_+ \times \underline{\underline{E}}$ measurable.

Now, for $k \in p\underline{\underline{A}}$, define $\bar{u}_t(x,k)$ to be $u_t(x,a\hat{k})$ where $\hat{k} = k/a$. By the special nature of $(\mathbb{E}, \underline{\underline{E}})$ again, by theorems on the existence of regular versions of conditional probabilities, we may take $\bar{u}_t(x,\cdot)$ to be a measure on $\underline{\underline{A}}$ while retaining the measurability of the mapping $(t,x) \to \bar{u}_t(x,A)$ for each $A \in \underline{\underline{A}}$. Finally, let

$$(3.41) \qquad u_t(x,A) = \bar{u}_t(x,A \cap \mathbb{A}), \qquad A \in \underline{\underline{E}}.$$

The statement (1.5) is true for this transition kernel u. □

The following is immediate from Proposition (1.5) applied to (3.39).

(3.42) THEOREM. For any $f \in p\underline{\underline{E}} \times \underline{\underline{E}}$ and $x \in \mathbb{E}$,

$$(3.43) \qquad E^x[f(Z_t^-,Z_t^+)] = \int_{\mathbb{E}} u_t(x,dy)a(y)f(y,y) + \int_{\mathbb{E}\times\mathbb{B}_t} U(x,dy,ds) \int_{\mathbb{E}} L(y,dz, \mathbb{R}_{t-s})f(y,z)$$

for (Lebesgue) almost every $t \in \mathbb{R}_+$. □

In view of Corollary (3.26), the second term on the right side of (3.43) is equal to the expectation of $f(Z_t^-,Z_t^+)$ on $\{R_t^+ > 0\}$. Hence, (3.43) implies that

$$(3.44) \quad Z_t^- = Z_t^+ \in \mathbb{A} \text{ a.s. on } \{R_t^+ = 0\}; \; E^x[g(Z_t^+); \; R_t^+ = 0] = u_t(x,ag)$$

for any $g \in p\underline{\underline{E}}$, $x \in \mathbb{E}$, and almost every $t \in \mathbb{R}_+$.

The method of the proof of Proposition (3.7) goes through to show

$$(3.45) \qquad P^x\{Z_t^- \in A,\ Z_t^+ \in B,\ R_t^- \in b + C,\ R_t^+ \in D\}$$

$$= \int_{A \times \mathbf{B}_t} U(x,dy,ds)1_C(t-s-b)L(y,B,t-s+D)$$

for all $A,B \in \underline{\underline{E}}$, all $C,D \in \underline{\underline{R}}_+$, all $t \in \mathbb{R}_+$, for $b > 0$. In particular, this yields

$$(3.46) \quad P^x\{Z_t^- = Z_t^+ \in \mathbb{A},\ R_t^- > 0,\ R_t^+ = 0\} = \int_{\mathbb{A}\times[0,t)} U(x,dy,ds)L^d(y,\{t-s\})$$

$$= \int_{\mathbb{A}\times[0,t)} u_s(x,dy)L^d(y,\{t-s\})ds \;=\; 0$$

since the function $s \to L^d(y,\{t-s\})$ is zero everywhere except on a countable set. Hence, for any t,

$$(3.47) \qquad R_t^- = 0 \quad \text{a.s. on } \{Z_t^- = Z_t^+ \in \mathbb{A};\ R_t^+ = 0\}.$$

Conversely, Corollary (3.26) and Theorem (3.42) show that

$$(3.48) \qquad R_t^+ = 0 \quad \text{a.s. on } \{Z_t^- = Z_t^+,\ R_t^- = 0\}.$$

It follows that a.s. on $\{Z_t^- = Z_t^+\}$, either $R_t^- = R_t^+ = 0$ or $R_t^- > 0$ and $R_t^+ > 0$.

(3.49) PROOF of Theorem (1.7) now follows from these observations put together with Theorem (3.42) and Corollary (3.26).

4. FROM ALMOST TO ALL

This section is devoted to showing that under certain reasonable conditions, when X is a regular step process, Theorem (1.7) can be strengthened so that (4.8) is true for every t (instead of for almost every t). Unfortunately, our technique does not generalize to arbitrary X.

We shall need the following facts concerning the case where X is trivial: that is, where $\mathbb{E}$ is a singleton $\{x\}$. Then, writing $\hat{u}_t = u_t(x,\{x\})$, $\hat{U}(ds) = U(x,\{x\},ds)$, etc., we obtain from Theorem (1.7) that

$$(4.1)\quad \hat{E}[f(R_t^-,R_t^+)] = \hat{u}_t\hat{a}f(0,0) + \int_{\mathbb{B}_t} \hat{U}(ds)\hat{L}(t-s+du)f(t-s,u)$$

for almost every t.

This is the result which KINGMAN (1973) obtained by inverting a triple Laplace transform in the case where $\hat{a} > 0$. If $\hat{a} > 0$, then $t \to \hat{u}_t$ is continuous and $\hat{U}(ds) = \hat{u}_s ds$, and (4.1) holds for every t (this is due to NEVEU (1961)). If $\hat{a} = 0$ and $\hat{L}(\mathbb{R}_0) = +\infty$, then $\hat{u}_t = 0$ and (4.1) holds for every t again (this is due to KESTEN (1969) essentially). If $\hat{a} = 0$ and $\hat{L}(\mathbb{R}_0) < \infty$, then $\hat{u}_t = 0$ but the restriction "almost every t" cannot be removed in the absence of further restrictions on the smoothness of $\hat{L}$. In fact, if $\hat{L}$ has a countable support and $a = 0$, $\hat{L}(\mathbb{R}_0) < \infty$, then (4.1) fails to hold at every t belonging to the group generated by that support.

In view of this, the only real restriction in the proposition below is on the process X. Define

$$(4.2)\qquad \mathbb{D} = \{x \in \mathbb{E}\colon a(x) > 0 \ \text{ or } L^d(x,\mathbb{R}_0) = \infty\}.$$

(4.3) PROPOSITION. Suppose X is a regular step process. Then, we may take $t \to u_t(x,A)$ to be continuous, and we have

$$(4.4)\quad E^x[g(Z_t^+,R_t^-,R_t^+);\ Z_t^- = Z_t^+ \in \mathbb{D}]$$

$$= \int_{\mathbb{D}} u_t(x,dy)a(y)g(y,0,0) + \int_{\mathbb{D}\times B_t} U(x,dy,ds)L^d(y,t-s+du)g(y,t-s,u)$$

for all $x \in \mathbb{E}$ and all $t \in \mathbb{E}$. □

(4.5) REMARK. Suppose X is a regular step process and (S_t) is strictly increasing (which means that (C_t) defined by (1.2) is continuous, which in turn means that the regeneration set M is without isolated points). In particular, GETOOR (1974) assumes this holds. (S_t) can be strictly increasing only if $\mathbb{D} = \mathbb{E}$. Hence, Proposition (4.3) applies to this important case.

PROOF of (4.3). The expectation on $\{R_t^+ > 0\}$ is equal to the second term on the right-hand side of (4.4) by the second statement of Theorem (1.7). Hence, we need only show that

$$(4.6) \qquad P^x\{Z_t^+ \in A,\ R_t^- = 0,\ R_t^+ = 0,\ Z_t^- = Z_t^+\}$$

is continuous in t for any $A \in \underline{\mathbb{D}}$. (This is equal to $u_t(x, a1_A)$ for almost every t; and therefore the density $t \to u_t(x, a1_A)$ can be taken continuous and equal to (4.6) for all t.)

Let τ be the time of first jump for X, and define $\tau_0 = 0$, $\tau_{n+1} = \tau_n + \tau \circ \theta_{\tau_n}$. Then, τ_n is the time of the n^{th} jump of X, and $\tau_n \to \zeta$ almost surely, and X remains constant on each interval $[\tau_n, \tau_{n+1})$. Therefore, (4.6) is equal to

$$(4.7) \qquad \sum_{n=0}^{\infty} P^x\{Z_t^- = Z_t^+ \in A,\ R_t^- = R_t^+ = 0,\ S_{\tau_n} \le t < S_{\tau_{n+1}}\}$$

$$= \sum_{n=0} P^x\{X_{\tau_n} \in A,\ R_t^- = R_t^+ = 0,\ S_{\tau_n} < t < S_{\tau_{n+1}}\}.$$

Note that, on $\{S_{\tau_n} = t - u\}$ we have $R_t^+ = R_u^+(\theta_{\tau_n})$, and $R_t^- = R_u^-(\theta_{\tau_n})$. By the strong Markov property at τ_n,

$$(4.8)\qquad P^x\{R_t^- = R_t^+ = 0,\ S_{\tau_n} < t < S_{\tau_{n+1}} | \underline{\underline{M}}_{\tau_n}\}$$

$$= f(X_{\tau_n}, t - S_{\tau_n}) I_{[0,t)}(S_{\tau_n})$$

where

$$(4.9)\qquad f(y,u) = P^y\{R_u^- = R_u^+ = 0,\ \tau_1 > u\}.$$

Starting at y, X stays there an exponential time with parameter $k(y) = K(y, \mathbb{E})$; and during that sojourn, S has the law of an increasing Lévy process with drift parameter a(y) and Lévy measure $L^d(y,\cdot)$. It follows from the results mentioned following (4.1) that

$$(4.10)\qquad f(y,t) = \begin{cases} 0 & \text{if } a(y) = 0 \\ r(y,t)a(y) & \text{if } a(y) > 0 \end{cases}$$

for all t, where $r(y,\cdot)$ is the density (which exists when $a(y) > 0$) of the potential measure $R(y,\cdot)$ with

$$(4.11)\qquad R(y,e_\lambda) = [\lambda a(y) + L^d(y,1-e_\lambda) + k(y)]^{-1}.$$

Putting (4.7)-(4.10) together, we see that (4.6) is equal to

$$(4.12)\qquad \sum_{n=0}^{\infty} E^x[a(X_{\tau_n})r(X_{\tau_n}, t - S_{\tau_n});\ X_{\tau_n} \in A \cap \mathbb{A};\ S_{\tau_n} < t]$$

$$= \int_{(A \cap \mathbb{A})\times[0,t)} V(x,dy,ds)a(y)r(y,t-s)$$

by an obvious definition for V. This is essentially a convolution, and the function $t \to r(y,t)$ is continuous (NEVEU (1961)). Hence, (4.12) is continuous in t, and the proof of Proposition (4.3) is complete. □

(4.13) REMARK. As mentioned before, the restriction "almost every t" cannot be removed on $\mathbb{E}\setminus\mathbb{D}$ without adding other conditions of smoothness. Similarly for equalities concerning expectations on the event $\{Z_t^- \neq Z_t^+\}$.

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