Jump Bidding and Budget Constraints in All-Pay Auctions and Wars of Attrition*

Eddie Dekel, Matthew O. Jackson, Asher Wolinsky†

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Abstract

We study all-pay auctions (or wars of attrition), where the highest bidder wins an object, but all bidders pay their bids. We consider such auctions when two bidders alternate in raising their bids and where all aspects of the auction are common knowledge including bidders' valuations. We analyze how the ability to "jump-bid," or raise bids by more than the minimal necessary increment affects the outcome of the auction. We also study the impact of budget caps on total bids. We show that both of these features, which are common in practice but absent from the previous literature, matter significantly in determining the outcome of the auctions.

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*Leininger (1991) contains results very similar to most of the results in this paper. Nevertheless, because there are some small differences in the model and the exposition is different, we are making this manuscript available, while emphasizing that versions of the main results herein were previously published by Leininger (1991).

†Dekel is at the Department of Economics, Tel Aviv University and Northwestern University, Evanston, IL 60208, USA, email: dekel@northwestern.edu. Jackson is at the Department of Economics, Stanford University, Stanford, CA 94305, email: jacksonm@stanford.edu. Wolinsky is at the Department of Economics, Northwestern University, Evanston, IL 60208, USA, email: a-wolinsky@northwestern.edu
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1 Introduction

A wide variety of economic interactions are usefully modeled as an all-pay auction or war-of-attrition.[1] The standard analysis of such competitions is one where bids or committed resources increase at a given rate per unit time, and players simply decide whether to stay in or drop out of the auction or contest. In this paper, we consider a dynamic version of this game that differs from the standard versions in that the players have discretion over how much they increase their bid by each time they move. Thus, the model can be thought of as an English-all-pay auction in which the bidders are allowed to "jump-bid" or as a war-of-attrition situation in which the players are not limited to investing at some fixed rate per unit of time, but can commit large amounts of resources at once to the contest. We analyze this model with and without the presence of budget caps on total expenditures by bidders. We show how the ability to make jump bids, and also how the presence of budget constraints, alter the equilibrium behavior.

The ability to jump-bid, or commit a large amount of resources at a time to the contest, changes the nature of equilibrium completely. For instance, when the players can only invest at a fixed rate, there are equilibria in which the player with the lower value wins with high probability. This equilibrium behavior changes dramatically if players are allowed to "jump bid". The simple intuition is that the higher valued player can preemptively bid an amount that the lower valued player would not wish to match. This guarantees that the higher valued player should have a positive expected payoff from the auction. In fact, this turns out to guarantee that the higher valued player wins the auction, even though such a preemptive bid is never made. The specifics of the equilibrium depend on the environment (e.g., what is known about players' valuations, and whether budgets are finite) in ways that we discuss in detail.

Budget caps affect the bidding in an interesting way. In the context of a sealed-bid all-pay auction setting, budget caps that exceed the bidders values do not affect the bidding. However, in a dynamic all-pay setting, even budget caps that exceed the bidders' valuations can be important. Since past bids become sunk costs, bidders may end up bidding more than their values, at least off-the-equilibrium-path, and budget caps may affect the equilibrium even when they exceed valuations. Indeed, our analysis shows that in some situations the bidder with the larger budget wins the auction rather than the bidder with the larger valuation, even when budgets greatly exceed valuations, as they alter play off the equilibrium path, which then influences equilibrium outcomes. As budget

[1] As is well known, the war of attrition is equivalent to a sealed bid, all-pay second-price auction.

limits operate off the equilibrium path, they might never be reached in equilibrium.

Most of our analysis concerns complete-information all-pay auctions, where bidders' valuations and any budget constraints are common knowledge. We do this for several reasons. First, it makes the role of jump-bidding and budget constraints very clear.[2] Second, the analysis of equilibrium is already rich and at times subtle, even with complete information. In the closing discussion we provide some results on the incomplete information case.

As emphasized in the opening footnote, Leininger (1991) already has results that are essentially the same as most of those in the present paper. His is a complete-information, sequential, all-pay auction with hetrogenous values and budgets. Since he does not restrict the bids to a grid (as we do), he considers ε-equilibria. He also seems to restrict attention to Markovian strategies. Although these differences do not matter much for the main results, the present paper might still be useful as the exposition here differs from that in Leininger's paper and as it includes a brief discussion of the incomplete-information case.[3]

2 Definitions

Most of our analysis is of a complete-information setting, where valuations for the object and budgets are common knowledge. When we depart from this assumption, we will make it clear.

2.1 Bidders, Values, and Budgets

We concentrate on auctions with two bidders and later comment on the extension to more bidders.

[2] Actually, there are at least two different roles of jump-bidding. The complete information analysis isolates the jump-bid as a "preemptive" tool. The incomplete information analysis also brings in the jump-bid as a signaling device.

[3] Oneil (1986) also analyzes a dynamic all-pay auction with budget constraints but focuses on the symmetric case with equal valuations and equal budgets. Harris and Vickers (1985) present a model of race which is also a dynamic all-pay contest with unlimited budgets. Dixit and Nalebuff (1991) present an example that has some aspects of preemptive investment. We have two other papers that analyze vote buying (Dekel, Jackson and Wolinsky (2006-a,b)) where we use related models. Those models are more complicated, since the bidders compete for buying the votes of a majority of the voters rather than just a single object, and so the impact of jump bidding and budget constraints are difficult to isolate, and affected by the distinct nature of the majority game.

Let V_i denote bidder i's private value for the object.

A bidder who bids a total of b_i and wins the object has a final utility of $V_i - b_i$. A bidder who bids a total of b_i and does not win the object has a final utility of $-b_i$.

Let B_i denote i's budget. B_i takes on values in $\Re_+ \cup \{\infty\}$.

A bidder is never allowed to bid more than his or her budget.

There is a smallest money unit $\varepsilon > 0$, so bids and budgets (when finite) are whole multiples of ε.

To simplify the analysis we assume that the V_i's are not multiples of ε. We will let $[V_i]_\varepsilon$ denote the maximal multiple of ε that is smaller than V_i.

We eliminate degenerate cases by assuming that $V_i > \varepsilon$ for each i.

2.2 A Description of the Auctions

We consider two versions of an all-pay auction. In each version, time proceeds in discrete periods $t \in \{1, 2, \ldots\}$. Bidders alternate in their moves, so that bidder 1 bids on odd dates and bidder 2 bids on even dates. Bids are nonnegative and can only be increased.

In a *jump-bidding all-pay auction*, a bidder who is called upon to move can choose to raise its bid to any higher feasible bid or to leave it unchanged. We will refer to the latter move as "dropping out" of the auction. The auction ends at the first time where either:[4]

(i) bidder 1 is called upon to bid and bidder 1's bid does not exceed bidder 2's last bid, in which case bidder 2 wins; or

(ii) bidder 2 is called upon to bid and bidder 2's bid does not match or exceed bidder 1's latest bid, in which case bidder 1 wins.

If the auction never ends, then each bidder gets a utility of $-\infty$.

A *no-jump all-pay auction* is as above except that a bidder can increase its outstanding bid by at most ε in each step. This is the discrete version of the "war-of-attrition."

The alternating moves and the tie-breaking in favor of bidder 2 introduce a slight asymmetry in the auctions.

Unless otherwise stated, the solution concept is subgame-perfect equilibrium.

[4] This is equivalent (in terms of equilibrium outcomes) to stopping the auction after successive rounds where each bidder has had a chance to bid and has not changed his or her bid.

2.3 A Useful Lemma on Dropping Out and Mixing

The following Lemma is useful in a number of our results. We note that it holds regardless of whether jumps are permitted and/or budgets are finite or infinite.

LEMMA 1 *Consider a subgame starting with a move by bidder i. If i bids (i.e., increases its standing bid) with positive probability in an equilibrium, then in the equilibrium continuation j drops out with positive probability at any node that follows a bid by i.*

Note that this implies that in any equilibrium, the only (decision) node on the equilibrium path where either bidder ever has a strictly positive expected payoff is at the first node. Note also that, if the bidding continues past the first node, it must involve mixing or dropping out completely at every subsequent node reached on the equilibrium path.

Proof of Lemma 1: Suppose to the contrary that there is a subgame starting at a node t and an equilibrium in that subgame where i makes a bid with positive probability, and following some bid that i makes with positive probability at t, j stays in at the next node, t', with probability one.

First, consider the case where the equilibrium strategies in this subgame are pure on the equilibrium path. Let t'' be the first subsequent node on the path where some bidder drops out (such a node exists as this is an equilibrium path and the payoff of an infinite play is negative infinity). The bidder who moves at t'' must have 0 expected continuation utility conditional on being at t'' (or the bidder would strictly prefer to stay in, and would not be exiting). Since by assumption the strategies are pure on the path, node t'' is reached with probability 1 in the continuation. If the bidder who moves at t'' is i, then i has a negative expected utility at t since i's payment increases between t and t'' only to attain 0 expected utility at t'' after having already sunk the incremental payment. If the bidder who moves at t'' is j, then by the same reasoning j has a negative expected utility conditional on making a bid at t' which precedes t''. Thus, we reach a contradiction in both cases.

Next, consider the case where the equilibrium strategies in this subgame may be mixed on the equilibrium path. In this case there is a set of paths which may occur with positive probability when the equilibrium is played. We construct a new equilibrium by selecting from these paths as follows. At node t bidder i makes the bid that leads to the node t' where player j stays in with certainty. At node t' bidder j chooses, from among the actions played with positive probability in the given equilibrium, an action that maximizes bidder i's expected payoff in the continuation of this equilibrium. At the next node i again selects an action that maximizes j's expected equilibrium payoff. The selection continues in this manner – with the bidder in control breaking own indifferences so as to benefit the other bidder – at the subsequent nodes up to the K^{th} node on this path from t, provided the game does not terminate prior to that. The number K is such that $K\varepsilon > 2\max\{V_1, V_2\}$. At all other nodes of the subgame, the behavior of the original equilibrium remains.

It follows almost directly from its construction that this selection of actions through K steps starting from t, together with the remaining strategies at other nodes from the original equilibrium, forms a subgame-perfect equilibrium in the subgame starting at node t. One can argue this by backwards induction starting from the most distant node out of the first K nodes on the path from t, using the fact that these are paths of the original given equilibrium.

Consider this new SPE. The first K moves in this equilibrium are pure. It must then be that at least one of the players drops out in these K moves, so that the equilibrium path itself is entirely pure. This follows since if play did last for K or more moves, then i's incremental payoff starting at node t would be at most $V_i - (K/2)\varepsilon < 0$, which is less the incremental payoff of quitting at t, which would be a contradiction. Therefore, the argument used above for the case of equilibrium strategies that are pure on the equilibrium path applies, and we conclude that, contrary to the initial hypothesis, j cannot stay at node t' with probability 1. ∎

3 Infinite Budgets

We begin by analyzing all-pay auctions when there are no limits to budgets.

3.1 No jumps and Infinite Budgets

We start with the benchmark case of no jump bidding and unlimited budgets. Without jump bidding, a bidder's strategy can simply be specified as a probability of remaining in the auction as a function of the time.

Proposition **1** *In the game with no jump-bidding and infinite budgets, the set of all equilibria can be characterized in terms of their equilibrium paths as follows. For any* $p \in [0, 1]$, *there exists an equilibrium in which bidder* 1 *drops with probability* p *in the first node. If* $p = 0$, *then there is a range of equilibria in which in the next node bidder* 2 *drops with probability* $q \geq \varepsilon/V_1$, *and if* $q < 1$ *then the bidders drop thereafter with* *probabilities ε/V_2 and ε/V_1, respectively. If $p \in (0,1)$, then in each subsequent node on the equilibrium path bidder 2 drops out with probability ε/V_1 and bidder 1 drops out with probability ε/V_2. If $p = 1$, then this is clearly the end of the equilibrium path.*

The proofs of all propositions appear in the appendix.

Note that in a subset of these equilibria, the bidder with the low valuation wins with higher probability, including an equilibrium in which the low valuation bidder wins with certainty. Three of the equilibria are Markov perfect (a subgame perfect equilibrium where each bidder's probability of staying in the auction is independent of time): the mixed equilibrium in which the bidders drop with probabilities ε/V_2 and ε/V_1 respectively at all nodes, and the two pure equilibria in which one bidder always continues with probability one and the other always drops out. The Markov-perfect mixed-strategy equilibrium is also the only symmetric equilibrium, in the sense that each bidder's behavior has the same functional form (as dependent on the pair of valuations). Notice that this equilibrium is one of those in which the low-value bidder wins with higher probability.

Some of the literature on the war of attrition uses reputation effects to refine away the multiplicity of equilibria.[5] Such effects can be introduced here by assuming that each one of the bidders has with small probability an obstinate type who never drops out. The reputation game that results has a unique equilibrium. This equilibrium involves mixed strategies and hence its outcome is a distribution. If the probability of the obstinate types is taken to zero, the equilibrium outcome distribution approaches the outcome in which the higher valuation bidder wins immediately. We show next that the simple ability to jump bid will directly single out this as the unique equilibrium outcome, without any need for augmenting the game with noise (and corresponding beliefs).

3.2 Jump-bidding and Infinite Budgets.

We now consider the case where jump-bidding is allowed and show that in all equilibria the high-value bidder wins for certain. Thus, the introduction of jump-bidding selects one out of the range of equilibrium outcomes that arise when jump-bidding is not allowed.

PROPOSITION **2** *In all equilibria of the game with jump-bidding and infinite budgets:*

(i) If $V_2 + \varepsilon \leq V_1$, then bidder 1 bids once to a price of ε and then bidder 2 drops out.

[5] See Fudenberg and Tirole (1986) and Abreu and Gul (1991).

(ii) If $V_1 + \varepsilon \leq V_2$, then bidder 1 drops out immediately and bidder 2 wins at a price of 0.

While the presence of jump-bidding narrows down the equilibrium set drastically, jump-bids are never used on the equilibrium path. It is simply the possibility that they could be used that is critical. To understand the role of jump-bids it is useful to examine how they preclude equilibria in which the lower value bidder wins. Suppose that $V_1 + \varepsilon \leq V_2$. Clearly, there is no SPE in which 1 wins by bidding ε, since by bidding $[V_2]_\varepsilon$ (recall that $[V_i]_\varepsilon$ denotes the largest whole multiple of ε that is still smaller than V_i), bidder 2 can force bidder 1 to drop out (since in order to stay in the auction bidder 1 would have to increase its bid by more than V_1). Since $[V_2]_\varepsilon < V_2$, bidder 2 would prefer to win in this way than to let bidder 1 win. But what if V_1 is close to V_2 and bidder 1 starts with a higher bid, say $[V_1]_\varepsilon$? If bidder 2 responds by bidding $[V_2]_\varepsilon$, bidder 1 may take a large lead again by increasing its bid by $[V_1]_\varepsilon$, so it is not immediately clear that bidder 2 must win in any SPE. To understand why this must be indeed the case, consider a race in which bidder 1 increases its standing bid by $[V_1]_\varepsilon$ in each round and bidder 2 increases its standing bid by $[V_2]_\varepsilon$ in each round. Eventually, there must come a point where in order to stay in the auction, bidder 1 must increase its bid by more than V_1. At this point, in any SPE, bidder 1 would drop out. Anticipating 2's response, bidder 1 would already drop out in the previous round, and inducting backwards, bidder 2 must win this race before it starts.

Finally, if $[V_2]_\varepsilon = [V_1]_\varepsilon$, then there exist multiple equilibria. For instance, there are equilibria where i always bids the minimal increment to stay in (if it does not exceed i's value) and j always exits.

4 Finite Budgets

Now, we consider the impact of limits on bidders' budgets, since this is an obvious feature of applications and so it is important to know if this feature has an impact on equilibrium outcomes. The analysis that follows applies when either of the bidders' budgets is finite (and it could still be that the other bidder's budget is infinite).

The following lemma points out an important difference between the finite and the infinite-budget cases, which will be useful in the subsequent analysis..

LEMMA **2** *Consider an arbitrary subgame in a game (with or without jump-bidding) where at least one bidder has a finite budget and starting with a move by bidder i. All* *equilibrium continuations are in pure strategies (on and off the equilibrium path).*

Proof of Lemma 2: Given that at least one bidder has a finite budget, the subgame starting at any node is finite. Consider any equilibrium continuation starting from an arbitrary node. Consider any *last* node (on or off the equilibrium path) where some player mixes, so that in all further subgames, pure strategies are played. By Lemma 1, it must be that if this player, say i, bids, then the other player must drop out at the next node (as the other player must drop with some probability by the Lemma, and is playing a pure strategy by hypothesis). Given that V_i is not a multiple of ε this means that for i to be willing to make such a bid, i must have a strictly positive payoff from the bid. Thus, i cannot mix at the node. ▮

Note that this Lemma is not in contradiction with Lemma 1. In fact, coupled together, they imply that when at least one of the bidders has a finite budget, then in any subgame (where i is the bidder who moves first), either bidder i drops out immediately and bidder j wins at no incremental cost, or bidder i bids enough to stay in the auction and then bidder j drops out at the next node.

4.1 No Jump-Bidding and Budget Limits

Lemmas 1 and 2 immediately imply that, when jump-bidding is not allowed, the equilibria in the presence of budget limits differ from those in the infinite budget case. Indeed, the following proposition establishes that, without jumps, the bidder with the largest budget wins, regardless of the values.

PROPOSITION **3** *In the unique equilibrium in the game with no jump-bidding and at least one finite budget, bidder* 1 *wins at a price of* ε *if* $B_1 > B_2$*, and bidder* 2 *wins at a price of 0 otherwise (i.e., if* $B_2 \geq B_1$*).*

The intuition here is straightforward and by backwards induction. Once the auction has reached bidding within ε of the lower budget constraint, it is clear that the higher budget player has an incentive to stay in, and the other player to drop out. Then, by induction, the same is true within ε of such a node, and this argument rolls backwards to the beginning of the game.

4.2 Jump-bidding with Budget Limits

The analysis of equilibria when there is jump-bidding and a cap on at least one bidder's budget is the most complex of the various cases that we have analyzed. There is a subtle, but intuitive, interaction between valuations and budgets that determines which bidder wins. Let us first state the characterization of the winning bidder, and then provide discussion of the characterization and a more detailed description of the equilibrium structure.

PROPOSITION **4** *In all equilibria in the game with jump-bidding and at least one finite budget:*

(i) If $\min\{B_1, [V_1]_\varepsilon\} > \min\{B_2, [V_2]_\varepsilon\}$*, bidder 1 wins.*

(ii) If $\min\{B_1, [V_1]_\varepsilon\} \leq \min\{B_2, [V_2]_\varepsilon\}$*, bidder 2 wins,* except *if (iii) below applies.*

(iii) If $B_1 > B_2$*,* $[V_1]_\varepsilon \leq [V_2]_\varepsilon$ *and* $B_2 < k[V_1]_\varepsilon$*, where* k *is the minimal integer such that* $(k-1)[V_2]_\varepsilon \geq k[V_1]_\varepsilon$ *(setting* $k = \infty$ *if* $[V_2]_\varepsilon = [V_1]_\varepsilon$*), then bidder 1 wins.*

Since bidder i cannot bid above B_i and would not like to bid above V_i, natural intuition would suggest that the winner is determined by the minimum of budgets and values. Parts (i) and (ii) of the proposition confirm that usually this is indeed the case, but Part (iii) describes an exception. Under certain circumstances, the first mover advantage of bidder 1 is translated to a win even though $\min\{B_1, [V_1]_\varepsilon\} < \min\{B_2, [V_2]_\varepsilon\}$, as discussed below.

To understand these results, it is helpful to first discuss the impact of finite budgets, and then examine the role of the jump-bidding. Given finite budgets, the auctions are finite games whether or not jumps are allowed. In the case of no jumps, we easily solve the game backwards. At a node where one bidder's budget would need to be exceeded in order to continue, that bidder must drop out. Knowing this, when we get within one bid of exhausting the lower budget, the bidder with the higher budget will continue and the bidder with the lower budget will drop out. Inducting on this logic, the bidder with the lower budget will drop out at the first opportunity.

Why do things change with jump-bidding? Here, even if a bidder has the lower budget, if that bidder's value and budget both exceed the other bidder's value, then that bidder can preemptively bid an amount that causes the other to drop out. This ability operates asymmetrically between the two bidders because of the first mover advantage of bidder 1.

Notice that 1's first mover advantage is not limited to situations in which bidder 1 can profitably preempt bidder 2 already in the first move. The scenarios covered by Part (iii) include ones in which $[V_1]_\varepsilon < [V_2]_\varepsilon$ and both valuations are substantially smaller than $B_2 < B_1$. In such a scenario, even the maximal bid that does not exceed 1's valuation, $[V_1]_\varepsilon$, does not preempt bidder 2 who can respond with a bid that exceeds $[V_1]_\varepsilon$ and is still below $[V_2]_\varepsilon$. To understand how bidder 1 might still preempt 2 in this situation, imagine that the bidders race each other by submitting in each round the maximal bids $[V_1]_\varepsilon$ and $[V_2]_\varepsilon$ that do not exceed their respective valuations. If there are no budget constraints, then since $[V_1]_\varepsilon < [V_2]_\varepsilon$ eventually they will reach a point where in order to surpass 2's bid, bidder 1 will have to bid more than $[V_1]_\varepsilon$. This will happen in round k described in Part (iii) of the proposition. In any equilibrium in this subgame, bidder 1 drops out. Thus, if this round k is reached before B_2 is depleted, then inducting backwards from this subgame, bidder 1 would already drop out in round $k-1$ of this race and hence in round $k-2$ and so on to the beginning. Alternatively, if B_2 would be depleted along this race before round k is reached, then bidder 1 would win this race once B_2 is depleted and inducting backwards, bidder 2 would drop out after 1 takes the first step in the race.

The following example illustrates the above points.

Example **1** *An Example where the low value wins.*

Let $V_1 = 10.5$, $V_2 = 13.5$, $B_2 = 15 < B_1 = 20$ and $\varepsilon = 1$.

If bidder 2 moved first, then it is clear that bidder 2 would win, as the bidder could preemptively jump-bid to 11, after which bidder 1 would drop out.

However, consider what happens when bidder 1 moves first. Suppose that bidder 1 bids 6 in the first round. Now consider any bid $b \geq 6$ by bidder 2, such that 2 stays in the auction. It must be that $b \leq [V_2]_\varepsilon = 13$. If 1 then responded with a bid of 16, then 2's budget will be exceeded and so 1 will win. This gives 1 a strictly positive incremental payoff (as 6 of the bid is already sunk) and so 1 would be willing to make such a bid, and can thus win the auction outright. While this is not the actual equilibrium continuation, it does show that 1 can win and thus must stay in the auction. It is then easy to argue that 1 will win in any continuation. Inducting, it makes no sense for 2 to make any bid $b \geq 6$ and so 2 drops out immediately.

The example points out another feature of the equilibrium behavior. Bidder 1 will actually bid 3 in the first period in all equilibria. If 1 bids 2 or less, then by jumping to 13, bidder 2 would surely win in the continuation. If 1 bids at least 3, then the maximal bid that 2 is willing to make still makes it worthwhile for 1 to stay in the auction at the next round, and then 1 is within striking distance of 2's budget. In the scenario in which jump-bidding is not allowed, the winner never pays more than ε. But, as the example shows, when jump-bids are allowed, the winner may have to bid more than the minimum. In the cases described in Part (ii) of the proposition, it is still the case that bidder 2 wins at 0 cost. But in the cases described in Parts (i) and (iii), bidder 1 might have to bid a significant amount. Of course, in all of these cases the loser does not bid. For example, in a scenario with $[V_1]_\varepsilon < [V_2]_\varepsilon$ that fits into Part (iii), if it is the case that B_2 is close to $k[V_1]_\varepsilon$, then in order to win bidder 1 would have to start with a bid that is close to $[V_1]_\varepsilon$. The observation that bidder 1 might submit a substantial bid in the first round might seem somewhat surprising given that there is complete information and a certain winner in all equilibria.

5 Discussion

Our analysis has focussed on alternating moves auctions and under complete information. We close with a brief discussion of alternative formulations.

5.1 Incomplete Information

As wars-of-attrition with incomplete information are well-studied in the context of fixed rates of bid increases, we examine the impact of jump-bidding. For simplicity, we examine a case with infinite budgets.

Let F_i denote the distribution of V_i, and suppose that the bidder's values are independently distributed. We assume that F_i is atomless and has connected support, and that the distributions have the same support.

For the purposes of this section, we let bidders bid any amount in $\Re_+$, rather than on a discrete grid. Play still alternates, and player 2 needs only to match player 1's bid to stay in the auction, while player 1 needs to exceed player 2's current bid. To keep the game well-defined, we presume that player 1 needs to bid at least $b_2 + \varepsilon$ after the first period in order to stay in the auction.

PROPOSITION **5** *There exists a sequential equilibrium, with the following behavior on the equilibrium path. In the first period, bidder* 1 *bids* $b(V_1)$ *where*

$$b(z) = zF_2(z) - \int_0^z F_2(t)dt.$$

Bidder 2 responds to b_1 by either dropping out if $V_2 < b^{-1}(b_1)$, and by bidding $b^{-1}(b_1)$ if $V_2 \geq b^{-1}(b_1)$, in which case 1 then drops out.

Since in this equilibrium the highest valuation bidder always wins, by revenue equivalence the total expected payments made by the bidders equal the expectation of the lower value among the two. The equilibrium has an interesting interpretation. Bidder 1 bids as if it is a sealed-bid, all-pay auction, where 1 will win if and only if 1 has the higher value. Bidder 2 then responds by dropping out if 2 has a value lower than bidder 1's revealed value, and bidding bidder 1's revealed value otherwise (in which case 1 subsequently drops out). Given that the actions are consistent with the high value winning, the equilibrium holds together. The efficient outcome means that the expected revenue had better be the same as that of a second-price (winner pays) auction, which it is.

There is no reason to suppose that this equilibrium is unique. In the context of a closely related model (Dekel, Jackson and Wolinsky (2006)), we showed that, when there is sufficiently "little" incomplete information, there is a perfect Bayesian equilibrium (PBE) outcome that is close to the complete-information outcome. The sense in which the incomplete information is small is that most of the mass of the distribution F_i is concentrated on the value $\tilde{V}_i$ where $\tilde{V}_2 > \tilde{V}_1 + \varepsilon$, and the sense in which equilibrium outcome is close to the complete information outcome is that with large probability bidder 2 wins at zero cost. The model there is different in some small details, so we cannot just state this result here without reproving it. However, the model here is sufficiently close to the model there to make it a bit superfluous to repeat the result here. So we only mention it informally as a result that is likely to hold here as well.

5.2 Simultaneous Moves

Suppose that the players move simultaneously in each period rather than sequentially. The results for the environment with no jump-bidding are essentially independent of whether the moves are simultaneous or sequential. For the environment with jump-bidding the analysis becomes much harder with simultaneous moves and it is not clear whether all the results survive, in particular whether the uniqueness results still hold. For example, consider the case in which budgets are infinite. We can show the following.

PROPOSITION **6** *In the simultaneous game with jump-bidding and infinite budgets:*[6]

[6] To keep the game as similar as possible to the previous analysis, we let bidder 2 win ties. The game

(i) *If* $V_2 + \varepsilon \leq V_1$*, then there exists an equilibrium where bidder* 1 *bids once to a price of* ε *and then bidder 2 drops out.*

(ii) *If* $V_1 + \varepsilon \leq V_2$*, then there exists an equilibrium where bidder 1 drops out immediately and bidder 2 wins at a price of 0.*

So, the equilibria of Proposition 2 extend to the simultaneous move game, but we have not been able to rule out the possibility that there are also other mixed-strategy equilibria. One of the key difficulties is that the proof of Lemma 1 relies on the sequentiality of the moves and hence the lemma cannot be invoked.

5.3 Many Bidders.

In the complete information analysis, we did not use the two-bidders assumption in an important way and all the results extend in a straightforward manner to the case of multiple bidders. Due to the complete information, the relevant competition is between the two strongest bidders and the presence of the others does not matter. In the incomplete information scenario, the presence of more bidders might affect the result as we know from other auction contexts.

6 References

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ends at the first node t where a bidder who would lose at the current standing bids (from $t-1$), does not make a bid that would beat the other bidder's bid from $t-1$. This is equivalent (in terms of equilibrium behavior) to having the auction end the first time that a bidder who is losing at a given node does not change his or her bid.

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7 Appendix

Proof of Proposition 1: Consider an equilibrium path along which play continues beyond the first two periods (i.e., beyond the first bid of each of the players) with positive probability. Let t be a period that is reached with positive probability along this path. Let π_i^t denote the probability that bidder i drops out in period t (in which i is called to move) along this path. Observe the following.

(i) If $t > 1$, then $\pi_i^t > 0$. This follows from Lemma **1** since otherwise t would not have been reached.

(ii) If i bids at $t > 2$, then $\pi_j^{t-1} \geq \varepsilon/V_i$. This follows from i's 0 incremental payoff from t on (implied by (i) above) and the fact that $i's$ expected benefit of not dropping out in $t-2$ is $\pi_j^{t-1}V_i - \varepsilon$.

(iii) If $\pi_j^{t-1} > \varepsilon/V_i$, then $\pi_i^{t-2} = 0$.

(iv) If $t > 3$, then $\pi_j^{t-1} = \varepsilon/V_i$. From observations (ii) and (iii) above, the only alternative is $\pi_j^{t-1} > \varepsilon/V_i$ that implies $\pi_i^{t-2} = 0$ that together with Lemma **1** implies $\pi_j^{t-3} = 1$ in contradiction to the assumption that the play on the path continues beyond $t-3$.

(v) If $t > 2$, then $\pi_i^t < 1$, since $\pi_i^t = 1$ would imply $\pi_j^{t-1} = 0$ in contradiction to (ii) above.

Thus, if play continues beyond the second period, then (v) implies that the mixing goes on forever and (iv) implies $\pi_2^2 \geq \varepsilon/V_i$ and for $t \geq 3$, $\pi_j^t = \varepsilon/V_i$. This characterizes all such equilibria and it immediate to verify that they are indeed equilibria.

It is also easy to verify that there also exists an equilibrium in which play stops in the first period with bidder 1 dropping out, i.e., $\pi_1^1 = 1$, and an equilibrium in which play stops in the second period, i.e., $\pi_1^1 = 0$, $\pi_2^2 = 1$. ▮

Proof of Proposition 2: Suppose that $V_i \geq V_j + \varepsilon$. Let b_i and b_j denote the latest bids by i and j.

We prove by induction that, for any $k \geq 0$, if $b_i \geq b_j + k\varepsilon$ then bidder j quits in any SPE in a subgame starting with j's move.

Initial step: The above holds for k such that $k\varepsilon = [V_j]_\varepsilon + (j-1)\varepsilon$. Obviously, if $b_i \geq b_j + [V_j]_\varepsilon + (j-1)\varepsilon$, bidder j quits in any SPE in a subgame starting with j's move. Notice that $(j-1)\varepsilon$ reflects the asymmetry between 1 and 2, whereby to win 1 has to bid strictly more than 2 while 2 has just to match.

Induction step: Suppose that the result holds for some $\dot{k} > 0$, then it also holds for $k-1$; that is, if $b_i \geq b_j + (k-1)\,\varepsilon$ then j quits in any SPE in a subgame starting with its move.

Suppose to the contrary that $b_i \geq b_j + (k-1)\varepsilon$, and j bids with positive probability when called upon to move. Let b'_j be a bid in the support of bids by j. Clearly $b'_j \leq b_j + [V_j]_\varepsilon$ (as otherwise it is better for j to quit). If i were to respond with $b'_i = b_i + [V_i]_\varepsilon$, then

$$b'_i \geq b_j + (k-1)\varepsilon + [V_i]_\varepsilon.$$

Since $[V_i]_\varepsilon \geq [V_j]_\varepsilon + \varepsilon$, it follows that

$$b'_i \geq b_j + k\varepsilon + [V_j]_\varepsilon \geq b'_j + k\varepsilon.$$

By the inductive hypothesis, j drops out and i wins. Thus, i has a strategy which in response to b'_j wins for sure at the next node controlled by i and gives i positive incremental payoff. However, by Lemma 1, i must quit with positive probability in any SPE in the subgame that starts after b'_j, which means that i's incremental payoff from continuing is not positive, yielding a contradiction.

From the above induction, we can conclude that, if $V_i \geq V_j + \varepsilon$ and $b_i \geq b_j$, then j drops out with certainty in any SPE starting with its move.

The proposition follows from noting that this implies that, if $V_1 \geq V_2 + \varepsilon$, then by bidding ε in the first move bidder 1 induces 2 to drop out and wins; and if $V_2 \geq V_1 + \varepsilon$, then by matching bidder 1's first bid (which must clearly not exceed V_1 and hence be less than V_2), bidder 2 induces bidder 1 to drop out at the next node. Therefore, bidder 1 drops out immediately and bidder 2 wins at a price of 0. ▮

Proof of Proposition 3:

We offer the proof for the case where $B_1 \geq B_2 + \varepsilon$. The other case is analogous (noting that 2 needs only match 1's bid to keep the auction going, while 1 needs to exceed 2's bid).

Step 1. Consider a subgame where 1's outstanding bid $b_1 \geq B_2 - ([V_1]_\varepsilon - \varepsilon)$. In the unique equilibrium continuation: at any node where 1 moves, 1 remains in the auction, while at any node where 2 moves 2 quits.

1 is guaranteed a positive payoff by continuing to the point where $b_1' \geq B_2 + \varepsilon$. Thus, 1 will stay in at any node in the subgame. The result then follows from Lemma 2 and 1

Step 2. By induction: Suppose that in any subgame where $b_1 \geq B_2 - k([V_1]_\varepsilon - \varepsilon)$, at any node where 1 moves 1 remains in the auction, while at any node where 2 moves 2 quits. Then in the unique equilibrium continuation starting from any subgame where $b_1 \geq B_2 - (k+1)([V_1]_\varepsilon - \varepsilon)$, at any node where 1 moves 1 remains in the auction, while at any node where 2 moves 2 quits.

It follows from the inductive hypothesis that if i remains in the game until $b_1' = B_2 - k([V_1]_\varepsilon - \varepsilon)$ then 1 wins, and 2 quits. The incremental cost to i is at most $[V_1]_\varepsilon$ and so i is guaranteed a positive payoff by staying in, and again the result follows from Lemma 2 and 1. ▮

Proof of Proposition 4:

The following lemma is useful.

LEMMA **3** *Suppose that $B_2 < B_1$. Consider a subgame where outstanding bids are b_1 and b_2, the auction has not ended, and it is 1's turn to move. Let $k(b_1, b_2)$ be the minimal integer k such that $b_1 + k[V_1]_\varepsilon \leq b_2 + (k-1)[V_2]_\varepsilon$ (set it equal to ∞ if no such k exists). In any subgame perfect equilibrium if $b_1 + k(b_1, b_2)[V_1]_\varepsilon > B_2$, then 1 wins in one step (and 2 drops out immediately); if $b_1 + k(b_1, b_2)[V_1]_\varepsilon \leq B_2$, then 1 drops out immediately and 2 wins.*

Proof of Lemma 3: Note that we are in a case where $[V_1]_\varepsilon \leq [V_2]_\varepsilon$ and $b_1 \leq b_2$ (or the auction would have ended). Let us proceed by induction on the difference between B_2 and b_1. We first show that it is true if $B_2 - b_1 \leq [V_1]_\varepsilon$. In this case we may have $b_1 + k(b_1, b_2)[V_1]_\varepsilon \leq B_2$, only if $k(b_1, b_2) = 1$ which implies $b_1 + [V_1]_\varepsilon \leq b_2$ and hence 1 loses since it can at most match b_2 without exceeding its own value. If $b_1 + k(b_1, b_2)[V_1]_\varepsilon > B_2$, then $k(b_1, b_2) > 1$. From the minimality of $k(b_1, b_2)$ it follows that $b_1 + [V_1]_\varepsilon > b_2$ and, since by assumption $b_1 + [V_1]_\varepsilon \geq B_2$, it follows that by bidding $b_1 + [V_1]_\varepsilon$, 1 is sure to win. (This follows since if $b_1 + [V_1]_\varepsilon > B_2$ then 2 would not be able to match, while if $b_1 + [V_1]_\varepsilon = B_2$ then 2 would be able to just match and then lose in the next round.)

Furthermore, since 2 would drop out immediately following 1's bid of $b_1 + [V_1]_\varepsilon$, this bid guarantees a positive incremental payoff to 1. Therefore, in any SPE 1 is increasing its bid at this subgame so by Lemmas 1 and 2 bidder 2 drops out immediately following 1's bid. This establishes the lemma for $B_2 - b_1 \leq [V_1]_\varepsilon$.

Now, suppose that the lemma holds whenever $B_2 - b_1 \leq T\varepsilon$ and let us show that it is true when $B_2 - b_1 \leq (T+1)\varepsilon$. Suppose that $b_1 + k(b_1, b_2)[V_1]_\varepsilon > B_2$ and that 1 bids $b_1' = b_1 + [V_1]_\varepsilon$. 2's best response in the subgame must be either to drop out in which case 1 wins, or to bid $b_2' \leq b_2 + [V_2]_\varepsilon$. In that case, we are in a situation where $k(b_1', b_2') \geq k(b_1, b_2) - 1$ and hence $b_1' + k(b_1', b_2')[V_1]_\varepsilon \geq b_1 + k(b_1, b_2)[V_1]_\varepsilon > B_2$. Since $B_2 - b_1' \leq T\varepsilon$ it follows from the inductive assumption that 1 wins. Therefore, in any SPE following b_1', bidder 2 drops immediately. Therefore, b_1' brings positive incremental payoff to bidder 1. Therefore, in any SPE in the subgame starting with b_1 and b_2 bidder 1 increases its bid and, by Lemmas 1 and 2, bidder 2 drops out immediately. Suppose next that $b_1 + k(b_1, b_2)[V_1]_\varepsilon \leq B_2$. To any bid b_1' by 1 such that $b_1 < b_1' \leq b_1 + [V_1]_\varepsilon$, bidder 2 can respond with $b_2' = b_2 + [V_2]_\varepsilon$ so that $k(b_1', b_2') \leq k(b_1, b_2) - 1$. Therefore, $b_1' + k(b_1', b_2')[V_1]_\varepsilon \leq b_1 + k(b_1, b_2)[V_1]_\varepsilon \leq B_2$ and it follows from the inductive assumption that following b_2', bidder 1 drops immediately. Therefore, b_2' gives bidder 2 positive incremental payoff which means that in any SPE bidder 2 increases its bid at this point and by Lemmas 1 and 2 bidder 1 drops immediately. ∎

Let us now complete the proof of Proposition 4.

If $\min\{B_1, [V_1]_\varepsilon\} > \min\{B_2, [V_2]_\varepsilon\}$, then by bidding $\min\{B_1, [V_1]_\varepsilon\}$ bidder 1 would win for sure (as 2 could either not match the bid, or would not wish to) and so 1 must have a positive payoff at the first node (noting that $V_1 > [V_1]_\varepsilon$). Thus, 1 must bid in the first node with probability 1. Thus, 1 must win by Lemmas 1 and 2.

If $\min\{B_1, [V_1]_\varepsilon\} \leq \min\{B_2, [V_2]_\varepsilon\}$, and $B_1 \leq \min\{B_2, [V_2]_\varepsilon\}$, then 2 can win for sure at 2's first move and get a positive payoff just by bidding B_1 when called on to bid. The result again follows from Lemmas 1 and 2.

So, consider a case where $[V_1]_\varepsilon \leq \min\{B_2, [V_2]_\varepsilon\} < B_1$. The result then follows from Lemma 3. ∎

Proof of Proposition 5: Let us first completely specify the behavior off the equilibrium path.

If 1 bids according to $b(z)$ and 2 responds with a bid of more than $b^{-1}(b_1)$ at 2's first move, then 1 believes that $V_2 > V_1$ and drops out.

If 2 responds with a bid in the interval $[b_1, b^{-1}(b_1))$, then 1 believes that $V_2 < V_1$ and 1 then continues by bidding the minimal amount to stay in the auction at all subsequent nodes, subject to the incremental bid not exceeding V_1.

If 2 responds with a bid of at least $b^{-1}(b_1)$, and 1 does not drop out, then 2 believes that $V_1 = b^{-1}(b_1)$, and continues to match all of 1's subsequent bids, provided 2's incremental bid does not have to exceed V_2.

First, note that the function $b(z)$ has the property that

$$V_1 = \arg\max_z [V_1 F_2(z) - b(z)]$$

Next, given 1's behavior, note that 2's response is optimal, since if 2 bids in $[b(z), z)$, the only way to win subsequently, would be to jump by more than V_1, which 2 believes to be equal to z.

Consider now 1. Suppose it bids $b(z)$. Following this bid 2 believes that $V_1 = z$ and responds accordingly. If 2 responds with bid z and 1 then jumps again, 2 would continue like in the complete information and the only way 1 can win is by a jump that exceeds V_2. Thus, 1's expected payoff from bidding $b(z)$ is

$$V_1 F_2(z) - b(z) + [1 - F_2(z)]\,[\text{1's value of continuation after 2's bid of } z]$$

Now,

$$[\text{1's value of continuation after 2's bid of } z] \begin{cases} = & 0 & \text{if} \quad z \geq V_1 \\ \leq & \max\{0, \frac{F_2(V_1)-F_2(z)}{1-F_2(z)} V_1 - \int_z^{V_1} t \frac{dF_2(t)}{1-F_2(t)}\} & \text{if} \quad z < V_1 \end{cases}$$

The explanation is as follows. If 2 bid $z \geq V_1$, then $V_2 \geq z$ and 1 will have to jump by more than $z \geq V_1$ to win, so it optimal for 1 to quit and the value of continuation is 0. If $z < V_1$ and 2 responded with z, 1 will have at some point to jump by more than V_2 in order to win. This will happen only if $V_2 \leq V_1$, which occurs with the conditional probability $\frac{F_2(V_1)-F_2(z)}{1-F_2(z)}$ and the least 1 would pay would be V_2 whose (conditional) expectation is $\int_z^{V_1} t \frac{dF_2(t)}{1-F_2(t)}$.

It follows that

$$[\text{1's expected payoff from bidding } b(z)]$$
$$\begin{cases} = & V_1 F_2(z) - b(z) & \text{if} \quad z \geq V_1 \\ \leq & V_1 F_2(z) - b(z) + [1 - F_2(z)] \max\{0, \frac{F_2(V_1)-F_2(z)}{1-F_2(z)} V_1 - \int_z^{V_1} t \frac{dF_2(t)}{1-F_2(t)}\} & \text{if} \quad z < V_1 \end{cases}$$

Now, since

$$V_1 F_2(z) - b(z) + [1 - F_2(z)][\frac{F_2(V_1) - F_2(z)}{1 - F_2(z)} V_1 - \int_z^{V_1} t \frac{dF_2(t)}{1 - F_2(t)}] = V_1 F_2(V_1) - b(V_1)$$

and since, by construction,

$$V_1 = \arg\max_z [V_1 F_2(z) - b(z)]$$

it follows that 1's expected payoff from bidding $b(z)$ is maximized at $b(V_1)$. ∎

Proof of Proposition 6: We examine the case where $V_1 \geq V_2 + \varepsilon$. The other case is the similar, except that 2 does not need to bid with tied bids. We simply describe the bidding behavior (at an arbitrary node), which is easily checked to be a best response at any node. Let the current standing bids be b_1 and b_2.

- If $b_1 - b_2 > 0$, then 1 bids 0 and 2 bids 0 and 1 wins.
- If $b_1 - b_2 = 0$, then 1 bids 1 and 2 bids 0, and then at the next stage 1 wins.
- If $b_1 - b_2 = -k\varepsilon$, where $(k+1)\varepsilon < V_1$, then 1 bids $(k+1)\varepsilon$ and 2 bids 0, and then at the next stage 1 wins.
- Otherwise, 1 bids 0 and 2 bids 0, and 2 wins. ∎