GAMES IN COALITIONAL FORM

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Abstract. How should a coalition of cooperating players allocate payoffs to its members? This question arises in a broad range of situations and evokes an equally broad range of issues. For example, it raises technical issues in accounting, if the players are divisions of a corporation, but involves issues of social justice when the context is how people behave in society.

Despite the breadth of possible applications, coalitional game theory offers a unified framework and solutions for addressing such questions. This brief survey presents some of its major models and proposed solutions.

1. Introduction

In their seminal book, von Neumann and Morgenstern (1944) introduced two theories of games: strategic and coalitional. Strategic game theory concentrates on the selection of strategies by payoff-maximizing players. Coalitional game theory concentrates on coalition formation and the distribution of payoffs.

The next two examples illustrate situations in the domain of the coalitional approach.

1.1. Games with no strategic structure.

Example 1. ***Cost allocation of a shared facility***. *Three municipalities, E, W, and S, need to construct water purification facilities. Costs of individual and joint facilities are described by the cost function c:* $c(E) = 20$, $c(W) = 30$, *and* $c(S) = 50$; $c(E, W) = 40$, $c(E, S) = 60$, *and* $c(W, S) = 80$; $c(E, W, S) = 80$. *For example, a facility that serves the needs of W and S would cost $80 million.*

The optimal solution is to build, at the cost of 80, one facility that serves all three municipalities. How should its cost be allocated?

1.2. Games with many Nash equilibria.

Example 2. ***Repeated sales***. *A seller and a buyer play the following stage game on a daily basis. The seller decides on the quality level, H, M, or L, of the item sold (at a fixed price); without knowledge of the seller's selected quality, the buyer decides whether or not to buy. If she does not buy, the payoffs of both are zero; if she buys, the corresponding payoffs are (0,3), (3,2) or (4,0), depending on whether the quality is H, M, or L.*

Under perfect monitoring of past choices and low discounting of future payoffs, the folk theorem of repeated games states that any pair of numbers in the convex hull of (0,0),(0,3), (3,2), and (4,0) are Nash-equilibrium average payoffs. What equilibrium and what average payoffs should they select?

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We proceed with a short survey of the major models and selected solution concepts. More elaborate overviews are available in the entry Game Theory by Aumann (2008) in this dictionary, Myerson (1991), and other surveys mentioned below.

2. Types of coalitional games

In what follows, N is a fixed set of n *players*; the set of *coalitions* $\mathcal{C}$ consists of the nonempty subsets of N; $|S|$ denotes the number of players in a coalition S. The terms "profile" and "S-profile" denote vectors of items (payoffs, costs, commodities, etc.) indexed by the names of the players.

For every coalition S, R^S denotes the $|S|$-dimensional Euclidean space indexed by the names of the players; for single-player coalitions the symbol i replaces $\{i\}$. A profile $u^S \in R^S$ denotes *payoffs* u_i^S of the players $i \in S$.

Definition 1. *A* game *(also known as a game with no transferable utility, or NTU game) is a function V that assigns every coalition S a set $V(S) \subset R^S$.*

Remark 1. *The initial models of coalitional games were presented in von Neumann and Morgenstern (1944) for the special case of TU games described below, Nash (1950) for the special case of two-person games, and Aumann and Peleg (1960) for the general case.*

The interpretation is that $V(S)$ describes all the *feasible payoff profiles* that the coalition S can generate for its members. Under the assumption that the *grand coalition* N is formed, the central question is which payoff profile $u^N \in V(N)$ to select. Two major considerations come into play: the relative strength of different coalitions, and the relative strength of players within coalitions.

To separate these two issues, game theorists study the two simpler types of games defined below: TU games and bargaining games. In TU games the players in every coalition are symmetric, so only the relative strength of coalitions matters. In bargaining games only one coalition is active, so only the relative strength of players' within that coalition matters. Historically, solutions of games have been developed first for these simpler classes of games, and only then extended to general (NTU) games. For this reason, the literature on these simpler classes is substantially richer then the general theory of (NTU) games.

Definition 2. *V is a* transferable-utility game *(TU game) if for a real-valued function $v = (v(S))_{S\in\mathcal{C}}$, $V(S) = \{u^S \in R^S : \sum_i u_i^S \leq v(S)\}$.*

It is customary to identify a TU game by the function v instead of V.

TU games describe many interactive environments. Consider, for example, any environment with individual outcomes consisting of prizes p and monetary payoffs m, and individual utilities that are additive and separable in money ($u_i(p,m) = v_i(p) + m$). Under the assumption that the players have enough funds to make transfers, the TU formulation presents an accurate description of the situation.

Definition 3. *A* Nash *(1950)* bargaining game *is a two-person game. An n-person* bargaining game *is a game V in which $V(S) = \times_{i\in S} V(i)$ for every coalition $S \subsetneqq N$.*

Remark 2. Partition games *(Lucas and Thrall [1963]) use a more sophisticated function V to describe coalitional payoffs. For every partition of the set of players $\pi = (T_1, T_2, ..., T_m)$, $V_\pi(T_j)$ is the set of T_j's feasible payoff profiles, under the cooperation structure described by π. Thus, what is feasible for a coalition may* *depend on the strategic alignment of the opponents. The literature on partition games is not highly developed.*

3. Some Special Families of Games

Coalitional game theory is useful for analyzing special types of interactive environments. And conversely, such special environments serve as a laboratory to test the usefulness of game theoretic solutions. The following are a few examples.

3.1. **Profit sharing and cost allocation.** Consider a partnership that needs to distribute its total profits, $v(N)$, to its n individual partners. A profit-distribution formula should consider the potential profits $v(S)$ that coalitions of partners S can generate on their own. A TU game is a natural description of the situation.

A cost allocation problem, like Example 1, can be turned into a natural TU game by defining the worth of a coalition to be the savings obtained by joining forces: $v(S) = \sum_{i \in S} c(i) - c(S)$.

Examples of papers on cost allocation are Shubik (1962) and Billera, Heath, and Raanan (1978). See Young (1994) for an extensive survey.

3.2. **Markets and auctions.** Restricting this discussion to simple exchange, consider an environment with n traders and m commodities. Each trader i starts with an initial *bundle* ω_i^0, an m-dimensional vector that describes the quantities of each commodity he owns. The utility of player i for a bundle ω_i is described by $u_i(\omega_i)$. An S-profile of bundles $\omega = (\omega_i)_{i \in S}$ is feasible for the coalition S if $\sum_{i \in S} \omega_i = \sum_{i \in S} \omega_i^0$.

Definition 4. *A game V is a* market game, *if for such an exchange environment (with assumed free-disposal of utility),*

$$V(S) = \{u^S \in R^S : \text{for some } S\text{-feasible profile of bundles } \omega,\ u_i^S \leq u_i(\omega_i) \text{ for every } i \in S\}.$$

Under the assumptions discussed earlier (additively separable utility and sufficient funds) the market game has the more compact TU description: $v(S) = max_\omega \sum_{i \in S} u_i(\omega_i)$, with the max taken over all S-feasible profiles ω.

As discussed below, market games play a central role in several areas of game theory.

Definition 5. *An* auction game *is a market game with a seller whose initial bundle consists of items to be sold, and bidders whose initial bundles consist of money.*

3.3. **Matching games.** Many theoretical and empirical studies are devoted to the subject of efficient and stable matching: husbands with wives, sellers with buyers, students with schools, donors with receivers, and more; see the Matching entry by Niederle, Roth, and Sonmez (2008) in this dictionary. The first of these was introduced by Gale and Shapley in their pioneering study (1962) using the following example.

Consider a matching environment with q males and q females. Payoff functions $u_m(f)$ and $u_m(none)$ describe the utilities of male m paired with female f or with no one; $u_f(m)$ and $u_f(none)$ describe the corresponding utilities of the females. A pairing P_S of a coalition S is a specification of male-female pairs from S, with the remaining S members being unpaired.

Definition 6. *A game* V *is a* marriage game *if for such an environment,* $V(S) = \{u^S \in R^S : \textit{for some pairing } P_S,\ u_i^S \leq u_i(P_S) \textit{ for every } i \in S\}$.

Solutions of marriage games that are efficient and stable (i.e., no divorce) can be computed by Gale-Shapley algorithms.

3.4. **Optimization games.** Optimization problems from operations research have natural extensions to multiperson coalitional games, as the following examples illustrate.

3.4.1. *Spanning tree games.* A cost-allocation TU spanning-tree game (Bird [1976]) is described by an undirected connected graph, with one node designated as the center C and every other node corresponding to a player. Every arc has an associated nonnegative connectivity cost. The cost of a coalition S, $c(S)$, is defined to be the minimum sum of all the arc costs, taken over all subgraphs that connect all the members of S to C.

3.4.2. *Flow games.* A TU flow game (Kalai and Zemel [1982b]) is described by a directed graph, with two nodes, s and t, designated as the *source* and the *sink*, respectively. Every arc has an associated capacity and is owned by one of the n players. For every coalition S, $v(S)$ is the maximal s-to-t flow that the coalition S can generate through the arcs owned by its members.

3.4.3. *Linear programming games.* Finding minimal-cost spanning trees and maximum flow can be described as special types of linear programs. Linear (and nonlinear) programming problems have been generalized to multiperson games (see Owen [1975], Kalai and Zemel [1982a], and Dubey and Shapley [1984]). The following is a simple example.

Fix a $p \times q$ matrix A and a q-dimensional vector w, to consider standard linear programs of the form max wx s.t. $Ax \leq b$. Endow each player i with a p-dimensional vector b_i, and define the linear-programming TU game v by $v(S) = max_x\ wx$ s.t. $Ax \leq \sum_{i \in S} b_i$.

3.5. **Simple games and voting games.** A TU game is simple if for every coalition S, $v(S)$ is either zero or one. Simple games are useful for describing the power of coalitions in political applications. For example, if every player is a party in a certain parliament, then $v(S) = 1$ means that under the parliamentary rules the parties in the coalition S have the ability to pass legislation (or *win*) regardless of the positions of the parties not in S; $v(S) = 0$ (or S *loses*) otherwise.

In applications like the one above, just formulating the game may already offer useful insights into the power structure. For example, consider a parliament that requires 50 votes in order to pass legislation, with three parties that have 12 votes, 38 votes, and 49 votes, respectively. Even though the third party seems strongest, a simple formulation of the game yields the symmetric *three-person majority game*: any coalition with two or more parties wins; single-party coalitions lose.

Beyond the initial stage of formulation, standard solutions of game theory offer useful insights into the power structure of such institutions and other political structures; see, for example, Shapley and Shubik (1954), Riker and Shapley (1968), and Brams et al. (1983).

4. Solution Concepts

When cooperation is beneficial, which coalitions will form and how would coalitions allocate payoffs to their members? Given the breadth of situations for which this question is relevant, game theory offers several different solutions that are motivated by different criteria. In this brief survey, we concentrate on the Core and on the Shapley value.

Under the assumptions that utility functions can be rescaled, that lotteries over outcomes can be performed, and that utility can be freely disposed of, we restrict the discussion to games V with the following properties.

Every $V(S)$ is a compact convex subset of the nonnegative orthant R^S_+, and it satisfies the following property: if $w^S \in R^S_+$ with $w^S \leq u^S$ for some $u^S \in V(S)$, then $w^S \in V(S)$. And for single player coalitions, assume $V(i) = \{0\}$. For TU games this means that every $v(S) \geq 0$, the corresponding $V(S) = \{u^S \in R^S_+ : \sum_{i \in S} u^S_i \leq v(S)\}$, and for each i, $v(i) = 0$.

In addition, we assume that the games are *superadditive*: for any pair of disjoint coalitions T and S, $V(T \cup S) \supseteq V(T) \times V(S)$; for TU games this translates to $v(T \cup S) \geq v(T) + v(S)$. Under superadditivity, the maximal possible payoffs are generated by the grand coalition N. Thus, the discussion turns to how the payoffs of the grand coalition should be allocated, ignoring the question of which coalitions would form.

A payoff profile $u \in R^N$ is *feasible for a coalition* S, if $u_S \in V(S)$, where u_S is the projection of u to R^S. The translation to TU games is that $u(S) \equiv \sum_{i \in S} u_i \leq v(S)$. A profile $u \in R^N$ can be *improved upon* by the coalition S if there is an S-feasible profile w with $w_i > u_i$ for all $i \in S$.

Definition 7. *An* imputation *of a game is a grand-coalition-feasible payoff profile that is both* individually rational *(i.e., no individual player can improve upon it) and* Pareto optimal *(i.e., the grand coalition cannot improve upon it).*

Given the uncontroversial nature of individual rationality and Pareto optimality, solutions of a game are restricted to the selection of imputations.

4.1. **The core.**

Definition 8. *The* core *of a game (see Shapley [1952] and Gillies [1953] for TU, and Aumann [1961] for NTU) is the set of imputations that cannot be improved upon by any coalition.*

The core turns out to be a compact set of imputations that may be empty. In the case of TU games it is a convex set, but in general games (NTU) it may even be a disconnected set. The core induces stable cooperation in the grand coalition because no subcoalition of players can reach a consensus to break away when a payoff profile is in the core.

Remark 3. *More refined notions of stability give rise to alternative solution concepts, such as the* stable sets *of von Neumann and Morgenstern (1944), and the* kernel *and* bargaining sets *of Davis and Maschler (1965). The* nucleolus *of Schmeidler (1969), with its NTU extension in Kalai (1975), offers a "refinement" of the core. It consists of a finite number of points (exactly one for TU games) and belongs to the core when the core is not empty. For more on these solutions, see Maschler (1992) and the entry Game Theory by Aumann (2008) in this dictionary.*

Unfortunately, games with an empty core are not unusual. Even the simple three-person majority game described in 3.5 has an empty core (since among any three numbers that sum to one there must be a pair that sums to less than one, there are always two players who can improve their payoffs).

4.1.1. *TU games with nonempty cores.* Given the coalitional stability obtained under payoff profiles in the core, it is desirable to know in which games the core is nonempty.

Bondareva (1963) and Shapley (1967) consider "part-time coalitions" that meet the availability constraints of their members. In this sense, a collection of nonnegative coalitional weights $\lambda = (\lambda_S)_{S \in \mathcal{C}}$ is *balanced*, if for every player i, $\sum_{S:i \in S} \lambda_S = 1$. They show that a game has a nonempty core if and only if the game is *balanced*: for every balanced collection λ, $\sum_S \lambda_S v(S) \leq v(N)$.

As Scarf (1967) demonstrates, all market games have nonempty cores and even the stronger property of having nonempty subcores: For every coalition S, consider the subgame v^S which is restricted to the players of S and their subcoalitions. The game v *has nonempty subcores*, if all its subgames v^S have nonempty cores.

By applying the balancedness condition repeatedly, one concludes that a game has nonempty subcores if and only if the balancedness condition holds for all its subgames v^S. Games with this property are called *totally balanced*.

Since Shapley and Shubik (1969a) demonstrate the converse of Scarf's result, a game is thus totally balanced if and only if it is a market game. Interestingly, the following description offers yet a different characterization of this family of games.

A game w is *additive* if there is a profile $u \in R^N$ such that for every coalition S, $w(S) = \sum_{i \in S} u_i$. A game v is the *minimum of a finite collection of games* (w^r) if for every coalition S, $v(S) = min_r w^r(S)$.

Kalai and Zemel (1982b) show that a game has nonempty subcores if and only if it is the minimum of a finite collection of additive games. Moreover, a game is such a minimum if and only if it is a flow game (as defined in 3.4.2).

In summary, a game v in this important class of TU games can be characterized by any of the following five equivalent statements: (1) v has nonempty subcores, (2) v is totally balanced, (3) v is the minimum of additive games, (4) v is a market game, (5) v is a flow game.

Scarf (1967), Billera and Bixby (1973), and the follow-up literature extend some of the results above to general (NTU) games.

4.2. **The Shapley TU Value.**

Definition 9. *The* Shapley *(1953)* value *of a TU game v is the payoff allocation $\varphi(v)$ defined by*

$$\varphi_i(v) = \sum_{S:i \in S} \frac{(|S|-1)!(|N|-|S|)!}{N!} [v(S) - v(S \backslash i)].$$

This expression describes the expected marginal contribution of player i to a random coalition. To elaborate, imagine the players arriving at the game in a random order. When player i arrives and joins the coalition of earlier arrivers S, he is paid his *marginal contribution* to that coalition, i.e., $v(S \cup i) - v(S)$. His Shapley value $\varphi_i(v)$ is the expected value of this marginal contribution when all orders of arrivals are equally likely.

Owen (1972) describes a parallel continuous-time process in which each player arrives at the game gradually. Owen extends the payoff function v to coalitions with "fractionally present" players, and considers the instantaneous marginal contributions of each player i to such fractional coalitions. The Shapley value of player i is the integral of his instantaneous marginal contributions, when all the players arrive simultaneously at a constant rate over the same fixed time interval.

This continuous-time arrival model, when generalized to coalitional games with infinitely many players, leads to the definition of *Aumann-Shapley prices*. These are useful for the allocation of production costs to different goods produced in a nonseparable joint production process (see Tauman [1988] and Young [1994]).

A substantial literature is devoted to extensions and variations of the axioms that Shapley (1953) used to justify his value. These include extensions to infinitely many players and to general (NTU) games (discussed briefly below), and to nonsymmetric values (see Weber [1988], Kalai and Samet [1987], and Levy and McLean [1991], among others).

Is the Shapley value in the core of the game? Not always. But as Shapley (1971) shows, if the game is *convex*, meaning that $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for every pair of coalitions S and T, then the Shapley value and all the $n!$ profiles of marginal contributions (obtained under different orders of arrival) are in the core. Moreover, Ichiishi (1981) shows that the converse is also true.

We will turn to notions of value for NTU games after we describe solutions to the special case of two-person NTU games, i.e., the Nash bargaining problem.

4.3. **Solutions to Nash bargaining games.** Nash (1950) pioneered the study of NTU games when he proposed a model of a two-person bargaining game and, using a small number of appealing principles, axiomatized the solution below.

Fix a two-person game V and for every imputation u define the *payoff gain* of player i by $gain_i(u) = u_i - v(i)$, with $v(i)$ being the highest payoff that player i can obtain on his own, i.e., in his $V(i)$.

Definition 10. *The* Nash bargaining solution *is the unique imputation u that maximizes the product of the gains of the two players, $gain_1(u) \cdot gain_2(u)$.*

Twenty five years later, Kalai and Smorodinsky (1975) and others showed that other appealing axioms lead to alternative solutions, like the two defined below.

The *ideal gain* of player i is $I_i = max_u gain_i(u)$, the maximum taken over all imputations u.

Definition 11. *The* Kalai-Smorodinsky *solution is the unique imputation u with payoff gains proportional to the players' ideal gains, $gain_1(u)/gain_2(u) = I_1/I_2$.*

Definition 12. *The* egalitarian *solution of Kalai (1977) is the unique imputation u that equalizes the gains of the players, $gain_1(u) = gain_2(u)$.*

For additional solutions, including these of Raiffa (1953) and Perles and Maschler (1981), see the comprehensive surveys of Lensberg and Thomson (1989) and Thomson (1994).

4.4. **Values of NTU games.** Three different extensions of the Shapley TU value have been proposed for NTU games: the *Shapley value* (extension), proposed by Shapley (1969) and axiomatized by Aumann (1985); the *Harsanyi value*, proposed by Harsanyi (1963) and axiomatized by Hart (1985); and the *egalitarian value*, proposed and axiomatized by Kalai and Samet (1985).

All three proposed extensions coincide with the original Shapley value on the class of TU games. For the class of NTU bargaining games, however, the (extended) Shapley value and the Harsanyi value coincide with the Nash bargaining solution, while the egalitarian value coincides with the egalitarian bargaining solution.

For additional material (beyond the brief discussion below) on these and related solutions, see McLean (2002).

4.5. **Axiomatic characterizations.** The imposition of general principles, or axioms, often leads to a unique determination of a solution. This approach is repeatedly used in game theory, as illustrated by the short summary below.

4.5.1. *Nash's axioms.* Nash (1950) characterizes his bargaining solution by the following axioms: individual rationality, symmetry, Pareto optimality, invariance to utility scale, and independence of irrelevant alternatives (IIA).

Invariance to utility scale means that changing the scale of the utility of a player does not change the solution. But this axiom goes further by disallowing all methods that use information extraneous to the game, even if such methods are invariant to scale.

Nash's *IIA* axiom requires that a solution that remains feasible when other payoff profiles are removed from the feasible set should not be altered.

4.5.2. *Shapley's axioms.* Shapley (1953) characterizes his TU value by the following axioms: symmetry, Pareto optimality, additivity, and dummy player.

A value is *additive* if in a game that is the sum of two games, the value of each player equals the sum of his values in the two component games.

A *dummy player*, i.e., one who contributes nothing to any coalition, should be allocated no payoff.

4.5.3. *Monotonicity axioms.* Monotonicity axioms describe notions of fairness and induce incentives to cooperate. The following are a few examples.

Kalai and Smorodinsky (1975) characterize their bargaining solution using *individual monotonicity*: a player's payoff should not be reduced if the set of imputations is expanded to improve his possible payoffs.

Kalai (1977) and Kalai and Samet (1985) characterize their egalitarian solutions using *coalitional monotonicity*: expanding the feasible set of one coalition should not reduce the payoffs of any of its members.

Thomson (1983) uses *population monotonicity* to characterize the n-person Kalai-Smorodinsky solution: in dividing fixed resources among n players, no player should benefit if more players are added to share the same resources.

Perles and Maschler (1981) characterize their bargaining solution using *superadditivity* (used also in Myerson [1981]): if a bargaining problem is to be randomly drawn, all the players benefit by reaching agreement prior to knowing the realized game.

Young (1985) shows that Shapley's TU additivity axiom can be replaced by *strong monotonicity*: a player's payoff can only depend on his marginal contributions to his coalitions, and it has to be monotonically nondecreasing in these.

4.5.4. *Axiomatizations of NTU values.* The NTU Shapley value is axiomatized in Aumann (1985) by adapting Shapley's TU axioms to the NTU setting, and combining them with Nash's IIA axiom. Different adaptations lead to an axiomatization of the Harsanyi (1963) value, as illustrated in Hart (1985). Kalai and Samet

(1985) use coalitional monotonicity and a weak version of additivity to axiomatize the NTU egalitarian value.

For more information on axiomatizations of NTU values, see McLean (2002).

4.5.5. *Consistency axioms.* Consistency axioms relate the solution of a game to the solutions of "subgames" obtained when some of the players leave the game with their share of the payoff. Authors who employ consistency axioms include: Davis and Maschler (1965) for the bargaining set, Peleg (1985, 1986, and 1992) for the core, Lensberg (1988) for the Nash n-person bargaining solution, Kalai and Samet (1987) and Levy and McLean (1991) for TU- and NTU-weighted Shapley values, Hart and Mas-Colell (1989) for the TU Shapley value, and Bhaskar and Kar (2004) for cost allocation in spanning trees.

5. Bridging strategic and coalitional models

Several theoretical bridges connect strategic and coalitional models. Aumann (1961) offers two methods for reducing strategic games to coalitional games. Such reductions allow one to study specific strategic games, such as repeated games, from the perspectives of various coalitional solutions, such as the core.

One substantial area of research is the Nash program, designed to offer strategic foundations for various coalitional solution concepts. In Nash (1953), he began by constructing a strategic bargaining procedure, and showing that the strategic solution coincides with the coalitional Nash bargaining solution. We refer the reader to the entry on the Nash Program in this dictionary (Serrano [2008]) for a survey of the extensive literature that followed.

Network games and coalition formation are the subjects of a growing literature. Amending a TU game with a communication graph, Myerson (1977) develops a appropriate extension of the Shapley Value. Using this extended value, Aumann and Myerson (1988) construct a dynamic strategic game of links formation that gives rise to stable communication graphs. For a survey of the large follow-up literature in this domain, see the entry Network Formation in this dictionary (Jackson [2008]).

Networks also offer a tool for the study of market structures. For example, Kalai, Postlewaite, and Roberts (1979) compare a market game with no restrictions to a star-shaped market, where all trade must flow through one middleman. Somewhat surprisingly, their comparisons of the cores of the corresponding games reveal the existence of economies in which becoming a middleman can only hurt a player.

Recent studies of strategic models of auctions point to interesting connections with the coalitional model. For example, empirical observations suggest that the better performing auctions are the ones with outcomes in the core of the corresponding coalitional game. For related references, see Bikhchandani and Ostroy (2006), De Vries, Schummer, and Vohra (2007), and Day and Milgrom (2007).

6. Large cooperative games

When the number of players is large, the exponential number of possible coalitions makes the coalitional analysis difficult. On the other hand, in games with many players each individual has less influence and the laws of large numbers reduce uncertainties.

Unfortunately, the substantial fascinating literature on games with many players is too large to survey here, so the reader is referred to Aumann and Shapley (1974) and Neyman (2002) for the theory of the Shapley value of large games, and to Shapley and Shubik (1969a), Wooders and Zame (1984), Anderson (1992), Kannai (1992), and the entry Core Convergence (Anderson [2008]) in this dictionary, for the theory of cores of large games .

A surprising discovery drawn from the above literature is a phenomenon unique to large market games that has become known as the equivalence theorem: when applied to large market games, the predictions of almost all (with the notable exception of the von Neumann Morgenstern stable sets) major solution concepts (in both coalitional and strategic game theory) coincide. Moreover, they all prescribe the economic price equilibrium as the solution for the game. This theorem presents the culmination of many papers, including Debreu and Scarf (1963), Aumann (1964), Shapley (1964), Shapley and Shubik (1969a) and Aumann (1975).

7. Directions for future work

Consider, for example, the task of constructing of a profit-sharing formula for a large consulting firm that has many partners with different expertise, located in offices around the world. While a coalitional approach should be suitable for the task, several current shortcomings limit its applicability. These include:

1. Incomplete information. Partners may have incomplete differential information about the feasible payoffs of different coalitions. While coalitional game theory has some literature on this subject (see Harsanyi and Selten [1972], Myerson [1984], and the follow-up literature), it is not nearly as developed as its strategic counterpart.

2. Dynamics. Although the feasible payoffs of coalitions vary with time, coalitional game theory is almost entirely static.

3. Computation. Even with a moderate number of players, the information needed for describing a game is very demanding. The literature on the complexity of computing solutions (as in Deng and Papadimitriou [1994] and Nisan et al. [2007]) is growing. But overall, coalitional game theory is still far from offering readily computable solution concepts for complex problems like the profit-sharing formula in the situation described above.

Further research on the topics above would be an invaluable contribution to coalitional game theory.

8. References

The list below includes more than the relatively small number of papers discussed in this entry, but due to space limitations, many important contributions do not appear here.

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